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A further study about the behaviour of foundation piles and beams in a Winkler–Pasternak soil

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Abstract

The natural vibrations and critical loads of foundation beams embedded in a soil simulated with two elastic parameters through the Winkler–Pasternak (WP) model are analysed. General end supports of the beam are considered by introducing elastic constraints to transversal displacements and rotations. The solution is tackled by means of a direct variational methodology previously developed by the authors who named it as whole element method. The solution is stated by means of extended trigonometric series. This method gives rise to theoretically exact natural frequencies and critical loads. A particular behaviour arises from the analysis of the lateral soil influence. It is found that the boundary conditions of the beam are influenced by the soil at the left and right sides of the beam. The possible alternatives are that the soil be cut or dragged by the non-fixed ends of the beam. In the standard WP model, the lateral soil influence is not considered. Natural frequencies and critical load numerical values are reported for beams and piles elastically supported and for various soil parameters. The results are found with arbitrary precision depending on the number of terms taken in the series. Some unexpected modes and eigenvalues are found when the different alternatives are studied. It should be noted that this special behaviour is present only when the Pasternak contribution is taken into account. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The behaviour of beams embedded in an elastic foundation is an engineering subject of practical and theoretical interest. The Winkler model [1] is frequently adopted for the soil. Its simplicity

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allows closed-form solutions to be found for various problems. However, a more realistic hypothesis is considered in the elastic soil model with two parameters developed by Pasternak [2] though first introduced by Kerr [3]. The dynamics and stability of beams in Winkler elastic soil have been extensively analysed [4–9]. The case of beams on a two-parameter soil has received less attention, probably due to the model complexity and due to the difficulties in the estimation of the parameter values. Some of the studies have been published in Ref. [10–14].

The main idea of this study is to analyse the free vibrations or the buckling of a beam embedded in a soil modelled with two elastic parameters from an unusual viewpoint. A particular behaviour arises from the analysis of the lateral soil influence. It is found that the boundary conditions of the beam are influenced by the soil at the left and right sides of the beam. The possible alternatives are that the soil be cut or dragged by the unfixed ends of the beam. In the standard Winkler–Pasternak (WP) model, the lateral soil influence is not considered. The problem is considered—in a first stage—studying a stepped, infinite length beam. Then the analysis is concentrated to the central span and it is possible to find a consistent behaviour of the soil adjacent to the beam, not found in the bibliography to the authors' knowledge. Critical loads and natural frequency results are presented and discussed for various possible cases of Bernoulli–Euler beams elastically restrained to translation and rotation at its ends. Due to its algebraic simplicity the exact results are obtained by means of a methodology developed previously by the authors and named whole element method (WEM) [15–19]. The conclusions will be valid, of course, for other structural elements in the same situation.

2. Statement of the problem

Fig. 1 shows the beam embedded in a WP soil. It is modelled at a first stage as an infinite length, stepped beam. The elastic characteristics of the soil are given by w^* (Winkler) and p^* (Pasternak) parameters. As is commonly done the soil reaction is assumed purely elastic and not massive. Let us non-dimensionalise with respect to L and refer all the geometric and physical data to the central span, i.e. setting $x \equiv X/L$ and $(\cdot)' \equiv \mathrm{d}(\cdot)/\mathrm{d}x$, etc. The transversal displacement functions for each

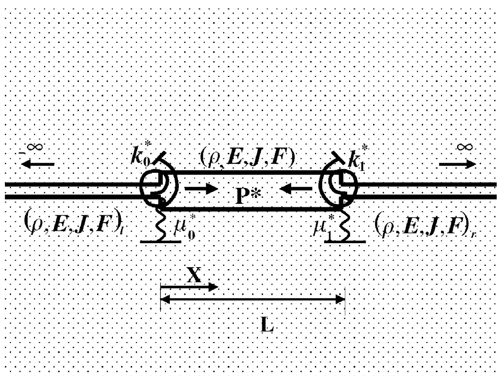


Fig. 1. Stepped beam embedded in a Winkler–Pasternak soil.

beam span are written as

$$\hat{v}_l(X) = \hat{v}_l(Lx) = v_l(x) \quad (-\infty < x \leqslant 0), \tag{1a}$$

$$\hat{v}(X) = \hat{v}(Lx) = v(x) \quad (0 < x \leqslant 1), \tag{1b}$$

$$\hat{v}_r(X) = \hat{v}_r(Lx) = v_r(x) \quad (1 < x \leqslant \infty). \tag{1c}$$

Whatever the sign of the displacements are, the medium always reacts similarly. Subscripts l and r stand for left and right spans, respectively. Also normal vibrations of circular frequency ω are accepted. The maximum strain and kinetic energies are noted as $U_T = U_S + U_B + U_E$ and $K_T = K_B$, respectively, where U_S, U_B and U_E are, respectively, the strain energies of the soil, of the beam and of the end springs at $x = 0$ and $x = 1$. K_B is the kinetic energy of the beam.

$$\begin{aligned} 2U_S = {} & w\left[\int_{-\infty}^{0} v_l^2(x)\,\mathrm{d}x + \int_0^1 v^2(x)\,\mathrm{d}x + \int_1^{\infty} v_r^2(x)\,\mathrm{d}x\right] \\ & + p\left[\int_{-\infty}^{0} v_l'^2(x)\,\mathrm{d}x + \int_1^{\infty} v_r'^2(x)\,\mathrm{d}x\right] + (p-P)\int_0^L v'^2(x)\,\mathrm{d}x, \end{aligned} \tag{2}$$

$$2U_B = j_l\int_{-\infty}^{0} v_l''^2(x)\,\mathrm{d}x + \int_0^1 v''^2(x)\,\mathrm{d}x + j_R\int_1^{\infty} v_r''^2(x)\,\mathrm{d}x, \tag{3}$$

$$2U_E = \mu_0 v^2(0) + k_0 v'^2(0) + \mu_1 v^2(1) + k_1 v'^2(1), \tag{4}$$

$$\frac{2K_B}{\Omega^2} = f_l\int_{-\infty}^{0} v_l^2(x)\,\mathrm{d}x + \int_0^1 v^2(x)\,\mathrm{d}x + f_r\int_1^{\infty} v_r^2(x)\,\mathrm{d}x, \tag{5}$$

The following non-dimensional parameters were introduced:

$$w \equiv \frac{w^* L^4}{EJ}, \quad p \equiv \frac{p^* L^2}{EJ}, \quad P \equiv \frac{P^* L^2}{EJ}, \quad \Omega^2 \equiv \frac{\rho F}{EJ}\omega^2 L^4,$$

$$j_l \equiv \frac{E_l J_l}{EJ}, \quad j_r \equiv \frac{E_r J_r}{EJ}, \quad f_l \equiv \frac{\rho_l F_l}{\rho F}, \quad f_r \equiv \frac{\rho_r F_r}{\rho F},$$

$$\mu_0 \equiv \frac{\mu_0^* L^3}{EJ}, \quad \mu_1 \equiv \frac{\mu_1^* L^3}{EJ}, \quad k_0 \equiv \frac{k_0^* L}{EJ}, \quad k_1 \equiv \frac{k_1^* L}{EJ},$$

where ρ is the density, E is the Young's modulus, J is the moment of inertia of the cross-section and F is the cross-sectional area. The parameters written as $(\,\cdot\,), (\,\cdot\,)_l, (\,\cdot\,)_r$ correspond to the central span, left span and right span, respectively. Other parameters may be found in Fig. 1. As may be observed the WP soil is assumed as constituted by elastic springs that react to displacements and rotations (see Ref. [11]).

The application of the minimum of the total energy principle leads to the general equations and their boundary conditions, i.e.

$$\delta[U_T - K_T] = 0, \tag{6}$$

where δ denotes the first variations of the functionals. From the independent variations $\delta v_l(x)$, $\delta v_r(x)$ and $\delta v(x)$ the following equations result:

$$j_l v_l'''' - p v_l'' + (w - \Omega^2 f_l) v_l = 0, \tag{7a}$$

$$j_r v_r'''' - p v_r'' + (w - \Omega^2 f_r) v_r = 0, \tag{7b}$$

$$v'''' - \bar{p} v'' + \bar{w} v = 0 \tag{7c}$$

with

$$\bar{p} \equiv p - P, \quad \bar{w} \equiv w - \Omega^2 \tag{8}$$

and the boundary conditions at this stage are

$$\begin{aligned} &p[|v_l' \,\delta v_l|_{-\infty}^{0} + |v_r' \delta v_r|_1^{\infty}] + \bar{p}|v' \,\delta v|_0^1 - j_l |v_l''' \delta v_l|_{-\infty}^{0} \\ &-|v''' \,\delta v|_0^1 - j_r |v_r''' \,\delta v_r|_1^{\infty} + \mu_0 v(0)\,\delta v(0) + \mu_1 v(1)\,\delta v(1) = 0 \end{aligned} \tag{9}$$

$$j_l |v_l'' \delta v_l'|_{-\infty}^{0} + |v'' \,\delta v|_0^1 - j_r |v_r'' \delta v_r'|_1^{\infty} + k_0 v'(0)\,\delta v'(0) + k_1 v'(1)\,\delta v'(1) = 0. \tag{10}$$

3. Assumption: $p \neq 0$

The problem represented by Eqs (7), (9) and (10) may be solved straightforwardly, even when other complexities such as Timoshenko, non-uniform beams are considered. The general idea has been described above. Now, as was anticipated, our interest is the influence and the behaviour of the medium at the left and right of the central span. This is achieved by setting

$$j_l = f_l = j_r = f_r = 0. \tag{11}$$

That is we assume infinite length lateral beams of negligible stiffness and mass. It is a valid case due to the arbitrariness of these parameters. Then the general solution of the differential equations (7a) and (7b) is simply

$$v_l(x) = A_l \sinh(cx) + B_l \cosh(cx), \tag{12a}$$

$$v_r(x) = A_r \sinh(cx) + B_r \cosh(cx). \tag{12b}$$

As $p \neq 0$ has been accepted, we may introduce

$$c^2 \equiv \frac{w}{p}. \tag{13}$$

Note that the case $p = 0$ would lead instead to $v_l(x) = v_r(x) \equiv 0$.

Elastica (12) may be understood as the displacement of the WP soil adjacent to the central beam. We also know that

$$x \to -\infty \quad v_l \to 0 \ (\delta v_l \to 0), \tag{14a}$$

$$x \to \infty \quad v_r \to 0 \ (\delta v_r \to 0) \tag{14b}$$

with which and due to solutions (12) the following statements hold:

$$A_l = B_l, \tag{15a}$$

$$A_r = -B_r. \tag{15b}$$

That is the shapes of $v_l(x)$ and $v_r(x)$ are

$$v_l(x) = A_l \exp(cx), \tag{16a}$$

$$v_r(x) = A_r \exp(-cx) \tag{16b}$$

and then

$$v_l'(x) = cv_l(x), \tag{17a}$$

$$v_r'(x) = -cv_r(x). \tag{17b}$$

Consequently, it is also verified that

$$x \to -\infty \quad v_l' \to 0 \ \ (\delta v_l' \to 0), \tag{18a}$$

$$x \to \infty \quad v_r' \to 0 \ \ (\delta v_r' \to 0). \tag{18b}$$

Now after these statements the boundary conditions (9) and (10) turn to be

$$pv_l'(0)\delta v_l(0) - [\bar{p}v'(0) - v'''(0) - \mu_0 v(0)]\delta v(0) = 0, \tag{19a}$$

$$pv_r'(1)\delta v_r(1) - [\bar{p}v'(1) - v'''(1) - \mu_1 v(1)]\delta v(1) = 0, \tag{19b}$$

$$[v''(0) - k_0 v'(0)]\delta v'(0) = 0, \tag{20a}$$

$$[v''(1) - k_1 v'(1)]\delta v'(1) = 0. \tag{20b}$$

3.1. Discussion of the problem

If the springs are assumed to be of finite stiffness (i.e. constants k_0, μ_0, k_1 and $\mu_1 < \infty$), it is consistent to accept that, *in general*, $v(0)$, $v'(0)$, $v(1)$ and $v'(1)$ are not null. Let us start discussing possible variants of the behaviour at the soil-beam interface at the ends of the beam.

3.1.1. Possibility I (PI)

We suppose that

$$(\mathrm{PI}) \quad v_l'(0) = v'(0), \quad \begin{array}{ll} (\mathrm{PI})_1 & v_l(0) = v(0), \\ (\mathrm{PI})_2 & v_l(0) \neq v(0). \end{array} \tag{21}$$

It is not included but it is easy to demonstrate that $(\mathrm{PI})_1$ should be disregarded since it conduces to the particular condition $v'(0) = cv(0)$, which may only be verified with a very special combination of the soil parameters. Also, $(\mathrm{PI})_2$ leads to a violation of the boundary conditions ($v'(0) = 0$). Then (PI) is not possible.

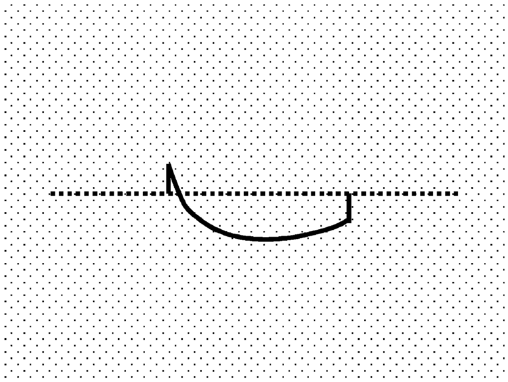


Fig. 2. Case 1. The soil is "cut" at both ends.

3.1.2. Possibility II (PII)

Then, since it is not possible for (PI) to be true, possibility (PII) should hold.

$$\text{(PII)} \quad v_l'(0) \neq v'(0) \quad \begin{array}{ll} (\text{PII})_1 & v_l(0) = v(0), \\ & \\ (\text{PII})_2 & v_l(0) \neq v(0). \end{array} \tag{22}$$

Although not shown alternatives $(\text{PII})_1$ and $(\text{PII})_2$ are valid. Condition $(\text{PII})_1$ introduces a "drag" of the soil at $x = 0$; meanwhile, condition $(\text{PII})_2$, also possible, produces a "cut" of the soil at the same place. A similar analysis may evidently be carried out at $x = 1$ with analogous conclusions. Finally from above, four independent cases may appear, as will be described in what follows.

3.2. Cases

Case 1 (C1): The soil is "cut" at $x = 0$ and 1 as depicted in Fig. 2. This is the only case that has been considered by the related works in the bibliography. It corresponds to $(\text{PII})_2$ at both ends of the beam. The conditions are

$$v_l(x) = v_l'(x) = v_r(x) = v_r'(x) = 0. \tag{23}$$

Case 2 (C2): The soil is "dragged" at $x = 0$ and 1 as shown in Fig. 3, assuming $(\text{PII})_1$ at both ends of the beam. That is

$$v_l(x) = v(0)\exp(cx), \tag{24a}$$

$$v_l'(x) = cv(0)\exp(cx), \tag{24b}$$

$$v_r(x) = v(1)\exp[c(1-x)], \tag{24c}$$

$$v_r'(x) = -cv(1)\exp[c(1-x)]. \tag{24d}$$

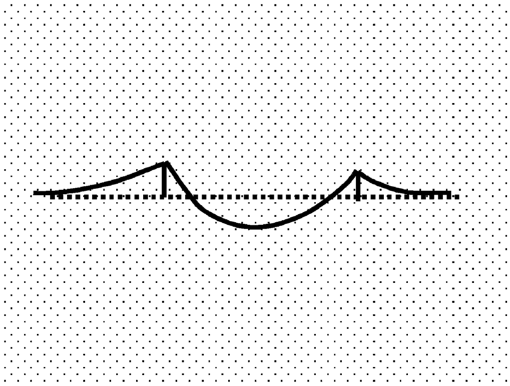


Fig. 3. Case 2. The soil is "dragged" at both ends.

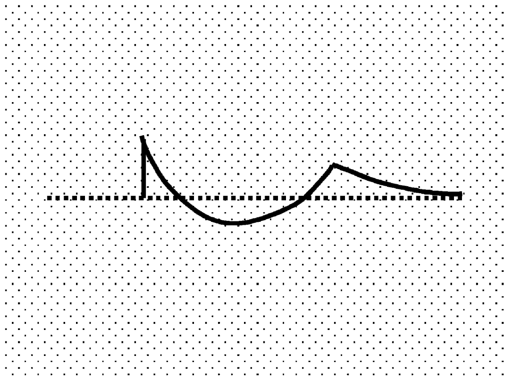


Fig. 4. Case 3. The soil is "cut" at the left and "dragged" at the right.

As was said before

$$v_l(0) = v(0), \tag{25a}$$

$$v_l'(0) = cv(0) \neq v'(0), \tag{25b}$$

$$v_r(1) = v(1), \tag{26a}$$

$$v_r'(1) = -cv(1) \neq v'(1). \tag{26b}$$

Case 3 (C3): The soil is "cut" at $x = 0$ and "dragged" at $x = 1$ as shown in Fig. 4—$(\mathrm{PII})_2$ at the left of the beam and $(\mathrm{PII})_1$ at the right of the beam. That is

$$v_l(x) = v_l'(x) = 0, \tag{27a}$$

$$v_r(x) = v(1)\exp[c(1-x)], \tag{27b}$$

$$v_r'(x) = -cv(1)\exp[c(1-x)]. \tag{27c}$$

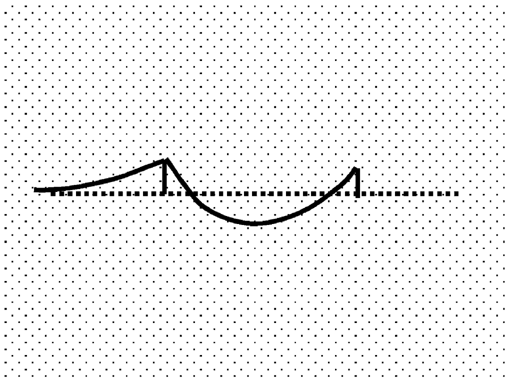


Fig. 5. Case 4. The soil is "dragged" at the left and "cut" at the right.

Then

$$v_r(1) = v(1), \tag{28a}$$

$$v_r'(1) = -cv(1) \neq v'(1). \tag{28b}$$

Case 4 (C4): The soil is "dragged" at $x = 0$ and "cut" at $x = 1$. See Fig. 5. That is

$$v_l(x) = v(0)\exp(cx), \tag{29a}$$

$$v_l'(x) = cv(0)\exp(cx), \tag{29b}$$

$$v_r(x) = v_r'(x) = 0. \tag{29c}$$

and

$$v_l(0) = v(0), \tag{30a}$$

$$v_l'(0) = cv(0) \neq v'(0). \tag{30b}$$

If two constants λ_0 and λ_1 which assume value 0 or 1, are introduced

Case	1	2	3	4
λ_0	0	1	0	1
λ_1	0	1	1	0

the four boundary conditions for each of the four cases developed above may be written generically as follows:

$$v'''(0) - \bar{p}v'(0) + (\mu_0 + \lambda_0\sqrt{wp})v(0) = 0, \tag{31a}$$

$$v''(0) - k_0 v'(0) = 0, \tag{31b}$$

$$v'''(1) - \bar{p}v'(1) - (\mu_1 + \lambda_1\sqrt{wp})v(1) = 0, \tag{31c}$$

$$v''(1) + k_1 v'(1) = 0. \tag{31d}$$

4. Solution of the problem

Obviously, the solution of the differential equation (7c) + boundary conditions may be found using classical tools in an elementary way by solving a characteristic quartic equation. Its roots will be complex, in general, depending on the discriminant sign which in turn depend on P and Ω. It is a rather easy task to state then the four boundary conditions that correspond (C1 or C2 or C3 or C4) that will allow to find the eigenvalue P or Ω. However, the authors have preferred in this paper to make use of a tool developed before and named WEM [15–19] and that leads to exact eigenvalues and mode shapes. Actually, the method yields arbitrary precision solutions [19].

We are dealing with a direct variational method (not a Ritz one) that starts from an ad hoc functional. Then the extreme condition is imposed on the functional written with certain extremizing sequences that should satisfy only the eventual essential boundary conditions of the problem. In the present study, no restriction is imposed since all the four conditions at the boundary are natural. On the other hand—and this is another reason to use WEM—the functional (the total energy of the system, in this problem) may also be written in a single general statement for all the four cases described in the previous section, as follows:

$$F[v] = \int_0^1 [v''^2(x) + \bar{p}v'^2(x) + \bar{w}v^2(x)]\,\mathrm{d}x + k_0 v'^2(0) + k_1 v'^2(1) + (\mu_0 + \lambda_0\sqrt{wp})v^2(0) + (\mu_1 + \lambda_1\sqrt{wp})v^2(1). \tag{32}$$

Let us clarify the meaning of the terms outside the integral of Eq. (32) and which do not correspond to the springs represented in expression (4). These terms arise from the introduction of solutions (16) and (17) in the improper integrals of Eq. (2), taking into account expressions (23) or (24) or (27) or (29), according to the case. There is an energy contribution of the adjacent soil (unless it is "cut" at $x = 0$ and $x = 1$, $\lambda_0 = \lambda_1 = 0$) that modifies, in appearance, the stiffness of the end springs. If $w = 0$ and/or $p = 0$ this contribution disappears, as is the commonly accepted case in the bibliography.

Also, it may be seen that from the condition $\delta F[v] = 0$ for arbitrary δv the differential equation (7c) and the boundary conditions (31) are obtained.

Now the practical application of WEM is summarised in the following steps:

- The extremizing sequences $v_M(x), v'_M(x)$ and $v''_M(x)$ are defined. They are extended trigonometric series [18]. The authors have demonstrated (the theorems may be found in Ref. [17]) that the essential functions are uniformly convergent towards the classical solution. The essential functions are those involving the function and derivatives up to order $k-1$, where $2k$ is highest derivative order in the differential equation. In this problem, the essential functions are $v(x)$ and $v'(x)$. Eventual essential boundary conditions are required to the complete sequence and not to each co-ordinate function.
- The sequences are replaced in $F[v]$ obtaining

$$F_M = F[v_M]. \tag{33}$$

- The stationary condition of F_M is imposed with respect to the unknown introduced by the extremizing sequences

$$\bar{\delta} F_M = 0; \tag{34}$$

thus, arriving at the solution of the problem.

The extremizing sequences assumed in this study are (they are not unique)

$$v''_M(x) = \sum_{i=1}^{M} A_i s_i, \tag{35a}$$

$$v'_M(x) = -\sum_{i=1}^{M} \frac{A_i c_i}{\alpha_i} + A_0, \tag{35b}$$

$$v_M(x) = -\sum_{i=1}^{M} \frac{A_i s_i}{\alpha_i^2} + A_0 x + a_0, \tag{35c}$$

where $\{A_i\}$ $(i = 1, 2, \ldots, M)$, A_0 and a_0 are the unknowns, while

$$s_k \equiv \sin \alpha_k x, \tag{36a}$$

$$c_k \equiv \cos \alpha_k x \tag{36b}$$

with $\alpha_k \equiv k\pi$ $(k = 1, 2, \ldots, M)$. This selection guarantees the convergence in L_2 (in the mean) for $v''(x)$ but, as was said, uniform convergence for $v'(x)$ and $v(x)$ as well as exactness achieved with arbitrary precision for the eigenvalues Ω (frequency) and P (critical load). As may be observed from Eq. (31), there are only natural boundary conditions. Then no restriction has to be imposed to the sequences. Now, once series (35) have been introduced, condition (34) implies the following $(M + 2)$ conditions

$$\frac{\partial F_M}{\partial A_i} = 0 \quad (i = 1, 2, \ldots, M), \tag{37a}$$

$$\frac{\partial F_M}{\partial A_0} = 0, \tag{37b}$$

$$\frac{\partial F_M}{\partial a_0} = 0. \tag{37c}$$

Eqs. (37) give place to

$$A_i D_i = 2\bar{w}\alpha_i^2 (A_0 L_i^1 + a_0 L_i^0) + 2\alpha_i^3 [k_0(A_0 - S) + (-1)^i k_1 (A_0 - Z)] \ (i = 1, 2, \ldots, M), \tag{38a}$$

$$A_0 D + a_0 C = \bar{w}\tau_1 + k_0 S + k_1 Z, \tag{38b}$$

$$A_0 C + a_0 R = \bar{w}\tau_0, \tag{38c}$$

where

$$D_i = \alpha_i^4 + \bar{p}\alpha_i^2 + \bar{w}, \quad L_k^m = \int_0^1 x^m s_k \, \mathrm{d}x \quad (i, k = 1, 2, \ldots, M), \quad (m = 0, 1),$$

$$S \equiv \sum_{p=1}^{M} \frac{A_p}{\alpha_p}, \quad Z \equiv \sum_{p=1}^{M} \frac{(-1)^p A_p}{\alpha_p}, \quad \tau_m \equiv \sum_{p=1}^{M} \frac{A_p L_p^m}{\alpha_p^2},$$

$$D=\bar{p}+\frac{\bar{w}}{3}+k_0+k_1+\mu_1+\lambda_1\sqrt{wp},\quad C=\frac{\bar{w}}{2}+\mu_1+\lambda_1\sqrt{wp},$$
$$R=\bar{w}+\mu_0+\mu_1+(\lambda_0+\lambda_1)\sqrt{wp}. \tag{39}$$

5. Numerical results

The problem of finding the frequencies Ω for a beam subjected to a load P lower than the critical one (including $P=0$), or obtaining P_{crit} when $\Omega=0$ is reduced to an algebraic manipulation. In turn the modal shapes are calculated by means of Eq. (35).

In Table 1, values of critical loads for a cantilever beam with different combinations of the two elastic parameters are shown. Since one of the ends of the beam is free, two alternatives are possible: Case 1 = Case 4 and Case 2 = Case 3. We may observe that in Case 1 ("cut" of the soil) the values of the loads are coincident with the results of Ref. [14]. This is due to the fact that both solutions are exact and "dragging" is not considered in the known bibliography. It is interesting to analyse the modal shapes. There is coincidence for different cases. Since the "dragging" case ($\lambda_1=0$) depends on the geometric mean of the parameters w and p, this drag behaviour disappears when one of the parameters is null. This is the reason why the critical loads do not vary with the Case in the first row and column of Table 1.

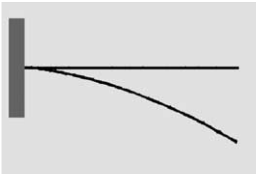
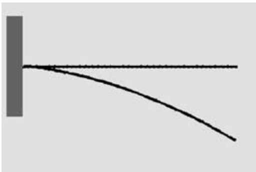
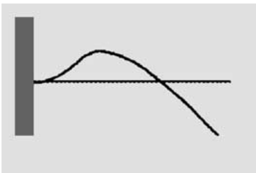
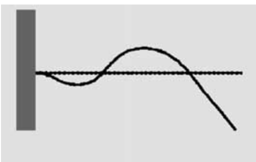
Table 2 depicts values of several natural frequencies for a free beam and various combinations of the parameters of the soil, w and p. According to the combinations of these parameters, frequencies lower than the ones found in the bibliography are obtained. This is due to the possibility of "cut" or "drag" of the soil at the lateral ends of the beam.

Also, Table 3 shows original results both for critical loads according to the Case and for fundamental frequencies of axially compressed cantilever beams. Again, the existence of "cut" or "drag" of the soil at the free end leads to these original results.

6. Comments and conclusions

One could think that the frequencies and critical loads found with the four cases described above, correspond to a succession of eigenvalues and modal shapes of the same subset; however, this is not true. In effect, the modal shapes of a case are not orthogonal to the ones corresponding to the remaining cases. The orthogonality holds among the modal shapes of a particular case. The checking of this property is easily performed. Certain special combinations of $k_0, k_1, \mu_0, \mu_1, w$ and p might conduce, in particular, to the orthogonality of mode shapes of different cases. Finally, very careful tests simulating elastically supported beam embedded in a WP medium should be carried out in order to conclude the correct model, i.e. whether the soil at the ends is "dragged" or "cut". The possibility of behaviour changing from a case to another is unexpected. Furthermore, it could be concluded that the WP model would be fictitious since—though dealing with a linear problem—the uniqueness of the solution in the classical sense, is not ensured. On the other hand, two causes will avoid the existence of Cases 2–4. One of them is that the end boundaries of the beam restrict the transverse displacements. The other cause appears when one of the elastic parameters of the

Table 1
Critical loads $P(\Omega = 0)$. Cantilever beam. $\mu_0 = k_0 = 10^6(\rightarrow \infty)$; $\mu_1 = k_1 = 0$. $M = 10,000$[a]

w	Mode shape	$p = 0$		$p = 0.5\,\pi^2$		$p = \pi^2$		$p = 2.5\,\pi^2$	
		C1 = C4	C2 = C3	C1 = C4	C2 = C3	C1 = C4	C2 = C3	C1 = C4	C2 = C3
0		2.4674	2.4674	7.4023	7.4023	12.3371	12.3371	27.1415	27.1415
1		2.6500	2.6500	7.5848	9.3513	12.5196	15.0023	27.3240	31.1966
100		11.9964	11.9964	16.9312	23.4945	21.8660	30.0584	36.6704	47.0132
10,000		100.012	100.012	104.947	124.856	109.882	136.790	124.686	163.619

[a]Note: when $p = 0$ ($\forall w$) and $w = 0$ ($\forall p$) C1 = C4 = C2 = C3.

soil (or both) is null. As may be concluded from the above expression, there is a coupling and its magnitude is given by the geometric mean of the parameters when they are not null. It is a contribution formally identical to the action of extensional springs that modify the values of μ.

The discussion of the model was done for a Bernoulli–Euler beam, but evidently the conclusions are general. Similar findings may be got for the Timoshenko beams, non-uniform beams, plane plates, elements with intermediate masses, springs or supports, etc.

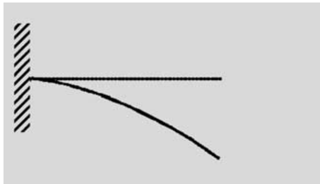
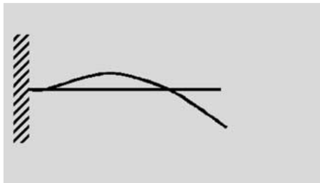
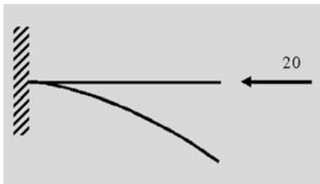
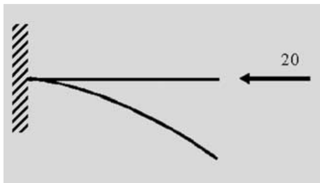
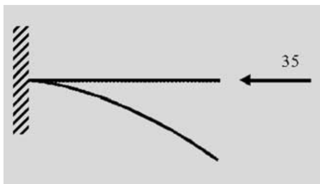
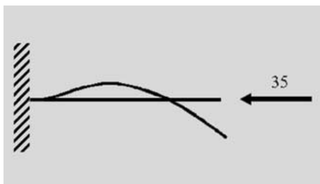
From the analysis of the results of frequencies and critical loads, one observes important variations on the values of critical loads for the same physical–geometric example depending on the possibility or not of "dragging" or "cut" of the soil. This relevant difference in the eigenvalues is also seen in Table 3, in the values of natural bending frequencies for columns subjected to an axial load.

Table 2
Natural frequency parameter Ω of a free beam ($\mu_0 = k_0 = \mu_1 = k_1 = 0$). $P = 0$, $M = 10,000$. Case 1, Case 2 and Case 3 = Case 4[a]

p	0			1			25			100		
w	C1	C2	C3 = C4	C1	C2	C3 = C4	C1	C2	C3 = C4	C1	C2	C3 = C4
0	22.3733	22.3733	22.3733	23.4507	23.4507	23.4507	41.0209	41.0209	41.0209	70.1390	70.1390	70.1390
	61.6728	61.6728	61.6728	62.5490	62.5490	62.5490	80.4621	80.4621	80.4621	118.324	118.324	118.324
	120.903	120.903	120.903	121.673	121.673	121.673	138.775	138.775	138.775	180.946	180.946	180.946
1							16.724[b]	17.473[b]				
	22.3956	22.3956	22.3956	23.4720	23.6412	23.5570	41.0331	41.4603	41.2479	70.1462	70.5531	70.3501
	61.6809	61.6809	61.6809	62.5570	62.6211	62.5891	80.4684	80.7151	80.5920	118.329	118.622	118.475
	120.907	120.907	120.907	121.677	121.710	121.694	138.779	138.926	138.853	180.949	181.163	181.056
25												8.2355[d]
											10.857[c]	
										32.727[b]	35.903[b]	34.374[b]
	22.9252	22.9252	22.9252	23.9778	24.8038	24.4000	41.3245	43.4228	42.4013	70.3170	72.3229	71.3314
	61.8752	61.8752	61.8752	62.7846	63.0960	63.0640	80.6173	81.8541	81.2416	118.430	119.893	119.166
	121.006	121.006	121.006	121.776	121.941	121.941	138.865	139.605	139.236	181.015	182.088	181.553
100								13.423[c]	11.394[c]		16.527[c]	13.208[c]
					13.039[b]	12.077[b]		24.858[b]		33.853[b]	39.582[b]	36.886[b]
	24.5064	24.5064	24.5064	25.4938	27.0453	26.3027	42.2222	46.2693	44.3446	70.8483	74.7678	72.8496
	62.4783	62.4783	62.4783	63.3434	63.9804	63.6640	81.0812	83.5528	82.3401	118.746	121.662	120.221
	121.316	121.316	121.316	122.083	122.413	122.248	139.135	140.620	139.883	181.222	183.372	183.304

[a]Note: when $p = 0$ ($\forall w$) and $w = 0$ ($\forall p$) C1 = C4 = C2 = C3.
[b]Antisymmetric mode.
[c]Symmetric mode.
[d]General mode.

Table 3
Critical loads (two first rows) and fundamental frequencies. Cantilever beam $\mu_0 = k_0 = 10^6$; $\mu_1 = k_1 = 0$. $M = 10,000$. $w = p = 25$.

Axial load P	Case	Fundamental frequency Ω	Mode shape
$P_{\mathrm{cr}_1} = 31.2941$	C4 = C1	0	
$P_{\mathrm{cr}_2} = 42.2873$	C3 = C2	0	
$P = 20 < P_{\mathrm{cr}_{1,2}}$	C4 = C1	7.6337	
	C3 = C2	11.3683	
$P = 35 > P_{\mathrm{cr}_1}$	C4 = C1	9.2243	
	C3 = C2	13.9300	

On the contrary, Table 2 shows that the difference is not so important when comparing the three cases. In this Table, it is worth noting a particular behaviour. As is known, a free beam without the influence of soil or in a soil with low values of w and p has a symmetric first mode shape. Special combinations of w and p values may conduce to an asymmetric, quasi-rigid first mode (e.g. $w=1$, $p=25$, C_1 and C_2). As the soil becomes stiffer (larger values of w and p), new and lower frequencies appear corresponding to quasi-rigid mode shapes (see caption of Table 2).

A final comment, beyond the academic character of the study and the eventual questioning of the classical WP model. Relevant changes on the behaviour of the beam with free or elastically restrained ends have been found when the drag effect was introduced. Further analysis should be carried out in order to confirm or disregard this hypothesis.

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