Linear Algebra and its Applications 496 (2016) 182–192

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Optimal inverses and abstract splines

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A R T I C L E I N F O

Article history:
Received 15 April 2015
Accepted 30 December 2015
Available online 8 February 2016
Submitted by P. Semrl

MSC:
41A65
46C05
47A05

Keywords:
Optimal inverses
Abstract spline problems
Compatibility

A B S T R A C T

We extend the notion of optimal inverse introduced by S.K. Mitra for matrices, to operators in Hilbert spaces. We obtain necessary and sufficient conditions for the existence of these inverses for a closed range operator and apply these results to characterize the solutions of abstract smoothing spline problems.

1. Introduction and notations

The main goal of this paper is to study the optimal inverses of a certain operator and the solutions of a given abstract smoothing spline problem by means of two algebraic-geometric tools, the compatibility property between positive operators and

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[1] Partially supported by UBACYT 20020100100250 (2011–2014), PIP 0757 (CONICET).

http://dx.doi.org/10.1016/j.laa.2015.12.028
0024-3795/

closed subspaces and the range additivity between two operators. Let us fix some notations. Throughout, $\mathcal{H}$, $\mathcal{K}$ are Hilbert spaces and $L(\mathcal{H},\mathcal{K})$ the space of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$. In particular, the algebra $L(\mathcal{H},\mathcal{H})$ is denoted by $L(\mathcal{H})$. A positive operator $A \in L(\mathcal{H})$ is a bounded linear operator such that $\langle Ah, h\rangle \geq 0$ for every $h \in \mathcal{H}$. We denote by $L(\mathcal{H})^+$ the cone of positive operators of $L(\mathcal{H})$. Given $C \in L(\mathcal{H},\mathcal{K})$, $R(C)$ denotes the range of C and $N(C)$ its nullspace. If $C \in L(\mathcal{H},\mathcal{K})$ has closed range, there exists a unique operator $C^\dagger \in L(\mathcal{K},\mathcal{H})$ such that $CC^\dagger C = C$, $C^\dagger CC^\dagger = C^\dagger$ and $CC^\dagger$, $C^\dagger C$ are selfadjoint; $C^\dagger$ is called the Moore–Penrose inverse of C.

Optimal inverses: Given two Hilbert spaces $\mathcal{H}$, $\mathcal{K}$ and operators $B \in L(\mathcal{H},\mathcal{K})$ and $A \in L(\mathcal{K} \oplus \mathcal{H})$ such that B has closed range and A is positive, an A-optimal inverse of B is an operator $G \in L(\mathcal{K},\mathcal{H})$ such that

$$\left\| \begin{pmatrix} BGk - k \\ Gk \end{pmatrix} \right\|_A = \min_{h\in\mathcal{H}} \left\| \begin{pmatrix} Bh - k \\ h \end{pmatrix} \right\|_A \tag{1.1}$$

for every $k \in \mathcal{K}$. Here $\|.\|_A$ denotes the seminorm defined by A: $\left\| \begin{pmatrix} k \\ h \end{pmatrix} \right\|_A = \left\| A^{1/2} \begin{pmatrix} k \\ h \end{pmatrix} \right\|$. The problem consists in providing conditions for the existence of optimal inverses and, in such cases, finding all such G's.

It was S.K. Mitra [15] who defined the optimal inverses for matrices. His goal was the search of the best approximate solutions of inconsistent linear systems under seminorms defined by positive semidefinite matrices. This notion had attracted some interest in the statistics community (see the comments in Mitra's paper). Here, we extend Mitra's concept to Hilbert space operators in order to apply the results to abstract interpolation theory.

Abstract interpolating splines and smoothing problems: Consider a partition of $[0, 1]$, $0 \leq t_1 < t_2 < \ldots < t_n \leq 1$ and (real or complex) numbers $z_1, \ldots, z_n$. An *interpolation problem* which appears frequently is to find $\sigma \in C^{(2)}[0, 1]$ such that $\sigma(t_k) = z_k$, $k = 1, \ldots, n$ and $\int_0^1 |\sigma^{(2)}(t)|dt$ is minimal. On the other hand, given a parameter $\rho > 0$, with the same data t_k, z_k, the *smoothing problem* consists in finding $\sigma \in C^{(2)}[0, 1]$ such that $\int_0^1 |\sigma^{(2)}(t)|dt + \rho \sum_{k=1}^n |\sigma(t_k) - z_k|^2$ is minimal (in statistics it is called a case of non-parametric regression). It turns out that the unique solution for both problems is provided by a natural cubic spline, which is a $\sigma \in C^{(2)}[0, 1]$ such that σ is a cubic polynomial on each interval $[t_k, t_{k+1}]$, $\sigma^{(2)}(t_1) = \sigma^{(2)}(t_n) = 0$ and σ is linear on $[0, t_1]$ and $[t_n, 1]$. These results, due to Schoenberg [16] and Holladay [14], started the so-called spline theory, where different families of relatively simple smooth functions were found in order to solve a variety of more complex interpolation and smoothing problems. After several attempts to unify the different approaches, Atteia [3] defined the notion of abstract splines.

Given Hilbert spaces $\mathcal{H}$, $\mathcal{E}$, $\mathcal{F}$, $f_0 \in \mathcal{F}$ and $T \in L(\mathcal{H}, \mathcal{E})$, $V \in L(\mathcal{H}, \mathcal{F})$ with closed range, an *abstract interpolating spline* (for these data) is $h_0 \in \mathcal{H}$ such that $Vh_0 = f_0$ and $\|Th_0\| \leq \|Th\|$ for every $h \in \mathcal{H}$ such that $Vh = f_0$. Denote by $spl(T, N(V), f_0)$ the set of such abstract interpolating splines, i.e.,

$$spl(T, N(V), f_0) = \{h_0 \in \mathcal{H} : Vh_0 = f_0, \|Th_0\| = \min_{Vh=f_0} \|Th\|\}. \tag{1.2}$$

Consider also a parameter $\rho > 0$ and define on $\mathcal{E} \times \mathcal{F}$ the inner product $\langle (e_1, f_1), (e_2, f_2)\rangle_\rho = \langle e_1, e_2\rangle + \rho\langle f_1, f_2\rangle$. An *abstract smoothing spline* for the data above is $h_\rho \in \mathcal{H}$ such that

$$\|(Th_\rho, Vh_\rho) - (0, f_0)\| \leq \|(Th, Vh) - (0, f_0)\|$$

for every $h \in \mathcal{H}$, where the norm is defined by the inner product $\langle\, ,\, \rangle_\rho$.

Again, as in the classical case, any smoothing spline is an interpolating spline. We refer the reader to the excellent expository papers [4,5]. Here, we slightly extend the definition of abstract splines so that we shall compare their performance with the optimal inverses of Mitra. We use the following definition, where the parameter ρ is supposed to be 1, just to simplify the exposition: with T, V as before and $f_0 \in \mathcal{F}$ an *abstract smoothing spline* is $h_0 \in \mathcal{H}$ such that

$$\|Th_0\|^2 + \|Vh_0 - f_0\|^2 \leq \|Th\|^2 + \|Vh - f_0\|^2$$

for all $h \in \mathcal{H}$.

Before stating our main results we need to introduce another notion. A closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$ and $A \in L(\mathcal{H})^+$ are said to be *compatible* if there exists an idempotent $E \in L(\mathcal{H})$ (an *oblique projection*) such that $R(E) = \mathcal{S}$ and $AE = E^*A$. This means that there exists a kind of "orthogonal projection" onto $\mathcal{S}$ if, instead of the original inner product of $\mathcal{H}$, we use the semi-inner product defined by A, i.e., $\langle h_1, h_2\rangle_A := \langle Ah_1, h_2\rangle$.

We describe now our main results. First, we prove that if $B \in L(\mathcal{H}, \mathcal{K})$ has closed range, $A \in L(\mathcal{K} \oplus \mathcal{H})^+$ has a block form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

with $A_1 \in L(\mathcal{K})^+$ and $A_2 \in L(\mathcal{H})^+$ such that $R(A_1)$ and $R(B) + N(A_1)$ are closed, then the following are equivalent:

i. B admits an A-optimal inverse;
ii. $R(B^*A_1B + A_2) = R(B^*A_1B) + R(A_2)$;
iii. A_2 and $N(A_1B)$ are compatible.

This is Theorem 2.3 below.

Second, consider a smoothing problem as before, with $T \in L(\mathcal{H}, \mathcal{E})$ and $V \in L(\mathcal{H}, \mathcal{F})$ such that $R(V)$ is closed. The following are equivalent:

1. for every $f_0 \in \mathcal{F}$, there is a solution of $\min_{h \in \mathcal{H}} (\|Th\|^2 + \|Vh - f_0\|^2)$;
2. $R(T^*T + V^*V) = R(T^*T) + R(V^*V)$;
3. T^*T and $N(V)$ are compatible;
4. there exists a *global solution* of the problem stated in item 1, i.e., there exists $G \in L(\mathcal{F}, \mathcal{H})$ such that

$$\|TGf\|^2 + \|VGf - f\|^2 = \min_{h \in \mathcal{H}} (\|Th\|^2 + \|Vh - f_0\|^2)$$

for every $f \in \mathcal{F}$.

Moreover, if $R(V) = \mathcal{H}$ and $R(T^*T) \cap R(V^*) = \{0\}$, then for every $f_0 \in \mathcal{F}$, $f_0 \neq 0$ the set of solutions of the problem in the above item 1 is

$$\{Gf_0 : G \text{ is a global solution}\}.$$

This is Theorem 4.2 together with Proposition 4.4.

2. Optimal inverses

In this section we obtain necessary and sufficient conditions for a closed range operator $B \in L(\mathcal{H}, \mathcal{K})$ to admit an A-optimal inverse, for some $A \in L(\mathcal{K} \oplus \mathcal{H})^+$.

The next result due to S.K. Mitra [15, Theorem 4.2] provides a condition for the existence of an A-optimal inverse of a closed range operator. The proof in the infinite dimensional case is similar, but we include it for the sake of completeness.

Theorem 2.1. *Consider two operators $B \in L(\mathcal{H}, \mathcal{K})$ with closed range and $A \in L(\mathcal{K} \oplus \mathcal{H})^+$. Then B admits an A-optimal inverse if and only if the equation*

$$(B^*A_{11}B + B^*A_{12} + A_{12}^*B + A_{22})X = B^*A_{11} + A_{12}^* \tag{2.1}$$

admits a solution. In this case, the set of A-optimal inverses of B is the set of solutions of (2.1).

Proof. By definition (see equation (1.1)), an operator $G \in L(\mathcal{K}, \mathcal{H})$ is an A-optimal inverse of B if and only if

$$\left\| \begin{pmatrix} BGy - y \\ Gy \end{pmatrix} \right\|_A \leq \left\| \begin{pmatrix} BGy - y \\ Gy \end{pmatrix} + \begin{pmatrix} B(z - Gy) \\ z - Gy \end{pmatrix} \right\|_A$$

for every $y \in \mathcal{K}$, $z \in \mathcal{H}$. Or equivalently,

$$\left\| \begin{pmatrix} BGy - y \\ Gy \end{pmatrix} \right\|_A \le \left\| \begin{pmatrix} BGy - y \\ Gy \end{pmatrix} - t \begin{pmatrix} Bw \\ w \end{pmatrix} \right\|_A$$

for every $y \in \mathcal{K}$, $w \in \mathcal{H}$, $t \in \mathbb{C}$. By a usual orthogonality argument, it is not difficult to see that this inequality is equivalent to

$$\langle \begin{pmatrix} BGy - y \\ Gy \end{pmatrix}, \begin{pmatrix} Bw \\ w \end{pmatrix} \rangle_A = 0$$

for every $y \in \mathcal{K}$, $w \in \mathcal{H}$, or, using the matricial form of A,

$$\langle \begin{pmatrix} A_{11}(BG - I)y + A_{12}Gy \\ A_{12}^*(BG - I)y + A_{22}Gy \end{pmatrix}, \begin{pmatrix} Bw \\ w \end{pmatrix} \rangle = 0$$

for every $y \in \mathcal{K}$, $w \in \mathcal{H}$. Then G is an A-optimal inverse of B if and only if

$$\langle B^*(A_{11}(BG - I) + A_{12}G)y, w\rangle + \langle (A_{12}^*(BG - I) + A_{22}G)y, w\rangle = 0 \tag{2.2}$$

for every $y \in \mathcal{K}$, $w \in \mathcal{H}$; this is equivalent to $B^*A_{11}(BG - I) + B^*A_{12}G + A_{12}^*(BG - I) + A_{22}G = 0$, i.e., G is a solution of the equation

$$(B^*A_{11}B + B^*A_{12} + A_{12}^*B + A_{22})X = B^*A_{11} + A_{12}^*. \qquad \square$$

Corollary 2.2. *Consider two operators $B \in L(\mathcal{H}, \mathcal{K})$ with closed range and $A \in L(\mathcal{K} \oplus \mathcal{H})^+$. Then B admits an A-optimal inverse if and only if*

$$R(B^*A_{11} + A_{12}^*) \subseteq R(B^*A_{11}B + B^*A_{12} + A_{12}^*B + A_{22}).$$

Proof. It follows from Theorem 2.1 and Douglas' theorem [11, Theorem 1]. $\square$

From now on, A is a diagonal weight on $\mathcal{K} \oplus \mathcal{H}$, i.e., $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ with $A_1 \in L(\mathcal{K})^+$, $A_2 \in L(\mathcal{H})^+$. In this case, by Theorem 2.1, G is an A-optimal inverse of B if and only if G is a solution of

$$(B^*A_1B + A_2)X = B^*A_1. \tag{2.3}$$

Several remarks are in order. Notice, first, that for given operators A, B as before, an A-optimal inverse need not exist and, if there exists one, it will not be unique, in general. In fact, any A-optimal inverse of B, for a diagonal weight A, can be written as

$$G = (B^*A_1B + A_2)^\dagger B^*A_1 + Z$$

where $Z \in L(\mathcal{K}, N(B^*A_1B + A_2))$.

Notice also that an A-optimal inverse of B is not necessarily a generalized inverse of B. For example, suppose that $A_2 \in L(\mathcal{H})^+$ is invertible. In this case, it follows from (2.3) that $G = (B^*A_1B + A_2)^{-1}B^*A_1$ is the unique A-optimal inverse of B and

$$\begin{aligned} B - BGB &= B - B(B^*A_1B + A_2)^{-1}B^*A_1B \\ &= B(I - (B^*A_1B + A_2)^{-1}B^*A_1B) \\ &= B(B^*A_1B + A_2)^{-1}(B^*A_1B + A_2 - B^*A_1B) \\ &= B(B^*A_1B + A_2)^{-1}A_2 \end{aligned}$$

so that $B - BGB = 0$ if and only if $B = 0$.

If A is a diagonal weight and G is an A-optimal inverse of B, then BG is A_1-selfadjoint. In fact, by equation (2.2), an operator G is an A-optimal inverse of B if and only if

$$\langle [B^*A_1(BG - I) + A_2G]y, w\rangle = 0$$

for all $y \in \mathcal{K}$ and $w \in \mathcal{H}$. In particular, if $w = Gy$ it follows that

$$\langle B^*A_1BGy, Gy\rangle - \langle B^*A_1y, Gy\rangle + \langle A_2Gy, Gy\rangle = 0$$

for all $y \in \mathcal{K}$, or equivalently, $\|BGy\|^2_{A_1} + \|Gy\|^2_{A_2} = \langle (BG)^*A_1y, y\rangle$, for all $y \in \mathcal{K}$. Therefore, $(BG)^*A_1 \in L(\mathcal{K})^+$, so that $A_1BG = (BG)^*A_1$. Hence, BG is A_1-selfadjoint.

The next result gives a different set of necessary and sufficient conditions for the existence of an A-optimal inverse for a closed range operator.

Theorem 2.3. *Let $B \in L(\mathcal{H}, \mathcal{K})$ be a closed range operator and suppose $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \in L(\mathcal{K} \oplus \mathcal{H})^+$, and $R(A_1)$ and $R(B) + N(A_1)$ are closed. Then the following statements are equivalent:*

1. *B admits an A-optimal inverse;*
2. *$R(B^*A_1B + A_2) = R(B^*A_1B) + R(A_2)$;*
3. *A_2 and $N(A_1B)$ are compatible.*

Proof. Since $R(A_1)$ and $R(B)$ are closed, if $R(B) + N(A_1)$ is closed, then the sum of their orthogonal complements $N(B^*) + R(A_1)$ is also closed, see [10, Theorem 2.13]. Then, by [10, Theorem 4.1] it holds that $R(B^*A_1)$ is closed. Hence, $\overline{R(B^*A_1)} = R(B^*A_1) \subseteq R(B^*A_1^{1/2}) \subseteq \overline{R(B^*A_1)}$ because $N(A_1B) = N(A_1^{1/2}B)$. Therefore $R(B^*A_1) = R(B^*A_1^{1/2})$ and it is closed so that $R(B^*A_1) = R(B^*A_1B)$.

1. $\leftrightarrow$ 2. By Corollary 2.2, B admits an A-optimal inverse if and only if $R(B^*A_1) \subseteq R(B^*A_1B + A_2)$, or $R(B^*A_1B) \subseteq R(B^*A_1B + A_2)$. Equivalently $R(B^*A_1B + A_2) = R(B^*A_1B) + R(A_2)$; for a study of pairs C, D of operators such that $R(C + D) = R(C) + R(D)$, see [1,2].

2. $\leftrightarrow$ 3. follows from [2, Theorem 4.4] since B^*A_1B has closed range and $N(A_1B) = N(B^*A_1B)$. $\square$

Remark 2.4.

i. The condition "$R(B) + N(A_1)$ closed" in the previous result is equivalent to the compatibility of the pair $(A_1, R(B))$ because $R(A_1)$ is closed, see [8, Theorem 6.2].

ii. It would be interesting to extend the theorem for a weight A not necessarily of diagonal type.

3. Smoothing problems

In this section, we study abstract smoothing problems in Hilbert spaces. The interpolating and smoothing problems were introduced by Atteia [3] and they are known as *abstract splines problems*, as mentioned in the introduction.

In what follows, we consider $T \in L(\mathcal{H}, \mathcal{E})$ and $V \in L(\mathcal{H}, \mathcal{F})$ such that $R(V)$ is closed.

In the next proposition we collect some results on abstract splines; the proofs can be found in [9, Proposition 3.1 and Theorem 3.2].

Proposition 3.1. *Let $A = T^*T$, $h_0 \in \mathcal{H}$ and $f_0 = Vh_0$. Then*

$$spl(T, N(V), f_0) = (h_0 + N(V)) \cap (A(N(V))^{\perp}.$$

Moreover, $spl(T, N(V), f_0)$ is not empty for every $f_0 \in R(V)$ if and only if A and $N(V)$ are compatible.

We state again the problem which is naturally associated to (1.2): finding the set of solutions of:

$$\min\ (\|Th\|^2 + \|Vh - f_0\|^2), \qquad \text{for } h \in \mathcal{H}, \tag{3.1}$$

where $f_0 \in \mathcal{F}$. This is known as a *smoothing problem*.

In [7, Theorem 6.4] it was proved that if V is surjective, then the compatibility of T^*T and $N(V)$ implies the existence of solutions of problem (3.1). In what follows, we prove that for any closed range operator V, the compatibility of T^*T and $N(V)$ is in fact equivalent to the existence of solutions of problem (3.1), for every $f_0 \in \mathcal{F}$.

In order to do so, define $K : \mathcal{H} \to \mathcal{E} \times \mathcal{F}$, $Kh = (Th, Vh)$, for $h \in \mathcal{H}$, and consider the following inner product on $\mathcal{E} \times \mathcal{F}$

$$\langle (e, f), (e', f') \rangle = \langle e, e' \rangle + \langle f, f' \rangle, \qquad \text{for } e, e' \in \mathcal{H},\ f, f' \in \mathcal{F},$$

and the associated norm $\|(e, f)\|^2 = \langle e, e \rangle + \langle f, f \rangle$, for $(e, f) \in \mathcal{E} \times \mathcal{F}$.

Then problem (3.1) can be restated as the following least squares problem: find the set of solutions of

$$\min_{h \in \mathcal{H}} \|Kh - (0, f_0)\|^2. \tag{3.2}$$

Theorem 3.2. *The following statements are equivalent:*

1. *Problem (3.1) admits a solution for all $f_0 \in \mathcal{F}$;*
2. *$R(T^*T + V^*V) = R(T^*T) + R(V^*V)$;*
3. *T^*T and $N(V)$ are compatible.*

Proof. 1. $\leftrightarrow$ 2.: Consider the operator K defined before. It is straightforward to check that the adjoint of K, $K^*: \mathcal{E} \times \mathcal{F} \to \mathcal{H}$, is given by $K^*(e, f) = T^*e + V^*f$, for $e \in \mathcal{E}$ and $f \in \mathcal{F}$.

Given $f_0 \in \mathcal{F}$, problem (3.2) (or equivalently, problem (3.1)) admits a solution if and only if the associated *normal equation* $K^*Kh = K^*(0, f_0)$ has a solution: observe that $\|Kh_0 - (0, f_0)\| \leq \|Kh - (0, f_0)\|$ for every $h \in \mathcal{H}$ if and only if $Kh_0 - (0, f_0) \in R(K)^\perp = N(K^*)$ (see [12,13]). But the last condition is equivalent to $K^*(Kh_0 - (0, f_0)) = 0$. Then, problem (3.2) has a solution if and only if for every $f_0 \in \mathcal{F}$ there exists $h \in \mathcal{H}$ such that $K^*(0, f_0) = K^*Kh$ or, equivalently, $V^*f_0 = T^*Th + V^*Vh$. But this means that $R(V^*) \subseteq R(T^*T + V^*V)$. Since $R(V)$ is closed it holds $R(V^*) = R(V^*V)$, so that (3.2) has a solution if and only if $R(V^*V) \subseteq R(T^*T + V^*V)$. Finally, this is equivalent to $R(T^*T + V^*V) = R(V^*V) + R(T^*T)$.

2. $\leftrightarrow$ 3.: It follows from [2, Theorem 4.4] noticing that $N(V^*V) = N(V)$. $\square$

From the proof of the above proposition it follows that problem (3.1) admits a solution for all $f_0 \in \mathcal{F}$ if and only if it admits a solution for all $f_0 \in R(V)$.

Corollary 3.3. *If $R(T^*T) \cap R(V^*V) = \{0\}$, then problem (3.1) admits a solution for all $f_0 \in \mathcal{F}$ if and only if $N(T) + N(V) = \mathcal{H}$.*

Proof. By Theorem 3.2, problem (3.1) admits a solution for every $f_0 \in \mathcal{F}$ if and only if $R(T^*T + V^*V) = R(T^*T) + R(V^*V)$, or equivalently, $\mathcal{H} = N(T^*T) + N(V^*V) = N(T) + N(V)$. $\square$

Corollary 3.4. *If $T \in L(\mathcal{H}, \mathcal{E})$ is a closed range operator, then problem (3.1) admits a solution for all $f_0 \in \mathcal{F}$ if and only if $N(T) + N(V)$ is closed.*

Proof. By Theorem 3.2, problem (3.1) admits a solution for all $f_0 \in \mathcal{F}$ if and only if T^*T and $N(V)$ are compatible, but this is equivalent to the condition $N(T) + N(V)$ closed, because T has closed range, see [8, Theorem 6.2]. $\square$

4. Global solutions of the smoothing problem

In what follows, we use the results on optimal inverses to find, for every $f \in \mathcal{F}$, a solution of the smoothing problem (3.1) which depends continuously on f.

Definition 4.1. An operator $G \in L(\mathcal{F}, \mathcal{H})$ is a global solution of problem (3.1) if

$$\|TGf\|^2 + \|VGf - f\|^2 = \min_{h \in \mathcal{H}}(\|Th\|^2 + \|Vh - f\|^2), \text{ for every } f \in \mathcal{F}.$$

Observe that if G is a global solution of problem (3.1), then Gf is a solution of the smoothing problem (3.1) for every $f \in \mathcal{F}$. The next theorem shows that the existence of a solution of problem (3.1) for every $f_0 \in \mathcal{F}$ is actually equivalent to the existence of a global solution. Then we can add another equivalent condition to those of Theorem 3.2.

Theorem 4.2. *The following statements are equivalent:*

1. *Problem* (3.1) *admits a solution for all* $f_0 \in \mathcal{F}$*;*
2. $R(T^*T + V^*V) = R(T^*T) + R(V^*V)$*;*
3. T^*T *and* $N(V)$ *are compatible;*
4. *problem* (3.1) *admits a global solution.*

Proof. 2. $\leftrightarrow$ 4.: By Theorem 2.3, $R(T^*T + V^*V) = R(T^*T) + R(V^*V)$ if and only if V admits an A-optimal inverse where $A_1 = I \in L(\mathcal{F})$, $A_2 = T^*T \in L(\mathcal{H})$, $A = A_1 \oplus A_2$ and $B = V$. Equivalently, problem (3.1) admits a global solution. In fact, $G \in L(\mathcal{F}, \mathcal{H})$ is an A-optimal inverse of V if for every $f \in \mathcal{F}$, it holds that

$$\|VGf - f\|_{A_1}^2 + \|Gf\|_{A_2}^2 \leq \|Vh - f\|_{A_1}^2 + \|h\|_{A_2}^2, \text{ for all } h \in \mathcal{H},$$

equivalently, for every $f \in \mathcal{F}$,

$$\|VGf - f\|^2 + \|TGf\|^2 \leq \|Vh - f\|^2 + \|Th\|^2, \text{ for all } h \in \mathcal{H},$$

i.e., G is a global solution of the smoothing problem (3.1). □

Remarks 4.3.

1. It follows from the above proof and (2.3) that the set of global solutions of problem (3.1) is the set of solutions of

$$(T^*T + V^*V)X = V^*.$$

   Moreover, it is not difficult to see that problem (3.1) admits a unique global solution if and only if $N(T) \cap N(V) = \{0\}$.

2. If T^*T and $N(V)$ are compatible and V is surjective, then for each $f_0 \in \mathcal{F}$ the solutions of problem (3.1) can be described as an abstract spline, see [7, Theorem 6.4]. In fact, it holds that for every $f_0 \in \mathcal{F}$ the set of solutions of problem (3.1) is

$$\operatorname{spl}(T, N(V), \tilde{f}_0)$$

where $\tilde{f}_0 = (P_{R(V)} + V^{\dagger^*} T^*T(I-E)V^\dagger)^\dagger f_0$, with E a projection such that $R(E) = N(V)$ and $T^*TE = E^*T^*T$.
In particular, if $R(T^*T) \cap R(V^*) = \{0\}$, then the solution of problem (3.1) is

$$\operatorname{spl}(T, N(V), f_0)$$

for every $f_0 \in \mathcal{F}$. In fact, if $h \in \mathcal{H}$ is a solution of problem (3.1), then, as observed in the proof of Theorem 3.2, it holds $V^*f_0 = T^*Th + V^*Vh$, so that $V^*f_0 - V^*Vh = T^*Th = 0$ because $R(T^*T) \cap R(V^*V) = \{0\}$ and $R(V^*V) = R(V^*)$. Therefore $V^*f_0 = V^*Vh$ and

$$f_0 = V^{*\dagger}V^*f_0 = V^{*\dagger}V^*Vh = Vh$$

because $VV^\dagger = I$. On the other hand, h is such that $Vh = \tilde{f}_0$. Hence $\tilde{f}_0 = f_0$.

Proposition 4.4. *Suppose T^*T and $N(V)$ are compatible, $R(T^*T) \cap R(V^*) = \{0\}$ and V is surjective. Then, for each $f_0 \in \mathcal{F}$, $f_0 \neq 0$, the set of solutions of (3.1) is*

$$\{Gf_0,\ G \text{ a global solution of } (3.1)\}.$$

Proof. By the remark above, the set of solutions of (3.1) is $\operatorname{spl}(T, N(V), f_0)$. Observe that $V^\dagger f_0 \neq 0$, because $N(V^\dagger) = N(V^*) = \{0\}$ and $f_0 \neq 0$. Let $A = T^*T$. Then, by [6, Proposition 4.24], it holds that

$$\operatorname{spl}(T, N(V), f_0) = \{(I - Z)V^\dagger f_0, Z \in \Pi(A, N(V))\}$$

where $\Pi(A, N(V)) = \{Z \in L(\mathcal{H}) : R(Z) \subseteq N(V), AZ = Z^*A, AZP_{N(V)} = AP_{N(V)}\}$. Therefore, every solution of (3.1) can be written as $(I-Z)V^\dagger f_0$ for some $Z \in \Pi(A, N(V))$. Finally, observe that $G = (I - Z)V^\dagger$ is a global solution of (3.1). The other inclusion follows from the definition of global solution. □

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