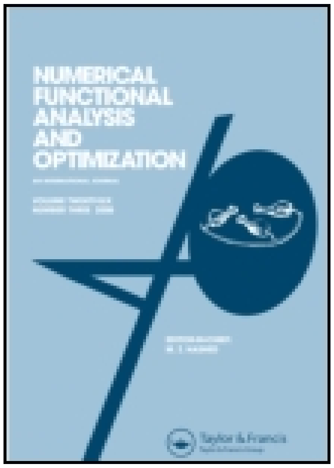

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BEST SIMULTANEOUS L^p-APPROXIMATION ON SMALL REGIONS

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□ *In this article, we study the behavior of best simultaneous L^p-approximation by algebraic polynomials on a union of intervals when the measure of them tend to zero. We also get an interpolation result.*

Keywords Algebraic polynomials; L^p-spaces; Simultaneous approximation.

Mathematics Subject Classification Primary 41A28; Secondary 41A10.

1. INTRODUCTION

Let $x_j \in \mathbb{R}$, $1 \leq j \leq k$, $k \in \mathbb{N}$, and let B_j be disjoint pairwise closed intervals centered at x_j and radius $\beta > 0$. Let $n \in \mathbb{N}$ and suppose that $n+1 = kc + d$, $c \in \mathbb{N} \cup \{0\}$, $0 \leq d < k$. We denote $\mathscr{C}^s(I)$, $s \in \mathbb{N} \cup \{0\}$, the space of real functions defined on $I := \cup_{j=1}^k B_j$, which are continuously differentiable up to order s on I. For simplicity, we write $\mathscr{C}(I)$ instead of $\mathscr{C}^0(I)$. Let Π^n be the class of algebraic polynomials of degree at most n.

If $\|\cdot\|$ denotes a norm defined on the space $\mathscr{C}(I)$ and $h \in \mathscr{C}(I)$, for each $0 < \epsilon \leq 1$, we write $\|h\|_\epsilon = \|h^\epsilon\|$, where $h^\epsilon(t) = h(\epsilon(t - x_j) + x_j)$, $t \in B_j$. We put $\|h\|_{p,I} := (\int_I |h(t)|^p \frac{dt}{|I|})^{1/p}, 1 < p < \infty$, where $|I|$ is the Lebesgue measure of I, and we denote $\|h\|_{p,\epsilon} = \|h^\epsilon\|_{p,I}$. If χ_{B_j} is the characteristic function of the set B_j, we write $\|h\|_{p,j} = \|h\chi_{B_j}\|_{p,I}$.

Given l functions $f_1, \ldots, f_l \in \mathscr{C}(I)$, set $P_\epsilon \in \Pi^n$, $0 < \epsilon \leq 1$, the best simultaneous approximation of them with respect to the semi-norm $\|\cdot\|_{p,\epsilon}$ (L^p-b.s.a.), that is,

$$\max_{1\leq i\leq l}\{\|f_i - P_\epsilon\|_{p,\epsilon}\} = \inf_{P\in\Pi^n} \max_{1\leq i\leq l}\{\|f_i - P\|_{p,\epsilon}\}.$$

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If the net $\{P_\epsilon\}$ has a limit in Π^n as $\epsilon \to 0$, this limit is called the *best simultaneous local L^p-approximation of f_i, $1 \le i \le l$, from Π^n on $\{x_1, \dots, x_k\}$* (L^p-b.s.l.a.).

Given $h, g \in \mathscr{C}(I)$, we will denote by $\gamma^+(h, g)$ the one-sided Gateaux derivative at h in the direction g, that is,

$$\gamma^+(h, g) := p \int_I |h(t)|^{p-1} sgn(h(t))g(t)\frac{dt}{|I|}. \tag{1}$$

For $c \in \mathbb{N}$ and $f_1, f_2 \in \mathscr{C}^{c-1}(I)$, we consider $A_j = \{i : 0 \le i \le c-1, f_1^{(i)}(x_j) \ne f_2^{(i)}(x_j)\}$, $1 \le j \le k$. If $A_j \ne \emptyset$ we write $m_j = \min A_j - 1$, otherwise $m_j = c - 1$. Now, we define $\overline{m} = \min\{m_j : 1 \le j \le k\}$. If $c = 0$, we define $\overline{m} = -1$. In the last case, no constrain over the derivatives of f_1 and f_2 is assumed. If $h \in \mathscr{C}^c(I)$, we will denote by

$$\mathscr{H}(h) := \{P \in \Pi^n : P^{(i)}(x_j) = h^{(i)}(x_j), 0 \le i \le c-1, 1 \le j \le k\},$$

$c \in \mathbb{N}$, and $\mathscr{H}(h) = \Pi^n$, $c = 0$. We also denote $\mathscr{M}(h)$ the set of polynomials $H \in \mathscr{H}(h)$ verifying

$$\sum_{i=1}^{k} |h^{(c)}(x_i) - H^{(c)}(x_i)|^p = \min_{P \in \mathscr{H}(h)} \sum_{i=1}^{k} |h^{(c)}(x_i) - P^{(c)}(x_i)|^p.$$

From the strictly convexity of the $l^p(\mathbb{R}^k)$-norm it is easy to prove that $\mathscr{M}(h)$ is a singleton.

The study of the behavior of best approximations on a small interval was introduced in [3] and [8]. In [2] and [7], the authors studied this problem for a single function and several intervals. Later, in [4, 5] it was considered the approximation simultaneous to two functions and one interval. In this article, we consider the last problem for many intervals. We have proved theorems of interpolation and existence of L^p-b.s.l.a., which generalize previous results of [4, 5, 7].

2. PRELIMINARY RESULTS

In this section, we obtain a general result about the asymptotic behavior of the error. We also get some lemmas, which will be used to obtain an interpolation result. We denote

$$E_\epsilon := \max\{\|f_1 - P_\epsilon\|_{p,\epsilon}^p, \|f_2 - P_\epsilon\|_{p,\epsilon}^p\}.$$

The following lemma was proved in [1].

Lemma 1. *Let $(X, \|\cdot\|)$ be a normed linear space. Let S be a finite dimensional subspace of X and let $f, g \in X$. If $p \in S$ is a b.s.a. to f and g, that is, p minimizes*

$$E(q) := \max\{\|f - q\|, \|g - q\|\}, \quad q \in S,$$

then $E(p) = \|f - p\| = \|g - p\|$ or p is a best approximation to the function where $E(p)$ is attained.

The following two results can be proved in a similar way to Theorems 2.1 and 2.6 in [5], respectively.

Theorem 2. *Let $f_1, f_2 \in \mathscr{C}^c(I)$, $0 < \epsilon \leq 1$ and let $P_\epsilon \in \Pi^n$ be the L^p-b.s.a. to f_1 and f_2 from Π^n. Then*

$$E_\epsilon^{1/p} = \frac{1}{2}\|H_1 - H_2\|_{p,\epsilon} + O(\epsilon^c), \quad \text{as } \epsilon \to 0,$$

where $H_i \in \mathscr{H}(f_i)$, $i = 1, 2$.

Lemma 3. *Let $f_1, f_2 \in \mathscr{C}^c(I)$, $0 < \epsilon \leq 1$ and let $P_\epsilon \in \Pi^n$ be the L^p-b.s.a. to f_1 and f_2 from Π^n. If $-1 \leq \overline{m} \leq c - 2$, there exists $\epsilon_0 > 0$ such that $\|f_1 - P_\epsilon\|_{p,\epsilon} = \|f_2 - P_\epsilon\|_{p,\epsilon}$, $0 < \epsilon \leq \epsilon_0$.*

We observe that if $-1 \leq \overline{m} \leq c - 2$, the hypotheses in [5], Proposition 2.2, are satisfied. Thus we have the following proposition.

Proposition 4. *Let $f_1, f_2 \in \mathscr{C}^c(I)$, $0 < \epsilon \leq 1$ and let $P_\epsilon \in \Pi^n$ be the L^p-b.s.a. to f_1 and f_2 from Π^n. If $-1 \leq \overline{m} \leq c - 2$, and $H_i \in \mathscr{H}(f_i)$, $i = 1, 2$, then*

$$\frac{1}{\epsilon^{\overline{m}+1}} \max_{1 \leq j \leq k} \max_{0 \leq i \leq c-1} \left| \left(P_\epsilon^\epsilon - \frac{(H_1 + H_2)^\epsilon}{2} \right)^{(i)} (x_j) \right| \to 0, \quad \text{as } \epsilon \to 0. \tag{2}$$

In particular, $(P_\epsilon^\epsilon)^{(i)}(x_j)$, $1 \leq j \leq k$, $0 \leq i \leq c - 1$, are bounded uniformly as $\epsilon \to 0$.

So, we have the following corollary.

Corollary 5. *Under the same hypotheses of Proposition 4, for all $1 \leq j \leq k$, we have*

a) $\lim_{\epsilon \to 0} P_\epsilon^{(i)}(x_j) = \frac{(f_1+f_2)^{(i)}(x_j)}{2}$, $0 \leq i \leq \overline{m} + 1$;
b) $\lim_{\epsilon \to 0} \epsilon^{i-\overline{m}-1} P_\epsilon^{(i)}(x_j) = 0$, $\overline{m} + 1 < i \leq c - 1$.

Now, replacing in [5], Corollary 2.4, x_1, m and $n + 1$ by x_j, $\overline{m}$ and c, respectively, we obtain the next corollary.

Corollary 6. *Let $f_1, f_2 \in \mathscr{C}^c(I)$, $0 < \epsilon \le 1$ and let $P_\epsilon \in \Pi^n$ be the L^p-b.s.a. to f_1 and f_2 from Π^n. If $-1 \le \overline{m} \le c-2$, then for $t \in B_j$, we have*

$$\lim_{\epsilon\to 0} \frac{(f_1^\epsilon - P_\epsilon^\epsilon)(t)}{\epsilon^{\overline{m}+1}} = \lim_{\epsilon\to 0} \frac{(P_\epsilon^\epsilon - f_2^\epsilon)(t)}{\epsilon^{\overline{m}+1}} = \frac{(f_1 - f_2)^{(\overline{m}+1)}(x_j)(t-x_j)^{\overline{m}+1}}{2(\overline{m}+1)!}, \quad (3)$$

where the convergence is uniform on B_j, $1 \le j \le k$. In addition, the equalities (3) hold replacing f_i by $H_i \in \mathscr{H}(f_i)$, $i = 1, 2$.

3. INTERPOLATION AND UNIFORM BOUNDEDNESS

In this section, we prove an interpolation result for the L^p-b.s.a. to two functions f_1 and f_2. As consequence we obtain that the net $\{P_\epsilon\}$ is uniformly bounded on compact sets.

We observe that if $\|f_1 - P_\epsilon\|_{p,\epsilon} = \|f_2 - P_\epsilon\|_{p,\epsilon}$, then the simultaneous approximation problem is equivalent to minimize the function $\|f_1 - P\|_{p,\epsilon}^p$, for $P \in \Pi^n$ with the constrain $\|f_1 - P\|_{p,\epsilon} = \|f_2 - P\|_{p,\epsilon}$. Given $P(x) = \sum_{i=0}^n a_i x^i$, we consider $\nabla(\|f_j - P\|_{p,\epsilon}^p) = \left(\frac{\partial \|f_j - P\|_{p,\epsilon}^p}{\partial a_0}, \ldots, \frac{\partial \|f_j - P\|_{p,\epsilon}^p}{\partial a_n}\right)$, $j = 1, 2$. If $\nabla(\|f_1 - P_\epsilon\|_{p,\epsilon}^p) \neq \nabla(\|f_2 - P_\epsilon\|_{p,\epsilon}^p)$, by the Lagrange multipliers method, there is a real number $\lambda(\epsilon)$ such that P_ϵ minimizes the real function

$$\|f_1 - P\|_{p,\epsilon}^p + \lambda(\epsilon)(\|f_1 - P\|_{p,\epsilon}^p - \|f_2 - P\|_{p,\epsilon}^p), \quad P \in \Pi^n. \quad (4)$$

From (1) and (4), we get

$$(\lambda(\epsilon)+1)\gamma^+((f_1 - P_\epsilon)^\epsilon, Q^\epsilon) - \lambda(\epsilon)\gamma^+((f_2 - P_\epsilon)^\epsilon, Q^\epsilon) = 0, \quad Q \in \Pi^n. \quad (5)$$

The next lemma immediately follows from Lemma 4.1 in [5] and Lemma 3.

Lemma 7. *Let $f_1, f_2 \in \mathscr{C}^c(I)$, $0 < \epsilon \le 1$ and let $P_\epsilon \in \Pi^n$ be the L^p-b.s.a. to f_1 and f_2 from Π^n. If $-1 \le \overline{m} \le c-2$, then there exists $\epsilon_1 > 0$ such that for all $0 < \epsilon \le \epsilon_1$ we have*

a) $\nabla(\|f_1 - P_\epsilon\|_{p,\epsilon}^p) \neq \nabla(\|f_2 - P_\epsilon\|_{p,\epsilon}^p)$, *and*
b) $-1 \le \lambda(\epsilon) \le 0$.

In addition, $\lambda(\epsilon) \to -\frac{1}{2}$ as $\epsilon \to 0$.

Next, we introduce some notation to prove an interpolation result.

Let $f_1, f_2 \in \mathscr{C}^c(I)$. Let $0 < \epsilon \le 1$, $y_i(\epsilon) := x_i + \epsilon\beta$, $y^i(\epsilon) := x_{i+1} - \epsilon\beta$, $1 \le i \le k-1$, and $I_\epsilon = \cup_{j=1}^k [x_j - \epsilon\beta, x_j + \epsilon\beta]$. If $g \in \mathscr{C}(I_\epsilon)$, we will denote

$$\mathscr{A}(g) = \{i : g(y_i(\epsilon))g(y^i(\epsilon)) < 0, 1 \le i \le k-1\} \quad \text{and} \quad k^\star(g) = \#\mathscr{A}(g),$$

where # denotes the cardinality of a set. If $k = 1$, $k^\star(g) = 0$. Let $\widetilde{f_1}, \widetilde{f_2} \in \mathscr{C}^c(co(I))$ extensions of f_1 and f_2, respectively, where $co(I)$ is the convex hull of I.

Set $h_\epsilon : co(I_\epsilon) \to \mathbb{R}$ the function

$$h_\epsilon := (\lambda(\epsilon) + 1)|\widetilde{f_1} - P_\epsilon|^{p-1} sgn(\widetilde{f_1} - P_\epsilon) - \lambda(\epsilon)|\widetilde{f_2} - P_\epsilon|^{p-1} sgn(\widetilde{f_2} - P_\epsilon), \quad (6)$$

where $\lambda(\epsilon)$, $0 < \epsilon \le \epsilon_1$, was introduced in (4).

Next, we establish the following result.

Theorem 8. *Let $f_1, f_2 \in \mathscr{C}^c(I)$, $0 < \epsilon \le 1$ and let $P_\epsilon \in \Pi^n$ be the L^p-b.s.a. to f_1 and f_2 from Π^n. If $-1 \le \overline{m} \le c - 2$ and $0 < \epsilon \le \epsilon_1$, then there are $n + 1$ different points of $co(I_\epsilon)$, say $z_1(\epsilon), \dots, z_{n+1}(\epsilon)$, such that at least $n + 1 - k^\star(h_\epsilon)$ points live in I_ϵ, and*

$$P_\epsilon(z_j(\epsilon)) = \delta(\epsilon)\widetilde{f_1}(z_j(\epsilon)) + (1 - \delta(\epsilon))\widetilde{f_2}(z_j(\epsilon)), \quad 1 \le j \le n + 1,$$

where $0 \le \delta(\epsilon) \le 1$.

Proof. If $|\{x \in I : P_\epsilon(x) = f_1(x)\}| > 0$ or $|\{x \in I : P_\epsilon(x) = f_2(x)\}| > 0$, the theorem is obvious, with $\delta(\epsilon) = 1$ or $\delta(\epsilon) = 0$, respectively.

Suppose that $|\{x \in I : P_\epsilon(x) = f_1(x)\}| = |\{x \in I : P_\epsilon(x) = f_2(x)\}| = 0$.

By (1), (5), and (6) we get

$$\int_{I_\epsilon} h_\epsilon(t) Q(t)\, dt = 0, \quad Q \in \Pi^n. \quad (7)$$

Suppose that h_ϵ exactly changes of sign in $z_1(\epsilon), \dots, z_s(\epsilon) \in I_\epsilon$, with $s < n + 1 - k^\star(h_\epsilon)$. We can choose $r_1(\epsilon), \dots, r_{k^\star(h_\epsilon)}(\epsilon)$, with $r_i(\epsilon) \in (y_i, y^i)$ such that $h_\epsilon(r_i(\epsilon)) = 0$, $i \in \mathscr{A}(h_\epsilon)$. Let $v := \eta \Pi_{i=1}^{s}(x - z_i(\epsilon))\Pi_{i \in \mathscr{A}(h_\epsilon)}(x - r_i(\epsilon))$, $\eta := \pm 1$ be such that v satisfies $h_\epsilon v \ge 0$ on I_ϵ and $h_\epsilon v > 0$ on a positive measure subset of I_ϵ. It contradicts (7), so $s \ge n + 1 - k^\star(h_\epsilon)$.

Let $x \in co(I_\epsilon)$ be such that $h_\epsilon(x) = 0$. Then

$$\begin{aligned} 0 = &(\lambda(\epsilon) + 1)|(\widetilde{f_1} - P_\epsilon)(x)|^{p-1} sgn((\widetilde{f_1} - P_\epsilon)(x)) \\ &+ (-\lambda(\epsilon))|(\widetilde{f_2} - P_\epsilon)(x)|^{p-1} sgn((\widetilde{f_2} - P_\epsilon)(x)). \end{aligned}$$

If $sgn((\widetilde{f_1} - P_\epsilon)(x)) = sgn((\widetilde{f_2} - P_\epsilon)(x))$, then

$$(\lambda(\epsilon) + 1)|(\widetilde{f_1} - P_\epsilon)(x)|^{p-1} + (-\lambda(\epsilon))|(\widetilde{f_2} - P_\epsilon)(x)|^{p-1} = 0.$$

By Lemma 7, $-1 \le \lambda(\epsilon) \le 0$. Therefore,

$$|(\widetilde{f_1} - P_\epsilon)(x)| = 0 = |(\widetilde{f_2} - P_\epsilon)(x)| \quad \text{and} \quad \widetilde{f_1}(x) = P_\epsilon(x) = \widetilde{f_2}(x).$$

If $sgn((\widetilde{f_1} - P_\epsilon)(x)) = -sgn((\widetilde{f_2} - P_\epsilon)(x))$, then

$$(\lambda(\epsilon)+1)^{\frac{1}{p-1}}((\widetilde{f_1} - P_\epsilon)(x)) = (-\lambda(\epsilon))^{\frac{1}{p-1}}((P_\epsilon - \widetilde{f_2})(x)), \text{ and it implies}$$

$$P_\epsilon(x) = \delta(\epsilon)\widetilde{f_1}(x) + (1-\delta(\epsilon))\widetilde{f_2}(x), \text{ where } \delta(\epsilon) = \frac{(\lambda(\epsilon)+1)^{\frac{1}{p-1}}}{(\lambda(\epsilon)+1)^{\frac{1}{p-1}} + (-\lambda(\epsilon))^{\frac{1}{p-1}}}. \qquad \square$$

We denote $l_j(\epsilon)$, $1 \le j \le k$, the cardinal of the set of points of B_j, where P_ϵ interpolates to the function $\delta(\epsilon)\widetilde{f_1} + (1-\delta(\epsilon))\widetilde{f_2}$. Then, we have the following corollary.

Corollary 9. *Under the same hypotheses of Theorem 8, there exists* j, $1 \le j \le k$, *such that* $l_j(\epsilon) \ge c$.

Proof. By Theorem 8, we get

$$l_1(\epsilon) + l_2(\epsilon) + \cdots + l_k(\epsilon) \ge n + m + 1 - k^\star(h_\epsilon) = kc + d - k^\star(h_\epsilon).$$

Suppose that $l_j(\epsilon) \le c - 1$ for all $1 \le j \le k$. Then

$$kc + d - k^\star(h_\epsilon) \le l_1(\epsilon) + l_2(\epsilon) + \cdots + l_k(\epsilon) \le k(c-1). \qquad (8)$$

Since $k^\star(h_\epsilon) \le k - 1$

$$k^\star(h_\epsilon) + k(c-1) \le kc - 1. \qquad (9)$$

From (8) and (9), we have $kc + d \le kc - 1$, a contradiction. $\square$

Next, we prove a result about uniform boundedness of a net of best simultaneous approximations.

Theorem 10. *Let* $f_1, f_2 \in \mathscr{C}^s(I)$, $s = \max\{c, n\}$, $0 < \epsilon \le 1$ *and let* $P_\epsilon \in \Pi^n$ *be the* L^p*-b.s.a. to* f_1 *and* f_2 *from* Π^n. *Then, the net* $\{P_\epsilon\}$ *is uniformly bounded on compact sets as* $\epsilon \to 0$.

Proof. Suppose that $\|f_1 - P_{\epsilon_s}\|_{p,\epsilon_s} \ne \|f_2 - P_{\epsilon_s}\|_{p,\epsilon_s}$, for some sequence $\epsilon_s \downarrow 0$. Without loss of generality we assume $\|f_2 - P_{\epsilon_s}\|_{p,\epsilon_s} < \|f_1 - P_{\epsilon_s}\|_{p,\epsilon_s}$. Then by Lemma 1 we have that P_{ϵ_s} are best approximations of the function f_2, and it is well known that P_{ϵ_s} converges to the only element of $\mathscr{M}(f_2)$ (see [7, Theorem 4]).

Now, we suppose that $\|f_1 - P_\epsilon\|_{p,\epsilon} = \|f_2 - P_\epsilon\|_{p,\epsilon}$ for $0 < \epsilon \le \epsilon_0$. We consider two cases.

a) If $-1 \le \overline{m} \le c-2$ and $0 < \epsilon \le \epsilon_1$, Theorem 8 implies there are $n+1$ points of $co(I_\epsilon)$, say $z_0(\epsilon) < \cdots < z_n(\epsilon)$, and $0 \le \delta(\epsilon) \le 1$ such that

$$P_\epsilon(z_i(\epsilon)) = \delta(\epsilon)\widetilde{f_1}(z_i(\epsilon)) + (1-\delta(\epsilon))\widetilde{f_2}(z_i(\epsilon)), \quad 0 \le i \le n.$$

Since the net $\{(z_0(\epsilon), \ldots, z_n(\epsilon))\}$ and $\delta(\epsilon)$ are bounded, we can find convergent subsequences. Suppose that $z_i(\epsilon_m) \to t_i$ and $\delta(\epsilon_m) \to \alpha$, as $\epsilon_m \to 0$. Clearly, $t_0 \le \cdots \le t_n$. Using the Newton's divided difference formula and the continuity of the divided differences we get

$$P_{\epsilon_m} \to H_{\{t_0,\ldots,t_n\}}(\alpha\widetilde{f_1} + (1-\alpha)\widetilde{f_2}),$$

where $H_{\{t_0,\ldots,t_n\}}(g)$ denotes the interpolation polynomial of g on $\{t_0, \ldots, t_n\}$.

b) If $\overline{m} = c-1$ and $H_1 \in \mathscr{H}(f_1)$, Theorem 2 implies $\|H_1 - P_\epsilon\|_{p,\epsilon} = O(\epsilon^c)$. We have $(H_1 - P_\epsilon)^\epsilon \in \Pi^n$ on B_j and

$$\|(H_1 - P_\epsilon)^\epsilon\|_{p,j} \le \|H_1 - P_\epsilon\|_{p,\epsilon}, \quad 1 \le j \le k. \tag{10}$$

By the equivalence of norms on Π^n, there exists $M > 0$ such that

$$\max_{0\le i\le n} |((H_1 - P_\epsilon)^\epsilon)^{(i)}(x_j)| \le M\|(H_1 - P_\epsilon)^\epsilon\|_{p,j}, \quad 1 \le j \le k.$$

From (10), it follows that

$$\begin{aligned}\max_{1\le j\le k}\max_{0\le i\le c} |H_1^{(i)}(x_j) - P_\epsilon^{(i)}(x_j)| &\le \max_{1\le j\le k}\max_{0\le i\le c} \epsilon^{i-c}|H_1^{(i)}(x_j) - P_\epsilon^{(i)}(x_j)| \\ &= \epsilon^{-c}\max_{1\le j\le k}\max_{0\le i\le c} |((H_1 - P_\epsilon)^\epsilon)^{(i)}(x_j)| \\ &\le M\|H_1 - P_\epsilon\|_{p,\epsilon}\epsilon^{-c} = O(1). \end{aligned} \tag{11}$$

In any case, we conclude that $\{P_\epsilon\}$ has a subsequence bounded, hence the net $\{P_\epsilon\}$ is uniformly bounded on compact sets as $\epsilon \to 0$. □

4. BEST SIMULTANEOUS LOCAL APPROXIMATION

In this section, we state results about convergence of b.s.a. Next theorem extends the one point result established in [4]. We consider a basis of Π^n, $\{u_{sv}\}_{\substack{1\le v\le k\\ 0\le s\le c-1}} \cup \{w_e\}_{1\le e\le d}$, which satisfies

$$u_{sv}^{(i)}(x_j) = \delta_{(i,j)(s,v)}, \quad w_e^{(i)}(x_j) = 0, \quad 0 \le i \le c-1, \quad 1 \le j \le k,$$

where δ is the Krönecker delta function. Let

$$A = \{Q \in \Pi^n : \lim_{n\to\infty} P_{\epsilon_n} = Q \text{ for some } \epsilon_n \downarrow 0\}.$$

Theorem 11. *Let $f_1, f_2 \in \mathscr{C}^c(I)$, $0 < \epsilon \leq 1$ and let $P_\epsilon \in \Pi^n$ be the L^p-b.s.a. to f_1 and f_2 from Π^n. Then*

a) *The set A is contained in the set $\mathscr{M}(f_1, f_2)$ of solutions of the following minimization problem:*

$$\min_{P\in\Pi^n} \max \left\{ \sum_{j=1}^{k} |(f_1 - P)^{(\overline{m}+1)}(x_j)|^p, \sum_{j=1}^{k} |(f_2 - P)^{(\overline{m}+1)}(x_j)|^p \right\} \qquad (12)$$

with the constrains $P^{(i)}(x_j) = \frac{(f_1+f_2)^{(i)}(x_j)}{2}$, $0 \leq i \leq \overline{m}$, $1 \leq j \leq k$.

b) *If $f_1, f_2 \in \mathscr{C}^s(I)$, where $s = \max\{c, n\}$, then $A \neq \emptyset$. In particular, if $\mathscr{M}(f_1, f_2)$ is unitary, this is the L^p-b.s.l.a. of f_1 and f_2 from Π^n on $\{x_1, \ldots, x_k\}$.*

Proof. a) Let $P_0 \in A$. By definition of A, there is a net $\epsilon \downarrow 0$ such that $P_\epsilon \to P_0$. By Theorem 2 and the definition of $\overline{m}$, we get $E_\epsilon^{1/p} = O(\epsilon^{\overline{m}+1})$. Therefore, if $H_1 \in \mathscr{H}(f_1)$, $\|H_1 - P_\epsilon\|_{p,\epsilon} = O(\epsilon^{\overline{m}+1})$ and

$$(f_1 - P_\epsilon)^{(i)}(x_j) = O(\epsilon^{\overline{m}+1-i}), \quad 0 \leq i \leq c-1, \ 1 \leq j \leq k. \qquad (13)$$

Similarly, we obtain $(f_2 - P_\epsilon)^{(i)}(x_j) = O(\epsilon^{\overline{m}+1-i})$, $0 \leq i \leq c-1$, $1 \leq j \leq k$.

From (13), we have

$$\lim_{\epsilon\to 0}(f_1 - P_\epsilon)^{(i)}(x_j)\epsilon^{i-\overline{m}-1} = d_{ij}, \quad 0 \leq i \leq \overline{m}, \ 1 \leq j \leq k \qquad (14)$$

for some subnet, that we again denote by ϵ. For $t \in B_j$ we have

$$\frac{(f_1 - P_\epsilon)^\epsilon(t)}{\epsilon^{\overline{m}+1}} = \sum_{i=0}^{\overline{m}} \frac{(f_1 - P_\epsilon)^{(i)}(x_j)}{i!}\epsilon^{i-(\overline{m}+1)}(t - x_j)^i + \frac{(f_1 - P_\epsilon)^{(\overline{m}+1)}(\epsilon(\xi_j(t) - x_j) + x_j)}{(\overline{m}+1)!}(t - x_j)^{\overline{m}+1},$$

where $\xi_j(t)$ belongs to the segment of ends t and x_j. From (14), we get

$$\lim_{\epsilon\to 0} \frac{(f_1 - P_\epsilon)^\epsilon(t)}{\epsilon^{\overline{m}+1}} = \sum_{i=0}^{\overline{m}} \frac{d_{ij}}{i!}(t - x_j)^i + \frac{(f_1 - P_0)^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!}(t - x_j)^{\overline{m}+1},$$

uniformly on B_j. Therefore,

$$\lim_{\epsilon\to 0}\left\|\frac{(f_1-P_\epsilon)^\epsilon}{\epsilon^{\overline{m}+1}}\right\|_{p,I}^p = \sum_{j=1}^k \left\|\sum_{i=0}^{\overline{m}} \frac{d_{ij}}{i!}(t-x_j)^i + \frac{(f_1-P_0)^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!}(t-x_j)^{\overline{m}+1}\right\|_{p,j}^p$$
$$\geq \sum_{j=1}^k \left|\frac{(f_1-P_0)^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!}\right|^p J_j^p, \tag{15}$$

where $J_j = \inf_{Q\in\Pi^{\overline{m}}} \|(t-x_j)^{\overline{m}+1} - Q(t)\|_{p,j}$. Clearly, (15) holds for f_2 instead of f_1. From (14), we can assume $P_0^{(i)}(x_j) = f_1^{(i)}(x_j)$ for $1 \leq j \leq k$, $0 \leq i \leq \overline{m}$, so we can write

$$P_0 = \sum_{v=1}^k \sum_{s=0}^{\overline{m}} f_1^{(s)}(x_v) u_{sv} + \sum_{e=1}^d \overline{b}_e w_e + \sum_{v=1}^k \sum_{s=\overline{m}+1}^{c-1} \overline{c}_{sv} u_{sv},$$

for some real numbers $\{\overline{b}_e\}_{1\leq e\leq d}$ and $\{\overline{c}_{sv}\}_{\substack{1\leq v\leq k\\ 0\leq s\leq c-1}}$. Given two sets of real numbers (independent of ϵ), say $\{c_{sv}\}_{\substack{1\leq v\leq k\\ 0\leq s\leq c-1}}$ and $\{b_e\}_{1\leq e\leq d}$, consider the following net of polynomials in Π^n,

$$Q_\epsilon = \sum_{v=1}^k \sum_{s=0}^{\overline{m}} (f_1^{(s)}(x_v) - c_{sv}\epsilon^{\overline{m}+1-s}) u_{sv} + \sum_{e=1}^d b_e w_e + \sum_{v=1}^k \sum_{s=\overline{m}+1}^{c-1} c_{sv} u_{sv}.$$

We observe that $Q_\epsilon^{(i)}(x_j) = f_1^{(i)}(x_j) - c_{ij}\epsilon^{\overline{m}+1-i}$, $1 \leq j \leq k$, $0 \leq i \leq \overline{m}$.

Let $h = \sum_{v=1}^k \sum_{s=0}^{\overline{m}} f_1^{(s)}(x_v) u_{sv} + \sum_{e=1}^d b_e w_e + \sum_{v=1}^k \sum_{s=\overline{m}+1}^{c-1} c_{sv} u_{sv}$. Expanding $(f_1 - Q_\epsilon)^\epsilon$ by its Taylor polynomial at x_j up to order $\overline{m}$, we obtain

$$\frac{(f_1-Q_\epsilon)^\epsilon(t)}{\epsilon^{\overline{m}+1}} = \sum_{i=0}^{\overline{m}} \frac{c_{ij}}{i!}(t-x_j)^i + \frac{(f_1-h)^{(\overline{m}+1)}(\epsilon(\xi_j(t)-x_j)+x_j)}{(\overline{m}+1)!}(t-x_j)^{\overline{m}+1}$$
$$+\sum_{v=1}^k\sum_{s=0}^{\overline{m}} \frac{c_{sv}\epsilon^{\overline{m}+1-s} u_{sv}^{(\overline{m}+1)}(\epsilon(\xi_j(t)-x_j)+x_j)}{(\overline{m}+1)!}(t-x_j)^{\overline{m}+1}, \quad t\in B_j,$$

where $\xi_j(t)$ belongs to the segment of ends t and x_j. Since $\lim_{\epsilon\to 0} \frac{(f_1-Q_\epsilon)^\epsilon(t)}{\epsilon^{\overline{m}+1}} = \sum_{i=0}^{\overline{m}} \frac{c_{ij}}{i!}(t-x_j)^i + \frac{(f_1-h)^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!}(t-x_j)^{\overline{m}+1}$, uniformly on B_j, we have

$$\lim_{\epsilon\to 0}\left\|\frac{(f_1-Q_\epsilon)^\epsilon}{\epsilon^{\overline{m}+1}}\right\|_{p,I}^p = \sum_{j=1}^k \left\|\sum_{i=0}^{\overline{m}} \frac{c_{ij}}{i!}(t-x_j)^i + \frac{(f_1-h)^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!}(t-x_j)^{\overline{m}+1}\right\|_{p,j}^p.$$

Let c_{ij}, $1 \le j \le k$, $0 \le i \le \overline{m}$, be such that $\sum_{i=0}^{\overline{m}} \frac{c_{ij}}{i!}(t-x_j)^i$ is the best approximation to $\frac{(f_1-h)^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!}(t-x_j)^{\overline{m}+1}$ respect to $\|\cdot\|_{p,j}$. Then

$$\lim_{\epsilon\to 0}\left\|\frac{(f_1-Q_\epsilon)^\epsilon}{\epsilon^{\overline{m}+1}}\right\|_{p,I}^p = \sum_{j=1}^k \left|\frac{(f_1-h)^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!}\right|^p J_j^p, \tag{16}$$

and, similarly, we get

$$\lim_{\epsilon\to 0}\left\|\frac{(f_2-Q_\epsilon)^\epsilon}{\epsilon^{\overline{m}+1}}\right\|_{p,I}^p = \sum_{j=1}^k \left|\frac{(f_2-h)^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!}\right|^p J_j^p. \tag{17}$$

From (15)–(17), and the continuity of the function $\max\{|x|,|y|\}$, we have

$$\begin{aligned}
&\max\left\{\sum_{j=1}^k \left|\frac{(f_1-P_0)^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!}\right|^p J_j^p, \sum_{j=1}^k \left|\frac{(f_2-P_0)^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!}\right|^p J_j^p\right\}\\
&\le \liminf_{\epsilon\to 0}\frac{E_\epsilon}{\epsilon^{(\overline{m}+1)p}} \le \lim_{\epsilon\to 0}\max\left\{\left\|\frac{(f_1-Q_\epsilon)^\epsilon}{\epsilon^{\overline{m}+1}}\right\|_{p,I}^p, \left\|\frac{(f_2-Q_\epsilon)^\epsilon}{\epsilon^{\overline{m}+1}}\right\|_{p,I}^p\right\}\\
&= \max\left\{\sum_{j=1}^k \left|\frac{(f_1-h)^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!}\right|^p J_j^p, \sum_{j=1}^k \left|\frac{(f_2-h)^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!}\right|^p J_j^p\right\},
\end{aligned} \tag{18}$$

for all $h = \sum_{v=1}^k \sum_{s=0}^{\overline{m}} f_1^{(s)}(x_v)u_{sv} + \sum_{e=1}^d b_e w_e + \sum_{v=1}^k \sum_{s=\overline{m}+1}^{c-1} c_{sv}u_{sv}$. On the other hand, J_j is a non null constant, in fact, $J_j = \inf_{Q\in\Pi^{\overline{m}}} \left(\int_{-\beta}^{\beta} |y^{\overline{m}+1} - Q(y)|^p \frac{dy}{|I|}\right)^{\frac{1}{p}}$, so it can be eliminated in (18). In addition, as $f_1^{(i)}(x_j) = f_2^{(i)}(x_j)$, $0 \le i \le \overline{m}$, $1 \le j \le k$, then $P_0^{(i)}(x_j) = \frac{(f_1+f_2)^{(i)}(x_j)}{2}$. The proof of a) is complete.

b) If $f_1, f_2 \in \mathscr{C}^s(I)$, where $s = \max\{c, n\}$, by Theorem 10 the net $\{P_\epsilon\}$ is uniformly bounded on compact sets, then there exists $P_0 \in \Pi^n$ such that $P_0 \in A$. From a) $P_0 \in \mathscr{M}(f_1, f_2)$. In particular, if $\mathscr{M}(f_1, f_2)$ is unitary, this is the L^p-b.s.l.a. of f_1 and f_2 from Π^n on $\{x_1, \ldots, x_k\}$. □

The following theorem gives sufficient conditions for that $\mathscr{M}(f_1, f_2)$ is a unitary set.

Theorem 12. *Let $f_1, f_2 \in \mathscr{C}^c(I)$, a) $\overline{m} = c-2$ and $d = 0$ or b) $\overline{m} = c-1$, then $\mathscr{M}(f_1, f_2)$ is a unitary set.*

Proof. We observe that the problem (12) is equivalent to find the b.s.a. to $(f_l^{(\overline{m}+1)}(x_1), \ldots, f_l^{(\overline{m}+1)}(x_k))$, $l = 1, 2$, respect to the $l^p(\mathbb{R}^k)$ norm from the convex set

$$A := \{(P^{(\overline{m}+1)}(x_1), \ldots, P^{(\overline{m}+1)}(x_k)) :$$
$$P \in \Pi^n \text{ and } P^{(i)}(x_j) = \frac{f_1^{(i)}(x_j) + f_2^{(i)}(x_j)}{2},\ 0 \leq i \leq \overline{m},\ 1 \leq j \leq k\}.$$

It is well known that this problem has a unique solution. Therefore, if $P_1, P_2 \in \Pi^n$ verify (12), we have $P_1^{(i)}(x_j) = P_2^{(i)}(x_j)$, $0 \leq i \leq \overline{m}+1$, $1 \leq j \leq k$. These conditions univocally determine the polynomial in the cases a) or b). So, $P_1 = P_2$. □

Remark 13. If a) and b) are not satisfying, it is easy to prove that $\mathcal{M}(f_1, f_2)$ is not a unitary set.

Theorem 14. *Let $f_1, f_2 \in \mathcal{C}^c(I)$, $0 < \epsilon \leq 1$ and let $P_\epsilon \in \Pi^n$ be the L^p-b.s.a. to f_1 and f_2 from Π^n. If $\|f_1 - P_{\epsilon_s}\|_{p,\epsilon_s} \neq \|f_2 - P_{\epsilon_s}\|_{p,\epsilon_s}$, for some sequence $\epsilon_s \downarrow 0$, then $\mathcal{M}(f_1, f_2) = \mathcal{M}(f_1)$ or $\mathcal{M}(f_1, f_2) = \mathcal{M}(f_2)$.*

Proof. Without loss of generality we suppose $\|f_1 - P_{\epsilon_s}\|_{p,\epsilon_s} < \|f_2 - P_{\epsilon_s}\|_{p,\epsilon_s}$, for some sequence $\epsilon_s \downarrow 0$, by Lemma 1 P_{ϵ_s} is the best approximation to f_2. Then, P_{ϵ_s} converges to the only element of $\mathcal{M}(f_2)$. Hence, Theorem 11 implies $\mathcal{M}(f_2) \subset \mathcal{M}(f_1, f_2)$.

On the other hand, from Lemma 3 we have $\overline{m} = c - 1$. Therefore, by Theorem 12, $\mathcal{M}(f_1, f_2)$ has an only element, in consequence $\mathcal{M}(f_1, f_2) = \mathcal{M}(f_2)$. □

5. CASE $p = 2$

Let $\mathcal{M}(f_1) = \{H_1\}$ and $\mathcal{M}(f_2) = \{H_2\}$ we prove that the L^2-b.s.a. to f_1 and f_2 from Π^n, say P_ϵ, converge to a convex combination of H_1 and H_2, as $\epsilon \to 0$. It is well known (see [6]), that there exists $\alpha_\epsilon \in [0, 1]$ such that

$$P_\epsilon = \alpha_\epsilon P_\epsilon^1 + (1 - \alpha_\epsilon) P_\epsilon^2, \tag{19}$$

where P_ϵ^l is the best approximation to f_l, $l = 1, 2$, from Π^n.

Theorem 15. *Let $f_1, f_2 \in \mathcal{C}^c(I)$, $0 < \epsilon \leq 1$, and let $P_\epsilon \in \Pi^n$ be the L^2-b.s.a. to f_1 and f_2 from Π^n. If $-1 \leq \overline{m} \leq c - 2$ then $\lim_{\epsilon \to 0} P_\epsilon = \frac{H_1 + H_2}{2}$.*

Proof. Let j, $1 \leq j \leq k$, be such that $(f_1 - f_2)^{(i)}(x_j) = 0$, $0 \leq i \leq \overline{m}$, and $(f_1 - f_2)^{(\overline{m}+1)}(x_j) \neq 0$. Thus,

$$(H_1 - H_2)(x) = \sum_{i=\overline{m}+1}^{n} \frac{(H_1 - H_2)^{(i)}(x_j)}{i!}(x - x_j)^i, \quad x \in \mathbb{R}. \tag{20}$$

If $Q_0(x) = \frac{(H_1-H_2)^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!}(x - x_j)^{\overline{m}+1}$, since $\overline{m} \leq c-2$ we get $Q_0(x) = \frac{(f_1-f_2)^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!}(x - x_j)^{\overline{m}+1} \neq 0$. Hence, from (20), we obtain

$$\lim_{\epsilon \to 0} \frac{\|H_1 - H_2\|_{2,\epsilon}}{\epsilon^{\overline{m}+1}} = \|Q_0\|_{2,I} > 0. \tag{21}$$

Writing $f_1 = \alpha_\epsilon f_1 + (1 - \alpha_\epsilon)f_1$, and taking into account (19) we have,

$$\begin{aligned} &\left| \|f_1 - P_\epsilon\|_{2,\epsilon} - (1-\alpha_\epsilon)\|H_1 - H_2\|_{2,\epsilon} \right| \\ &\quad \leq \alpha_\epsilon \|f_1 - P_\epsilon^1\|_{2,\epsilon} + (1-\alpha_\epsilon)\|f_1 - H_1\|_{2,\epsilon} + (1-\alpha_\epsilon)\|P_\epsilon^2 - H_2\|_{2,\epsilon}. \end{aligned}$$

Now, it is clear that

$$\|f_1 - P_\epsilon\|_{2,\epsilon} = (1-\alpha_\epsilon)\|H_1 - H_2\|_{2,\epsilon} + O(\epsilon^c). \tag{22}$$

Furthermore, as $\{\alpha_\epsilon\}$ is bounded, we can assume that $\alpha_{\epsilon_s} \to \alpha$. Now, (21) and (22) imply

$$\lim_{\epsilon_s \to 0} \frac{\|f_1 - P_{\epsilon_s}\|_{2,\epsilon_s}}{{\epsilon_s}^{\overline{m}+1}} = (1-\alpha)\|Q_0\|_{2,I}. \tag{23}$$

Similarly, we can obtain

$$\lim_{\epsilon_s \to 0} \frac{\|f_2 - P_{\epsilon_s}\|_{2,\epsilon_s}}{{\epsilon_s}^{\overline{m}+1}} = \alpha\|Q_0\|_{2,I}. \tag{24}$$

By Lemma 3, (23) and (24) we have $(1-\alpha)\|Q_0\|_{2,I} = \alpha\|Q_0\|_{2,I}$, so $\alpha = \frac{1}{2}$. The theorem immediately follows from (19) and [7, Theorem 4]. □

Theorem 16. *Let $f_1, f_2 \in \mathscr{C}^c(I)$, $0 < \epsilon \leq 1$ and let $P_\epsilon \in \Pi^n$ be the L^2-b.s.a. to f_1 and f_2 from Π^n, and $\overline{m} = c-1$.*

a) *If $\sum_{j=1}^{k} |(f_l - H_{l'})^{(c)}(x_j)|^2 \leq \sum_{j=1}^{k} |(f_{l'} - H_{l'})^{(c)}(x_j)|^2$, $1 \leq l,\ l' \leq 2$, $l \neq l'$, then $\lim_{\epsilon \to 0} P_\epsilon = H_{l'}$.*

b) *If a) does not hold, then $\lim_{\epsilon \to 0} P_\epsilon = \alpha H_1 + (1-\alpha)H_2$, where α is determined by*

$$\sum_{j=1}^{k} |(f_1 - (\alpha H_1 + (1-\alpha)H_2))^{(c)}(x_j)|^2 = \sum_{j=1}^{k} |(f_2 - (\alpha H_1 + (1-\alpha)H_2))^{(c)}(x_j)|^2.$$

Proof. By [7, Theorem 4], and (19), $A \neq \emptyset$. Since $\overline{m} = c-1$, Theorem 12 implies there exists $P_0 \in \Pi^n$ such that $\lim_{\epsilon\to 0} P_\epsilon = P_0$. In addition, $P_0 \in \mathscr{H}(f_1) = \mathscr{H}(f_2)$ and $(P_0^{(c)}(x_1), \dots, P_0^{(c)}(x_k))$ is the b.s.a. to the vectors $(f_l^{(c)}(x_1), \dots, f_l^{(c)}(x_k))$, $l = 1, 2$, respect to the $l^2(\mathbb{R}^k)$ norm from $\mathscr{H}(f_1)$. Let $F_l := f_l - \frac{H_1+H_2}{2}$, $l = 1, 2$, and let $Q_l := H_l - \frac{H_1+H_2}{2}$ be its best approximation from $\mathscr{H}(0)$. If Q_0 is the b.s.a. of F_1 and F_2 from $\mathscr{H}(0)$, we have $P_0 = Q_0 + \frac{H_1+H_2}{2}$. Since $Q_0 = \alpha Q_1 + (1-\alpha)Q_2$ for some $\alpha \in [0,1]$, we have $Q_0^{(c)}(x_j) = \alpha Q_1^{(c)}(x_j) + (1-\alpha)Q_2^{(c)}(x_j)$, $1 \le j \le k$. Now, a) and b) immediately follows from [6, p. 526]. □

6. CASE $p > 2$

If $\overline{m} = c-1$, by Theorem 11 and 12 there exists the L^p-b.s.l.a. of f_1 and f_2, from Π^n on $\{x_1, \dots, x_k\}$, and this is the only element of $\mathscr{M}(f_1, f_2)$. Now, we suppose $-1 \le \overline{m} \le c-2$, $\beta = 1$ and $2 < p < \infty$. Lemmas 3 and 7 imply that there exists $\epsilon_1 > 0$, such that for all $0 < \epsilon \le \epsilon_1$ we have $\|f_1 - P_\epsilon\|_{p,\epsilon} = \|f_2 - P_\epsilon\|_{p,\epsilon}$ and $\nabla(\|f_1 - P_\epsilon\|_{p,\epsilon}^p) \neq \nabla(\|f_2 - P_\epsilon\|_{p,\epsilon}^p)$. We consider the function $G(t) = |t|^{p-1} sgn(t)$, $t \in \mathbb{R}$. By the Mean Value Theorem we have $G(x) - G(y) = \mu(x,y)(x-y)$, where $\mu(x,y)$ is a continuous function defined by $\mu(x,y) = (p-1)|\eta_{x,y}|^{p-2}$, $x \neq y$, with $\eta_{x,y}$ in the segment of extremes x and y, and $\mu(x,x) = (p-1)|x|^{p-2}$. For $0 < \epsilon \le \epsilon_1$, we denote

$$F_{1,\epsilon} := (\lambda(\epsilon)+1)^{\frac{1}{p-1}} \frac{f_1 - P_\epsilon}{\epsilon^{\overline{m}+1}} \quad \text{and}$$
$$F_{2,\epsilon} := (-\lambda(\epsilon))^{\frac{1}{p-1}} \frac{P_\epsilon - f_2}{\epsilon^{\overline{m}+1}},$$

where $\lambda(\epsilon)$ was introduced earlier. Let $\xi_\epsilon := \max\{\| |F_{1,\epsilon}^\epsilon|^{p-2}\|_{\infty,I}, \| |F_{2,\epsilon}^\epsilon|^{p-2}\|_{\infty,I}\}$ and $w_\epsilon := \mu(F_{1,\epsilon}, F_{2,\epsilon})\xi_\epsilon^{-1}$. We consider the semi-norms on $\mathscr{C}(I)$ defined by

$$\|u\|_{w_\epsilon,2} := \left(\int_{I_\epsilon} |u(t)|^2 \frac{w_\epsilon(t)}{\epsilon} dt\right)^{\frac{1}{2}} \quad \text{and} \quad \|u\|_{w_\epsilon^\epsilon,2,j} := \left(\int_{B_j} |u(t)|^2 w_\epsilon^\epsilon(t)\, dt\right)^{\frac{1}{2}}.$$

Theorem 17. *Let $f_1, f_2 \in \mathscr{C}^c(I)$, $0 < \epsilon \le \epsilon_1$ and let $P_\epsilon \in \Pi^n$ be the L^p-b.s.a. to f_1 and f_2 from Π^n, then there exists $\alpha_\epsilon \in [0,1]$ such that P_ϵ is the best approximation to $\alpha_\epsilon f_1 + (1-\alpha_\epsilon) f_2$ respect to $\|\cdot\|_{w_\epsilon,2}$. In addition, $\alpha_\epsilon \to \frac{1}{2}$ and $\|\alpha_\epsilon f_1 + (1-\alpha_\epsilon) f_2 - P_\epsilon\|_{w_\epsilon,2} = O(\epsilon^c)$, as $\epsilon \to 0$.*

Proof. By Lemma 7, P_ϵ minimizes

$$\|f_1 - P\|_{p,\epsilon}^p + \lambda(\epsilon)(\|f_1 - P\|_{p,\epsilon}^p - \|f_2 - P\|_{p,\epsilon}^p), \quad P \in \Pi^n, \; 0 < \epsilon \le \epsilon_1.$$

From (5), the definition of G and (1) we get $\int_I (G(F^{\epsilon}_{1,\epsilon}) - G(F^{\epsilon}_{2,\epsilon}))Q^{\epsilon} = 0$, that is, $\int_I \mu(F^{\epsilon}_{1,\epsilon}, F^{\epsilon}_{2,\epsilon})(F_{1,\epsilon} - F_{2,\epsilon})^{\epsilon} Q^{\epsilon} = 0, Q \in \Pi^n$. Therefore, $\int_{I_\epsilon} (h_\epsilon - P_\epsilon)Q\, w_\epsilon = 0$, where $h_\epsilon := \alpha_\epsilon f_1 + (1-\alpha_\epsilon) f_2$ and $\alpha_\epsilon = \frac{(\lambda(\epsilon)+1)^{\frac{1}{p-1}}}{(\lambda(\epsilon)+1)^{\frac{1}{p-1}} + (-\lambda(\epsilon))^{\frac{1}{p-1}}}$. In consequence P_ϵ is the best approximation to h_ϵ respect to $\|\cdot\|_{w_\epsilon,2}$. Further, Lemma 7 implies $\alpha_\epsilon \to \frac{1}{2}$ as $\epsilon \to 0$.

The continuity of μ implies that there is $M_1 > 0$ such that $|w^{\epsilon}_{\epsilon}| \le M_1$ on I. Furthermore, there exists $M_2 > 0$ such that $|h^{\epsilon}_{\epsilon} - H^{\epsilon}_{\epsilon}| \le M_2 \epsilon^c$, on I, where $H_\epsilon \in \mathscr{H}(h_\epsilon)$. Then,

$$\frac{\|h_\epsilon - P_\epsilon\|_{w_\epsilon,2}}{\epsilon^c} \le \frac{\|h_\epsilon - H_\epsilon\|_{w_\epsilon,2}}{\epsilon^c} = \left(\int_I \left|\frac{h^{\epsilon}_{\epsilon} - H^{\epsilon}_{\epsilon}}{\epsilon^c}\right|^2 w^{\epsilon}_{\epsilon}\right)^{\frac{1}{2}} \le 2kM_1^{\frac{1}{2}} M_2. \qquad \square$$

Lemma 18. *Let $f_1, f_2 \in \mathscr{C}^c(I)$, and let $P_\epsilon \in \Pi^n$ be the L^p-b.s.a. to f_1 and f_2 from Π^n, then w^{ϵ}_{ϵ} uniformly converges on I to*

$$w(t) := \sum_{r=1}^{k} \frac{(p-1)|(f_1 - f_2)^{(\overline{m}+1)}(x_r)|^{p-2}}{\max_{1\le l\le k} |(f_1 - f_2)^{(\overline{m}+1)}(x_l)|^{p-2}} |t - x_r|^{(\overline{m}+1)(p-2)} \chi_{B_r}(t). \qquad (25)$$

Proof. Corollary 6 and Lemma 7 imply

$$\lim_{\epsilon\to 0} F^{\epsilon}_{1,\epsilon}(t) = \lim_{\epsilon\to 0} F^{\epsilon}_{2,\epsilon}(t) = \sum_{j=1}^{k} \left(\frac{1}{2}\right)^{\frac{1}{p-1}} \frac{(f_1 - f_2)^{(\overline{m}+1)}(x_j)}{2(\overline{m}+1)!} (t - x_j)^{\overline{m}+1} \chi_{B_j}(t),$$

uniformly on I. In addition,

$$\begin{aligned}\lim_{\epsilon\to 0} \xi_\epsilon &= \max_{t\in I} \left| \sum_{j=1}^{k} \left(\frac{1}{2}\right)^{\frac{1}{p-1}} \frac{(f_1 - f_2)^{(\overline{m}+1)}(x_j)}{2(\overline{m}+1)!} (t - x_j)^{\overline{m}+1} \chi_{B_j}(t) \right|^{p-2} \\ &= \max_{1\le j\le k} \left| \left(\frac{1}{2}\right)^{\frac{1}{p-1}} \frac{(f_1 - f_2)^{(\overline{m}+1)}(x_j)}{2(\overline{m}+1)!} \right|^{p-2}.\end{aligned}$$

On the other hand, by the continuity of the function μ, we have

$$\begin{aligned}&\lim_{\epsilon\to 0} \mu(F^{\epsilon}_{1,\epsilon}(t), F^{\epsilon}_{2,\epsilon}(t)) \\ &= \left| \sum_{j=1}^{k} \left(\frac{1}{2}\right)^{\frac{1}{p-1}} \frac{(f_1 - f_2)^{(\overline{m}+1)}(x_j)}{2(\overline{m}+1)!} (t - x_j)^{\overline{m}+1} \chi_{B_j}(t) \right|^{p-2} (p-1).\end{aligned}$$

Now, the lemma follows from the definition of w_ϵ. $\square$

If $\mathscr{C}_i := \{\sum_{m=0,m\neq i}^{n} c_m(x-x_j)^m : c_m \in \mathbb{R}\}$, let $Q_{i,j,\epsilon} \in \mathscr{C}_i$ be such that $0 < \|(x-x_j)^i - Q_{i,j,\epsilon}\|_{w_\epsilon^\epsilon,2,j} = \inf_{P\in\mathscr{C}_i} \|(x-x_j)^i - P\|_{w_\epsilon^\epsilon,2,j}$.

Lemma 19. *If $f_1^{(\overline{m}+1)}(x_j) \neq f_2^{(\overline{m}+1)}(x_j)$, then there exist ϵ', $0 < \epsilon' < \epsilon_1$, and N_j, such that*

$$|P^{(i)}(x_j)| \leq \frac{i!N_j}{\epsilon^i}\|P\|_{w_\epsilon,2}, \quad P \in \Pi^n, \ \ 0 \leq i \leq n, \ \ 0 < \epsilon < \epsilon'. \tag{26}$$

Proof. Let $P \in \Pi^n$, $0 \leq i \leq n$ and $\epsilon < \epsilon_1$. We consider $Q(x) = P(\epsilon(x - x_j) + x_j) = \sum_{i=0}^{n} c_i(x-x_j)^i$, and we assume $c_i \neq 0$. Since $|c_i|\|(x-x_j)^i - Q_{i,j,\epsilon}\|_{w_\epsilon^\epsilon,2,j} \leq \|Q\|_{w_\epsilon^\epsilon,2,j} \leq \|P\|_{w_\epsilon,2}$, if $R_{i,j,\epsilon} := \frac{1}{\|(x-x_j)^i - Q_{i,j,\epsilon}\|_{w_\epsilon^\epsilon,2,j}}$ then

$$\frac{\epsilon^i|P^{(i)}(x_j)|}{i!} = |c_i| \leq R_{i,j,\epsilon}\|P\|_{w_\epsilon,2}, \tag{27}$$

By Lemma 18 we can choose ϵ', $0 < \epsilon' < \epsilon_1$, $0 < \delta < 1$, and $\kappa > 0$ such that for all $t \in D_j := B_j - (x_j - \delta, x_j + \delta)$ and $0 < \epsilon < \epsilon'$, $w_\epsilon^\epsilon(t) \geq \kappa$. Thus, $\frac{1}{R_{i,j,\epsilon}} \geq \kappa^{\frac{1}{2}} \inf_{S\in\mathscr{C}_i}\left(\int_{D_j}|(x-x_j)^i - S|^2\right)^{\frac{1}{2}} =: \frac{1}{N_{i,j}}$, $0 < \epsilon < \epsilon'$. Finally, from (27) we obtain (26), with $N_j = \max_{0\leq i\leq n} N_{i,j}$. □

Lemma 20. *Let $f_1, f_2 \in \mathscr{C}^c(I)$, $0 < \epsilon \leq 1$, and let $P_\epsilon \in \Pi^n$ be the L^p-b.s.a. to f_1 and f_2 from Π^n. If $f_1^{(\overline{m}+1)}(x_j) \neq f_2^{(\overline{m}+1)}(x_j)$, then*

$$|(\alpha_\epsilon f_1 + (1-\alpha_\epsilon)f_2 - P_\epsilon)^{(i)}(x_j)| = O(\epsilon^{c-i}), \quad 0 \leq i \leq c-1, \ \ as\ \epsilon \to 0, \tag{28}$$

where α_ϵ was defined in Theorem 17.

Proof. By Theorem 17 we have $\|\alpha_\epsilon f_1 + (1-\alpha_\epsilon)f_2 - P_\epsilon\|_{w_\epsilon,2} = O(\epsilon^c)$, and $\|\alpha_\epsilon H_1 + (1-\alpha_\epsilon)H_2 - P_\epsilon\|_{w_\epsilon,2} = O(\epsilon^c)$, where $H_l \in \mathscr{H}(f_l)$, $l = 1,2$. Then, Lemma 19 implies (28). □

We denote $\|\cdot\|_{w,2,j}$, the norm on B_j given by

$$\|u\|_{w,2,j} := \left(\int_{B_j}|u(t)|^2 w(t)\,dt\right)^{\frac{1}{2}}, \quad u \in \mathscr{C}(I).$$

Let $\tau := \{j : 1 \leq j \leq k \text{ and } f_1^{(\overline{m}+1)}(x_j) \neq f_2^{(\overline{m}+1)}(x_j)\}$.

Now, we state a result about convergence of L^p-b.s.a., when $p > 2$ and $\#(\tau) = k$.

Theorem 21. *Let $f_1, f_2 \in \mathscr{C}^c(I)$, $0 < \epsilon \leq 1$ and let $P_\epsilon \in \Pi^n$ be the L^p-b.s.a. to f_1 and f_2 from Π^n. If $\#(\tau) = k$, then*

a) *The set A is contained in the set of solutions of the following minimization problem:*

$$\min_{P\in\Pi^n}\sum_{j=1}^{k}\left|\left(\frac{f_1+f_2}{2}-P\right)^{(c)}(x_j)\right|^2|(f_1-f_2)^{(\overline{m}+1)}(x_j)|^{p-2} \tag{29}$$

with the constrains $P^{(i)}(x_j)=\frac{(f_1+f_2)^{(i)}(x_j)}{2}$, $0\le i\le c-1,\ 1\le j\le k.$

b) *If $f_1,f_2\in\mathscr{C}^s(I)$, where $s=\max\{c,n\}$, then $A\ne\emptyset$. In particular, if the problem (29) has a unique solution, this is the L^p-b.s.l.a. of f_1 and f_2 from Π^n on $\{x_1,\dots,x_k\}$.*

Proof. a) Let $P_0\in A$. By definition of A, there is a net $\epsilon\downarrow 0$ such that $P_\epsilon\to P_0$. By Theorem 17, there exists $\alpha_\epsilon\in[0,1]$ such that P_ϵ is the best approximation to $\alpha_\epsilon f_1+(1-\alpha_\epsilon)f_2$ respect to $\|\cdot\|_{w_\epsilon,2}$, and $\alpha_\epsilon\to\frac{1}{2}$. If $h_\epsilon=\alpha_\epsilon f_1+(1-\alpha_\epsilon)f_2$, then $\lim_{\epsilon\to0}h_\epsilon=\frac{f_1+f_2}{2}=:h$, uniformly on I. For $1\le j\le k$, Lemma 20 implies

$$\lim_{\epsilon\to0}P_\epsilon^{(i)}(x_j)=h^{(i)}(x_j)\quad\text{and}\quad\lim_{\epsilon\to0}(h_\epsilon-P_\epsilon)^{(i)}(x_j)\epsilon^{i-c}=d_{ij},$$

for some subsequence, that we again denote ϵ, $0\le i\le c-1$.

Now, using Lemma 18, the theorem may be proved in the same way as Theorem 11 with h_ϵ, c, $\|\cdot\|_{w_\epsilon,2}$, $J_{j,w}$ instead of f_1, $\overline{m}+1$, $\|\cdot\|_{p,I}$, J_j, respectively, where $J_{j,w}=\min_{P\in\Pi^{c-1}}\|(t-x_j)^c-P(t)\|_{w,2,j}$. □

The following result shows that the problem (29) has a unique solution.

Theorem 22. *Let $f_1,f_2\in\mathscr{C}^c(I)$, $0<\epsilon\le1$ and let $P_\epsilon\in\Pi^n$ be the L^p-b.s.a. to f_1 and f_2 from Π^n. If $\#(\tau)=k$, then the problem (29) has a unique solution.*

Proof. The problem (29) is equivalent to find the best approximation to $\left(\frac{(f_1+f_2)^{(c)}(x_1)}{2},\dots,\frac{(f_1+f_2)^{(c)}(x_k)}{2}\right)$ respect to

$$\|(y_1,\dots,y_k)\|=\sum_{j=1}^{k}|y_j|^2|f_1^{(\overline{m}+1)}(x_j)-f_2^{(\overline{m}+1)}(x_j)|^{p-2},$$

from the convex set

$$A:=\{(P^{(c)}(x_1),\dots,P^{(c)}(x_k)):$$
$$P\in\Pi^n\text{ and }P^{(i)}(x_j)=\frac{f_1^{(i)}(x_j)+f_2^{(i)}(x_j)}{2},\ 0\le i\le c-1,\ 1\le j\le k\}.$$

If P_1 and P_2 are two solutions of (29), we have $P_1^{(i)}(x_j) = P_2^{(i)}(x_j)$, $0 \leq i \leq c$, $1 \leq j \leq k$. Now, as $n + 1 = kc + d$, it follows that $P_1 = P_2$. $\square$

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