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Representation Equivalence and p-Spectrum of Constant Curvature Space Forms

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Abstract We study the p-spectrum of a locally symmetric space of constant curvature $\Gamma\backslash X$, in connection with the right regular representation of the full isometry group G of X on $L^2(\Gamma\backslash G)_{\tau_p}$, where τ_p is the complexified p-exterior representation of $\mathrm{O}(n)$ on $\bigwedge^p(\mathbb{R}^n)_{\mathbb{C}}$. We give an expression of the multiplicity $d_\lambda(p,\Gamma)$ of the eigenvalues of the p-Hodge–Laplace operator in terms of multiplicities $n_\Gamma(\pi)$ of specific irreducible unitary representations of G.

As a consequence, we extend results of Pesce for the spectrum on functions to the p-spectrum of the Hodge–Laplace operator on p-forms of $\Gamma\backslash X$, and we compare p-isospectrality with τ_p-equivalence for $0 \le p \le n$. For spherical space forms, we show that τ-isospectrality implies τ-equivalence for a class of τ's that includes the case $\tau = \tau_p$. Furthermore, we prove that $p-1$ and $p+1$-isospectral implies p-isospectral.

For nonpositive curvature space forms, we give examples showing that p-isospectrality is far from implying τ_p-equivalence, but a variant of Pesce's result remains true. Namely, for each fixed p, q-isospectrality for every $0 \le q \le p$ implies τ_q-equivalence for every $0 \le q \le p$. As a byproduct of the methods we obtain several results relating p-isospectrality with τ_p-equivalence.

Keywords Representation equivalent · p-Spectrum · Constant curvature · Space forms · Isospectrality

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1 Introduction

Let $X = G/K$ be a homogeneous Riemannian manifold where $G = \mathrm{Iso}(X)$ is the full isometry group and where $K \subset G$ is a compact subgroup. We shall consider discrete *cocompact* subgroups Γ of G acting on X without fixed points, so that $\Gamma\backslash X$ is a compact Riemannian manifold. Under the right regular representation R_Γ of G, $L^2(\Gamma\backslash G)$ splits as a direct sum

$$L^2(\Gamma\backslash G) = \sum_{\pi\in\widehat{G}} n_\Gamma(\pi) H_\pi$$

of closed irreducible subspaces H_π with finite multiplicity $n_\Gamma(\pi)$. Here $\widehat{G}$ denotes the unitary dual of G. Let (τ, V_τ) be a finite-dimensional complex unitary representation of K and consider the associated vector bundle

$$E_\tau := G \underset{\tau}{\times} V_\tau \longrightarrow G/K \tag{1.1}$$

endowed with a G-invariant inner product (see Sect. 2.1). Let $\Delta_{\Gamma,\tau}$ denote the Laplace operator acting on sections of the bundle $\Gamma\backslash E_\tau \to \Gamma\backslash X$ (see Sect. 2.1).

In [15], Pesce considers spectra of Laplace operators on $\Gamma\backslash X$, in connection with the right regular representations $R_{\Gamma,\tau}$ of G on the space

$$L^2(\Gamma\backslash G)_\tau := \sum_{\pi\in\widehat{G}_\tau} n_\Gamma(\pi) H_\pi, \tag{1.2}$$

where $\widehat{G}_\tau = \{\pi \in \widehat{G} : \mathrm{Hom}_K(\tau, \pi) \neq 0\}$. In the terminology in [15], two subgroups Γ_1, Γ_2 of G, are said to be τ*-representation equivalent* or simply τ*-equivalent*, if the representations $L^2(\Gamma_1\backslash G)_\tau$ and $L^2(\Gamma_2\backslash G)_\tau$ are equivalent, that is, $n_{\Gamma_1}(\pi) = n_{\Gamma_2}(\pi)$ for any $\pi \in \widehat{G}_\tau$. In the case when $\tau = \mathbf{1}$, the trivial representation of K, Pesce calls such groups K*-equivalent*. In analogy, $\Gamma_1\backslash X$ and $\Gamma_2\backslash X$ are said to be τ-isospectral if the spectra of the Laplace operators $\Delta_{\Gamma_1,\tau}$, $\Delta_{\Gamma_2,\tau}$ are the same.

The question of comparing equivalence (resp., τ-equivalence) of representations with isospectrality (resp., τ-isospectrality) has been studied by several authors in recent years (see, for instance, [2, 3, 5, 8, 14, 15, 18]). One has that if two groups Γ_1, Γ_2 are τ-equivalent, then $\Gamma_1\backslash X$ and $\Gamma_2\backslash X$ are τ-isospectral (see [15, App. Prop. 2] or Proposition 2.5). Furthermore, Pesce has shown for constant sectional curvature space forms that the converse holds for $\tau = \mathbf{1}$, that is, if the manifolds $\Gamma_1\backslash X$ and $\Gamma_2\backslash X$ are isospectral on functions, then Γ_1 and Γ_2 are K-equivalent (see [15, Sect. 3, Prop. 2]).

In this paper, again in the context of spaces of constant sectional curvature, that is, of compact manifolds covered by S^n, $\mathbb{R}^n$, or $\mathbb{H}^n$, we will study the case when $\tau = \tau_p$, the complexified p-exterior representation of $\mathrm{O}(n)$ on $\bigwedge^p(\mathbb{R}^n)_\mathbb{C}$; thus, $\Delta_{\Gamma,\tau}$ is the Hodge–Laplace operator acting on p-forms. That is, we study the p-spectrum of $\Gamma\backslash X$ in connection with the representation $L^2(\Gamma\backslash G)_{\tau_p}$. A main tool will be the following formula, valid for any compact locally symmetric space $\Gamma\backslash X$ and any representation τ of K, expressing the multiplicity of an eigenvalue λ of $\Delta_{\Gamma,\tau}$ in terms of the coefficients $n_\Gamma(\pi)$ for $\pi \in \widehat{G}_\tau$:

$$d_\lambda(\tau,\Gamma) = \sum_{\pi\in\widehat{G}:\lambda(C,\pi)=\lambda} n_\Gamma(\pi)\dim\bigl(\operatorname{Hom}_K\bigl(V_\tau^*, H_\pi\bigr)\bigr).$$

Here $\lambda(C,\pi)$ denotes a scalar depending only on π (see Sect. 2.1). In the case at hand this formula reduces to

$$d_\lambda(\tau,\Gamma) = \sum_{\pi\in\widehat{G}_{\tau,\lambda}} n_\Gamma(\pi), \tag{1.3}$$

where $\widehat{G}_{\tau,\lambda} = \widehat{G}_\tau \cap \{\pi \in \widehat{G} : \lambda(C,\pi) = \lambda\}$. Therefore, $\operatorname{Spec}_\tau(\Gamma\backslash X)$ is determined by the multiplicities $n_\Gamma(\pi)$ for π in the sets $\widehat{G}_{\tau,\lambda}$.

We will use a general approach that applies to the three cases to be considered. In light of formula (1.3), the goal is to determine the sets $\widehat{G}_\tau$, then compute $\lambda(C,\pi)$ in each case, and then, for each given $\lambda \in \mathbb{R}$, to find the set $\widehat{G}_{\tau,\lambda}$. For general $\tau \in \widehat{K}$ this can be complicated, but it can be carried out for some choices of τ.

As a consequence of the method, by choosing $\tau = \tau_p$, we will give a generalization of results in [15] for the p-spectrum of the Hodge–Laplace operator of $\Gamma\backslash X$, comparing p-isospectrality with τ_p-equivalence. We shall see that, for nonpositive curvature, p-isospectrality is far from implying τ_p-equivalence, but a variant of Pesce's result remains true. We shall consider separately the three cases, spherical, flat, and hyperbolic space forms, although they will all share common features.

The case when X has positive curvature has been studied by several authors. Most of the results in this case are included or implicit in the work of Ikeda–Taniguchi [11], Ikeda [9], Pesce [14, 15], Gornet–McGowan [8], and others. However, we will give a comprehensive presentation that allows us to extend the results to other choices of τ (see Proposition 3.3) and illuminates the cases when the curvature is zero and negative. Let $\mathbb{N}_0$ denote the nonnegative integers and, for $1 \le p \le n$ and $k \in \mathbb{N}$, set

$$\lambda_{p,k} = k^2 + k(n-1) + (p-1)(n-p), \qquad \mathcal{E}_p = \{\lambda_{p,k} : k \in \mathbb{N}\} \tag{1.4}$$

and $\mathcal{E}_0 = \mathcal{E}_{n+1} = \{0\}$.

Theorem 1.1 *Let Γ be a finite subgroup of* $\mathrm{O}(n+1)$ *acting freely on an odd-dimensional sphere S^n with $n = 2m-1$ and let $0 \le p \le n$.*

If $\lambda \in \operatorname{Spec}_p(\Gamma\backslash S^n)$ then $\lambda \in \mathcal{E}_p \cup \mathcal{E}_{p+1}$ and $\mathcal{E}_p$ and $\mathcal{E}_{p+1}$ are disjoint sets. Furthermore, for each $\lambda \in \mathcal{E}_p \cup \mathcal{E}_{p+1}$, we have

$$d_\lambda(p,\Gamma) = \begin{cases} n_\Gamma(\pi_{\Lambda_{k,p},\delta}) & \text{if } \lambda \in \mathcal{E}_p, \\ n_\Gamma(\pi_{\Lambda_{k,p+1},\delta}) & \text{if } \lambda \in \mathcal{E}_{p+1}. \end{cases}$$

Here $\pi_{\Lambda_{k,p},\delta}$ is a specific irreducible representation of $\mathrm{O}(n+1)$ (*see* (2.7), (2.8)) *where $\pi_{\Lambda_{k,p},\delta}|_{\mathrm{SO}(n+1)}$ has highest weight $\Lambda_{k,p} = k\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_p$, k is given by* (3.6), *and $\delta \in \{0, \pm 1\}$ is uniquely determined by λ.*

In particular, if $\lambda \in \mathrm{Spec}_0(\Gamma\backslash S^n)$ *then* $\lambda \in \{k(k+n-1) : k \in \mathbb{N}_0\}$ *with*

$$d_\lambda(0, \Gamma) = n_\Gamma(\pi_{k\varepsilon_1,\delta}), \tag{1.5}$$

where $\pi_{k\varepsilon_1,\delta}$ *restricted to* $\mathrm{SO}(n+1)$ *has highest weight* $k\varepsilon_1$ (*see Sect*. 2.2).

As a direct consequence:

Corollary 1.2 *Let* Γ_1, Γ_2 *be finite subgroups of* $\mathrm{O}(n+1)$ *acting freely on* $X = S^n$. *Then*

(i) (*see* [8, 11, 15]) $\Gamma_1\backslash X$ *and* $\Gamma_2\backslash X$ *are* p*-isospectral if and only if* Γ_1 *and* Γ_2 *are* τ_p*-equivalent.*
(ii) *If* $\Gamma_1\backslash X$ *and* $\Gamma_2\backslash X$ *are* $p-1$*-isospectral and* $p+1$*-isospectral*, *then they are* p*-isospectral.*

We note that both in the theorem and in the corollary, the discrete subgroups Γ and Γ_i, $i = 1, 2$ act freely on S^n; hence, they must necessarily be included in $\mathrm{SO}(n+1)$; thus, all the manifolds occurring are orientable.

In [9], Ikeda constructed for each p, lens spaces q-isospectral for every $0 \le q \le p$ but not $p+1$-isospectral. More recently, Gornet and McGowan [8] gave a very useful survey on the results of Pesce and Ikeda and, by computer methods using Ikeda's approach, exhibited a rich list of lens spaces that are p-isospectral for some values of p only. Their list (see p. 274) again shows no simple 'holes' in the set of values of p for which there is p-isospectrality. This is consistent with the assertion in (ii) of the corollary that shows that this is valid in general for all spherical space forms. As noted in [8], the examples in [8] are τ_p-equivalent for these values of p only.

By following the general method described above we shall prove the following results for flat and negative curvature compact locally symmetric spaces.

Theorem 1.3 *Let* Γ *be a Bieberbach group*, *that is*, Γ *is a discrete*, *cocompact subgroup of* $\mathrm{Iso}(\mathbb{R}^n) \simeq \mathrm{O}(n) \ltimes \mathbb{R}^n$ *acting without fixed points on* $\mathbb{R}^n$. *Let* Λ *denote the translation lattice of* Γ *and let* Λ^* *be the dual lattice of* Λ. *The multiplicity of the eigenvalue* $\lambda = 4\pi^2\|v\|^2$, $v \in \Lambda^*$, *is given by*

$$d_\lambda(\tau_p, \Gamma) = \begin{cases} n_\Gamma(\widetilde{\tau}_p) = \beta_p(\Gamma\backslash\mathbb{R}^n) & \text{if } \lambda = 0, \\ n_\Gamma(\pi_{\sigma_p,\sqrt{\lambda}/2\pi}) + n_\Gamma(\pi_{\sigma_{p-1},\sqrt{\lambda}/2\pi}) & \text{if } \lambda > 0. \end{cases} \tag{1.6}$$

Here σ_p *is the* p*-exterior representation of* $\mathrm{O}(n-1)$ *and* $\widetilde{\tau}_p$ *and* $\pi_{\sigma_p,r}$ *are certain unitary irreducible representations of* $\mathrm{Iso}(\mathbb{R}^n)$ (*see* (4.2)).

Theorem 1.4 *Let* $G = \mathrm{SO}(n, 1)$, $K = \mathrm{O}(n)$, $\Gamma \subset G$ *be a discrete subgroup acting without fixed points on* $\mathbb{H}^n$.

If $0 \le p \le n$, *and* $\lambda = 0$, *then*

$$d_0(\tau_p, \Gamma) = \beta_p\big(\Gamma\backslash\mathbb{H}^n\big) = \begin{cases} n_\Gamma(J_{\sigma_p,\rho_p}) + n_\Gamma(J_{\sigma_{p-1},\rho_{p-1}}) & \text{if } p \neq \frac{n}{2}, \\ n_\Gamma(D^+_{\frac{n}{2}} \oplus D^-_{\frac{n}{2}}) & \text{if } p = \frac{n}{2}. \end{cases}$$

If $\lambda \neq 0$, *then*

$$d_\lambda(\tau_p, \Gamma) = \begin{cases} n_\Gamma(\pi_{\sigma_p, \sqrt{\rho_p^2-\lambda}}) + n_\Gamma(\pi_{\sigma_{p-1}, \sqrt{\rho_{p-1}^2-\lambda}}) & \text{if } p \neq \frac{n}{2}, \\ n_\Gamma(\pi_{\sigma_m, \sqrt{1/4-\lambda}}) + n_\Gamma(\pi_{\sigma_{m-1}, \sqrt{1/4-\lambda}}) & \text{if } p = \frac{n}{2} = m. \end{cases}$$

In the expressions above, σ_p *is the* p*-exterior representation of* $M \simeq \mathrm{O}(n-1)$, $\rho_p := \frac{n-1}{2} - \min(p, n-1-p)$ *and* $\pi_{\sigma_p,\nu}$, $J_{\sigma_p,\nu}$, *and* $D^+_{\frac{n}{2}} \oplus D^-_{\frac{n}{2}}$ *denote specific unitary irreducible representations of* G (*see Sect.* 5).

In the proofs of Theorems 1.3 and 1.4 we use the description of the unitary duals of G in terms of induced representations. It will turn out that, generically, there will be at most two irreducible representations in $\widehat{G}$ contributing to the multiplicity of a given eigenvalue λ and these multiplicities will be linked to each other for p and $p+1$. Using this fact, one first shows that 0-isospectrality implies τ_0-equivalence, then one realizes that 0- and 1-isospectrality, taken together, imply τ_0- and τ_1-equivalence, taken together. In this way, one can build an interval from 0 to p and obtain the assertion in the following theorem that gives a generalization of Pesce's result for nonpositive curvature space forms.

Theorem 1.5 *Let* $X = G/K$ *be a simply connected symmetric space of constant nonpositive curvature where* G *is the full isometry group of* X. *Let* Γ_1, Γ_2 *be discrete cocompact subgroups of* G *acting without fixed points on* X. *For each* $0 \leq p \leq n$, $\Gamma_1\backslash X$ *and* $\Gamma_2\backslash X$ *are* q*-isospectral for every* $0 \leq q \leq p$ *if and only if* Γ_1 *and* Γ_2 *are* τ_q*-equivalent for every* $0 \leq q \leq p$.

From Theorem 1.5 and its proof, one can derive several consequences relating p-isospectrality and τ_p-equivalence (see Proposition 4.5, Corollary 4.6 in the flat case and in the negative curvature case). Denote by $\beta_p(M)$ the p-th Betti number of M. If $X = \mathbb{R}^n$ or $X = \mathbb{H}^n$, given Γ_1, Γ_2 discrete cocompact subgroups of $G = \mathrm{Iso}(X)$ acting without fixed points on X, we show

- If Γ_1, Γ_2 are τ_1-equivalent, then $\Gamma_1\backslash X$ and $\Gamma_2\backslash X$ are 0- and 1-isospectral.
  Example 4.8 gives two 4-dimensional compact flat manifolds that are 1-isospectral but not 0-isospectral; hence, Γ_1, Γ_2 are not τ_1-equivalent.
- If Γ_1, Γ_2 are τ_{p+1}-equivalent (or τ_{p-1}-equivalent) and $\Gamma_1\backslash X$ and $\Gamma_2\backslash X$ are p-isospectral, then Γ_1 and Γ_2 are τ_p-equivalent.
- If Γ_1, Γ_2 are τ_{p-1} and τ_{p+1}-equivalent and $\beta_p(\Gamma_1\backslash X) = \beta_p(\Gamma_2\backslash X)$, then Γ_1, Γ_2 are τ_p-equivalent. Hence $\Gamma_1\backslash X$ and $\Gamma_2\backslash X$ are $p-1$, p and $p+1$-isospectral.
- If $\Gamma_1\backslash X$ and $\Gamma_2\backslash X$ are p-isospectral for every $p \in \{1, 2, \dots, k\}$ and they are not 0-isospectral then Γ_1, Γ_2 are not τ_p-equivalent for any $p \in \{0, 1, 2, \dots, k+1\}$.
  In Example 4.10 we give two flat 8-manifolds that are p-isospectral for $p = 1, 2, 3, 5, 6, 7$ but not for $0, 4, 8$; hence, the corresponding groups cannot be τ_p-equivalent for any $p \in \{0, 1, 2, \dots, 8\}$. Similarly, Example 4.9 gives two flat 4-manifolds that are p-isospectral for $p = 1, 3$ only. Thus, these pairs of Bieberbach groups cannot be τ_p-equivalent for any $0 \leq p \leq 4$.

The examples we give in the flat case show that, a priori, the theorems cannot be improved substantially. In the hyperbolic case, similar examples should exist but their construction seems much more difficult. In general, little is known about the multiplicities $n_\Gamma(\pi)$.

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2 General Setting and Preliminaries

Let $X = G/K$ be a simply connected Riemannian symmetric space, where G is the full isometry group of X and K is the isotropy subgroup of a point in X. Let $\Gamma \subset G$ be a discrete *cocompact* subgroup acting freely on X; thus, the manifold $\Gamma\backslash X$ inherits a locally G-invariant Riemannian structure. We shall be interested in the cases when X is a space of constant sectional curvature:

- $X = S^n$, $(G, K) = (\mathrm{O}(n+1), \mathrm{O}(n))$;
- $X = \mathbb{R}^n$, $(G, K) = (\mathrm{O}(n) \ltimes \mathbb{R}^n, \mathrm{O}(n))$;
- $X = \mathbb{H}^n$, $(G, K) = (\mathrm{SO}(n, 1), \mathrm{O}(n))$.

The embedding of $\mathrm{O}(n)$ in $\mathrm{SO}(n, 1)$ in the third case is the standard one in $\mathrm{S}(\mathrm{O}(n) \times \mathrm{O}(1))$.

2.1 Homogeneous Vector Bundles

Given (τ, V_τ), a unitary representation of K, we consider the *homogeneous vector bundle* $E_\tau = G \times_\tau V_\tau$ of X. This is constructed as the quotient of $G \times V_\tau$ under the right action of K given as $(x, v) \cdot k = (xk, \tau(k^{-1})v)$. We denote by $[x, v]$ the class of $(x, v) \in G \times V_\tau$ in E_τ and by $(E_\tau)_{xK} = \{[x, v] \in E_\tau : v \in V_\tau\}$ the fiber of xK. The full isometry group G of X acts on E_τ by $g[x, v] = [gx, v]$ and sends $(E_\tau)_{xK}$ to $(E_\tau)_{gxK}$ linearly. We equip E_τ with the unique unitary structure which, at the fiber of eK, coincides with the unitary structure of V_τ and such that the action of G is unitary. This homogeneous vector bundle is *natural* in the sense that an isometry g of X gives an isomorphism of the complex vector spaces $(E_\tau)_{xK}$ and $(E_\tau)_{gxK}$ that preserves the unitary structure.

Let $\Gamma^\infty(E_\tau)$ denote the space of smooth sections of E_τ. Given $\psi \in \Gamma^\infty(E_\tau)$, we have that $\psi(xK) = [x, f(x)]$, with f in $C^\infty(G; \tau)$, the set of smooth functions $f : G \to V_\tau$ such that $f(xk) = \tau(k^{-1})f(x)$. Conversely, any $f \in C^\infty(G; \tau)$ defines an element $\psi \in \Gamma^\infty(E_\tau)$. The group G acts on $\Gamma^\infty(E_\tau)$ on the left by $(g \cdot \psi)(xK) := g\psi(g^{-1}xK) = g[g^{-1}x, f(g^{-1}x)] = [x, f(g^{-1}x)]$, and hence on $C^\infty(G; \tau)$ by $(g \cdot f)(x) = f(g^{-1}x)$.

Let Γ be a discrete cocompact subgroup of G that acts freely on X. We restrict to Γ the left actions of G on $X = G/K$, E_τ, $\Gamma^\infty(E_\tau)$, and $C^\infty(G; \tau)$. The space $\Gamma\backslash X$ is a compact Riemannian manifold and $\Gamma\backslash E_\tau$ is a natural homogeneous vector bundle over $\Gamma\backslash X$. The space of smooth sections $\Gamma^\infty(\Gamma\backslash E_\tau)$ of this vector bundle is isomorphic to the space $C^\infty(\Gamma\backslash G; \tau)$ of left Γ-invariant functions in $C^\infty(G; \tau)$.

We denote by $L^2(\Gamma\backslash E_\tau)$ the closure of $C^\infty(\Gamma\backslash G/K;\tau)$ with respect to the inner product

$$(f_1, f_2) = \int_{\Gamma\backslash X} \langle f_1(x), f_2(x)\rangle \mathrm{d}x.$$

The Lie algebra $\mathfrak{g}$ of G acts on $C^\infty(G;\tau)$ by

$$(X \cdot f)(x) = \frac{d}{dt}\bigg|_0 f\big(\exp(-tX)x\big)$$

for $X \in \mathfrak{g}$ and $f \in C^\infty(G/K;\tau)$. This action induces a representation of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. If G is semisimple we let $C = \sum X_i^2 \in U(\mathfrak{g})$ where $X_1, \ldots, X_n$ is an orthonormal basis of $\mathfrak{g}$. In this case, C is called the *Casimir* element. When $G = \mathrm{Iso}(\mathbb{R}^n)$, thus $X = \mathbb{R}^n$, we let $C = \sum_{i=1}^n X_i^2 \in U(\mathfrak{g})$, where $X_1, \ldots, X_n$ is an orthonormal basis of $\mathbb{R}^n$. In both cases, the element C does not depend on the basis.

The element C defines a *differential operator* Δ_τ on $C^\infty(G;\tau)$. This operator commutes with the left action of G on $C^\infty(G;\tau)$, in particular with elements in Γ, thus Δ_τ induces a differential operator $\Delta_{\tau,\Gamma}$ acting on smooth sections of $\Gamma\backslash E_\tau$.

Proposition 2.1 *Let $X = G/K$ be an irreducible simply connected Riemannian symmetric space of constant curvature and denote by $(\tau_p, \bigwedge^p(\mathbb{C}^n))$ the p-exterior representation of $K = \mathrm{O}(n)$, for any $0 \le p \le n$. Then $\Delta_{\tau_p,\Gamma}$ coincides with the Hodge–Laplace operator on complex-valued differential forms of degree p.*

We now recall some notions from the Introduction that will be the main object of this paper.

Definition 2.2 Let τ be a unitary representation of K. Let Γ_1 and Γ_2 be two cocompact discrete subgroups of G acting freely on X. The spaces $\Gamma_1\backslash X$ and $\Gamma_2\backslash X$ are said to be *τ-isospectral* if the Laplace type operators Δ_{τ,Γ_1} and Δ_{τ,Γ_2} have the same spectrum. Here, we shall just say that the spaces are *p-isospectral* if $\tau = \tau_p$.

Given Γ a discrete cocompact subgroup of G acting freely on X, we consider the *right regular representation* $R_\Gamma = \mathrm{Ind}_\Gamma^G(1_\Gamma)$ of G on $L^2(\Gamma\backslash G)$. This representation decomposes as an orthogonal direct sum of closed invariant subspaces of finite multiplicity

$$L^2(\Gamma\backslash G) = \sum_{\pi\in\widehat{G}} n_\Gamma(\pi) H_\pi \tag{2.1}$$

where $\widehat{G}$ is the unitary dual of G and, for each $\pi \in \widehat{G}$, $n_\Gamma(\pi)$ denotes the multiplicity of π in this decomposition. Note that if G is noncompact then, generically, H_π will be infinite dimensional.

Following the notation in [15], we let $\widehat{G}_\tau = \{\pi \in \widehat{G} : \mathrm{Hom}_K(\tau,\pi) \neq 0\}$ and we let $R_{\Gamma,\tau}$ be the unitary subrepresentation of R_Γ given by

$$L^2(\Gamma\backslash G)_\tau = \sum_{\pi\in\widehat{G}_\tau} n_\Gamma(\pi) H_\pi. \tag{2.2}$$

Definition 2.3 (see [15]) Let τ be an irreducible unitary representation of K. Let Γ_1 and Γ_2 be two discrete subgroups of G acting freely on G/K. Then Γ_1 and Γ_2 are said to be *τ-equivalent* if the representations $R_{\Gamma_1,\tau}$ and $R_{\Gamma_2,\tau}$ are equivalent, that is, if $n_{\Gamma_1}(\pi) = n_{\Gamma_2}(\pi)$ for every $\pi \in \widehat{G}_\tau$.

Proposition 2.4 *If $\lambda \in \mathbb{R}$, the multiplicity $d_\lambda(\tau, \Gamma)$ of the eigenvalue λ of $\Delta_{\tau,\Gamma}$ is given by*

$$d_\lambda(\tau, \Gamma) = \sum_{\pi \in \widehat{G}:\lambda(C,\pi)=\lambda} n_\Gamma(\pi) \dim\big(\mathrm{Hom}_K\big(V_\tau^*, H_\pi\big)\big). \tag{2.3}$$

Proof This result is well known. We sketch the proof for completeness. One has a map $\phi : C^\infty(\Gamma\backslash G) \times V_\tau \longrightarrow C^\infty(\Gamma\backslash G, V_\tau)$ given by $\phi(f, v) = f(g)v$. Thus ϕ induces a homomorphism $\overline{\phi} : C^\infty(\Gamma\backslash G) \otimes V_\tau \to C^\infty(\Gamma\backslash G, V_\tau)$ that is actually an isomorphism and preserves the K-action. Indeed,

$$\begin{aligned}\phi\big(R_k f, \tau(k)v\big)(g) &= f(gk)\tau(k)(v) = \tau(k) f(gk)v \\ &= \tau(k)\phi(f, v)(gk) = \big(k \cdot \phi(f, v)\big)(g).\end{aligned}$$

Hence, $\overline{\phi}$ sends K-invariants isomorphically onto K-invariants; thus,

$$\big(C^\infty(\Gamma\backslash G) \otimes V_\tau\big)^K \simeq C^\infty(\Gamma\backslash G, V_\tau)^K = C^\infty(\Gamma\backslash G; \tau) \simeq \Gamma^\infty(\Gamma\backslash E_\tau).$$

Now,

$$\big(C^\infty(\Gamma\backslash G) \otimes V_\tau\big)^K = \sum_{\pi\in\widehat{G}} n_\Gamma(\pi)\big(H_\pi^\infty \otimes V_\tau\big)^K \simeq \sum_{\pi\in\widehat{G}} n_\Gamma(\pi)\,\mathrm{Hom}_K\big(V_\tau^*, H_\pi^\infty\big),$$

where H_π^∞ denotes the set of smooth elements in H_π. Thus, the eigenspace $L^2(\Gamma\backslash E_\tau)_\lambda$ of C in $L^2(\Gamma\backslash E_\tau)$ can be written as

$$L^2(\Gamma\backslash E_\tau)_\lambda \simeq \sum_{\pi\in\widehat{G}:\lambda(C,\pi)=\lambda} n_\Gamma(\pi)\,\mathrm{Hom}_K\big(V_\tau^*, H_\pi\big).$$

□

From formula (2.3) one sees that the only representations in $\widehat{G}$ that can contribute to the multiplicity of the eigenvalue λ are those in $\widehat{G}_{\tau^*}$. As a direct consequence we have that:

Proposition 2.5 *Let Γ_1 and Γ_2 be discrete cocompact subgroups of G acting freely on X. If Γ_1 and Γ_2 are τ-equivalent then $\Gamma_1\backslash X$ and $\Gamma_2\backslash X$ are τ^*-isospectral.*

2.2 Unitary Dual Group of the Orthogonal Group

If X is a symmetric space of constant curvature, then either $X = S^n$, $X = \mathbb{R}^n$, or $X = \mathbb{H}^n$. In all three cases we have $K \simeq \mathrm{O}(n)$. We will need some well-known facts about the irreducible representations of $\mathrm{O}(n)$.

Table 1 Root systems for $\mathfrak{so}(n)$

	$\mathfrak{so}(2m+1)$	$\mathfrak{so}(2m)$
Roots	$\pm\varepsilon_i \pm \varepsilon_j$ $(i \neq j)$ $\pm\varepsilon_i$	$\pm\varepsilon_i \pm \varepsilon_j$ $(i \neq j)$
Positive roots	$\varepsilon_i \pm \varepsilon_j$ $(i < j)$ ε_i	$\varepsilon_i \pm \varepsilon_j$ $(i < j)$
Simple roots	$\varepsilon_i - \varepsilon_{i+1}$ $(1 \le i < m)$ ε_m	$\varepsilon_i - \varepsilon_{i+1}$ $(1 \le i < m)$ $\varepsilon_{m-1} + \varepsilon_m$

We first recall the root system of the complex simple Lie algebra $\mathfrak{so}(n,\mathbb{C})$. Let

$$\mathfrak{h} = \left\{ H = \sum_{j=1}^{m} i h_j (E_{2j-1,2j} - E_{2j,2j-1}) : h_j \in \mathbb{C} \right\},$$

where $E_{i,j}$ denotes the matrix with entries 1 in the (i,j)-th place and 0 elsewhere. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{so}(2m,\mathbb{C})$ and also of $\mathfrak{so}(2m+1,\mathbb{C})$ if we add a zero row and a zero column at the end. For $H \in \mathfrak{h}$, set $\varepsilon_j(H) = h_j$ for $1 \le j \le m$. We consider the inner product $\langle , \rangle$ on $\mathfrak{h}_{\mathbb{R}}$ obtained by $\frac{1}{2(n-1)}$ times the restriction of the Killing form on $\mathfrak{g}$, and its dual form on $h_{\mathbb{R}}^*$. The root systems of $\mathfrak{so}(2m+1,\mathbb{C})$ and $\mathfrak{so}(2m,\mathbb{C})$ with respect to $\mathfrak{h}$ and $\langle , \rangle$ are of type B_m and D_m, respectively. We list the roots in Table 1.

The finite-dimensional irreducible representations of a complex simple Lie algebra are characterized by their corresponding highest weights. We will denote them by $\mathcal{P}(\mathfrak{g})$.

We have

$$\mathcal{P}\big(\mathfrak{so}(2m)\big) = \left\{ \sum_{i=1}^{m} c_i\varepsilon_i : \begin{array}{l} c_i \in \mathbb{Z}\ \forall i \text{ or } c_i \in \frac{1}{2} + \mathbb{Z}\ \forall i, \text{ and} \\ c_1 \ge c_2 \ge \cdots \ge c_{m-1} \ge |c_m|. \end{array} \right\},$$

$$\mathcal{P}\big(\mathfrak{so}(2m+1)\big) = \left\{ \sum_{i=1}^{m} c_i\varepsilon_i : \begin{array}{l} c_i \in \mathbb{Z}\ \forall i \text{ or } c_i \in \frac{1}{2} + \mathbb{Z}\ \forall i, \text{ and} \\ c_1 \ge c_2 \ge \cdots \ge c_{m-1} \ge c_m \ge 0. \end{array} \right\}.$$

The irreducible representations of $\mathfrak{so}(n)$ are in a one-to-one correspondence with those of the simply connected Lie group $\mathrm{Spin}(n)$. In the case of $\mathrm{SO}(n)$, the highest weights of the irreducible representations are given by

$$\mathcal{P}\big(\mathrm{SO}(n)\big) = \left\{ \sum_{i=1}^{m} c_i\varepsilon_i \in \mathcal{P}\big(\mathfrak{so}(n)\big) : c_i \in \mathbb{Z}\ \forall i \right\}. \tag{2.4}$$

Example 2.6 Set $\Lambda_p = \sum_{i=1}^{\min(p,n-p)} \varepsilon_i \in \mathcal{P}(\mathrm{SO}(n))$ for $0 \le p \le n$. If $p \neq \frac{n}{2}$, then Λ_p is the highest weight of the p-exterior representation on $\bigwedge^p \mathbb{C}^n$ of $\mathrm{SO}(n)$. These representations are irreducible. The m-exterior power representation $\bigwedge^m(\mathbb{C}^{2m})$ of $\mathrm{SO}(2m)$ decomposes as $\bigwedge_+^m(\mathbb{C}^{2m}) \oplus \bigwedge_-^m(\mathbb{C}^{2m})$, where $\bigwedge_\pm^m(\mathbb{C}^{2m})$ are irreducible and have highest weights $\sum_{j=1}^{m-1} \varepsilon_j \pm \varepsilon_m$.

We now describe the irreducible representations of the full orthogonal group $\mathrm{O}(n)$ in terms of the irreducible representations of the special orthogonal group $\mathrm{SO}(n)$. Let

$$g_0 = \begin{cases} -Id_n & \text{if } n \text{ is odd,} \\ \begin{bmatrix} Id_{n-1} & \\ & -1 \end{bmatrix} & \text{if } n \text{ is even.} \end{cases} \tag{2.5}$$

Then $\mathrm{O}(n) = \mathrm{SO}(n) \cup g_0\,\mathrm{SO}(n)$, thus we will define the representations of $\mathrm{O}(n)$ on each component, $\mathrm{SO}(n)$ and $g_0\,\mathrm{SO}(n)$.

For $\Lambda \in \mathcal{P}(\mathrm{SO}(2m+1))$ and $\delta = \pm 1$, let (π_Λ, V) be the representation of $\mathrm{SO}(2m+1)$ with highest weight Λ. Then we may define a representation $(\pi_{\Lambda,\delta}, V)$ of $\mathrm{O}(2m+1)$ on V by setting, for $g \in \mathrm{O}(2m+1)$,

$$\pi_{\Lambda,\delta}(g)(v) = \begin{cases} \pi_\Lambda(g)(v) & \text{if } g \in \mathrm{SO}(2m+1), \\ \delta \pi_\Lambda(g_0 g)(v) & \text{if } g \in g_0\,\mathrm{SO}(2m+1). \end{cases} \tag{2.6}$$

For $\Lambda = \sum_{j=1}^m c_j \varepsilon_j \in \mathcal{P}(\mathrm{SO}(2m))$ ($c_j \in \mathbb{Z}$ for all j and $c_1 \geq \cdots \geq c_{m-1} \geq |c_m|$), we denote by $\overline{\Lambda} = \sum_{j=1}^{m-1} c_j \varepsilon_j - c_m \varepsilon_m \in \mathcal{P}(\mathrm{SO}(2m))$. Let (π_Λ, V_Λ) be the irreducible representation of $\mathrm{SO}(2n)$ with highest weight Λ. If $I_{g_0}(g) = g_0 g g_0$, then I_{g_0} defines an automorphism of $\mathrm{SO}(2m)$ and one can see that $(\pi_\Lambda \circ I_{g_0}, V_\Lambda)$ has highest weight $\overline{\Lambda}$. Thus, there exists a unitary operator $T_\Lambda : V_\Lambda \to V_{\overline{\Lambda}}$ such that $T_\Lambda \circ (\pi_\Lambda \circ I_{g_0})(g) = \pi_{\overline{\Lambda}}(g) \circ T_\Lambda$ for every $g \in \mathrm{SO}(m)$. Furthermore, $(\pi_\Lambda \circ I_{g_0}, V_\Lambda)$ is equivalent to (π_Λ, V_Λ) if and only if $c_m = 0$.

If $\Lambda \in \mathcal{P}(\mathrm{SO}(2m))$ is such that $c_m = 0$ and $\delta \in \{\pm 1\}$, we define a representation $\pi_{\Lambda,\delta}$ of $\mathrm{O}(2m)$ on V_Λ as

$$\pi_{\Lambda,\delta}(g)(v) = \begin{cases} \pi_\Lambda(g)(v), & \text{if } g \in \mathrm{SO}(2m), \\ \delta T_\Lambda(\pi_\Lambda(g_0 g)(v)), & \text{if } g \in g_0\,\mathrm{SO}(2m). \end{cases} \tag{2.7}$$

Note that this definition depends on the choice of T_Λ since $-T_\Lambda$ is another intertwining operator between π_Λ and $\pi_{\overline{\Lambda}}$. However, we have $\pi_{\Lambda,\delta} \simeq \pi_{\Lambda,-\delta} \otimes \det$.

If $\Lambda \in \mathcal{P}(\mathrm{SO}(2m))$ is such that $c_m > 0$, we set $\delta = 0$ and define the representation $\pi_{\Lambda,0}$ of $\mathrm{O}(2m)$ on $V_\Lambda \oplus V_{\overline{\Lambda}}$ as follows.

$$\pi_{\Lambda,0}(g)\big(v, v'\big) = \begin{cases} (\pi_\Lambda(g)(v), \pi_{\overline{\Lambda}}(g)(v')), & \text{if } g \in \mathrm{SO}(2m), \\ (\pi_\Lambda(g_0 g)v', \pi_\Lambda(g_0 g)v), & \text{if } g \in g_0\,\mathrm{SO}(2m). \end{cases} \tag{2.8}$$

In particular, $\pi_{\Lambda,0}(g_0)(v, v') = (v', v)$; thus, $(V_\Lambda \oplus V_{\overline{\Lambda}}, \pi_{\Lambda,0})$ is irreducible.

In the next theorem we describe the unitary dual of $G = \mathrm{O}(n)$ (see, for instance, [7]).

Theorem 2.7 *We have*

$$\begin{aligned} \widehat{\mathrm{O}(2m+1)} &= \big\{\pi_{\Lambda,\delta} \textit{ as in } (2.6) : \Lambda \in \mathcal{P}\big(\mathrm{SO}(2m+1)\big), \delta = \pm 1\big\}, \\ \widehat{\mathrm{O}(2m)} &= \big\{\pi_{\Lambda,\delta} \textit{ as in } (2.7) : \Lambda \in \mathcal{P}\big(\mathrm{SO}(2m)\big), c_m = 0, \delta = \pm 1\big\} \\ &\quad \cup \big\{\pi_{\Lambda,0} \textit{ as in } (2.8) : \Lambda \in \mathcal{P}\big(\mathrm{SO}(2m)\big), c_m > 0\big\}. \end{aligned}$$

Example 2.8 We denote by $(\tau_p, \bigwedge^p(\mathbb{C}^n))$ the complexification of the p-exterior representation of the canonical representation of $\mathrm{O}(n)$ on $\mathbb{R}^n$. We have that τ_p is irreducible for every value of p. Furthermore, $\tau_p \simeq \tau_{n-p} \otimes \det$ for any $0 \le p \le n$, where the intertwining operator is given by the Hodge star operator.

Recall that $\Lambda_p = \sum_{i=1}^{\min(p,n-p)} \varepsilon_i \in \mathcal{P}(\mathrm{SO}(n))$. In the notation of Theorem 2.7, if n is odd we have $\tau_p \simeq \pi_{\Lambda_p,(-1)^p}$ and, for n even, $\tau_{\frac{n}{2}} \simeq \pi_{\Lambda_{\frac{n}{2}},0}$. To write $\tau_p \in \widehat{\mathrm{O}(2m)}$ as (2.7) for $p \neq m$, we must fix an intertwining operator T_{Λ_p}. For $0 \le p < m$, we write $\bigwedge^p(\mathbb{C}^{2m}) = W_0 \oplus W_1$, where W_0 (resp., W_1) is the subspace of $\bigwedge^p(\mathbb{C}^{2m})$ generated by $e_{i_1} \wedge \cdots \wedge e_{i_p}$ where $2m \notin \{i_1, \ldots, i_p\}$ (resp., $2m \in \{i_1, \ldots, i_p\}$). It is not hard to check that $T_{\Lambda_p} := Id_{W_0} \oplus (-Id_{W_1})$ satisfies $T_{\Lambda_p} \circ (\pi_{\Lambda_p} \circ I_{g_0})(g) = \pi_{\overline{\Lambda}_p}(g) \circ T_{\Lambda_p}$ for every $g \in \mathrm{SO}(2m)$. Finally, one has that $\tau_p \simeq \pi_{\Lambda_p,1}$ for $0 \le p < m$ and $\tau_p \simeq \pi_{\Lambda_p,-1}$ for $m < p \le n$.

We conclude this section by stating two branching laws for orthogonal groups that will be needed in the following sections (see [7], for instance).

Proposition 2.9 *Let τ_p and σ_p be the p-exterior representations of* $\mathrm{O}(n)$ *and* $\mathrm{O}(n-1)$, *respectively. Then, for any $0 \le p \le n$, we have*

$$\tau_p|_{\mathrm{O}(n-1)} = \sigma_p \oplus \sigma_{p-1}, \tag{2.9}$$

with the understanding that σ_{-1}, σ_n are the zero representations of $\mathrm{O}(n-1)$. *That is,* $\tau_0|_{\mathrm{O}(n-1)} = \sigma_0$ *and* $\tau_n|_{\mathrm{O}(n-1)} = \sigma_{n-1}$.

Lemma 2.10 *Let τ_p be the p-exterior representation of* $\mathrm{O}(2m-1)$ *and let* $\pi_{\Lambda,\delta} \in \widehat{\mathrm{O}(2m)}$ *in the notation of Theorem* 2.7. *Then $[\tau_p : \pi_{\Lambda,\delta}|_K] > 0$ if and only if*

$$\Lambda = k\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_p + c_{p+1}\varepsilon_{p+1} \tag{2.10}$$

with $k \in \mathbb{N}$, $c_{p+1} \in \{0, 1\}$, and where $\delta \in \{0, \pm 1\}$ has a specific value. More precisely, if $p = m-1, m$ and $c_m > 0$ then $\delta = 0$ while if $p \neq m-1, m$ or $p = m-1, m$ and $c_m = 0$, then $\delta = \pm 1$ and the sign depends on p and on the choice of the intertwining operator T_Λ. Moreover, $[\tau_p : \pi_{\Lambda,\delta}|_K] = 1$.

3 Compact Case

In this section we shall prove the assertions in Theorem 1.1 and Corollary 1.2 for constant curvature spaces of compact type, that is, for spherical space forms. We fix the following notation for this section.

$$\begin{aligned}
G &= \mathrm{O}(n+1) \simeq \mathrm{Iso}\big(S^n\big),\\
K &= \mathrm{O}(n) = \{g \in G : g.e_{n+1} = e_{n+1}\},\\
X &= G/K \simeq S^n.
\end{aligned}$$

Since even-dimensional spheres S^n cover only S^n and $\mathbb{R}P^n$, and their spectra are well known, we will look only at odd-dimensional spheres. Thus, we assume throughout this section that $n = 2m - 1$; then $G = \mathrm{O}(2m)$ and $K = \mathrm{O}(2m-1)$. We first describe the set $\widehat{G}_{\tau_p}$, in the notation of Theorem 2.7. Set, for $2 \le p \le n-2$ and $k \in \mathbb{N}$,

$$\Lambda_{k,p} = k\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{\min(p,n-p)} \tag{3.1}$$

and $\Lambda_{k,p} = k\varepsilon_1$ for $p = 1, n-1$ and $k \in \mathbb{N}_0$. In particular, $\Lambda_{1,p} = \Lambda_p$, as in Example 2.6.

Proposition 3.1 *Let τ_p be the p-exterior representation of K. If $0 < p < m-1$, then*

$$\widehat{G}_{\tau_0} = \{\pi_{\Lambda_{0,1},\delta}, \pi_{\Lambda_{k,1},\delta} : k \in \mathbb{N} \textit{ with } \delta \in \{\pm 1\}\},$$
$$\widehat{G}_{\tau_p} = \{\pi_{\Lambda_{k,p},\delta}, \pi_{\Lambda_{k,p+1},\delta} : k \in \mathbb{N} \textit{ with } \delta \in \{\pm 1\}\},$$
$$\widehat{G}_{\tau_{m-1}} = \{\pi_{\Lambda_{k,m-1},\delta}, \pi_{\Lambda_{k,m},0} : k \in \mathbb{N} \textit{ with } \delta \in \{\pm 1\}\}.$$

In each of the sets above, δ is uniquely determined by k, p, and T_Λ. Furthermore, if $m \le p \le 2m-1 = n$, then $\widehat{G}_{\tau_p} = \{\pi_{\Lambda,\delta} : \pi_{\Lambda,-\delta} \in \widehat{G}_{\tau_{n-p}}\}$. Moreover, for any $0 \le p < n$ and $k \in \mathbb{N}$, $\pi_{\Lambda_{k,p+1},\delta} \in \widehat{G}_{\tau_p} \cap \widehat{G}_{\tau_{p+1}}$.

Proof From Theorem 2.7 we see that $\widehat{\mathrm{O}(2m)}$ is the set of all representations $\pi_{\Lambda,\delta}$ where $\Lambda = \sum_{i=1}^m c_i\varepsilon_i \in \mathcal{P}(\mathrm{SO}(2m))$ (see (2.4)), $c_m \in \mathbb{N}_0$, and either $\delta = \pm 1$ if $c_m = 0$, or $\delta = 0$ if $c_m \in \mathbb{N}$. Also, from Example 2.8 we see that, if $p > 0$, $\tau_p = \tau_{\Lambda_p,\kappa} \in \widehat{K}$ as in (2.6), where $\Lambda_p = \sum_{j=1}^p \varepsilon_j$ and $\kappa = (-1)^p$.

Taking this into account, by using the branching law in Lemma 2.10 one checks that the description of $\widehat{G}_{\tau_p}$ is as stated in the proposition. □

Now we prove that, for a spherical space form, the multiplicity of each eigenvalue of the Hodge–Laplace operator on p-forms involves only one specific $n_\Gamma(\pi)$, that is to say, the sum in (2.3) has only one term. We recall the notation introduced in (1.4):

$$\begin{aligned} \mathcal{E}_0 &= \mathcal{E}_{n+1} = \{0\} \\ \mathcal{E}_p &= \{k^2 + k(n-1) + (p-1)(n-p) : k \in \mathbb{N}\} \end{aligned} \tag{3.2}$$

for $1 \le p \le n$. Note that $\mathcal{E}_p = \mathcal{E}_{n+1-p}$ for every $0 \le p \le n+1$.

Proof of Theorem 1.1 By Schur's lemma, the Casimir element C acts on any irreducible representation $\pi_{\Lambda,\delta}$ with $\Lambda = \sum_{i=1}^m c_i\varepsilon_i$ by multiplication by a scalar $\lambda(C,\pi)$ given by

$$\lambda(C,\pi_{\Lambda,\delta}) = \langle \Lambda+\rho, \Lambda+\rho\rangle - \langle\rho,\rho\rangle = \langle\Lambda,\Lambda\rangle + 2\langle\Lambda,\rho\rangle, \tag{3.3}$$

where $\rho = \sum_{i=1}^m (m-i)\varepsilon_i$. Note that the scalar $\lambda(C,\pi_{\Lambda,\delta})$ does not depend on δ.

We first assume that $p = 0$. By Proposition 3.1, the highest weights of representations in $\widehat{G}_{\tau_0}$ have the form $\Lambda = k\varepsilon_1$ with $k \in \mathbb{N}_0$ and

$$\lambda(C,\pi_{k\varepsilon_1,\delta}) = k^2 + 2k(m-1) = k(k+n-1). \tag{3.4}$$

Proposition 2.4 now implies that if $\lambda \notin \mathcal{E}_0 \cup \mathcal{E}_1$ then λ is not in $\mathrm{Spec}_0(\Gamma \backslash S^n)$, that is, $d_\lambda(\tau_p, \Gamma) = 0$. Moreover, since $k \mapsto k(k+n-1)$ is increasing for $k \geq 0$ hence $k = k_\lambda$ is clearly determined by $\lambda \in \mathcal{E}_0 \cup \mathcal{E}_1$. Actually,

$$k_\lambda = -\frac{n-1}{2} + \sqrt{\left(\frac{n-1}{2}\right)^2 + \lambda} = -(m-1) + \sqrt{(m-1)^2 + \lambda}. \tag{3.5}$$

Thus, in this case $d_\lambda(\tau_0, \Gamma) = n_\Gamma(\pi_{\Lambda_{k_\lambda \varepsilon_1}, \delta})$.

Now assume $0 < p < m$. By Proposition 3.1, if $\pi_{\Lambda,\delta} \in \widehat{G}_{\tau_p}$ then $\Lambda = \Lambda_{k,p}$ or $\Lambda = \Lambda_{k,p+1}$ and by (2.3), for each λ, we must consider $\pi_{\Lambda_{k,p},\delta}, \pi_{\Lambda_{k,p+1},\delta} \in \widehat{G}_{\tau_p}$ with $\lambda(C, \pi_{\Lambda_{k,p},\delta}) = \lambda$ or $\lambda(C, \pi_{\Lambda_{k,p+1},\delta}) = \lambda$. Since

$$\begin{aligned}
\lambda(C, \pi_{\Lambda_{k,p},\delta}) &= \langle \Lambda_{k,p}, \Lambda_{k,p} \rangle + 2\langle \Lambda_{k,p}, \rho \rangle \\
&= k^2 + p - 1 + 2k(m-1) + 2\textstyle\sum_{i=2}^{p}(m-i) \\
&= k^2 + k(n-1) + (p-1)(n-p)
\end{aligned}$$

lies in $\mathcal{E}_p$ and $\lambda(C, \pi_{\Lambda_{k,p+1},\delta}) = k^2 + k(n-1) + p(n-p-1) \in \mathcal{E}_{p+1}$, it follows that λ is not an eigenvalue of $\Delta_{\tau,\Gamma}$ if $\lambda \notin \mathcal{E}_p \cup \mathcal{E}_{p+1}$.

It is clear that for $\lambda \in \mathcal{E}_p$ or $\lambda \in \mathcal{E}_{p+1}$, k is uniquely determined by λ. Indeed, we have

$$k_\lambda = \begin{cases} -(m-1) + \sqrt{(m-1)^2 + \lambda - (p-1)(n-p)} & \text{if } \lambda \in \mathcal{E}_p, \\ -(m-1) + \sqrt{(m-1)^2 + \lambda - p(n-p-1)} & \text{if } \lambda \in \mathcal{E}_{p+1}. \end{cases} \tag{3.6}$$

It remains only to check that $\mathcal{E}_p$ and $\mathcal{E}_{p+1}$ are disjoint. Thus, let us assume that $\lambda(C, \pi_{\Lambda_{k,p},\delta}) = \lambda(C, \pi_{\Lambda_{h,p+1},\delta})$ for some $h, k \in \mathbb{N}$. Then

$$k^2 + k(n-1) + (p-1)(n-p) = h^2 + h(n-1) + p(n-p-1),$$

which implies that

$$(k-h)(k+h+n-1) = n - 2p. \tag{3.7}$$

Now since $n > 2p$, then $k > h$; thus, the left-hand side is at least $n + 1 > n - 2p$, hence (3.7) cannot hold. Thus, $\mathcal{E}_p \cap \mathcal{E}_{p+1} = \emptyset$ for $0 \leq p \leq n+1$.

It follows from this that for each $\lambda \in \mathcal{E}_p \cup \mathcal{E}_{p+1}$, the sum in Proposition 3.1 has only one term; indeed,

$$d_\lambda(\tau_p, \Gamma) = \begin{cases} n_\Gamma(\pi_{\Lambda_{k_\lambda,p},\delta}) & \text{if } \lambda \in \mathcal{E}_p, \\ n_\Gamma(\pi_{\Lambda_{k_\lambda,p+1},\delta}) & \text{if } \lambda \in \mathcal{E}_{p+1}. \end{cases}$$

The case when $m \leq p \leq 2m-1$ follows in the same way since $\pi_{\Lambda,\delta} \in \widehat{G}_{\tau_p}$ if and only if $\pi_{\Lambda,-\delta} \in \widehat{G}_{\tau_{n-p}}$ by Proposition 3.1 and C acts by the same scalar on both representations. $\square$

We can now prove Corollary 1.2, as an application of Theorem 1.1.

Proof of Corollary 1.2 (i) For each p, Theorem 1.1 yields that the multiplicities of the eigenvalues of the Hodge–Laplace operator on p-forms determine the multiplicity $n_\Gamma(\pi)$ of every $\pi \in \widehat{G}_{\tau_p}$; hence, Γ_1 and Γ_2 are τ_p-equivalent. The converse follows from Proposition 2.5.

(ii) Given Γ_1 and Γ_2, assume that $\Gamma_1\backslash X$ and $\Gamma_2\backslash X$ are $p-1$-isospectral and $p+1$-isospectral. Then, by (i), Γ_1 and Γ_2 are τ_{p-1} and τ_{p+1}-equivalent.

For $0 < p < n$, since $\widehat{G}_{\tau_p} \subset \widehat{G}_{\tau_{p-1}} \cup \widehat{G}_{\tau_{p+1}}$ by Proposition 3.1, it follows that $n_{\Gamma_1}(\pi) = n_{\Gamma_2}(\pi)$ for every $\pi \in \widehat{G}_{\tau_p}$ hence Γ_1 and Γ_2 are τ_p-equivalent, and as a consequence $\Gamma_1\backslash X$ and $\Gamma_2\backslash X$ are p-isospectral.

For $p = 0$ we have $\widehat{G}_{\tau_0} \subset \widehat{G}_{\tau_1} \cup \{\pi_{0,\delta}\}$; hence, Γ_1 and Γ_2 are τ_0-equivalent since $n_{\Gamma_i}(\pi_{0,\delta}) = \beta_0(\Gamma_i\backslash S^n) = 1$. Hence, $\Gamma_1\backslash X$ and $\Gamma_2\backslash X$ are 0-isospectral. For $p = n$ the argument is completely analogous. □

Remark 3.2 The p-spectrum of spherical space forms has been investigated by many authors. For instance, in [11], Ikeda and Taniguchi studied the p-spectrum of homogeneous spaces G/K from the point of view of representation theory, determining the eigenvalues and the eigenspaces in the case of S^n and $\mathbb{C}P^n$.

In the 1980s, A. Ikeda constructed several classes of examples of 0-isospectral spherical space forms, most of them having cyclic fundamental groups. Later, Gilkey [6] (1985), Ikeda [10] (1997) and Wolf [17] (2001) studied isospectrality by using the Sunada method as generalized in [5], producing many new examples. Isospectral manifolds arising from this method are always strongly isospectral; in particular, they are p-isospectral for all p. Wolf's construction yields a wealth of examples of strongly isospectral spherical space forms, which recovers many of the earlier examples.

In [9], Ikeda gave examples of a different type, finding, for every $0 \le p < \frac{n-1}{2}$, lens spaces that are q-isospectral for every $0 \le q \le p$ but are not $p+1$-isospectral. Note that these examples cannot arise from Sunada's method. In [15], Pesce considered the notion of τ-equivalent discrete subgroups and showed that τ-isospectral spherical space forms give τ-equivalent groups in the case when the real-eigenspaces in $L^2(S^n;\tau)$ are irreducible. In 2002, by computer methods, Gornet and McGowan found a rich family of examples of lens spaces that are τ_p-equivalent for some values of p only ([8]).

As mentioned in the Introduction, Corollary 1.2(i) can be extended to $\tau = \tau_{\mu,\kappa} \in \widehat{K}$, for more general choices of the highest weight μ.

Proposition 3.3 *Let Γ_1, Γ_2 be finite subgroups of $G = \mathrm{O}(n+1)$ acting freely on S^n. Let $\mu = \sum_{i=1}^{m-1} b_i\varepsilon_i \in \mathcal{P}(\mathrm{SO}(2m))$ be such that*

$$2 = b_1 \ge b_2 \ge \cdots \ge b_{m-1} \ge 0$$

and let $\kappa \in \{\pm 1\}$. If $\Gamma_1\backslash S^n$ and $\Gamma_2\backslash S^n$ are $\tau_{\mu,\kappa}$-isospectral, then Γ_1 and Γ_2 are $\tau_{\mu,\kappa}$-equivalent.

Proof As we noted in the proof of Corollary 1.2(i), it is sufficient to show that different representations of $\widehat{G}_{\tau_{\mu,\kappa}}$ have different Casimir eigenvalues. The proof will be divided into two cases:

(a) $\mu_p := 2\varepsilon_1 + \cdots + 2\varepsilon_p$ for some $1 \le p \le m-1$,
(b) $\mu_{p,q} := 2\varepsilon_1 + \cdots + 2\varepsilon_p + \varepsilon_{p+1} + \cdots + \varepsilon_q$ for some $1 \le p < q \le m-1$.

Case (a). By the branching law (see, for example, [14, Prop. I.5]) we have that $[\tau_{\mu,\kappa} : \pi_{\Lambda,\delta}|_K] > 0$ if and only if

$$\Lambda = \Lambda_{(k,a),p} := k\varepsilon_1 + 2\varepsilon_2 + \cdots + 2\varepsilon_p + a\varepsilon_{p+1},$$

where $k \ge 2$, $0 \le a \le 2$ and $\delta \in \{0, \pm 1\}$ has a specific value. Hence, the highest weights involved in $\widehat{G}_{\tau_{\mu_p,\kappa}}$ have the form $\Lambda_{(k,a),p}$ with $0 \le a \le 2$, for every p. We have

$$\lambda(C, \pi_{\Lambda_{(k,a),p},\delta}) = k(k+2m-2) + \sum_{i=2}^{p} 2(2+2m-2i) + a(a+2m-2p-2).$$

It remains to prove that $\pi_{\Lambda_{(k,a),p},\delta} = \pi_{\Lambda_{(h,b),p},\delta}$ if and only if $(k,a) = (h,b)$. Suppose $0 \le a < b \le 2$. Then

$$k(k+2m-2) + a(a+2m-2p-2) = h(h+2m-2) + b(b+2m-2p-2)$$
$$(k-h)(k+h+2m-2) = (b-a)(b+a+2m-2p-2).$$

If $k > h$, $b > a$, since $k+h+2m-2 > b+a+2m-2p-2$ then $0 < k-h < b-a$. Hence, $b-a=2$ and $k-h=1$; thus, we have a contradiction since the left-hand side is odd and the right-hand side is even. Therefore, necessarily, $k=h$ and $b=a$, as asserted.

Case (b). The proof is very similar to the previous one, so we only give the main ingredients. The highest weights involved in $\widehat{G}_{\tau_{\mu_{p,q},\kappa}}$ have the form

$$\Lambda_{(k,a_1,a_2),p} := k\varepsilon_1 + 2\varepsilon_2 + \cdots + 2\varepsilon_p + a_1\varepsilon_{p+1} + \varepsilon_{p+2} + \cdots + \varepsilon_q + a_2\varepsilon_{q+1},$$

where $k \ge 2$ and $0 \le a_1 \le 1 \le a_2 \le 2$. In this case,

$$\begin{aligned}\lambda(C, \pi_{\Lambda_{(k,a_1,a_2),p},\delta}) &= k(k+2m-2) + \sum_{i=2}^{p} 2(2+2m-2i) \\ &\quad + a_1(a_1+2m-2p-2) \\ &\quad + \sum_{i=p+2}^{q} (1+2m-2i) + a_2(a_2+2m-2q-2).\end{aligned}$$

Suppose $\lambda(C, \pi_{\Lambda_{(k,a_1,a_2),p},\delta}) = \lambda(C, \pi_{\Lambda_{(h,b_1,b_2),p},\delta})$ with $a_2 < b_2$, i.e., $a_2 = 0$ and $b_2 = 1$. One can check that

$$(k-h)(k+h+2m-2) = (b_1-a_1)(b_1+a_1+2m-2p-2) + 1 + 2m - 2q - 2.$$

In case $b_1 = a_1$ we arrive at a contradiction as above. If $a_1 = 1$ and $b_1 = 2$, then the right-hand side is equal to $4m - 2(p+q)$; hence, $k-h$ is an even positive integer, thus the right-hand side is greater than the left-hand side. If $a_1 = 2$ and $b_1 = 1$, the right-hand side is equal to $-2(q-p+1)$ and again we arrive at a contradiction as before. $\square$

Remark 3.4 Note that Proposition 3.3 follows again from the fact that, for any $\lambda \in \mathbb{R}$, in formula (2.3) at most one irreducible representation in $\widehat{G}_{\tau_{\mu,\kappa}}$ gives a contribution. This is not true generically for $\tau \in \widehat{K}$. For instance, for $\tau = \tau_{\mu,\kappa}$ with $\mu = 3\varepsilon_1$ and $\kappa = \pm 1$, set $\Lambda = 2m\varepsilon_1$ and $\Lambda' = (2m-1)\varepsilon_1 + 3\varepsilon_2$, thus $\pi_{\Lambda,\delta}, \pi_{\Lambda',\delta} \in \widehat{G}_\tau$ for a single value of δ and we have

$$\begin{aligned}\lambda(C, \pi_{\Lambda,\delta}) &= \langle \Lambda, \Lambda + 2\rho\rangle = 2m\bigl(2m + 2(m-1)\bigr) = 2n(n+1),\\ \lambda(C, \pi_{\Lambda',\delta}) &= (2m-1)\bigl(2m - 1 + 2(m-1)\bigr) + 3\bigl(3 + 2(m-2)\bigr)\\ &= n(n+n-1) + 3n = 2n(n+1).\end{aligned}$$

Therefore, the eigenspace of $\Delta_{\tau_{\mu,\kappa},\Gamma}$ for the eigenvalue $\lambda = 2n(n+1)$ is equal to $\pi_{\Lambda,\delta} \oplus \pi_{\Lambda',\delta}$, which is not irreducible.

Remark 3.5 Let $\Omega_p(M)$ denote the space of differential forms of degree p on a Riemannian compact manifold M. By the Hodge decomposition at degree p,

$$\Omega_p(M) = \mathcal{H}_p(M) \oplus \Omega'_p(M) \oplus \Omega''_p(M), \tag{3.8}$$

where $\mathcal{H}_p(M)$ denotes the p-harmonic forms and $\Omega'_p(M)$ and $\Omega''_p(M)$ denote the subspace of *exact* ($d\Omega_{p-1}(M)$) and *coexact forms* ($d^*\Omega_{p+1}(M)$), respectively. A subscript $\lambda \in \mathbb{R}$ in these sets will denote the restriction to the eigenspace associated with the eigenvalue λ. Clearly $\Omega_p(M)_0 = \mathcal{H}_p(M)$ and $\Omega_p(M)_\lambda = \Omega'_p(M)_\lambda \oplus \Omega''_p(M)_\lambda$ for any $\lambda \neq 0$.

In this case, Theorem 1.1 ensures that the sets $\Omega'_p(M)_\lambda$ and $\Omega''_p(M)_\lambda$ cannot both be nonempty. Moreover, the p-eigenspace associated with $\lambda \in \mathcal{E}_p$ (resp., $\mathcal{E}_{p+1}$) is contained in $\Omega'_p(M)_\lambda$ (resp., $\Omega''_p(M)_\lambda$) for every $0 < p < n$.

Gornet and McGowan introduced the notion of *half-isospectrality* (see [8, Rmk. 4.5]) meaning isospectrality with respect to $\Delta_{\tau_p,\Gamma}$ restricted to exact or coexact p-forms. They also showed several examples of half-isospectral lens spaces. In a way similar to Corollary 1.2(i), we can give an equivalent formulation in terms of representations as follows.

> $\Gamma_1\backslash S^n$ and $\Gamma_2\backslash S^n$ are isospectral on exact (resp., coexact) p-forms if and only if $n_{\Gamma_1}(\pi_{\Lambda_{k,p},\delta}) = n_{\Gamma_2}(\pi_{\Lambda_{k,p},\delta})$ (resp., $n_{\Gamma_1}(\pi_{\Lambda_{k,p+1},\delta}) = n_{\Gamma_2}(\pi_{\Lambda_{k,p+1},\delta})$) for every $k \in \mathbb{N}$.

The examples of p-isospectral and not $p+1$-isospectral lens spaces given in [9] and [8] are examples of manifolds $p+1$-isospectral on exact forms but not on coexact forms.

4 Flat Case

We now consider the flat case $X = \mathbb{R}^n$. Then

$$G = \mathrm{O}(n) \ltimes \mathbb{R}^n \simeq \mathrm{Iso}\bigl(\mathbb{R}^n\bigr), \tag{4.1}$$

and $K = \mathrm{O}(n)$. Let Γ be a discrete cocompact subgroup of G acting freely on $\mathbb{R}^n$, i.e., a *Bieberbach group*. Any element $\gamma \in \Gamma \subset G$ decomposes uniquely as $\gamma = BL_b$, with $B \in K$ and $b \in \mathbb{R}^n$. The matrix B is called the rotational part of γ and L_b is called the translational part. The subgroup L_Λ of pure translations in Γ is called the *translation lattice* of Γ and $F := \Lambda\backslash\Gamma$ is the *point group* (or the *holonomy group*) of Γ.

We need a description of the unitary dual of G. We will use Mackey's method (see [16, Sect. 5.4]). We identify $\widehat{\mathbb{R}^n}$ with $\mathbb{R}^n$ via the correspondence $\alpha \to \xi_\alpha(.) = e^{2\pi i\langle\alpha,.\rangle}$ for $\alpha \in \mathbb{R}^n$. The group G acts on $\widehat{\mathbb{R}^n}$ by $(g \cdot \xi_\alpha)(b) = \xi_\alpha(g^{-1}b)$. For $\alpha \in \mathbb{R}^n$ we consider $K_\alpha = \{k \in K : k \cdot \xi_\alpha = \xi_\alpha\}$, the stabilizer of ξ_α in K.

For $\alpha \in \mathbb{R}^n$ and $(\sigma, V_\sigma) \in \widehat{K_\alpha}$, we consider the induced representation of G given by

$$(\pi_{\sigma,\alpha}, W_{\sigma,\alpha}) := \mathrm{Ind}_{K_\alpha \ltimes \mathbb{R}^n}^{K \ltimes \mathbb{R}^n}(\sigma \otimes \xi_\alpha). \tag{4.2}$$

Here, the space $W_{\sigma,\alpha}$ is the completion of the space

$$C_{\sigma,\alpha} = \left\{ f : G \to V_\sigma \text{ cont.} : f\big((k,b)g\big) = \sigma(k)\xi_\alpha(b) f(g),\ \forall k \in K_\alpha, b \in \mathbb{R}^n \right\}$$

with respect to a canonical inner product. The action of G on $W_{\sigma,\alpha}$ is by right translations. Since $(\sigma \otimes \xi_\alpha, V_\sigma)$ is unitary, $(\pi_{\sigma,\alpha}, W_{\sigma,\alpha})$ is a unitary representation of G. It is well known that $\pi_{\sigma,\alpha}$ is irreducible and, furthermore, every unitary representation of G is unitarily equivalent to one of this form.

Note that if $\alpha = 0$, then $K_\alpha = K = \mathrm{O}(n)$. Furthermore, for $(\tau, V) \in \widehat{K}$, we have $\widetilde{\tau} := \pi_{\tau,0} \simeq \tau \otimes Id$, i.e., $\widetilde{\tau}(v) = \tau(v)$ for $v \in V$; therefore, $(\widetilde{\tau}, V) \in \widehat{G}$ is finite dimensional.

On the other hand, if $\alpha \neq 0$ and $\sigma \in \widehat{K_\alpha}$, then $\pi_{\sigma,\alpha} \simeq \pi_{\sigma,re_n}$ where $r = \|\alpha\|$. We shall abbreviate π_{σ,re_n} by writing $\pi_{\sigma,r}$ for $r \geq 0$. In this case, $K_\alpha = \begin{bmatrix} \mathrm{O}(n-1) & \\ & 1 \end{bmatrix} \simeq \mathrm{O}(n-1)$, when $r > 0$.

Summing up, a full set of representatives of $\widehat{G}$ is given by

$$\widehat{G} = \left\{\pi_{\sigma,r} : \sigma \in \widehat{\mathrm{O}(n-1)}, r > 0\right\} \cup \left\{\widetilde{\tau} : \tau \in \widehat{\mathrm{O}(n)}\right\}. \tag{4.3}$$

Now we determine $\widehat{G}_{\tau_p}$, that is, the representations in $\widehat{G}$ such that its restriction to $\mathrm{O}(n)$ contains the p-exterior representations τ_p of $\mathrm{O}(n)$. Recall that σ_p denotes the complexified p-exterior representations of $\mathrm{O}(n-1)$.

Lemma 4.1 *We have*

$$\widehat{G}_{\tau_p} = \{\pi_{\sigma_p,r}, \pi_{\sigma_{p-1},r} : r > 0\} \cup \{\widetilde{\tau}_p\}$$

for all p. Moreover $[\tau_p : \pi|_K] = 1$ for every $\pi \in \widehat{G}_{\tau_p}$.

Proof Let $\pi_{\sigma,r} \in \widehat{G}$ with $\sigma \in \widehat{\mathrm{O}(n-1)}$ and $r > 0$. Since $\pi_{\sigma,r}|_K = \mathrm{Ind}_{K_\alpha}^K(\sigma)$ and $[\tau_p : \mathrm{Ind}_{K_\alpha}^K(\sigma)] = [\sigma : \tau_p|_{K_\alpha}]$ by Frobenius reciprocity, we have that $[\tau_p : \pi_{\sigma,r}|_K] > 0$ if and only if $\sigma = \sigma_p, \sigma_{p-1}$ by Proposition 2.9.

Now if $\widetilde{\tau} \in \widehat{G}$ with $\tau \in \widehat{K}$, then $\widetilde{\tau}|_K = \tau$; it follows that $[\tau_p, \widetilde{\tau}|_K] > 0$ if and only if $\tau = \tau_p$. □

If $e_1, \dots, e_n$ is the canonical basis of $\mathbb{R}^n$, the operator $C = \sum_{i=1}^n e_i^2 \in U(\mathfrak{g})$ descends to the Hodge–Laplace operator $\Delta_{\tau_p,\Gamma}$ on p-forms of $\Gamma\backslash\mathbb{R}^n \simeq \Gamma\backslash \operatorname{Iso}(\mathbb{R}^n)/\operatorname{O}(n)$ (see Sect. 2.1). The following lemma tells us how $\Delta_{\tau_p,\Gamma}$ operates on any $\pi \in \widehat{G}$.

Lemma 4.2 *The element $C \in U(\mathfrak{g})$ acts on $\pi \in \widehat{G}$ by multiplication by a scalar $\lambda(C,\pi)$ given as follows.*

$$\lambda(C,\pi) = \begin{cases} 0 & \textit{for } \pi = \widetilde{\tau}, \\ -4\pi^2\|\alpha\|^2 & \textit{for } \pi = \pi_{\sigma,\alpha}, \alpha \neq 0. \end{cases}$$

Proof In the first case $\widetilde{\tau}(k,v) = \tau(k)$, for any $k \in K$, $v \in \mathbb{R}^n$. If $X \in \mathbb{R}^n$,

$$\widetilde{\tau}(X)(k,v) = \frac{d}{dt}\bigg|_0 \widetilde{\tau}(k, v+tX) = \frac{d}{dt}\bigg|_0 \tau(k) = 0.$$

If $\pi = \pi_{\sigma,\alpha}$ with $\alpha \neq 0$ and $f \in C_{\sigma,\alpha}$, then

$$\begin{aligned} \pi_{\sigma,\alpha}(X) f(k,v) &= \frac{d}{dt}\bigg|_0 f(k, v+tX) = \frac{d}{dt}\bigg|_0 f\big((1, tk\cdot X)\cdot(k,v)\big) \\ &= \frac{d}{dt}\bigg|_0 e^{2\pi i t\langle \alpha, (k\cdot X)\rangle} f(k,v) = 2\pi i \big\langle k^{-1}\cdot\alpha, X\big\rangle f(k,v). \end{aligned}$$

Thus $\pi_{\sigma,\alpha}(C) f(k,v) = -4\pi^2 \sum_{i=1}^n \langle k^{-1}\alpha, e_i\rangle^2 f(k,v) = -4\pi^2\|\alpha\|^2 f(k,v)$. □

Now we are in a condition to prove the results in the Introduction in the flat case.

Proof of Theorem 1.3 By Proposition 2.4, given an eigenvalue $\lambda \in \mathbb{R}$ of the Hodge–Laplace operator on p-forms $\Delta_{\tau_p,\Gamma}$ on $\Gamma\backslash\mathbb{R}^n$, the multiplicity $d_\lambda(\tau_p,\Gamma)$ is given by $\sum n_\Gamma(\pi)\ [\tau_p : \pi|_K]$, where the sum is over every $\pi \in \widehat{G}_{\tau_p}$ such that $-\lambda(C,\pi) = \lambda$. Now, by using Lemmas 4.1 and 4.2 we obtain that

$$d_\lambda(\tau_p,\Gamma) = \begin{cases} n_\Gamma(\widetilde{\tau}_p) & \text{if } \lambda = 0, \\ n_\Gamma(\pi_{\sigma_p,\sqrt{\lambda}/2\pi}) + n_\Gamma(\pi_{\sigma_{p-1},\sqrt{\lambda}/2\pi}) & \text{if } \lambda > 0, \end{cases} \tag{4.4}$$

and thus Theorem 1.3 follows. □

We will use the following lemma to prove Theorem 1.5 in the flat case and other consequences in Corollary 4.6.

Lemma 4.3 *If Γ_1 and Γ_2 are τ_{p-1}-equivalent (or τ_{p+1}-equivalent) and $\Gamma_1\backslash\mathbb{R}^n$ and $\Gamma_2\backslash\mathbb{R}^n$ are p-isospectral, then Γ_1 and Γ_2 are τ_p-equivalent.*

Proof Since Γ_1 and Γ_2 are τ_{p-1}-equivalent we have that $n_{\Gamma_1}(\pi_{\sigma_{p-1},r}) = n_{\Gamma_2}(\pi_{\sigma_{p-1},r})$ for every $r > 0$ by Proposition 3.1. On the other hand, since $\Gamma_1\backslash\mathbb{R}^n$ and $\Gamma_2\backslash\mathbb{R}^n$ are p-isospectral we have that $n_{\Gamma_1}(\widetilde{\tau}_p) = n_{\Gamma_2}(\widetilde{\tau}_p)$ and

$$n_{\Gamma_1}(\pi_{\sigma_p,r}) + n_{\Gamma_1}(\pi_{\sigma_{p-1},r}) = n_{\Gamma_2}(\pi_{\sigma_p,r}) + n_{\Gamma_2}(\pi_{\sigma_{p-1},r})$$

for any $r > 0$, by (4.4). These three facts taken together clearly imply τ_p-equivalence. The assertion assuming τ_{p+1}-equivalence follows in a similar way. $\square$

Proof of Theorem 1.5 (flat case) The fact that τ_p-equivalence implies p-isospectrality is clear in light of Proposition 2.5. For the converse assertion, we proceed by induction. Lemma 4.3 for $p = 0$ says that 0-isospectrality implies τ_0-equivalence. Now assume that the manifolds are q-isospectral for every $0 \le q \le p$; thus, we have that the groups are τ_q-equivalent for every $0 \le q \le p-1$ by the induction hypothesis. In particular, we have τ_{p-1}-equivalence and p-isospectrality; hence, Lemma 4.3 implies τ_p-equivalence, which completes the proof. $\square$

Remark 4.4 One can also prove the above result for intervals decreasing from n, that is, *q-isospectrality for every $p \le q \le n$ is equivalent to τ_q-equivalence for every $p \le q \le n$.*

We can also obtain from Theorem 1.3 several other consequences relating p-isospectrality and τ_p-equivalence. Given a compact n-manifold M, $\beta_p(M)$ denotes the p-th Betti number of M and one has that $\beta_p(M) = d_0(\tau_p, M)$, the multiplicity of the eigenvalue 0 of the Hodge–Laplace operator on p-forms of M.

Proposition 4.5 *Let Γ_1 and Γ_2 be Bieberbach groups and let $\Gamma_1\backslash\mathbb{R}^n$ and $\Gamma_2\backslash\mathbb{R}^n$ be the corresponding flat Riemannian manifolds. Then the following assertions hold.*

(i) *If Γ_1 and Γ_2 are τ_1-equivalent, then $\Gamma_1\backslash\mathbb{R}^n$ and $\Gamma_2\backslash\mathbb{R}^n$ are 0- and 1-isospectral.*
(ii) *If Γ_1 and Γ_2 are τ_{n-1}-equivalent and $\beta_n(\Gamma_1\backslash\mathbb{R}^n) = \beta_n(\Gamma_2\backslash\mathbb{R}^n)$, then $\Gamma_1\backslash\mathbb{R}^n$ and $\Gamma_2\backslash\mathbb{R}^n$ are n- and $n-1$-isospectral.*
(iii) *If Γ_1 and Γ_2 are τ_{p-1} and τ_{p+1}-equivalent and $\beta_p(\Gamma_1\backslash\mathbb{R}^n) = \beta_p(\Gamma_2\backslash\mathbb{R}^n)$, then Γ_1 and Γ_2 are also τ_p-equivalent; hence, $\Gamma_1\backslash\mathbb{R}^n$ and $\Gamma_2\backslash\mathbb{R}^n$ are $p-1$, p and $p+1$-isospectral.*

Proof We will use repeatedly the facts

$$\widehat{G}_{\tau_p} = \{\pi_{\sigma_p,r}, \pi_{\sigma_{p-1},r} : r > 0\} \cup \{\widetilde{\tau}_p\}, \tag{$*$}$$

$$d_\lambda(\tau_p, \Gamma_i) = \begin{cases} n_{\Gamma_i}(\widetilde{\tau}_p) & \text{if } \lambda = 0, \\ n_{\Gamma_i}(\pi_{\sigma_p,\sqrt{\lambda}/2\pi}) + n_{\Gamma_i}(\pi_{\sigma_{p-1},\sqrt{\lambda}/2\pi}) & \text{if } \lambda > 0 \end{cases} \tag{$**$}$$

from Lemma 4.1 and Theorem 1.3.

We first prove (i). Suppose that Γ_1 and Γ_2 are τ_1-equivalent; then $\Gamma_1\backslash\mathbb{R}^n$ and $\Gamma_2\backslash\mathbb{R}^n$ are 1-isospectral by Proposition 2.5. Since $\pi_{\sigma_0,r} \in \widehat{G}_{\tau_1}$ for $r > 0$, ($*$) and ($**$)

imply that $d_\lambda(\tau_0, \Gamma_1) = d_\lambda(\tau_0, \Gamma_2)$ for every $\lambda > 0$; hence, $\Gamma_1\backslash\mathbb{R}^n$ and $\Gamma_2\backslash\mathbb{R}^n$ are also 0-isospectral, since $d_0(\tau_0, \Gamma_1) = d_0(\tau_0, \Gamma_2) = 1$.

Assertion (ii) follows in a similar way by using that $d_0(\tau_n, \Gamma_i) = \beta_n(\Gamma_i\backslash\mathbb{R}^n)$.

Relative to (iii) if Γ_1 and Γ_2 are τ_{p-1} and τ_{p+1}-equivalent, then on the one hand, $n_{\Gamma_1}(\pi_{\sigma_{p-1},r}) = n_{\Gamma_2}(\pi_{\sigma_{p-1},r})$ for every $r > 0$ since $\pi_{\sigma_{p-1},r} \in \widehat{G}_{\tau_{p-1}}$ and, on the other hand, since $\pi_{\sigma_p,r} \in \widehat{G}_{\tau_{p+1}}$, then $n_{\Gamma_1}(\pi_{\sigma_p,r}) = n_{\Gamma_2}(\pi_{\sigma_p,r})$ for every $r > 0$. Finally, the equality of the p-th Betti numbers implies that $n_{\Gamma_1}(\widetilde{\tau}_p) = n_{\Gamma_2}(\widetilde{\tau}_p)$ by (∗∗); thus, Γ_1 and Γ_2 are τ_p-equivalent. $\square$

Note that the condition $\beta_n(\Gamma_1\backslash\mathbb{R}^n) = \beta_n(\Gamma_2\backslash\mathbb{R}^n)$ in Proposition 4.5(ii) is equivalent to $\Gamma_1\backslash\mathbb{R}^n$ and $\Gamma_2\backslash\mathbb{R}^n$ being both orientable or both non-orientable. A flat manifold $\Gamma\backslash\mathbb{R}^n$ is orientable if and only if $\Gamma \subset \mathrm{SO}(n) \ltimes \mathbb{R}^n$.

The next result follows immediately from Lemma 4.3 and will be applied in explicit examples.

Corollary 4.6 *Let Γ_1 and Γ_2 be Bieberbach groups. If $\Gamma_1\backslash\mathbb{R}^n$ and $\Gamma_2\backslash\mathbb{R}^n$ are p-isospectral for every $p \in \{1, 2, \dots, k\}$ and are not 0-isospectral, then Γ_1 and Γ_2 are not τ_p-equivalent for any $p \in \{0, 1, 2, \dots, k+1\}$. Similarly, if $\beta_n(\Gamma_1\backslash\mathbb{R}^n) = \beta_n(\Gamma_2\backslash\mathbb{R}^n)$ and $\Gamma_1\backslash\mathbb{R}^n, \Gamma_2\backslash\mathbb{R}^n$ are p-isospectral for every $p \in \{n-k, \dots, n-2, n-1\}$ and are not n-isospectral, then Γ_1 and Γ_2 are not τ_p-equivalent for any $p \in \{n-k-1, n-k, \dots, n\}$*

Remark 4.7 We now study the Hodge decomposition of a compact flat manifold as in Remark 3.5. In this case, Theorem 1.3 implies that $\mathcal{H}_p(M)_0$ is the 0-eigenspace associated with $\widetilde{\tau}_p$ and for $\lambda \neq 0$, again we have $\Omega_p(M)_\lambda = \Omega'_p(M)_\lambda \oplus \Omega''_p(M)_\lambda$, where both can be nonempty at the same time.

Unlike the notion of p-isospectrality, we have an equivalent definition of compact flat manifolds p-isospectral on exact forms (resp., coexact forms) in terms of representations. Namely, from Lemma 4.1 one can see that

> $\Gamma_1\backslash\mathbb{R}^n$ and $\Gamma_2\backslash\mathbb{R}^n$ are isospectral on exact (resp., coexact) p-forms if and only if $n_{\Gamma_1}(\pi_{\sigma_{p-1},r}) = n_{\Gamma_2}(\pi_{\sigma_{p-1},r})$ (resp., $n_{\Gamma_1}(\pi_{\sigma_p,r}) = n_{\Gamma_2}(\pi_{\sigma_p,r})$) for every $r > 0$.

It would be of interest to find a pair of compact flat manifolds that are *half-isospectral* but not isospectral, that is, manifolds isospectral on exact or coexact p-forms (for some fixed p) but not on both at the same time. There are several known examples that come close to this. In particular, manifolds that are 0-isospectral and not 1-isospectral (see [13, Examples 5.1, 5.5, 5.9]) are isospectral on exact 1-forms; however, one cannot claim that they are not isospectral on coexact 1-forms, since they have different first Betti number.

In the rest of this section we give several examples of compact flat manifolds satisfying some p-isospectralities or τ_p-equivalences for some values of p only. We denote by $\{e_1, \dots, e_n\}$ the canonical basis of $\mathbb{R}^n$.

We recall from [13, Thm. 3.1] that the multiplicity of the eigenvalue $4\pi^2\mu$ of $\Delta_{\tau_p,\Gamma}$ is given by

$$d_{4\pi^2\mu}(\tau_p,\Gamma)=|F|^{-1}\sum_{\gamma=BL_b\in\Lambda\backslash\Gamma}\operatorname{tr}_p(B)e_{\mu,\gamma}(\Gamma), \tag{4.5}$$

where $e_{\mu,\gamma}(\Gamma):=\sum_{v\in\Lambda_\mu^*:Bv=v}e^{-2\pi i v\cdot b}$, $\Lambda_\mu^*:=\{v\in\Lambda^*:\|v\|^2=\mu\}$ (Λ^* the dual lattice of Λ), and $\operatorname{tr}_p(B):=\operatorname{tr}(\tau_p(B))$. If $p=0$, (4.5) reads

$$d_{4\pi^2\mu}(\tau_0,\Gamma)=|F|^{-1}\sum_{\gamma=BL_b\in\Lambda\backslash\Gamma}\ \sum_{v\in\Lambda_\mu^*:Bv=v}e^{-2\pi i v\cdot b}. \tag{4.6}$$

Example 4.8 We first show a pair of non-isometric Klein bottles that are 1-isospectral but not 0-isospectral; hence, the corresponding Bieberbach groups cannot be τ_1-equivalent by Proposition 4.5(i).

Let $\Gamma=\langle\gamma,L_\Lambda\rangle$ and $\Gamma'=\langle\gamma',L_\Lambda\rangle$, where $\Lambda=\mathbb{Z}e_1\oplus\mathbb{Z}ce_2$ with $c>1$ and in *column notation*

$$\begin{array}{c}\gamma\\ \boxed{\begin{array}{r}1_{\frac12}\\ -1\end{array}}\end{array}\quad\text{and}\quad\begin{array}{c}\gamma'\\ \boxed{\begin{array}{r}-1\\ 1_{\frac12}\end{array}}\end{array}. \tag{4.7}$$

That means that $\gamma=BL_b$ and $\gamma'=B'L_{b'}$ where $B=\left[\begin{smallmatrix}1&\\&-1\end{smallmatrix}\right]$, $B'=\left[\begin{smallmatrix}-1&\\&1\end{smallmatrix}\right]$, $b=\frac12 e_1$, and $b'=\frac12 ce_2$, i.e., the column in (4.7) gives the rotation part of γ, γ' and the subindices indicate their translation vectors.

The manifolds $\Gamma\backslash\mathbb{R}^n$ and $\Gamma'\backslash\mathbb{R}^n$ are 1-isospectral in light of (4.5) since $\operatorname{tr}_1(B)=\operatorname{tr}_1(B')=0$. However, they are not 0-isospectral since, by using (4.6), one can see that the smallest eigenvalue for $\Gamma\backslash\mathbb{R}^n$, $\lambda=4\pi^2c^{-2}$, has multiplicity 2 while λ is not an eigenvalue for $\Gamma'\backslash\mathbb{R}^n$.

The Klein bottles just defined are homeomorphic. However, it is not hard to give a pair of *non-homeomorphic* compact flat 4-manifolds that are 1-isospectral but not 0-isospectral. We define $\Gamma=\langle\gamma,L_{\mathbb{Z}^4}\rangle$ and $\Gamma'=\langle\gamma',L_{\mathbb{Z}^4}\rangle$ where, in column notation,

$$\begin{array}{c}\gamma\\ \boxed{\begin{array}{r}1_{\frac12}\\ 1\\ -1\\ -1\end{array}}\end{array}\quad\text{and}\quad\begin{array}{c}\gamma'\\ \boxed{\begin{array}{r}1_{\frac12}\\ J\\ -1\end{array}}\end{array}.$$

Here $J=\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$ and $\gamma'=B'L_{b'}$ with $B'=\operatorname{diag}(1,J,-1)\in\mathrm{GL}(4,\mathbb{R})$ and $b'=(1/2,0,0,0)^t\in\mathbb{R}^4$.

These manifolds are 1-isospectral because, again, $\operatorname{tr}_1(B)=\operatorname{tr}_1(B')=0$. They are not 0-isospectral. Indeed, it follows easily from (4.6) that the smallest nonzero eigenvalue is $4\pi^2$ for both manifolds, but it has multiplicity $\frac12(8+0)=2$ for the first one and $\frac12(8-2)=3$ for the second one.

One can show, by using the theory of Bieberbach groups, that these manifolds cannot be homeomorphic since the holonomy representations are not semiequivalent.

Example 4.9 We now give a pair of 4-dimensional compact flat manifolds that are p-isospectral for $p = 1, 3$ and they are not p-isospectral for $p = 0, 2, 4$. The corresponding Bieberbach groups cannot be τ_p-equivalent for any p, by Proposition 2.5, for p even and by Proposition 4.5(i)–(ii), for p odd.

The manifolds mentioned are called M_{24} and M_{25} in the notation in [4, Example 4.8], and can be described as $\Gamma = \langle \gamma_1, \gamma_2, L_{\mathbb{Z}^4}\rangle$ and $\Gamma' = \langle \gamma_1', \gamma_2', L_{\mathbb{Z}^4}\rangle$ where

$$\begin{array}{cc} \gamma_1 & \gamma_2 \\ \hline \multicolumn{1}{|r|}{-1} & \multicolumn{1}{r|}{1} \\ \multicolumn{1}{|r|}{-1} & \multicolumn{1}{r|}{-1_{\frac{1}{2}}} \\ \multicolumn{1}{|r|}{1} & \multicolumn{1}{r|}{-1} \\ \multicolumn{1}{|r|}{1_{\frac{1}{2}}} & \multicolumn{1}{r|}{1_{\frac{1}{2}}} \\ \hline \end{array} \qquad \begin{array}{cc} \gamma_1' & \gamma_2' \\ \hline \multicolumn{1}{|r|}{-1} & \multicolumn{1}{r|}{1_{\frac{1}{2}}} \\ \multicolumn{1}{|r|}{-1} & \multicolumn{1}{r|}{-1_{\frac{1}{2}}} \\ \multicolumn{1}{|r|}{1} & \multicolumn{1}{r|}{-1} \\ \multicolumn{1}{|r|}{1_{\frac{1}{2}}} & \multicolumn{1}{r|}{1} \\ \hline \end{array}.$$

The manifolds $\Gamma\backslash\mathbb{R}^4$ and $\Gamma'\backslash\mathbb{R}^4$ are non-homeomorphic since they have different homology over $\mathbb{Z}_2$. Indeed, one has that $\beta_1^{\mathbb{Z}_2}(M_{24}) = 4 \neq \beta_1^{\mathbb{Z}_2}(M_{25}) = 3$ and $\beta_2^{\mathbb{Z}_2}(M_{24}) = 6 \neq \beta_2^{\mathbb{Z}_2}(M_{25}) = 4$.

Example 4.10 This is a peculiar example of two 8-dimensional flat manifolds which are p-isospectral for every $p \in \{1, 2, 3, 5, 6, 7\}$ but not for $p \in \{0, 4, 8\}$. According to Corollary 4.6, the corresponding Bieberbach groups cannot be τ_p-equivalent for any p.

We define $\Gamma = \langle \gamma, L_{\mathbb{Z}^8}\rangle$ and $\Gamma' = \langle \gamma', L_{\mathbb{Z}^8}\rangle$, where

$$\begin{array}{ccc} \gamma & \gamma^2 & \gamma^3 \\ \hline \multicolumn{1}{|r|}{\widetilde{J}} & \multicolumn{1}{r|}{-I} & \multicolumn{1}{r|}{-\widetilde{J}} \\ \multicolumn{1}{|r|}{\widetilde{J}} & \multicolumn{1}{r|}{-I} & \multicolumn{1}{r|}{-\widetilde{J}} \\ \multicolumn{1}{|r|}{1_{\frac{1}{4}}} & \multicolumn{1}{r|}{1_{\frac{1}{2}}} & \multicolumn{1}{r|}{1_{\frac{3}{4}}} \\ \multicolumn{1}{|r|}{1} & \multicolumn{1}{r|}{1} & \multicolumn{1}{r|}{1} \\ \multicolumn{1}{|r|}{-1} & \multicolumn{1}{r|}{1} & \multicolumn{1}{r|}{-1} \\ \multicolumn{1}{|r|}{-1} & \multicolumn{1}{r|}{1} & \multicolumn{1}{r|}{-1} \\ \hline \end{array} \qquad \begin{array}{ccc} \gamma' & \gamma'^2 & \gamma'^3 \\ \hline \multicolumn{1}{|r|}{\widetilde{J}} & \multicolumn{1}{r|}{-I} & \multicolumn{1}{r|}{-\widetilde{J}} \\ \multicolumn{1}{|r|}{\widetilde{J}} & \multicolumn{1}{r|}{-I} & \multicolumn{1}{r|}{-\widetilde{J}} \\ \multicolumn{1}{|r|}{1_{\frac{1}{4}}} & \multicolumn{1}{r|}{1_{\frac{1}{2}}} & \multicolumn{1}{r|}{1_{\frac{3}{4}}} \\ \multicolumn{1}{|r|}{1_{\frac{1}{2}}} & \multicolumn{1}{r|}{1} & \multicolumn{1}{r|}{1_{\frac{1}{2}}} \\ \multicolumn{1}{|r|}{-1} & \multicolumn{1}{r|}{1} & \multicolumn{1}{r|}{-1} \\ \multicolumn{1}{|r|}{-1} & \multicolumn{1}{r|}{1} & \multicolumn{1}{r|}{-1} \\ \hline \end{array}.$$

Here $\widetilde{J} = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ and I is the 2×2 identity matrix. The elements γ and γ' have order 4; thus, the manifolds $\Gamma\backslash\mathbb{R}^8$ y $\Gamma'\backslash\mathbb{R}^8$ have holonomy group isomorphic to $\mathbb{Z}_4$. We also include the elements γ^2, γ^3, γ'^2, and γ'^3 to facilitate the computation of the multiplicities of the eigenvalues. Note that the only difference between their generators lies in the sixth coordinate of the translational part; in particular, we have $B = B'$ and $\gamma^2 = \gamma'^2$.

We shall compare the spectra of $\Gamma\backslash\mathbb{R}^8$ and $\Gamma'\backslash\mathbb{R}^8$ by using the formula (4.5) for the multiplicities of the eigenvalues of the Hodge–Laplace operator on p-forms. The manifolds are 1-isospectral since $\mathrm{tr}_1(B^k) = 0$ for $k = 1, 2, 3$. One also has that $\mathrm{tr}_2(B) = \mathrm{tr}_2(B^3) = 0$ (resp., $\mathrm{tr}_3(B) = \mathrm{tr}_3(B^3) = 0$), which implies that the manifolds are 2-isospectral (resp., 3-isospectral) since the equality in (4.5) follows from the fact that $\gamma^2 = \gamma'^2$. We will carry out the verification for $\mathrm{tr}_2(B)$; the vanishing of $\mathrm{tr}_2(B^3)$ is entirely similar.

Fix a basis $f_1, f_2, \ldots, f_8$ of $\mathbb{C}^8$ such that B diagonalizes in this basis with $Bf_j = if_j$ for $j = 1, 3$, $Bf_j = -if_j$ for $j = 2, 4$, $Bf_j = -f_j$ for $j = 5, 6$, and $Bf_j = f_j$ for $j = 7, 8$. Then, on $\Lambda^2(\mathbb{C}^8)$, B diagonalizes with respect to the basis $f_i \wedge f_j$, with $1 \le i < j \le 8$. It is easy to see that the eigenvalues $\pm i$ both have multiplicity 8. Indeed, the i-eigenspace (resp., $-i$-eigenspace) is spanned by $f_i \wedge f_j$ with (i, j) equal to $(1, 7)$, $(1, 8)$, $(3, 7)$, $(3, 8)$, $(2, 5)$, $(2, 6)$, $(4, 5)$, and $(4, 6)$ (resp., $(1, 5)$, $(1, 6)$, $(2, 7)$, $(2, 8)$, $(3, 5)$, $(3, 6)$, $(4, 7)$, and $(4, 8)$). On the other hand, one checks that the eigenvalues 1, -1 both have multiplicity 6. Indeed, the eigenspace of eigenvalue 1 (resp., -1) is spanned by $f_i \wedge f_j$ with (i, j) equal to $(1, 2)$, $(3, 4)$, $(1, 4)$, $(2, 3)$, $(5, 6)$, and $(7, 8)$ (resp., $(1, 3)$, $(2, 4)$, $(5, 7)$, $(5, 8)$, $(6, 7)$, and $(6, 8)$). This clearly implies that $\mathrm{tr}_2(B) = 0$. The vanishing of $\mathrm{tr}_3(B)$ and $\mathrm{tr}_3(B^3)$ follow by a similar argument.

The manifolds cannot be 0-isospectral since the first nonzero eigenvalue $\lambda = 4\pi^2$ has different multiplicity in both cases. Indeed, $d_\lambda(\tau_0, \Gamma) = 6 \neq 4 = d_\lambda(\tau_0, \Gamma')$. Since $\det(B) = 1$ the manifolds are orientable and then the previous reasoning is valid for $p = 5, 6, 7, 8$. Finally, they cannot be 4-isospectral since one checks that $\mathrm{tr}_4(B) = \mathrm{tr}_4(B^3) = -2$, $\mathrm{tr}_4(B^2) = 6$ and then, by (4.5), we obtain that the first nonzero eigenvalue $\lambda = 4\pi^2$ has multiplicities $d_\lambda(\tau_0, \Gamma) = 284 \neq 288 = d_\lambda(\tau_0, \Gamma')$.

These two compact flat manifolds are homeomorphic to each other, but it is not difficult to obtain a similar example with non-homeomorphic groups. Namely, we take

$$
\begin{array}{ccc}
\gamma & \gamma^2 & \gamma^3 \\
\begin{array}{|c|}\hline \widetilde{J} \\ \widetilde{J} \\ 1_{\frac14} \\ 1_{\frac14} \\ -1 \\ -1 \\ \hline\end{array} &
\begin{array}{c|}\hline -I \\ -I \\ 1_{\frac12} \\ 1_{\frac12} \\ 1 \\ 1 \\ \hline\end{array} &
\begin{array}{c|}\hline -\widetilde{J} \\ -\widetilde{J} \\ 1_{\frac34} \\ 1_{\frac34} \\ -1 \\ -1 \\ \hline\end{array}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\gamma' & \gamma'^2 & \gamma'^3 \\
\begin{array}{|c|}\hline \widetilde{J} \\ \widetilde{J} \\ J^{\frac12}_{0} \\ 1 \\ -1 \\ \hline\end{array} &
\begin{array}{c|}\hline -I \\ -I \\ I^{\frac12}_{\frac12} \\ 1 \\ 1 \\ \hline\end{array} &
\begin{array}{c|}\hline -\widetilde{J} \\ -\widetilde{J} \\ J^{0}_{\frac12} \\ 1 \\ -1 \\ \hline\end{array}
\end{array}.
$$

5 Negative Curvature Case

The goal of this section is to consider the p-spectrum of compact hyperbolic manifolds in connection with τ_p-isospectrality. We set $G = \mathrm{SO}(n, 1)$, $K = \mathrm{O}(n)$, $X \simeq \mathbb{H}^n$, and $X_\Gamma \simeq \Gamma\backslash\mathrm{SO}(n, 1)/K$; thus, $X = \mathbb{H}^n$ the n-dimensional hyperbolic space. Let $\Gamma \subset \mathrm{SO}(n, 1)$ be a discrete cocompact subgroup acting without fixed points on X. We recall that $\mathrm{SO}(n, 1)$ is the group of linear transformations on $\mathbb{R}^{n+1}$ preserving the Lorentzian form of signature $(n, 1)$ and determinant one.

We will need a description of $\widehat{G}$. We will first introduce the principal series representation of G. The group G has an Iwasawa decomposition $G = NAK$, with a corresponding decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ at the Lie algebra level, where N is nilpotent and A is Abelian of dimension one. Let M be the centralizer of A in K.

One has $M \simeq \mathrm{O}(n-1)$. The Lie subgroup $P = MAN$ of G is a minimal parabolic subgroup of G.

If $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, then $\xi_\nu(a) = a^\nu = \exp(\nu \log(a))$ defines a character of A. We set $\rho_{\mathfrak{a}} = \frac{1}{2}(\dim \mathfrak{g}_\alpha)\alpha = \frac{n-1}{2}\alpha$ where α is the simple root of the pair $(\mathfrak{g}, \mathfrak{a})$. If $(\sigma, V_\sigma) \in \widehat{M}$ and $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, then we let $C_{\sigma,\nu}$ be the space

$$\left\{f \text{ cont.} : G \to V_\sigma : f(mang) = a^{\nu+\rho_{\mathfrak{a}}}\sigma(m) f(g), \forall m \in M, a \in A, n \in N\right\}.$$

If $\langle , \rangle$ is an M-invariant inner product on V_σ, for $f_1, f_2 \in C_{\sigma,\nu}$ set

$$\langle f_1, f_2 \rangle = \int_{M\backslash K} \big\langle f_1(k), f_2(k)\big\rangle dk.$$

Then $(C_{\sigma,\nu}, \langle , \rangle)$ is a pre-Hilbert space and the Hilbert space completion is denoted by $H_{\sigma,\nu}$. The action of G by right translations on $C_{\sigma,\nu}$ extends to $H_{\sigma,\nu}$ defining a continuous series of representations of G, $(\pi_{\sigma,\nu}, H_{\sigma,\nu})$, which is unitary if $\nu \in i\mathfrak{a}^*$. It is called the *principal series representations* of G. They are generically irreducible and play a main role in the description of the irreducible representations of G.

One usually identifies $\mathfrak{a}_{\mathbb{C}}^*$ with $\mathbb{C}$ via the map $\nu \to \nu(H_0)$, where $H_0 \in \mathfrak{a}$ satisfies $\alpha(H_0) = 1$, in such a way that $\alpha \to 1$ and $\rho_{\mathfrak{a}} \to \frac{n-1}{2}$. In this way, $\pi_{\sigma,\nu}$ is unitary if $\nu \in i\mathbb{R}$, as mentioned above.

A Hilbert representation (π, H) of G is said to be *square integrable* if any K-finite matrix coefficient lies in $L^2(G)$. These representations were classified by Harish–Chandra and form the so-called *discrete series representations of* G, denoted $\widehat{G}_d$.

The determination of the irreducible unitary representations of a general noncompact semisimple Lie group is an open problem, but is known in the particular case of Lie groups of real rank one (see [1] and also [12]). In the case at hand of $G = \mathrm{SO}(n, 1)$ one has:

Theorem 5.1 *The unitary dual of $G = \mathrm{SO}(n, 1)$ consists of*

(i) *the unitary principal series $\pi_{\sigma,\nu}$ for $\nu \in i\mathbb{R}_{\geq 0}$, $\sigma \in \widehat{M}$,*
(ii) *the complementary series $\pi_{\sigma,\nu}$ for $0 \leq \nu < \rho_\sigma$, where $\rho_\sigma = \rho - q$ with $q \in \mathbb{N}_0$, $q \leq \rho$, depending only on the highest weight of σ,*
(iii) *unitarizable Langlands quotients J_{σ,ρ_σ},*
(iv) *$\widehat{G}_d$, the discrete series representations of G. For n odd one has $\widehat{G}_d = \emptyset$.*

The following theorem gives a description of the subset $\widehat{G}_{\tau_p}$ of $\widehat{G}$, which is all we need for the purpose of this paper.

Proposition 5.2 *Let τ_p and σ_p be the complexified p-exterior representations of $K \simeq \mathrm{O}(n)$ and $M \simeq \mathrm{O}(n-1)$, respectively. If $0 \leq p \leq n$ and $p \neq \frac{n}{2}$, then*

$$\begin{aligned} \widehat{G}_{\tau_p} = &\left\{\pi_{\sigma_p,\nu} : \nu \in i\mathbb{R}_{\geq 0} \cup (0, \rho_p)\right\} \\ &\cup \left\{\pi_{\sigma_{p-1},\nu} : \nu \in i\mathbb{R}_{\geq 0} \cup (0, \rho_{p-1})\right\} \cup \{J_{\sigma_p,\rho_p}, J_{\sigma_{p-1},\rho_{p-1}}\}. \end{aligned}$$

Here $\rho_p = \rho_{\mathfrak{a}} - \min(p, n-1-p)$ *and* $\rho_{\mathfrak{a}} = \frac{n-1}{2}$. *In particular,*

$$\widehat{G}_{\tau_0} = \widehat{G}_{\mathbf{1}} = \left\{\pi_{\mathbf{1},\nu} : \nu \in i\mathbb{R}_{\geq 0} \cup (0, \rho_{\mathfrak{a}})\right\} \cup \{\mathbf{1}\}.$$

In the case $n = 2m$ *and* $p = m$, *one has*

$$\widehat{G}_{\tau_m} = \left\{\pi_{\sigma_{m-1},\nu} : \nu \in i\mathbb{R}_{\geq 0} \cup (0, \frac{1}{2})\right\}$$
$$\cup \left\{\pi_{\sigma_m,\nu} : \nu \in i\mathbb{R}_{\geq 0} \cup (0, \frac{1}{2})\right\} \cup \left\{D_m^+ \oplus D_m^-\right\}.$$

Here $D_{\frac{n}{2}}^+ \oplus D_{\frac{n}{2}}^-$ *is the sum of the two discrete series* $D_{\frac{n}{2}}^{\pm}$ *of* $\mathrm{SO}(n,1)_0$ *having lowest* K*-types* $\tau_{\frac{n}{2}}^{\pm}$.

Proof The spherical case, $p = 0$, is well known, so we assume $p > 0$. As mentioned, the unitarizable Langlands quotients $J_{\sigma,\nu}$ occur only at the endpoints of complementary series $\nu = \rho_\sigma$.

Since $\tau_p|_M = \sigma_p \oplus \sigma_{p-1}$ by Proposition 2.9, Frobenius reciprocity implies that $\pi_{\sigma,\nu}|_K$ contains τ_p if and only if $\sigma = \sigma_p$ or $\sigma = \sigma_{p-1}$.

Now for $n = 2m+1$ and $0 \leq p \leq m$ we have complementary series $\pi_{\sigma_p,\nu}$ for $0 < \nu < \rho_p = m - p$ (see [12, Prop. 49]) and a Langlands quotient J_{σ_p,ρ_p} containing τ_p. For the M-type σ_{p-1} we have the same description.

We note that in the extreme cases $p = 0$ and $p = n$, one gets $J_{\sigma_0,\rho_{\mathfrak{a}}} = \mathbf{1}$ and $J_{\sigma_n,\rho_{\mathfrak{a}}} = \det$.

For $p > m$, $\pi_{\sigma_p,\nu}$ has complementary series for $0 < \nu < \rho_p = p - m$ and a Langlands quotient J_{σ_p,ρ_p} at the endpoint, with lowest K-type τ_p. Since $\widehat{G}_d = \emptyset$, the description of $\widehat{G}_{\tau_p}$ for n odd is complete.

We now assume $n = 2m$. If $0 \leq p \leq m-1$ we have complementary series $\pi_{\sigma_p,\nu}$ again for $0 < \nu < \rho_p = m - \frac{1}{2} - p$ (see [12, Prop. 50]) and a Langlands quotient J_{σ_p,ρ_p}, both containing τ_p, with a similar description for σ_{p-1} in place of σ_p. For $p \geq m+1$, again $\pi_{\sigma_p,\nu}$ has complementary series for $0 < \nu < \rho_p = p - (m - \frac{1}{2})$ and a Langlands quotient at the endpoint. Furthermore, $\widehat{G}_{\tau_p} \cap \widehat{G}_d = \emptyset$ if $p \neq m$; hence, the description of $\widehat{G}_{\tau_p}$ is complete in this case.

Finally, if $p = m$, then $\widehat{G}_{\tau_p} \cap \widehat{G}_d = \{D_m^+ \oplus D_m^-\}$ and the unitary representations that contain τ_m are the unitary principal series and the complementary series $\pi_{\sigma,\nu}$ for $\sigma = \sigma_{m-1}, \sigma_m$ and $\nu \in i\mathbb{R} \cup (0, \frac{1}{2})$. Furthermore, at the endpoint $\frac{1}{2}$, the representations $\pi_{\sigma_m,\frac{1}{2}}$ and $\pi_{\sigma_{m-1},\frac{1}{2}}$ are reducible and the K-type τ_m is a K-type of the irreducible subrepresentation $D_m^+ \oplus D_m^-$ with multiplicity 1. This completes the proof. $\square$

Proposition 5.3 *For* $\nu \in \mathbb{C}$, *the Casimir eigenvalue for the representation* $\pi_{\sigma_p,\nu}$ *is given by*

$$\lambda(C, \pi_{\sigma_p,\nu}) = -\nu^2 + \rho_p^2 = -\nu^2 + \big(\rho_{\mathfrak{a}} - \min(p, n-1-p)\big)^2. \tag{5.1}$$

In particular, $\lambda(C, J_{\sigma_p,\rho_p}) = 0$ *for every* p. *Furthermore,* $\lambda(C, D_{\frac{n}{2}}^{\pm}) = 0$.

Proof It is well known that the Casimir eigenvalue for the principal series is given by

$$\lambda(C, \pi_{\sigma,\nu}) = -\nu^2 + \rho_{\mathfrak{a}}^2 - c_\sigma \tag{5.2}$$

where $c_\sigma = \langle \Lambda_\sigma + \rho_M, \Lambda_\sigma + \rho_M \rangle - \langle \rho_M, \rho_M \rangle$, Λ_σ is the highest weight of σ and

$$\rho_M = \begin{cases} \sum_{j=1}^{m} (m-j)\varepsilon_j & \text{if } n = 2m+1, \\ \sum_{j=1}^{m-1} (m-j-\frac{1}{2})\varepsilon_j & \text{if } n = 2m. \end{cases}$$

Furthermore, for $\sigma = \sigma_p \in \widehat{\mathrm{O}(n-1)}$ we have $\Lambda_{\sigma_p} = \sum_{j=1}^{\min(p,n-1-p)} \varepsilon_j$ (see Example 2.6).

Assume first that $0 \le p \le [\frac{n}{2}] = m$ and $p \ne \frac{n}{2}$. By a calculation one can see that

$$c_{\sigma_p} = \begin{cases} p + 2\sum_{j=1}^{p} (m-j) = p + 2mp - p(p+1) & \text{if } n \text{ is odd}, \\ p + 2\sum_{j=1}^{p} (m - \frac{1}{2} - j) = p + 2(m-\frac{1}{2})p - p(p+1) & \text{if } n \text{ is even}. \end{cases}$$

Thus, in light of (5.2),

$$\lambda(C, \pi_{\sigma_p,\nu}) = \begin{cases} -\nu^2 + (m-p)^2 & \text{if } n = 2m+1, \\ -\nu^2 + (m-p-\frac{1}{2})^2 & \text{if } n = 2m, \end{cases}$$

which establishes the formula.

On the other hand, in the cases when $p > [\frac{n}{2}]$, one has that $\lambda(C, \pi_{\sigma_p,\nu}) = \lambda(C, \pi_{\sigma_{n-1-p},\nu})$ and for $n = 2m$, $\lambda(C, D_m^{\pm}) = \lambda(C, \pi_{\sigma_m,\frac{1}{2}}) = 0$, as asserted. □

After all this preparation, we can prove the results in the Introduction for negatively curved manifolds.

Proof of Theorem 1.4 For each λ, set $\widehat{G}_{\tau_p,\lambda} = \{\pi \in G_{\tau_p} : \lambda(C, \pi) = \lambda\}$. If $p = 0$, then the representations in $\widehat{G}_{\mathbf{1},\lambda}$ for any fixed λ have the form $\pi_{\mathbf{1},\nu}$ with $\nu \in i\mathbb{R}_{\ge 0} \cup (0, \rho_{\mathfrak{a}})$ and the equality $\lambda(C, \pi_{\mathbf{1},\nu}) = -\nu^2 + \rho_{\mathfrak{a}}^2 = \lambda$ determines $\nu = \sqrt{\rho_{\mathfrak{a}}^2 - \lambda}$, where $\nu \in i\mathbb{R}_{\ge 0}$ if $\lambda \ge \rho_{\mathfrak{a}}^2$ and $\nu \in (0, \rho_{\mathfrak{a}}]$ otherwise.

Assume now that $0 < p \le [\frac{n}{2}]$. For $\lambda = 0$ we have

$$\widehat{G}_{\tau_p,0} = \begin{cases} \{J_{\sigma_p,\rho_p}, J_{\sigma_{p-1},\rho_{p-1}}\} & \text{if } p \ne \frac{n}{2}, \\ \{D_{\frac{n}{2}}^{+} \oplus D_{\frac{n}{2}}^{-}\} & \text{if } p = \frac{n}{2}. \end{cases}$$

Therefore,

$$d_0(\tau_p, \Gamma) = \begin{cases} n_\Gamma(J_{\sigma_p,\rho_p}) + n_\Gamma(J_{\sigma_{p-1},\rho_{p-1}}) & \text{if } p \ne \frac{n}{2}, \\ n_\Gamma(D_{\frac{n}{2}}^{+} \oplus D_{\frac{n}{2}}^{-}) & \text{if } p = \frac{n}{2}. \end{cases} \tag{5.3}$$

Now, let $\lambda > 0$. Since $\lambda(C, \pi_{\sigma_p,\nu}) = -\nu^2 + \rho_p^2 = \lambda$, then $\nu = \sqrt{\rho_p^2 - \lambda}$ where $\nu \in (0, \rho_p) \cup i\mathbb{R}_{\geq 0}$ and similarly for $\lambda(C, \pi_{\sigma_{p-1},\nu}) = \lambda$. Thus, we get

$$\widehat{G}_{\tau_p,\lambda} = \{\pi_{\sigma_p,\sqrt{\rho_p^2-\lambda}}, \ \pi_{\sigma_{p-1},\sqrt{\rho_{p-1}^2-\lambda}}\}$$

and

$$d_\lambda(\tau_p, \Gamma) = \begin{cases} n_\Gamma(\pi_{\sigma_p,\sqrt{\rho_p^2-\lambda}}) + n_\Gamma(\pi_{\sigma_{p-1},\sqrt{\rho_{p-1}^2-\lambda}}) & \text{if } p \neq \frac{n}{2}, \\ n_\Gamma(\pi_{\sigma_m,\sqrt{1/4-\lambda}}) + n_\Gamma(\pi_{\sigma_{m-1},\sqrt{1/4-\lambda}}) & \text{if } p = \frac{n}{2} = m. \end{cases} \tag{5.4}$$

This completes the proof for $p \leq [\frac{n}{2}]$. The case $p > [\frac{n}{2}]$ is similar. □

The following lemma is the analogue of Lemma 4.3 in the flat case.

Lemma 5.4 *Let Γ_1 and Γ_2 be discrete cocompact subgroups of* $\mathrm{SO}(n,1)$ *acting freely on $\mathbb{H}^n$. If Γ_1 and Γ_2 are τ_{p-1}-equivalent (or τ_{p+1}-equivalent) and the manifolds $\Gamma_1\backslash\mathbb{H}^n$ and $\Gamma_2\backslash\mathbb{H}^n$ are p-isospectral, then Γ_1 and Γ_2 are τ_p-equivalent. In particular,* 0*-isospectrality implies τ_0-equivalence.*

Proof Assume that $p \notin \{\frac{n}{2}, \frac{n}{2}+1\}$. Since Γ_1 and Γ_2 are τ_{p-1}-equivalent, we have

$$n_{\Gamma_1}(J_{\sigma_{p-1},\rho_{p-1}}) = n_{\Gamma_2}(J_{\sigma_{p-1},\rho_{p-1}})$$
$$n_{\Gamma_1}(\pi_{\sigma_{p-1},\nu}) = n_{\Gamma_2}(\pi_{\sigma_{p-1},\nu})$$

for every $\nu \in i\mathbb{R}_{\geq 0} \cup (0, \rho_{p-1})$ by Proposition 5.2. Now, by p-isospectrality we have that $d_\lambda(\tau_p, \Gamma_1) = d_\lambda(\tau_p, \Gamma_2)$ for every λ, thus (5.3) implies that $n_{\Gamma_1}(J_{\sigma_p,\rho_p}) = n_{\Gamma_2}(J_{\sigma_p,\rho_p})$ and (5.4) implies $n_{\Gamma_1}(\pi_{\sigma_p,\nu}) = n_{\Gamma_2}(\pi_{\sigma_p,\nu})$ for every $\nu \in i\mathbb{R}_{\geq 0} \cup (0, \rho_p)$. By Proposition 5.2, these equations imply τ_p-equivalence.

The remaining cases are proved similarly. □

Proof of Theorem 1.5 (noncompact case) The proof is exactly as in the flat case, since Lemmas 4.3 and 5.4 have exactly the same statements. □

Remark 5.5 One can also prove the above result for intervals decreasing from n, that is, *q-isospectrality for every $p \leq q \leq n$ is equivalent to τ_q-equivalence for every $p \leq q \leq n$.*

Remark 5.6 We now consider the Hodge decomposition of compact hyperbolic manifolds as in Remarks 3.5 and 4.7. One obtains here results that are very similar to those in the flat case. Namely,

> $\Gamma_1\backslash\mathbb{H}^n$ and $\Gamma_2\backslash\mathbb{H}^n$ are isospectral on exact (resp., coexact) p-forms if and only if $n_{\Gamma_1}(\pi_{\sigma_{p-1},\nu}) = n_{\Gamma_2}(\pi_{\sigma_{p-1},\nu})$ (resp., $n_{\Gamma_1}(\pi_{\sigma_p,\nu}) = n_{\Gamma_2}(\pi_{\sigma_p,\nu})$) for every $\nu \in i\mathbb{R}_{\geq 0} \cup (0, \rho_{p-1})$ (resp., $\nu \in i\mathbb{R}_{\geq 0} \cup (0, \rho_p)$).

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