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Polar varieties, Bertini's theorems and number of points of singular complete intersections over a finite field [☆]

Antonio Cafure [a,b,c], Guillermo Matera [b,c,*], Melina Privitelli [a,c,d]

[a] Ciclo Básico Común, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón III (1428), Buenos Aires, Argentina
[b] Instituto del Desarrollo Humano, Universidad Nacional de General Sarmiento, J.M. Gutiérrez 1150 (B1613GSX), Los Polvorines, Buenos Aires, Argentina
[c] National Council of Science and Technology (CONICET), Argentina
[d] Instituto de Ciencias, Universidad Nacional de General Sarmiento, J.M. Gutiérrez 1150 (B1613GSX), Los Polvorines, Buenos Aires, Argentina

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ABSTRACT

Let $V \subset \mathbb{P}^n(\bar{\mathbb{F}}_q)$ be a complete intersection defined over a finite field $\mathbb{F}_q$ of dimension r and singular locus of dimension at most s, and let $\pi : V \dashrightarrow \mathbb{P}^{s+1}(\bar{\mathbb{F}}_q)$ be a generic linear mapping. We obtain an effective version of the Bertini smoothness theorem concerning π, namely an explicit upper bound of the degree of a proper Zariski closed subset of $\mathbb{P}^{s+1}(\bar{\mathbb{F}}_q)$ which contains all the points defining singular fibers of π. For this purpose we make use of the concept of polar variety associated with the set of exceptional points of π. As a consequence, we obtain results of existence of smooth rational points of V, that is, conditions on q which imply that V has a smooth $\mathbb{F}_q$-rational point. Finally, for $s = r - 2$ and $s = r - 3$ we estimate the number of $\mathbb{F}_q$-rational points and smooth $\mathbb{F}_q$-rational points of V.

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* Corresponding author at: Instituto del Desarrollo Humano, Universidad Nacional de General Sarmiento, J.M. Gutiérrez 1150 (B1613GSX), Los Polvorines, Buenos Aires, Argentina.
E-mail addresses: acafure@ungs.edu.ar (A. Cafure), gmatera@ungs.edu.ar (G. Matera), mprivitelli@conicet.gov.ar (M. Privitelli).

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Multihomogeneous Bézout theorem
Deligne estimate
Hooley–Katz estimate

1. Introduction

Let $\mathbb{F}_q$ be the finite field of q elements and let $\overline{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$. We denote by $\mathbb{P}^n := \mathbb{P}^n(\overline{\mathbb{F}}_q)$ and $\mathbb{A}^n := \mathbb{A}^n(\overline{\mathbb{F}}_q)$ the n-dimensional projective and affine spaces defined over $\overline{\mathbb{F}}_q$ respectively. For any affine or projective variety V defined over $\mathbb{F}_q$, we denote by $V(\mathbb{F}_q)$ the set of $\mathbb{F}_q$-rational points of V, i.e., the set of points of V with coordinates in $\mathbb{F}_q$, and by $|V(\mathbb{F}_q)|$ its cardinality. Observe that, for any $r \geq 0$, we have

$$p_r := \left|\mathbb{P}^r(\mathbb{F}_q)\right| = q^r + \cdots + q + 1.$$

Let $V \subset \mathbb{P}^n$ be an ideal-theoretic complete intersection defined over $\mathbb{F}_q$, of dimension r, multidegree $\boldsymbol{d} := (d_1, \ldots, d_{n-r})$ and singular locus of dimension at most $s \geq 0$. In this paper we obtain estimates on $|V(\mathbb{F}_q)|$ and conditions on q which imply that $V(\mathbb{F}_q)$ is not empty. All these estimates and conditions will be expressed in terms of r, $\boldsymbol{d}$ and s.

In a fundamental work [13], P. Deligne has shown that if V is nonsingular, then

$$\left|\left|V(\mathbb{F}_q)\right| - p_r\right| \leq b'_r(n, \boldsymbol{d}) q^{\frac{r}{2}}, \tag{1}$$

where $b'_r(n, \boldsymbol{d})$ is the rth primitive Betti number of any nonsingular complete intersection of $\mathbb{P}^n$ of dimension r and multidegree $\boldsymbol{d}$ (see, e.g., [16, Theorem 4.1] for an explicit expression of $b'_r(n, \boldsymbol{d})$ in terms of n, r and $\boldsymbol{d}$).

This result has been extended by C. Hooley and N. Katz to singular complete intersections. More precisely, in [22] it is proved that if the singular locus of V has dimension at most $s \geq 0$, then

$$\left|V(\mathbb{F}_q)\right| = p_r + \mathcal{O}\left(q^{\frac{r+s+1}{2}}\right), \tag{2}$$

where the constant implied by the $\mathcal{O}$-notation depends only on n, r and $\boldsymbol{d}$, and it is not explicitly given.

In [16] (see also [17]), S. Ghorpade and G. Lachaud have obtained the following explicit version of (2):

$$\left|\left|V(\mathbb{F}_q)\right| - p_r\right| \leq b'_{r-s-1}(n-s-1, \boldsymbol{d}) q^{\frac{r+s+1}{2}} + C(n, r, \boldsymbol{d}) q^{\frac{r+s}{2}}, \tag{3}$$

where $C(n, r, \boldsymbol{d}) := 9 \cdot 2^{n-r}((n-r)d + 3)^{n+1}$ and $d := \max_{1 \leq i \leq n-r} d_i$.

From the point of view of the potential applications of (3), the fact that $C(n, r, \boldsymbol{d})$ depends exponentially on n may be inconvenient. This is particularly the case if V is a hypersurface, because $C(n,r,\boldsymbol{d})$ becomes exponential in the degree of V. For this reason, in [9] it is shown that, for V normal,

$$\big||V(\mathbb{F}_q)| - p_r\big| \leq b_1'(n-r+1,\boldsymbol{d})q^{r-\frac{1}{2}} + 2\big((n-r)d\delta\big)^2 q^{r-1}, \tag{4}$$

provided that $q > 2(n-r)d\delta+1$, where $\delta = d_1 \cdots d_{n-r}$ is the degree of V. This solves the exponential dependency on n of the error term in (3) for $s = r-2$ and q large enough.

1.1. Our contributions

A fundamental tool for our work is an effective version of the Bertini smoothness theorem. The Bertini smoothness theorem asserts that a generic $(r-s-1)$-dimensional linear section of a variety $V \subset \mathbb{P}^n$ of dimension r and singular locus of dimension at most s is nonsingular. With notations and assumptions as above, an effective version of this result establishes a threshold $C(n,r,s,\boldsymbol{d})$ such that, if V is a singular complete intersection, then for $q > C(n,r,s,\boldsymbol{d})$ there exists a nonsingular linear section of V of dimension $r-s-1$ defined over $\mathbb{F}_q$. In this paper we show the following result (see Theorem 6.4 and Corollary 6.6).

Theorem 1.1. *Let $V \subset \mathbb{P}^n$ be a complete intersection defined over $\mathbb{F}_q$, of dimension r, degree δ, multidegree $\boldsymbol{d} := (d_1,\ldots,d_{n-r})$ and singular locus of dimension at most $s \geq 0$. Let $D := \sum_{i=1}^{n-r}(d_i-1)$. Then for $q > (n+1)^2 D^{r-s-1}\delta$ there exist nonsingular $(r-s-1)$-dimensional linear sections of V defined over $\mathbb{F}_q$.*

We remark that [1] and [9] provide effective versions of the Bertini smoothness theorem for hypersurfaces and normal complete intersections respectively. Theorem 1.1 significantly improves and generalizes both results.

The linear sections underlying Theorem 1.1 are obtained as (the Zariski closure of) fibers of a "generic" linear mapping $\pi : V \dashrightarrow \mathbb{P}^{s+1}$. For this purpose, it is necessary to analyze the set S of critical points of π. Our treatment of the set S relies on the notion of polar varieties. Polar varieties are a classical concept of projective geometry which, in its modern formulation, was introduced in the 1930's by F. Severi and J. Todd. Around 1975 a renewal of the theory of polar varieties took place with essential contributions due to R. Piene [29], B. Teissier [35] and others (see [36] for a historical account and references). Our main result in connection with polar varieties is a genericity condition on π which implies that the polar variety associated with the exceptional locus of π has the expected dimension (Theorem 4.5).

More precisely, let $\boldsymbol{\lambda} \in (\mathbb{P}^n)^{s+2}$ denote the matrix of coefficients of the linear forms defining π. We show that there exists a hypersurface of $(\mathbb{P}^n)^{s+2}$ which contains all the points $\boldsymbol{\lambda}$ for which the exceptional locus of π has not the expected dimension. To bound the degree of this hypersurface we use tools from intersection theory for products of projective spaces, as a multiprojective version of the Bézout theorem (see, e.g., [11, Theorem 1.11]). Combining this with the results on the number of $\mathbb{F}_q$-rational points of multiprojective hypersurfaces of Section 3 we obtain suitable bounds on the number of nonsingular linear sections of V defined over $\mathbb{F}_q$.

Next we obtain conditions on q which imply that V has a smooth $\mathbb{F}_q$-rational point. A classical problem is that of establishing conditions which imply that a variety has an $\mathbb{F}_q$-rational point. Nevertheless, in several number-theoretical applications is not just an $\mathbb{F}_q$-rational point what is required, but a *smooth* $\mathbb{F}_q$-rational point (see, e.g., [26,38,39]).

A standard approach to this question consists of combining a lower bound for the number of $\mathbb{F}_q$-rational points of V with an upper bound for the number of singular $\mathbb{F}_q$-rational points of V. Instead of doing this, we use our effective Bertini theorem, that is, we obtain a condition on q which implies that there exists a nonsingular $(r-s-1)$-dimensional linear section S of V defined over $\mathbb{F}_q$, and apply Deligne's estimate (1) to this section. As S is contained in the smooth locus $V_{\mathrm{sm}} := V \setminus \mathrm{Sing}(V)$, the existence of an $\mathbb{F}_q$-rational point of S implies that of a smooth $\mathbb{F}_q$-rational point of V. More precisely, we obtain the following result (see Corollaries 7.3 and 7.4).

Theorem 1.2. *Let $V \subset \mathbb{P}^n$ be a complete intersection defined over $\mathbb{F}_q$, of dimension r, degree δ, multidegree $\boldsymbol{d}$ and singular locus of dimension at most s. Let $D := \sum_{i=1}^{n-r}(d_i-1)$. If either $s = r-2$ and $q > 2(D+2)^2\delta^2$, or $s = r-3$ and $q > 3D(D+2)^2\delta$, then V has a smooth $\mathbb{F}_q$-rational point.*

Finally, we estimate the number of $\mathbb{F}_q$-rational points and smooth $\mathbb{F}_q$-rational points of a complete intersection with a singular locus of dimension at most $r-2$ or $r-3$. For this purpose, assuming that there exists a linear mapping $\pi : V \dashrightarrow \mathbb{P}^{s+1}$ defined over $\mathbb{F}_q$ which is generic in the sense above, we express V as the union of $p_{s+1} := |\mathbb{P}^{s+1}(\mathbb{F}_q)|$ linear sections of V of dimension $r-s-1$, namely the Zariski closure of the fibers of the points of $\mathbb{P}^{s+1}(\mathbb{F}_q)$ under π. "Most" fibers will be nonsingular and thus Deligne's estimate can be applied to them, while the $\mathbb{F}_q$-rational points lying in the remaining fibers do not make a significant contribution to the estimate. Summarizing, we obtain the following result (see Corollaries 8.3 and 8.4).

Theorem 1.3. *Let $V \subset \mathbb{P}^n$ be a complete intersection defined over $\mathbb{F}_q$, of dimension r, degree δ, multidegree $\boldsymbol{d}$ and singular locus of dimension at most $s \in \{r-3, r-2\}$. Let $D := \sum_{i=1}^{n-r}(d_i-1)$. Then, for $s = r-2$,*

$$\big||V(\mathbb{F}_q)| - p_r\big| \le \big(\delta(D-2)+2\big)q^{r-1/2} + 14D^2\delta^2 q^{r-1},$$
$$\big||V_{\mathrm{sm}}(\mathbb{F}_q)| - p_r\big| \le \big(\delta(D-2)+2\big)q^{r-1/2} + 11(r+1)D^2\delta^2 q^{r-1}.$$

On the other hand, for $s = r-3$,

$$\big||V(\mathbb{F}_q)| - p_r\big| \le 14D^3\delta^2 q^{r-1},$$
$$\big||V_{\mathrm{sm}}(\mathbb{F}_q)| - p_r\big| \le (34r-20)D^3\delta^2 q^{r-1}.$$

Our estimates have the same pattern as (3) for $s = r - 2$ or $s = r - 3$, but differ from (3) in that the exponential dependency on n is not present. In Section 8 we show that Theorem 1.3 yields a more accurate estimate than (3) when $s = r-2$ and $s = r-3$ for varieties of large dimension, say $r \geq (n+1)/2$, or small degree, say $\delta \leq (2(n-r))^{n-r}$. On the other hand, (3) may be preferable to Theorem 1.3 for varieties of small dimension and large degree. In this sense, we may say that Theorem 1.3 complements (3) for $s = r - 2$ and $s = r - 3$. Finally, Theorem 1.3 improves (4) for normal varieties since it holds without restrictions on q.

2. Notions, notations and preliminary results

We use standard notions and notations of commutative algebra and algebraic geometry as can be found in, e.g., [18], [25] or [33].

2.1. Basic notions

Let K be any of the fields $\mathbb{F}_q$ or $\overline{\mathbb{F}}_q$. We denote by $\mathbb{A}^n$ the n-dimensional affine space $\overline{\mathbb{F}}_q^n$ and by $\mathbb{P}^n$ the n-dimensional projective space over $\overline{\mathbb{F}}_q^{n+1}$. Both spaces are endowed with their respective Zariski topologies over K, for which a closed set is the zero locus of a set of polynomials of $\mathsf{K}[X_1, \dots, X_n]$, or of a set of homogeneous polynomials of $\mathsf{K}[X_0, \dots, X_n]$.

We say that a subset $V \subset \mathbb{P}^n$ is a *projective variety defined over* K (or a projective K-variety for short) if it is the set of common zeros in $\mathbb{P}^n$ of a family of homogeneous polynomials $F_1, \dots, F_m \in \mathsf{K}[X_0, \dots, X_n]$. Correspondingly, an *affine variety of* $\mathbb{A}^n$ *defined over* K (or an affine K-variety for short) is the set of common zeros in $\mathbb{A}^n$ of polynomials $F_1, \dots, F_m \in \mathsf{K}[X_1, \dots, X_n]$. We think a projective or affine K-variety to be equipped with the induced Zariski topology. We shall frequently denote by $V(F_1, \dots, F_m)$ or $\{F_1 = 0, \dots, F_m = 0\}$ the affine or projective K-variety consisting of the common zeros of the polynomials $F_1, \dots, F_m$.

In what follows, unless otherwise stated, all results referring to varieties in general should be understood as valid for both projective and affine varieties.

A K-variety V is K-*irreducible* if it cannot be expressed as a finite union of proper K-subvarieties of V. Further, V is *absolutely irreducible* if it is $\overline{\mathbb{F}}_q$-irreducible as an $\overline{\mathbb{F}}_q$-variety. Any K-variety V can be expressed as an irredundant union $V = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_s$ of irreducible (absolutely irreducible) K-varieties, unique up to reordering, which are called the *irreducible* (*absolutely irreducible*) K-*components* of V.

For a K-variety V contained in $\mathbb{P}^n$ or $\mathbb{A}^n$, we denote by $I(V)$ its *defining ideal*, namely the set of polynomials of $\mathsf{K}[X_0, \dots, X_n]$, or of $\mathsf{K}[X_1, \dots, X_n]$, vanishing on V. The *coordinate ring* $\mathsf{K}[V]$ of V is defined as the quotient ring $\mathsf{K}[X_0, \dots, X_n]/I(V)$ or $\mathsf{K}[X_1, \dots, X_n]/I(V)$. The *dimension* $\dim V$ of V is the length r of the longest chain $V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r$ of nonempty irreducible K-varieties contained in V. We call V

equidimensional if all the irreducible K-components of V are of the same dimension. In this case, we say that V has *pure dimension* r if V is equidimensional of dimension r.

A K-variety in $\mathbb{P}^n$ or $\mathbb{A}^n$ of pure dimension $n-1$ is called a K-*hypersurface*. A K-hypersurface in $\mathbb{P}^n$, or $\mathbb{A}^n$, is the set of zeros of a single nonzero polynomial of $\mathsf{K}[X_0,\ldots,X_n]$, or of $\mathsf{K}[X_1,\ldots,X_n]$.

Degree. The *degree* $\deg V$ of an irreducible K-variety V is the maximum number of points lying in the intersection of V with a linear space L of codimension $\dim V$, for which $V\cap L$ is a finite set. More generally, following [19] (see also [15]), if $V=\mathcal{C}_1\cup\cdots\cup\mathcal{C}_s$ is the decomposition of V into irreducible K-components, we define the degree of V as

$$\deg V := \sum_{i=1}^{s}\deg\mathcal{C}_i.$$

The degree of a K-hypersurface V is the degree of a polynomial of minimal degree defining V. Another property is that the degree of a dense open subset of a K-variety V is equal to the degree of V.

An important tool for our estimates is the following *Bézout inequality* (see [19,15,37]): if V and W are K-varieties of the same ambient space, then

$$\deg(V\cap W)\le \deg V\cdot\deg W. \tag{5}$$

Another result we shall use concerns the behavior of degree under linear mappings. Let $V\subset\mathbb{P}^m$ and $W\subset\mathbb{P}^n$ be K-varieties and let $\phi: V\to W$ be a regular linear map. Then (see, e.g., [9, Lemma 2.1])

$$\deg\overline{\phi(V)}\le\deg V, \tag{6}$$

where $\overline{\phi(V)}$ is the Zariski closure of $\phi(V)$ in $\mathbb{P}^n$, $\deg\overline{\phi(V)}$ denotes the degree of $\overline{\phi(V)}$ as a K-subvariety of $\mathbb{P}^n$ and $\deg V$ denotes the degree of V as a K-subvariety of $\mathbb{P}^m$.

Singular locus. Let $V\subset\mathbb{A}^n$ be a K-variety and let $I(V)\subset\mathsf{K}[X_1,\ldots,X_n]$ be its defining ideal. Let x be a point of V. The *dimension* $\dim_x V$ *of* V *at* x is the maximum of the dimensions of the irreducible K-components of V that contain x. If $I(V)=(F_1,\ldots,F_m)$, the *tangent space* T_xV *to* V *at* x is the kernel of the Jacobian matrix $(\partial F_i/\partial X_j)_{1\le i\le m,1\le j\le n}(x)$ of the polynomials $F_1,\ldots,F_m$ with respect to $X_1,\ldots,X_n$ at x. We have (see, e.g., [33, p. 94])

$$\dim T_xV\ge\dim_x V.$$

The point x is *regular* if $\dim T_xV=\dim_x V$. Otherwise, the point x is called *singular*. The set of singular points of V is the *singular locus* $\mathrm{Sing}(V)$ of V; this is a closed K-subvariety of V. A variety is called *nonsingular* if its singular locus is empty. For a projective variety, the concepts of tangent space, regular and singular point can be defined by considering an affine neighborhood of the point under consideration.

Mappings. Regular maps will be represented by solid arrows $\rightarrow$, while partial rational maps will be indicated with dashed arrows $\dashrightarrow$. Let V and W be irreducible affine K-varieties of the same dimension and let $f : V \rightarrow W$ be a regular map for which $\overline{f(V)} = W$, where $\overline{f(V)}$ is the closure of $f(V)$ with respect to the Zariski topology of W. Such a map is called *dominant*. Then f induces a ring extension $\mathsf{K}[W] \hookrightarrow \mathsf{K}[V]$ by composition with f. We say that the dominant map f is a *finite morphism* if this extension is integral, i.e., each element $\eta \in \mathsf{K}[V]$ satisfies a monic equation with coefficients in $\mathsf{K}[W]$. A basic fact is that a dominant finite morphism is necessarily closed. Another fact concerning dominant finite morphisms we shall use is that the preimage $f^{-1}(S)$ of an irreducible closed subset $S \subset W$ is equidimensional of dimension $\dim S$ (see, e.g., [12, §4.2, Proposition]).

2.2. Rational points

We denote by $\mathbb{A}^n(\mathbb{F}_q)$ the n-dimensional $\mathbb{F}_q$-vector space $\mathbb{F}_q^n$ and by $\mathbb{P}^n(\mathbb{F}_q)$ the set of 1-dimensional subspaces of the $(n+1)$-dimensional $\mathbb{F}_q$-vector space $\mathbb{F}_q^{n+1}$. For a projective variety $V \subset \mathbb{P}^n$ or an affine variety $V \subset \mathbb{A}^n$, we denote by $V(\mathbb{F}_q)$ the set of $\mathbb{F}_q$-rational points of V, namely $V(\mathbb{F}_q) := V \cap \mathbb{P}^n(\mathbb{F}_q)$ in the projective case and $V(\mathbb{F}_q) := V \cap \mathbb{A}^n(\mathbb{F}_q)$ in the affine case.

For a projective variety V of dimension r and degree δ we have the upper bound (see [17, Proposition 12.1] or [9, Proposition 3.1])

$$\left|V(\mathbb{F}_q)\right| \leq \delta p_r. \tag{7}$$

On the other hand, if V is an affine variety of dimension r and degree δ, then (see, e.g., [8, Lemma 2.1])

$$\left|V(\mathbb{F}_q)\right| \leq \delta q^r. \tag{8}$$

2.3. Complete intersections

A K-variety V of dimension r in an n-dimensional (affine or projective) space is an (ideal-theoretic) *complete intersection* if its ideal $I(V)$ over K can be generated by $n-r$ polynomials. If $V \subset \mathbb{P}^n$ is a complete intersection defined over K, of dimension r and degree δ, and $F_1, \dots, F_{n-r}$ is a system of homogeneous generators of $I(V)$, the degrees $d_1, \dots, d_{n-r}$ depend only on V and not on the system of generators. Arranging the d_i in such a way that $d_1 \geq d_2 \geq \cdots \geq d_{n-r}$, we call $\boldsymbol{d} := (d_1, \dots, d_{n-r})$ the *multidegree* of V.

According to the Bézout inequality (5), if $V \subset \mathbb{P}^n$ is a complete intersection defined over K of multidegree $\boldsymbol{d} := (d_1, \dots, d_{n-r})$, then $\deg V \leq \prod_{i=1}^{n-r} d_i$. Actually, a much stronger result holds, which is sometimes called the *Bézout theorem* (see, e.g., [18, Theorem 18.3] or [34, §5.5, p. 80]):

$$\deg V = d_1 \cdots d_{n-r}.$$

In what follows we shall deal with a particular class of complete intersections, which we now define. A K-variety V is *regular in codimension* m if its singular locus $\mathrm{Sing}(V)$ has codimension at least $m+1$ in V, i.e., $\dim V - \dim \mathrm{Sing}(V) \geq m+1$. A complete intersection V which is regular in codimension 1 is called *normal* (actually, normality is a general notion that agrees on complete intersections with the one we define here). A fundamental result for projective complete intersections is the Hartshorne connectedness theorem (see, e.g., [25, Theorem VI.4.2]), which we now state. If $V \subset \mathbb{P}^n$ is a complete intersection defined over K and $W \subset V$ is any K-subvariety of codimension at least 2, then $V \setminus W$ is connected in the Zariski topology of $\mathbb{P}^n$ over K. Applying the Hartshorne connectedness theorem with $W := \mathrm{Sing}(V)$, one deduces the following result.

Theorem 2.1. *If $V \subset \mathbb{P}^n$ is a normal complete intersection, then V is absolutely irreducible.*

2.4. Multiprojective space

Let $\mathbb{N} := \mathbb{Z}_{\geq 0}$ be the set of nonnegative integers. For $\boldsymbol{n} := (n_1, \ldots, n_m) \in \mathbb{N}^m$, we define $|\boldsymbol{n}| := n_1 + \cdots + n_m$ and $\boldsymbol{n}! := n_1! \cdots n_m!$. Given $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^m$, we write $\boldsymbol{\alpha} \geq \boldsymbol{\beta}$ whenever $\alpha_i \geq \beta_i$ holds for $1 \leq i \leq m$. For $\boldsymbol{d} := (d_1, \ldots, d_m) \in \mathbb{N}^m$, the set $\mathbb{N}_{\boldsymbol{d}}^{\boldsymbol{n}+\boldsymbol{1}} := \mathbb{N}_{d_1}^{n_1+1} \times \cdots \times \mathbb{N}_{d_m}^{n_m+1}$ consists of the elements $\boldsymbol{a} := (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m) \in \mathbb{N}^{n_1+1} \times \cdots \times \mathbb{N}^{n_m+1}$ with $|\boldsymbol{a}_i| = d_i$ for $1 \leq i \leq m$.

We denote by $\mathbb{P}^{\boldsymbol{n}}$ the multiprojective space $\mathbb{P}^{\boldsymbol{n}} := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$. For $1 \leq i \leq m$, let $\boldsymbol{X}_i := \{X_{i,0}, \ldots, X_{i,n_i}\}$ be group of $n_i + 1$ variables and let $\boldsymbol{X} := \{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m\}$. For $\mathsf{K} := \overline{\mathbb{F}}_q$ or $\mathsf{K} := \mathbb{F}_q$, a *multihomogeneous* polynomial $F \in \mathsf{K}[\boldsymbol{X}]$ of multidegree $\boldsymbol{d} := (d_1, \ldots, d_m)$ is a polynomial which is homogeneous of degree d_i in $\boldsymbol{X}_i$ for $1 \leq i \leq m$. An ideal $I \subset \mathsf{K}[\boldsymbol{X}]$ is *multihomogeneous* if it is generated by a family of multihomogeneous polynomials. For such an ideal, we denote by $V(I) \subset \mathbb{P}^{\boldsymbol{n}}$ the variety defined over K (K-variety for short) as its set of common zeros. In particular, a hypersurface in $\mathbb{P}^{\boldsymbol{n}}$ defined over K is the set of zeros of a multihomogeneous polynomial of $\mathsf{K}[\boldsymbol{X}]$. The notions of irreducible variety and dimension of a subvariety of $\mathbb{P}^{\boldsymbol{n}}$ are defined as in the projective space.

Now we discuss the concept of mixed degree of a multiprojective variety and a few properties and results concerning mixed degrees. For this purpose, we follow the exposition in [11]. Let $V \subset \mathbb{P}^{\boldsymbol{n}}$ be an irreducible $\overline{\mathbb{F}}_q$-variety of dimension r and let $I(V) \subset \overline{\mathbb{F}}_q[\boldsymbol{X}]$ denote its multihomogeneous ideal. The quotient ring $\overline{\mathbb{F}}_q[\boldsymbol{X}]/I(V)$ is multigraded and its part of multidegree $\boldsymbol{b} \in \mathbb{N}^m$ is denoted by $(\overline{\mathbb{F}}_q[\boldsymbol{X}]/I(V))_{\boldsymbol{b}}$. The *Hilbert–Samuel* function of V is the function $H_V : \mathbb{N}^m \to \mathbb{N}$ defined as $H_V(\boldsymbol{b}) := \dim(\overline{\mathbb{F}}_q[\boldsymbol{X}]/I(V))_{\boldsymbol{b}}$. It turns out that there exist $\boldsymbol{\delta_0} \in \mathbb{N}^m$ and a unique polynomial $P_V \in \mathbb{Q}[z_1, \ldots, z_m]$ of degree r such that $P_V(\boldsymbol{\delta}) = H_V(\boldsymbol{\delta})$ for every $\boldsymbol{\delta} \in \mathbb{N}^m$ with $\boldsymbol{\delta} \geq \boldsymbol{\delta_0}$ (see [11, Proposition 1.8]). For $\boldsymbol{b} \in \mathbb{N}_r^m$, we define the *mixed degree of* V *of index* $\boldsymbol{b}$ as the nonnegative integer

$$\deg_{\boldsymbol{b}}(V) := \boldsymbol{b}! \operatorname{coeff}_{\boldsymbol{b}}(P_V).$$

This notion can be extended to equidimensional varieties and, more generally, to equidimensional cycles (formal linear combinations with integer coefficients of subvarieties of equal dimension) by linearity.

The Chow ring of $\mathbb{P}^{\boldsymbol{n}}$ is the graded ring

$$A^*(\mathbb{P}^{\boldsymbol{n}}) := \mathbb{Z}[\theta_1, \ldots, \theta_m]/(\theta_1^{n_1+1}, \ldots, \theta_m^{n_m+1}),$$

where each θ_i denotes the class of the inverse image of a hyperplane of $\mathbb{P}^{n_i}$ under the projection $\mathbb{P}^{\boldsymbol{n}} \to \mathbb{P}^{n_i}$. Given a variety $V \subset \mathbb{P}^{\boldsymbol{n}}$ of pure dimension r, its class in the Chow ring is

$$[V] := \sum_{\boldsymbol{b}} \deg_{\boldsymbol{b}}(V) \theta_1^{n_1-b_1} \cdots \theta_m^{n_m-b_1} \in A^*(\mathbb{P}^{\boldsymbol{n}}),$$

where the sum is over all $\boldsymbol{b} \in \mathbb{N}_r^m$ with $\boldsymbol{b} \leq \boldsymbol{n}$. This is a homogeneous element of degree $|\boldsymbol{n}| - r$. In particular, if $\mathcal{H} \subset \mathbb{P}^{\boldsymbol{n}}$ is a hypersurface and $F \in \overline{\mathbb{F}}_q[\boldsymbol{X}]$ is a polynomial of minimal degree defining $\mathcal{H}$, then

$$[\mathcal{H}] := \sum_{i=1}^{m} \deg_{\boldsymbol{X}_i}(F)\theta_i \tag{9}$$

(see [11, Proposition 1.10]).

A fundamental tool for estimates of mixed degrees of intersections of multiprojective varieties is the following multiprojective version of the Bézout theorem, called the *multihomogeneous Bézout theorem* (see [11, Theorem 1.11]). If $V \subset \mathbb{P}^{\boldsymbol{n}}$ is a multiprojective variety of pure dimension $r > 0$ and $F \in \overline{\mathbb{F}}_q[\boldsymbol{X}]$ is a multihomogeneous polynomial such that $V \cap V(F)$ is of pure dimension $r - 1$, then

$$[V \cap V(F)] = [V] \cdot [V(F)]. \tag{10}$$

Finally, we mention the following result, which shows that mixed degrees are monotonic with respect to linear projections. Let $\boldsymbol{l} := (l_1, \ldots, l_m) \in \mathbb{N}^m$ be an m-tuple with $\boldsymbol{l} \leq \boldsymbol{n}$ and let $\pi : \mathbb{P}^{\boldsymbol{n}} \dashrightarrow \mathbb{P}^{\boldsymbol{l}}$ be the linear projection which takes the first l_i coordinates of each coordinate $\boldsymbol{x}_i$ of each point $\boldsymbol{x} := (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m) \in \mathbb{P}^{\boldsymbol{n}}$, namely

$$\pi(x_{i,j} : 1 \leq i \leq m,\ 0 \leq j \leq n_i) := (x_{i,j} : 1 \leq i \leq m,\ 0 \leq j \leq l_i).$$

This rational map induces the following injective $\mathbb{Z}$-linear map:

$$\jmath : A^*\big(\mathbb{P}^{\boldsymbol{l}}\big) \to A^*\big(\mathbb{P}^{\boldsymbol{n}}\big), \quad \jmath(P) := \boldsymbol{\theta}^{\boldsymbol{n}-\boldsymbol{l}}P.$$

If $V \subset \mathbb{P}^{\boldsymbol{n}}$ is an equidimensional variety and $\overline{\pi(V)}$ is of pure dimension $\dim V$, then (see [11, Proposition 1.16])

$$\jmath\big(\big[\overline{\pi(V)}\big]\big) \leq [V]. \tag{11}$$

3. Number of zeros of multihomogeneous hypersurfaces

Let $\boldsymbol{n} := (n_1, \ldots, n_m) \in \mathbb{N}^m$ and let $\mathbb{P}^{\boldsymbol{n}}$ be the corresponding multiprojective space. By $\mathbb{P}^{\boldsymbol{n}}(\mathbb{F}_q)$ we denote the set of $\mathbb{F}_q$-rational points of $\mathbb{P}^{\boldsymbol{n}}$. For $1 \leq i \leq m$, let $\boldsymbol{X}_i := \{X_{i,0}, \ldots, X_{i,n_i}\}$ be a group of $n_i + 1$ variables and let $\boldsymbol{X} := \{\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m\}$. Let $F \in \overline{\mathbb{F}}_q[\boldsymbol{X}]$ be a multihomogeneous polynomial of multidegree $\boldsymbol{d} := (d_1, \ldots, d_m)$. In this section we establish two basic results concerning the number N of $\mathbb{F}_q$-rational zeros of F in $\mathbb{P}^{\boldsymbol{n}}(\mathbb{F}_q)$.

The first result is a nontrivial upper bound on N, which generalizes (7) to the multiprojective setting. This bound shall be used to estimate the number of smooth $\mathbb{F}_q$-rational points of a singular complete intersection (Theorem 8.2). The second result is a sufficient condition for the existence of a point in $\mathbb{P}^{\boldsymbol{n}}(\mathbb{F}_q)$ which does not annihilates F and will be used in the proof of our effective version of the Bertini smoothness theorem (Corollary 6.6).

For $\boldsymbol{\alpha} \in \mathbb{N}^m$, we use the notations $\boldsymbol{d}^{\boldsymbol{\alpha}} := d_1^{\alpha_1} \cdots d_m^{\alpha_m}$ and $p_{\boldsymbol{n}-\boldsymbol{\alpha}} := p_{n_1-\alpha_1} \cdots p_{n_m-\alpha_m}$ for $\boldsymbol{n} \geq \boldsymbol{\alpha}$. Further, let

$$\eta_m(\boldsymbol{d}, \boldsymbol{n}) := \sum_{\boldsymbol{\varepsilon} \in \{0,1\}^m \setminus \{\boldsymbol{0}\}} (-1)^{|\boldsymbol{\varepsilon}|+1} \boldsymbol{d}^{\boldsymbol{\varepsilon}} p_{\boldsymbol{n}-\boldsymbol{\varepsilon}}.$$

Observe that $\eta_m(\boldsymbol{d}, \boldsymbol{n}) \leq p_{n_1} \cdots p_{n_m} = |\mathbb{P}^{\boldsymbol{n}}(\mathbb{F}_q)|$ if $q \geq \max_{1 \leq i \leq m} d_i$, while this inequality may not hold for $q < \max_{1 \leq i \leq m} d_i$. We have the following result.

Proposition 3.1. *Let $F \in \overline{\mathbb{F}}_q[\boldsymbol{X}]$ be a multihomogeneous polynomial of multidegree $\boldsymbol{d}$ with $\max_{1 \leq i \leq m} d_i \leq q$ and let N be the number of zeros of F in $\mathbb{P}^{\boldsymbol{n}}(\mathbb{F}_q)$. Then*

$$N \leq \eta_m(\boldsymbol{d}, \boldsymbol{n}).$$

Proof. We argue by induction on m. The case $m = 1$ is (7).

Suppose that the statement holds for $m - 1$ and let $F \in \overline{\mathbb{F}}_q[\boldsymbol{X}]$ be an m-homogeneous polynomial of multidegree $\boldsymbol{d} := (d_1, \ldots, d_m)$. Let N be the number of zeros of F in $\mathbb{P}^{\boldsymbol{n}}(\mathbb{F}_q)$, and let Z_m be the set of elements $\boldsymbol{x}_m$ in $\mathbb{P}^{n_m}(\mathbb{F}_q)$ such that the substitution $F(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{m-1}, \boldsymbol{x}_m)$ of $\boldsymbol{x}_m$ for $\boldsymbol{X}_m$ in F yields the zero polynomial of $\overline{\mathbb{F}}_q[\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{m-1}]$. Consider F as an element of $\overline{\mathbb{F}}_q[\boldsymbol{X}_m][\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{m-1}]$ and let $A \in \overline{\mathbb{F}}_q[\boldsymbol{X}_m]$ be a nonzero homogeneous polynomial of degree d_m which occurs as the coefficient of a monomial $\boldsymbol{X}_1^{\boldsymbol{\alpha}_1} \cdots \boldsymbol{X}_{m-1}^{\boldsymbol{\alpha}_{m-1}}$ in the dense representation of F. Then Z_m is contained in the set of zeros in $\mathbb{P}^{n_m}(\mathbb{F}_q)$ of A. Therefore, by (7) we have $|Z_m| \leq d_m p_{n_m-1}$.

Since $d_m \leq q$ by hypothesis, it follows that $|Z_m| \leq d_m p_{n_m-1} < p_{n_m} = |\mathbb{P}^{n_m}(\mathbb{F}_q)|$, which implies that $\mathbb{P}^{n_m}(\mathbb{F}_q) \setminus Z_m$ is nonempty. Fix $\boldsymbol{x}_m \in \mathbb{P}^{n_m}(\mathbb{F}_q) \setminus Z_m$ and denote by N_{m-1} the number of zeros of $F(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{m-1}, \boldsymbol{x}_m)$ in $\mathbb{P}^{n_1}(\mathbb{F}_q) \times \cdots \times \mathbb{P}^{n_{m-1}}(\mathbb{F}_q)$. Combining the inductive hypothesis and the fact that $\max_{1 \leq i \leq m-1} d_i \leq q$, we see that

$$N_{m-1} \leq \eta_{m-1}(\boldsymbol{d}^*, \boldsymbol{n}^*) \leq p_{n_1} \cdots p_{n_{m-1}},$$

where $\boldsymbol{d}^* := (d_1, \ldots, d_{m-1})$ and $\boldsymbol{n}^* := (n_1, \ldots, n_{m-1})$. As a consequence,

$$\begin{aligned} N &\leq |Z_m| p_{n_1} \cdots p_{n_{m-1}} + (p_{n_m} - |Z_m|)\eta_{m-1}(\boldsymbol{d}^*, \boldsymbol{n}^*) \\ &= |Z_m|\big(p_{n_1} \cdots p_{n_{m-1}} - \eta_{m-1}(\boldsymbol{d}^*, \boldsymbol{n}^*)\big) + \eta_{m-1}(\boldsymbol{d}^*, \boldsymbol{n}^*) p_{n_m} \leq \eta_m(\boldsymbol{d}, \boldsymbol{n}). \end{aligned}$$

This completes the proof of the proposition. □

In the proof of Proposition 3.1 we use the upper bound (7) in order to bound the number of zeros of a given homogeneous polynomial of $\overline{\mathbb{F}}_q[\boldsymbol{X}_m]$. Given a homogeneous polynomial $F \in \mathbb{F}_q[X_0, \ldots, X_n]$ of degree $\delta < q$, the number N of zeros of F in $\mathbb{P}^n(\mathbb{F}_q)$ can be bounded using the well-known Serre bound (see [32]):

$$N \leq \delta q^{n-1} + p_{n-2}. \tag{12}$$

Although (12) is stated for polynomials with coefficients in $\mathbb{F}_q$, it is easy to see that it also holds for polynomials with coefficients in $\overline{\mathbb{F}}_q$, as it is asserted in the following result.

Lemma 3.2. *Let $F \in \overline{\mathbb{F}}_q[X_0, \ldots, X_n]$ be a nonzero homogeneous polynomial of degree $\delta < q$ and let N be the number of zeros of F in $\mathbb{P}^n(\mathbb{F}_q)$. Then the following upper bound holds:*

$$N \leq \delta q^{n-1} + p_{n-2}.$$

Proof. Let K be the finite field extension of $\mathbb{F}_q$ defined by the coefficients of F and let $\{\alpha_1, \ldots, \alpha_r\}$ be a basis of K as an $\mathbb{F}_q$-vector space. Then there exist unique polynomials $F_1, \ldots, F_r \in \mathbb{F}_q[X_0, \ldots, X_n]$, which are homogeneous of degree δ or zero, such that $F = \alpha_1 F_1 + \cdots + \alpha_r F_r$ holds. Assume without loss of generality that $F_1 \neq 0$. Then it is clear that the set of zeros of F in $\mathbb{P}^n(\mathbb{F}_q)$ is contained in the set of zeros of F_1 in $\mathbb{P}^n(\mathbb{F}_q)$. Let N and N_1 denote the number of zeros of F and F_1 in $\mathbb{P}^n(\mathbb{F}_q)$. By the Serre bound (12) it follows that $N \leq N_1 \leq \delta q^{n-1} + p_{n-2}$, finishing thus the proof of the lemma. □

Using Lemma 3.2, the upper bound of Proposition 3.1 can be slightly improved. In particular, it may be worthwhile to remark that, if $\boldsymbol{d}, \boldsymbol{n} \in \mathbb{N}^m$ are of the form $\boldsymbol{d} = (d, \ldots, d)$ and $\boldsymbol{n} = (n, \ldots, n)$ with $d < q$, then combining Lemma 3.2 with the proof of Proposition 3.1 we obtain

$$N \leq p_n^m - \big(q^n - (d-1)q^{n-1}\big)^m. \tag{13}$$

We finish this section with a sufficient condition for the existence of a point in $\mathbb{P}^{\boldsymbol{n}}(\mathbb{F}_q)$ not annihilating F. This condition significantly improves the one which is deduced by a direct application of (8) to F, considering F as a homogeneous polynomial of $\overline{\mathbb{F}}_q[\boldsymbol{X}]$ of degree $d_1 + \cdots + d_m$.

Corollary 3.3. *Let $F \in \overline{\mathbb{F}}_q[\boldsymbol{X}]$ be a multihomogeneous polynomial of multidegree $\boldsymbol{d}$ and let $d := \max_{1 \leq i \leq m} d_i$. If $q > d$, then there exists $\boldsymbol{x} \in \mathbb{P}^{\boldsymbol{n}}(\mathbb{F}_q)$ with $F(\boldsymbol{x}) \neq 0$.*

Proof. Let N be the number of $\mathbb{F}_q$-rational zeros of F in $\mathbb{P}^{\boldsymbol{n}}(\mathbb{F}_q)$. Proposition 3.1 shows that the number $N_{\neq 0}$ of elements in $\mathbb{P}^{\boldsymbol{n}}(\mathbb{F}_q)$ not annihilating F is bounded as follows:

$$N_{\neq 0} = p_{\boldsymbol{n}} - N \geq p_{\boldsymbol{n}} - \eta_m(\boldsymbol{d}, \boldsymbol{n}) = \sum_{\boldsymbol{\varepsilon} \in \{0,1\}^m} (-1)^{|\boldsymbol{\varepsilon}|} \boldsymbol{d}^{\boldsymbol{\varepsilon}} p_{\boldsymbol{n}-\boldsymbol{\varepsilon}} = \prod_{i=1}^{m} (p_{n_i} - d_i p_{n_i - 1}).$$

Since $q > d$, we have $p_{n_i} > d_i p_{n_i-1}$ for $1 \leq i \leq m$, which yields the corollary. □

4. Polar varieties

Let $V \subset \mathbb{P}^n$ be a variety of pure dimension r. Let $\Sigma \subset V$ denote the singular locus of V, let $V_{\mathrm{sm}} := V \setminus \Sigma$ and let $L \subset \mathbb{P}^n$ be a linear variety of dimension $n - s - 2$. For each integer s with $0 \leq s \leq r - 2$ and $x \in V_{\mathrm{sm}}$, the linear variety L meets $T_x V \subset \mathbb{P}^n$ in dimension at least $r - s - 2$. The set of points $x \in V_{\mathrm{sm}}$ such that the dimension of $T_x V \cap L$ is at least $r - s - 1$ is called the sth *polar variety* of V with respect to L and is denoted by $\mathsf{M}(L)$:

$$\mathsf{M}(L) := \big\{x \in V_{\mathrm{sm}} : \dim(T_x V \cap L) \geq r - s - 1\big\}.$$

This classical notion of projective geometry shall play a critical role in our approach to the Bertini smoothness theorem. Indeed, as we explain in Lemma 4.1 below, the polar variety $\mathsf{M}(L)$ is the set of points of V_{sm} which are critical for the linear projection associated with L. Furthermore, by means of the polar variety $\mathsf{M}(L)$ we shall be able to obtain a useful description of the set of singular points of the linear section $V \cap L$ of V. To establish these properties, we fix some notations that will be kept throughout the paper.

Set $X := (X_0, \ldots, X_n)$. For $\mu := (\mu_0 : \cdots : \mu_n) \in \mathbb{P}^n$, we shall use the notation $\mu \cdot X := \mu_0 X_0 + \cdots + \mu_n X_n$. Let $\lambda_0, \ldots, \lambda_{s+1}$ be linearly independent elements of $\mathbb{P}^n$, let

$\boldsymbol{\lambda} := (\lambda_0, \ldots, \lambda_{s+1}) \in (\mathbb{P}^n)^{s+2}$ and let $L \subset \mathbb{P}^n$ be the following linear space of dimension $n-s-2$ associated with $\boldsymbol{\lambda}$:

$$L := \{x \in \mathbb{P}^n : \lambda_0 \cdot x = \cdots = \lambda_{s+1} \cdot x = 0\}. \tag{14}$$

Let $Y_i := \lambda_i \cdot X$ for $0 \leq i \leq s+1$ and consider the rational mapping from V to $\mathbb{P}^{s+1}$ defined by $Y_0, \ldots, Y_{s+1}$, that is,

$$\begin{aligned} \pi : V &\dashrightarrow \mathbb{P}^{s+1} \\ x &\mapsto (\lambda_0 \cdot x : \cdots : \lambda_{s+1} \cdot x). \end{aligned} \tag{15}$$

This mapping is well-defined outside its *exceptional locus* E, namely the set of points $x \in V$ with $\lambda_0 \cdot x = \cdots = \lambda_{s+1} \cdot x = 0$. In other words, π is well-defined in $V \setminus L$ and $\mathsf{E} = V \cap L$.

Additionally, for $x \in V_{\mathrm{sm}}$ we shall consider the rational mapping

$$\begin{aligned} \pi_x : T_x V &\dashrightarrow \mathbb{P}^{s+1} \\ v &\mapsto (\lambda_0 \cdot v : \cdots : \lambda_{s+1} \cdot v). \end{aligned} \tag{16}$$

The exceptional locus E_x of π_x is the set of elements $v \in T_x V$ with $\lambda_0 \cdot v = \cdots = \lambda_{s+1} \cdot v = 0$. Observe that π_x may also be seen as the differential mapping of the morphism $C_V \to \mathbb{A}^{s+2}$ defined by $Y_0, \ldots, Y_{s+1}$, where $C_V \subset \mathbb{A}^{n+1}$ is the affine cone of V.

With these notations, we have the following result.

Lemma 4.1. *Let $V \subset \mathbb{P}^n$ be a variety of pure dimension r and let Σ be its singular locus. Let $L \subset \mathbb{P}^n$ be the linear variety of dimension $n-s-2$ defined in (14) and let π and π_x be defined as in (15) and (16). Then:*

1. *The polar variety $\mathsf{M}(L)$ coincides with the set of points $x \in V_{\mathrm{sm}}$ such that the dimension of E_x is at least $r-s-1$.*
2. $\mathrm{Sing}(V \cap L) = (\Sigma \cap L) \cup (\mathsf{M}(L) \cap L)$.

Proof. We prove the first assertion. By definition, for $x \in V_{\mathrm{sm}}$ the exceptional locus E_x of π_x is the set of points $v \in T_x V$ such that $\lambda_0 \cdot v = \cdots = \lambda_{s+1} \cdot v = 0$, i.e., $\mathsf{E}_x = T_x V \cap L$. As a consequence, $\dim \mathsf{E}_x \geq r-s-1$ if and only if $\dim(T_x V \cap L) \geq r-s-1$. Therefore, the polar variety $\mathsf{M}(L)$ is the set of points $x \in V_{\mathrm{sm}}$ for which $\dim \mathsf{E}_x \geq r-s-1$.

Now we consider the second assertion. According to [16, Lemma 1.1.],

$$\mathrm{Sing}(V \cap L) = (V \cap \mathrm{Sing}\, L) \cup (\Sigma \cap L) \cup N(V, L) = (\Sigma \cap L) \cup N(V, L),$$

where $N(V,L)$ is the set of points $x \in V_{\mathrm{sm}}$ where V and L do not meet transversely, that is, where $\dim T_xV \cap L > \dim T_xV - \operatorname{codim} L = r - s - 2$. By definition it follows that $N(V,L) = \mathsf{M}(L) \cap L$, which readily shows the second assertion of the lemma. □

The polar variety $\mathsf{M}(L)$ is empty or of pure dimension at least s. In fact, following [24, Section IV.B], for a generic L the polar variety $\mathsf{M}(L)$ has dimension s. We include a proof of this result for the sake of completeness (see also [29, Transversality Lemma 1.3]).

Proposition 4.2. *For a generic linear variety $L \subset \mathbb{P}^n$ of dimension $n - s - 2$, the polar variety $\mathsf{M}(L)$ has dimension s.*

Proof. Let $\mathbb{G}(r,n)$ denote the Grassmannian of r-planes in $\mathbb{P}^n$. We consider the Gauss map $\mathcal{G} : V_{\mathrm{sm}} \to \mathbb{G}(r,n)$, which maps a point x into the tangent space T_xV. Let $S \subset \mathbb{G}(r,n)$ be the Schubert variety

$$S := \big\{\Lambda \in \mathbb{G}(r,n) : \dim(\Lambda \cap L) \geq r - s - 1\big\}.$$

Observe that S has dimension $\dim \mathbb{G}(r,n) - (r - s)$ (see, e.g., [18, Example 11.42]). Furthermore, it is clear that $\mathsf{M}(L) = \mathcal{G}^{-1}(S \cap \mathcal{G}(V_{\mathrm{sm}}))$. Let $i : S \hookrightarrow \mathbb{G}(r,n)$ denote the standard inclusion mapping. We claim that the polar variety $\mathsf{M}(L)$ coincides with the fiber product $V_{\mathrm{sm}} \times_{\mathbb{G}(r,n)} S$. Indeed,

$$\begin{aligned} V_{\mathrm{sm}} \times_{\mathbb{G}(r,n)} S &= \big\{(x,\Lambda) \in V_{\mathrm{sm}} \times S \,:\, T_xV = \Lambda\big\} \\ &= \big\{x \in V_{\mathrm{sm}} \,:\, \dim(T_xV \cap L) \geq r - s - 1\big\} = \mathsf{M}(L). \end{aligned}$$

The general linear group acts transitively on $\mathbb{G}(r,n)$, and with respect to this action S is in general position, namely the fiber of a general translate of S is equidimensional of the expected dimension, because L is so by hypothesis. Therefore, [23, Theorem 2] shows that $\mathsf{M}(L)$ is of pure dimension

$$\dim \mathsf{M}(L) = \dim V_{\mathrm{sm}} + \dim S - \dim \mathbb{G}(r,n) = s.$$

This finishes the proof of the proposition. □

One may think that it is natural to describe the linear $(n - s - 2)$-dimensional variety $L \subset \mathbb{P}^n$ of (14) defining the polar variety $\mathsf{M}(L)$ as a point of the Grassmannian $\mathbb{G}(n - s - 2, n)$ of $(n - s - 2)$-planes in $\mathbb{P}^n$, and not by means of a point $\boldsymbol{\lambda}$ in the multiprojective space $(\mathbb{P}^n)^{s+2}$. The reason why we choose the latter is that we have the tools provided by multiprojective elimination theory, as summarized in Section 2.4. In particular, the multihomogeneous Bézout theorem and the behavior of mixed degrees under linear projections will allow us to bound the degree of the variety of points $\boldsymbol{\lambda} \in (\mathbb{P}^n)^{s+2}$ for which the corresponding polar variety $\mathsf{M}(L)$ has not the expected dimension.

4.1. Polar varieties of complete intersections

From now on we consider polar varieties associated with a complete intersection V. We shall see that the polar variety $\mathsf{M}(L)$ associated with V and a linear variety L can be expressed in terms of the vanishing of certain minors involving the partial derivatives of the polynomials defining V and L. This will allow us to obtain an explicit system of equations defining the polar variety $\mathsf{M}(L)$. In the series of papers [3–6,2] polar varieties of complete intersections are locally described by regular sequences consisting of the polynomials defining V and certain well-determined maximal minors of their Jacobian in the context of efficient real elimination.

Let $V \subset \mathbb{P}^n$ be a complete intersection defined by homogeneous polynomials $F_1, \dots, F_{n-r} \in \mathbb{F}_q[X] := \mathbb{F}_q[X_0, \dots, X_n]$ of degrees $d_1 \geq \cdots \geq d_{n-r} \geq 2$ respectively. Denote $\Sigma := \operatorname{Sing} V$ and suppose that there exists $0 \leq s \leq r-2$ with $\dim \Sigma \leq s$. In particular, V is a normal complete intersection and then absolutely irreducible (Theorem 2.1). Finally, denote $\delta := \deg V = d_1 \cdots d_{n-r}$ and $D := \sum_{i=1}^{n-r}(d_i - 1)$.

Let $x \in V$. Since $F_1, \dots, F_{n-r}$ define the radical ideal of V, by, e.g., [25, §VI, Proposition 1.5], the tangent space T_xV is the linear variety

$$T_xV = \big\{v \in \mathbb{P}^n : \nabla F_1(x) \cdot v = \cdots = \nabla F_{n-r}(x) \cdot v = 0\big\}. \tag{17}$$

For $x \in V_{\mathrm{sm}}$, the gradients $\nabla F_1(x), \dots, \nabla F_{n-r}(x)$ are linearly independent and T_xV has dimension r.

Let $\lambda_i := (\lambda_{i,0} : \cdots : \lambda_{i,n})$ $(0 \leq i \leq s+1)$ be linearly independent elements of $\mathbb{P}^n$, let $\boldsymbol{\lambda} := (\lambda_0, \dots, \lambda_{s+1})$ and consider the $(n-s-2)$-dimensional linear variety $L \subset \mathbb{P}^n$ defined as in (14). Further, consider the matrix

$$\mathsf{M}(X, \boldsymbol{\lambda}) := \begin{pmatrix} \frac{\partial F_1}{\partial X_0} & \cdots & \frac{\partial F_1}{\partial X_n} \\ \vdots & & \vdots \\ \frac{\partial F_{n-r}}{\partial X_0} & \cdots & \frac{\partial F_{n-r}}{\partial X_n} \\ \lambda_{0,0} & \cdots & \lambda_{0,n} \\ \vdots & & \vdots \\ \lambda_{s+1,0} & \cdots & \lambda_{s+1,n} \end{pmatrix}. \tag{18}$$

For $x \in V_{\mathrm{sm}}$, the dimension of $T_xV \cap L$ is equal to $r-s-2$ if and only if $\mathsf{M}(x, \boldsymbol{\lambda})$ has maximal rank. Equivalently, $\mathsf{M}(x, \boldsymbol{\lambda})$ is not of full rank if and only if the dimension of $T_xV \cap L$ is at least $r-s-1$. As a consequence, if $\Delta_1(x, \boldsymbol{\lambda}), \dots, \Delta_N(x, \boldsymbol{\lambda})$ denote the maximal minors of $\mathsf{M}(x, \boldsymbol{\lambda})$, then

$$\mathsf{M}(L) = \big\{x \in V_{\mathrm{sm}} : \Delta_1(x, \boldsymbol{\lambda}) = \cdots = \Delta_N(x, \boldsymbol{\lambda}) = 0\big\}. \tag{19}$$

As established in Proposition 4.2, for generic L the dimension of $\mathsf{M}(L)$ is equal to s. Next we obtain conditions on $\boldsymbol{\lambda} := (\lambda_0, \ldots, \lambda_{s+1}) \in (\mathbb{P}^n)^{s+2}$ which imply that the corresponding polar variety $\mathsf{M}(L)$ has dimension s.

For $0 \leq i \leq s+1$, we denote by $\Lambda_i := (\Lambda_{i,0}, \ldots, \Lambda_{i,n})$ a group of $n+1$ variables and set $\boldsymbol{\Lambda} := (\Lambda_0, \ldots, \Lambda_{s+1})$. We consider the so-called *generic polar variety*, namely

$$W := (V_{\mathrm{sm}} \times \mathcal{U}) \cap \big\{\Delta_1(X, \boldsymbol{\Lambda}) = \cdots = \Delta_N(X, \boldsymbol{\Lambda}) = 0\big\}, \tag{20}$$

where $\mathcal{U} \subset (\mathbb{P}^n)^{s+2}$ is the Zariski open subset of all the $(s+2) \times (n+1)$-matrices of maximal rank and $\Delta_1, \ldots, \Delta_N$ are the maximal minors of the generic version $\mathsf{M}(X, \boldsymbol{\Lambda})$ of the matrix $\mathsf{M}(X, \boldsymbol{\lambda})$ of (18).

Proposition 4.3. *Let $t := n(s+2)$. Then W is an irreducible variety of $V_{\mathrm{sm}} \times \mathcal{U}$ of dimension $t+s$.*

Proof. Let $\pi_1 : W \to V_{\mathrm{sm}}$ be the linear projection $\pi_1(x, \boldsymbol{\lambda}) := x$. Fix $x \in V_{\mathrm{sm}}$ and consider the fiber $\pi_1^{-1}(x)$. We have $\pi_1^{-1}(x) = \{x\} \times \mathcal{L}$, where $\mathcal{L} \subset \mathcal{U}$ is the set of matrices $\boldsymbol{\lambda} := (\lambda_0, \ldots, \lambda_{s+1})$ for which the matrix $\mathsf{M}(x, \boldsymbol{\lambda})$ is not of full rank. This is the same as saying that

$$\langle \lambda_0, \ldots, \lambda_{s+1} \rangle \cap \big\langle \nabla F_1(x), \ldots, \nabla F_{n-r}(x) \big\rangle \neq \emptyset,$$

where $\langle v_0, \ldots, v_m \rangle \subset \mathbb{A}^{n+1}$ is the linear variety spanned by $v_0, \ldots, v_m$. Equivalently, $\lambda_0, \ldots, \lambda_{s+1}$ are linearly dependent in the quotient $\overline{\mathbb{F}}_q$-vector space

$$\mathbb{V} := \mathbb{A}^{n+1} / \big\langle \nabla F_1(x), \ldots, \nabla F_{n-r}(x) \big\rangle.$$

This $\mathbb{F}_q$-vector space has dimension $r+1$ because $x \in V_{\mathrm{sm}}$.

The affine cone $(\mathbb{A}^{n+1})^{s+2}$ of $(\mathbb{P}^n)^{s+2}$ can be identified with the $\overline{\mathbb{F}}_q$-vector space $\mathrm{Hom}_{\overline{\mathbb{F}}_q}(\mathbb{A}^{s+2}, \mathbb{A}^{n+1})$ of linear homomorphisms from $\mathbb{A}^{s+2}$ to $\mathbb{A}^{n+1}$. In particular, the Zariski open subset $\mathcal{U}_{\mathrm{aff}} \subset (\mathbb{A}^{n+1})^{s+2}$ of matrices of full rank is the affine cone of $\mathcal{U} \subset (\mathbb{P}^n)^{s+2}$, and can be identified with the open subset of homomorphisms of full rank of $\mathrm{Hom}_{\overline{\mathbb{F}}_q}(\mathbb{A}^{s+2}, \mathbb{A}^{n+1})$:

$$L_{s+2}^{=}\big(\mathbb{A}^{s+2}, \mathbb{A}^{n+1}\big) := \big\{f \in \mathrm{Hom}_{\overline{\mathbb{F}}_q}\big(\mathbb{A}^{s+2}, \mathbb{A}^{n+1}\big) : \mathrm{rank}(f) = s+2\big\}.$$

The quotient map $\mathbb{A}^{n+1} \to \mathbb{V}$ induces a surjective map

$$\Phi : \mathrm{Hom}_{\overline{\mathbb{F}}_q}\big(\mathbb{A}^{s+2}, \mathbb{A}^{n+1}\big) \to \mathrm{Hom}_{\overline{\mathbb{F}}_q}\big(\mathbb{A}^{s+2}, \mathbb{V}\big).$$

With a slight abuse of notation we denote the image of $L_{s+2}^{=}(\mathbb{A}^{s+2}, \mathbb{A}^{n+1})$ under Φ by $\Phi(\mathcal{U}_{\mathrm{aff}})$.

From the above, if $\pi_1^{-1}(x) = \{x\} \times \mathcal{L}$, then the affine cone of $\mathcal{L}$ is, modulo $\langle \nabla F_1(x), \ldots, \nabla F_{n-r}(x)\rangle$, isomorphic to the Zariski open set $L_{s+1}(\mathbb{A}^{s+2}, \mathbb{V}) \cap \Phi(\mathcal{U}_{\mathrm{aff}})$, where

$$L_{s+1}\big(\mathbb{A}^{s+2}, \mathbb{V}\big) := \big\{f \in \mathrm{Hom}_{\overline{\mathbb{F}}_q}\big(\mathbb{A}^{s+2}, \mathbb{V}\big) : \mathrm{rank}(f) \leq s+1\big\}.$$

According to [7, Proposition 1.1], $L_{s+1}(\mathbb{A}^{s+2}, \mathbb{V})$ is an irreducible variety of dimension $(s+1)(r+2)$. Since we are considering subspaces of $\mathbb{A}^{n+1}$ of dimension $s+2$ modulo a subspace $\langle \nabla F_1(x), \ldots, \nabla F_{n-r}(x)\rangle$ of dimension $n-r$, the affine cone of $\mathcal{L}$ is an irreducible variety of $\mathcal{U}_{\mathrm{aff}}$ of dimension $(s+1)(r+2) + (n-r)(s+2) = (n+1)(s+2) + s - r$. We may rephrase this conclusion saying that $\pi_1^{-1}(x) = \{x\} \times \mathcal{L}$ is an irreducible subvariety of $V_{\mathrm{sm}} \times \mathcal{U}$ of dimension $t + s - r$.

We claim that the projection $V_{\mathrm{sm}} \times \mathcal{U} \to V_{\mathrm{sm}}$ is closed. Indeed, this is the case if $\mathcal{U}$ is a complete variety (see, e.g., [12, Chapter 2, §3]). A well-known fact is that a projective variety is complete (see, e.g., [12, Chapter 2, §3.3]). Furthermore, if there exists a proper map from a quasiprojective variety to a complete variety, then the former is complete (see, e.g., [12, Chapter 2, §3.2]). In our case, it is not hard to see that the mapping $\mathcal{U} \to \mathbb{G}(s+1, n)$ defined by the Plücker coordinates is proper. Since $\mathbb{G}(s+1, n)$ is a projective variety and thus complete, the claim follows.

Let $W = \bigcup_j \mathcal{C}_j$ be the decomposition of W into irreducible components. Our previous arguments show that $\pi_1 : W \to V_{\mathrm{sm}}$ is surjective. As $V_{\mathrm{sm}} \times \mathcal{U} \to V_{\mathrm{sm}}$ is closed, we have $\pi_1(W) = V_{\mathrm{sm}} = \bigcup_j \pi_1(\mathcal{C}_j)$, where each $\pi_1(\mathcal{C}_j)$ is a closed subset of V_{sm}. Recall that V is a normal complete intersection and thus irreducible (Theorem 2.1). Then V_{sm} is irreducible and there exists j with $V_{\mathrm{sm}} = \pi_1(\mathcal{C}_j)$.

Now the proof repeats *mutatis mutandis* the second and third paragraph of the proof of [33, §I.6.3, Theorem 8] to conclude that W is an irreducible subvariety of $V_{\mathrm{sm}} \times \mathcal{U}$.

Finally, by the theorem on the dimension of fibers (see, e.g., [33, §I.6.3, Theorem 7]), for any $x \in V_{\mathrm{sm}}$ we have

$$t + s - r = \dim \pi_1^{-1}(x) = \dim W - \dim V_{\mathrm{sm}} = \dim W - r.$$

This shows that $\dim W = t + s$ and finishes the proof of the proposition. □

As the polar variety $\mathsf{M}(L)$ defined by a generic linear variety L has dimension s (Proposition 4.2), the second projection $\pi_2 : W \to \mathbb{P}^{n(s+2)}$ is a dominant mapping. As we shall see, by the theorem on the dimension of fibers it follows that, for any $\boldsymbol{\lambda}$ in a Zariski open subset of $\mathbb{P}^{n(s+2)}$, the corresponding polar variety has dimension s. The main result of this section asserts that there exists a closed subset of $\mathbb{P}^{n(s+2)}$ of “low” degree containing the fibers of π_2 of dimension greater than s.

For this purpose we shall use the following technical lemma, which shows how we obtain such a closed subset. Although the general technique is well-known, we state and prove it here due to lack of a suitable reference.

Lemma 4.4. *Let $\mathcal{W} \subset \mathbb{P}^{\boldsymbol{n}}$ be a multiprojective variety of pure dimension e and let $\mathcal{W}_1 \subset \mathbb{P}^{\boldsymbol{n}}$ be a subvariety of $\mathcal{W}$ of dimension at most $e_1 < e$. Suppose that there exist multihomogeneous polynomials $H_1, \ldots, H_M \in \overline{\mathbb{F}}_q[\boldsymbol{X}]$ of multidegree $\boldsymbol{e}$ such that*

$$\mathcal{W} \cap \{H_1 = \cdots = H_M = 0\} = \mathcal{W}_1. \tag{21}$$

Then there exist linear combinations $H^1, \ldots, H^{e-e_1}$ of $H_1, \ldots, H_M$ such that $\mathcal{W} \cap \{H^1 = \cdots = H^{e-e_1} = 0\}$ contains $\mathcal{W}_1$ and is of pure dimension e_1.

Proof. We show by induction that for $1 \leq i \leq e - e_1$ there exist linear combinations $H^1, \ldots, H^i$ of $H_1, \ldots, H_M$ such that $\mathcal{W} \cap \{H^1 = \cdots = H^i = 0\}$ contains $\mathcal{W}_1$ and is of pure dimension $e - i$. The assertion for $i = e - e_1$ is the statement of the lemma.

We start with the step $i = 1$. Set $\mathcal{W}^0 := \mathcal{W}$ and let $\mathcal{W}^0 = \bigcup_{j=1}^{t} \mathcal{C}_{0,j}$ be the decomposition of $\mathcal{W}^0$ into irreducible components. Observe that $\dim \mathcal{C}_{0,j} = e$ for $1 \leq j \leq t$. Since $\dim(\mathcal{W}_1) \leq e_1 < e$, there exists $\boldsymbol{x}_{0,j} \in \mathcal{C}_{0,j} \setminus \mathcal{W}_1$ for $1 \leq j \leq t$.

Let $\Gamma := (\Gamma_1, \ldots, \Gamma_M)$ be a vector of indeterminates over $\overline{\mathbb{F}}_q$ and let $\mathcal{H}_1 \in \mathbb{F}_q[\Gamma]$ be the following polynomial:

$$\mathcal{H}_1 := \prod_{j=1}^{t} \big(\Gamma_1 H_1(\boldsymbol{x}_{0,j}) + \cdots + \Gamma_M H_M(\boldsymbol{x}_{0,j})\big).$$

Since $\boldsymbol{x}_{0,j} \in \mathcal{W}^0 \setminus \mathcal{W}_1$ for $1 \leq j \leq t$, by (21) we see that for each j there exists H_{i_j} with $H_{i_j}(\boldsymbol{x}_{0,j}) \neq 0$. Then $\mathcal{H}_1$ is a nonzero polynomial and there exists $\gamma_1 := (\gamma_{1,1}, \ldots, \gamma_{1,M}) \in \overline{\mathbb{F}}_q^M$ with $\mathcal{H}_1(\gamma_1) \neq 0$. In particular, the polynomial $H^1 := \sum_{k=1}^{M} \gamma_{1,k} H_k \in \overline{\mathbb{F}}_q[\boldsymbol{X}]$ is multihomogeneous of multidegree $\boldsymbol{e}$ and does not vanish on $\boldsymbol{x}_{0,j}$ for $1 \leq j \leq t$. Therefore, the multiprojective variety $\mathcal{C}_{0,j} \cap \{H^1 = 0\}$ is of pure dimension $e - 1$ for $1 \leq j \leq t$. This implies that $\mathcal{W}^1 := \mathcal{W}^0 \cap \{H^1 = 0\}$ is of pure dimension $e - 1$. From (21) we have that H^1 vanishes identically on $\mathcal{W}_1$, and hence $\mathcal{W}_1 \subset \mathcal{W}^1$. This finishes the proof of the first step of our inductive argument.

Now, given i with $1 < i \leq e - e_1$, assume that there exist linear combinations $H^1, \ldots, H^{i-1}$ of $H_1, \ldots, H_M$ such that $\mathcal{W}^{i-1} := \mathcal{W} \cap \{H^1 = \cdots = H^{i-1} = 0\}$ is of pure dimension $e - i + 1$ and $\mathcal{W}_1 \subset \mathcal{W}^{i-1}$. Let $\mathcal{W}^{i-1} = \bigcup_{j=1}^{t'} \mathcal{C}_{i-1,j}$ be the decomposition of $\mathcal{W}^{i-1}$ into irreducible components. We have $\dim \mathcal{C}_{i-1,j} = e - i + 1 > e - e_1 = \dim \mathcal{W}_1$ for $1 \leq j \leq t'$. Then the argument of the first step of the inductive argument works *mutatis mutandis* and shows that there exists a linear combination H^i of $H_1, \ldots, H_M$ such that $\mathcal{W}^i := \mathcal{W} \cap \{H^1 = \cdots = H^i = 0\}$ is of pure dimension $e - i$ and $\mathcal{W}_1 \subset \mathcal{W}^i$.

Setting $i = e - e_1$ we obtain the assertion of the lemma. □

Now we are in a position to prove our result concerning the points $\boldsymbol{\lambda} \in (\mathbb{P}^n)^{s+2}$ for which the polar variety $\mathsf{M}(L)$ has dimension greater than expected, where $L \subset \mathbb{P}^n$ is the linear variety associated with $\boldsymbol{\lambda}$ as in (14).

Theorem 4.5. *There exists a hypersurface $\mathcal{H}_1 \subset (\mathbb{P}^n)^{s+2}$, defined by a multihomogeneous polynomial of degree at most $(n-s)(r-s)D^{r-s-1}\delta+1$ in each group of variables Λ_i, such that for any $\boldsymbol{\lambda} \in (\mathbb{P}^n)^{s+2} \setminus \mathcal{H}_1$ the polar variety $\mathsf{M}(L)$ has dimension at most s.*

Proof. Let $\pi_2 : W \to \mathcal{U}$ be projection $\pi_2(x, \boldsymbol{\lambda}) := \boldsymbol{\lambda}$, where $W \subset V_{\mathrm{sm}} \times \mathcal{U}$ is the generic polar variety of (20). For $\boldsymbol{\lambda} \in \pi_2(W)$, we have $\pi_2^{-1}(\boldsymbol{\lambda}) = \mathsf{M}(L)$. According to Proposition 4.2, for a generic point $\boldsymbol{\lambda} \in \mathcal{U}$ the polar variety $\mathsf{M}(L)$ is of pure dimension $s \geq 0$. Then π_2 is dominant. On the other hand, by Proposition 4.3 the generic polar variety W is irreducible of dimension $t+s$, where $t := n(s+2)$. Hence, the theorem on the dimension of fibers (see, e.g., [33, §I.6.3, Theorem 7]) shows that for any $\boldsymbol{\lambda} \in \pi_2(W)$ and any irreducible component $\mathcal{C}$ of the fiber $\pi_2^{-1}(\boldsymbol{\lambda})$, we have

$$\dim \mathcal{C} \geq \dim W - \dim \mathcal{U} = t+s-t = s.$$

Furthermore, there exists a Zariski open subset of $\mathcal{U}$ where equality holds.

The fact that π_2 is dominant implies that the field extension $\overline{\mathbb{F}}_q(\boldsymbol{\Lambda}) \hookrightarrow \overline{\mathbb{F}}_q(W)$ has transcendence degree $s+1$, and there exist indices $i_0, \ldots, i_s$ such that the coordinate functions of $\overline{\mathbb{F}}_q(W)$ defined by $X_{i_0}, \ldots, X_{i_s}$ form a transcendence basis of this field extension.

Fix $i \in \Gamma := \{0, \ldots, n\} \setminus \{i_0, \ldots, i_s\}$ and consider the linear mapping $\pi_i : W \dashrightarrow \mathbb{P}^{s+1} \times (\mathbb{P}^n)^{s+2}$ defined by $X_{i_0}, \ldots, X_{i_s}, X_i$ and $\boldsymbol{\Lambda}$.

Claim. *The Zariski closure $W_i \subset \mathbb{P}^{s+1} \times (\mathbb{P}^n)^{s+2}$ of $\pi_i(W)$ is a hypersurface.*

Proof. Since the coordinate functions of $\overline{\mathbb{F}}_q(W)$ defined by $X_{i_0}, \ldots, X_{i_s}$ form a transcendence basis of the field extension $\overline{\mathbb{F}}_q(\boldsymbol{\Lambda}) \hookrightarrow \overline{\mathbb{F}}_q(W)$, for each $i \in \Gamma$ there exists a polynomial $m_i \in \overline{\mathbb{F}}_q[X_{i_0}, \ldots, X_{i_s}, \boldsymbol{\Lambda}, T]$ of minimal degree $D_i > 0$ in T, which is primitive as an element of $\overline{\mathbb{F}}_q[X_{i_0}, \ldots, X_{i_s}, \boldsymbol{\Lambda}][T]$, such that the coordinate function defined by X_i in $\overline{\mathbb{F}}_q(W)$ vanishes identically. Let $A_i \in \overline{\mathbb{F}}_q[X_{i_0}, \ldots, X_{i_s}, \boldsymbol{\Lambda}]$ be the (nonzero) polynomial appearing as the coefficient of T^{D_i} in m_i, considering m_i as an element of $\overline{\mathbb{F}}_q[X_{i_0}, \ldots, X_{i_s}, \boldsymbol{\Lambda}][T]$. Finally, let $A_\Gamma := X_{i_0} \prod_{i \in \Gamma} A_i$. Since $A_\Gamma \in \overline{\mathbb{F}}_q[X_{i_0}, \ldots, X_{i_s}, \boldsymbol{\Lambda}]$, the coordinate functions of W defined by $X_{i_0}, \ldots, X_{i_s}$ are algebraically independent over $\overline{\mathbb{F}}_q(\boldsymbol{\Lambda})$ and W is irreducible, we conclude that $W \cap \{A_\Gamma \neq 0\}$ is a nonempty Zariski open dense subset of W.

Fix $(x, \boldsymbol{\lambda}) \in W \cap \{A_\Gamma \neq 0\}$. Then $\pi_i(x, \boldsymbol{\lambda})$ is well-defined and its fiber has dimension zero, since all the polynomials $m_j(X_{i_0}, \ldots, X_{i_s}, \boldsymbol{\lambda}, X_j)$ with $j \in \Gamma$ vanish on any point $(\widetilde{x}, \boldsymbol{\lambda}) \in \pi_i^{-1}(\pi_i(x, \boldsymbol{\lambda}))$. By the theorem on the dimension of fibers we have

$$0 = \dim \pi_i^{-1}\big(\pi_i(x, \boldsymbol{\lambda})\big) \geq \dim W - \dim \pi_i(W).$$

It follows that $\dim \pi_i(W) \geq \dim W$, which implies $\dim \pi_i(W) = \dim W = n(s+2)+s$. Since $\pi_i(W)$ is irreducible, its Zariski closure W_i is a hypersurface. This finishes the proof of the claim. $\square$

Observe that $F_1, \dots, F_{n-r}$ define a subvariety of $(\mathbb{P}^n)^{s+3}$ of pure dimension $t + r$, namely the variety $V \times (\mathbb{P}^n)^{s+2}$. Furthermore, by the definition of W in (20) we have

$$W \subset \left(V \times \left(\mathbb{P}^n\right)^{s+2}\right) \cap \{\Delta_1 = \cdots = \Delta_N = 0\},$$

where $\Delta_1, \dots, \Delta_N$ are the maximal minors the generic version $\mathsf{M}(X, \boldsymbol{\Lambda})$ of the matrix of (18). Proposition 4.3 shows that W is a subvariety of codimension $r-s$ of $V \times (\mathbb{P}^n)^{s+2}$. Hence, one might expect that $r - s$ generic linear combinations of $\Delta_1, \dots, \Delta_N$ cut out a variety of pure dimension $t + s$ containing W. Indeed, we have the following claim.

Claim. *There exist linear combinations $\Delta^1, \dots, \Delta^{r-s}$ of $\Delta_1, \dots, \Delta_N$ such that $F_1, \dots, F_{n-r}, \Delta^1, \dots, \Delta^{r-s}$ define a subvariety W' of $(\mathbb{P}^n)^{s+3}$ of pure dimension $t + s$ containing W.*

Proof. Observe that $\Sigma \times (\mathbb{P}^n)^{s+2}$ has dimension at most $t + s$. On the other hand, the affine cone of $\mathcal{U} \subset (\mathbb{P}^n)^{s+2}$ represents the Zariski open set of $(s + 2) \times (n + 1)$-matrices with entries in $\overline{\mathbb{F}}_q$ of full rank. The affine cone of $(\mathbb{P}^n)^{s+2} \setminus \mathcal{U}$ is then the closed set of matrices of rank at most $s+1$, that is, $L_{s+1}(\mathbb{A}^{s+2}, \mathbb{A}^{n+1})$. By [7, Proposition 1.1], it is an irreducible subvariety of $(\mathbb{A}^{n+1})^{s+2}$ of dimension $(s+1)(n+2) = t-n+s+s+2$. Therefore, $(\mathbb{P}^n)^{s+2} \setminus \mathcal{U}$ is a multiprojective variety of dimension $t - n + s$ and $V \times ((\mathbb{P}^n)^{s+2} \setminus \mathcal{U})$ has dimension $t + r - n + s < t + s$. We conclude that

$$W'' := W \cup \left(\Sigma \times \left(\mathbb{P}^n\right)^{s+2}\right) \cup \left(V \times \left(\left(\mathbb{P}^n\right)^{s+2} \setminus \mathcal{U}\right)\right) \tag{22}$$

has dimension $t + s$.

For $(x, \boldsymbol{\lambda}) \in V \times (\mathbb{P}^n)^{s+2}$, either $x \in \Sigma$, or $\boldsymbol{\lambda} \in (\mathbb{P}^n)^{s+2} \setminus \mathcal{U}$, or $(x, \boldsymbol{\lambda}) \in V_{\mathrm{sm}} \times \mathcal{U}$. In the first two cases, $(x, \boldsymbol{\lambda}) \in W''$ and $\Delta_j(x, \boldsymbol{\lambda}) = 0$ for $1 \le j \le N$. In the last case, $(x, \boldsymbol{\lambda}) \in W''$ if and only if $\Delta_j(x, \boldsymbol{\lambda}) = 0$ for $1 \le j \le N$. As a consequence,

$$W'' = \left(V \times \left(\mathbb{P}^n\right)^{s+2}\right) \cap \{\Delta_1 = \cdots = \Delta_N = 0\}, \tag{23}$$

and thus W'' is a Zariski closed subset of $(\mathbb{P}^n)^{s+3}$ of dimension $t + s$.

Now we apply Lemma 4.4 to $\mathcal{W} := V \times (\mathbb{P}^n)^{s+2}$ and $\mathcal{W}_1 := W''$. Since W'' has codimension $r-s$ in $V \times (\mathbb{P}^n)^{s+2}$, by Lemma 4.4 there exist linear combinations $\Delta^1, \dots, \Delta^{r-s}$ of $\Delta_1, \dots, \Delta_N$ such that the multiprojective variety $W' := (V \times (\mathbb{P}^n)^{s+2}) \cap \{\Delta^1 = \cdots = \Delta^{r-s} = 0\}$ is of pure dimension $t + s$ and contains W'', and thus W. This finishes the proof of the claim. $\square$

Denote by W'_i the union of the irreducible components of W' for which the Zariski closure of its image under π_i is a hypersurface of $\mathbb{P}^{s+1} \times (\mathbb{P}^n)^{s+2}$. Since $\overline{\pi_i(W)}$ has codimension 1 and $W \subset W'$, such a union is nonempty. Then $\overline{\pi_i(W'_i)}$ is a hypersurface of $\mathbb{P}^{s+1} \times (\mathbb{P}^n)^{s+2}$ which contains W_i.

Next we estimate the multidegree of W' and hence of W_i' and W_i. For this purpose, we consider the class $[W']$ of W' in the Chow ring $\mathcal{A}^*((\mathbb{P}^n)^{s+3})$ of $(\mathbb{P}^n)^{s+3}$. Denote by θ_{j-2} the class of the inverse image of a hyperplane of $\mathbb{P}^n$ under the jth canonical projection $(\mathbb{P}^n)^{s+3} \to \mathbb{P}^n$ for $1 \leq j \leq s+3$. In particular, θ_{-1} is associated with the projection $\pi_1 : \mathbb{P}^n \times (\mathbb{P}^n)^{s+2} \to \mathbb{P}^n$ considered before. Observe that $W' := \{F_1 = \cdots = F_{n-r} = 0, \Delta^1 = \cdots = \Delta^{r-s} = 0\}$. From (9) it follows that

$$\begin{aligned} \big[V(F_i)\big] &\leq d_i\theta_{-1} \quad (1 \leq i \leq n-r), \\ \big[V(\Delta^i)\big] &\leq D\theta_{-1} + \theta_0 + \cdots + \theta_{s+1} \quad (1 \leq i \leq r-s). \end{aligned}$$

Then the multihomogeneous Bézout theorem (10) shows that

$$\begin{aligned} \big[W'\big] &\leq \prod_{i=1}^{n-r}(d_i\theta_{-1}) \prod_{k=1}^{r-s}(D\theta_{-1} + \theta_0 + \cdots + \theta_{s+1}) \\ &= \delta D^{r-s-1}\big(D(\theta_{-1})^{n-s} + (r-s)(\theta_{-1})^{n-s-1}(\theta_0 + \cdots + \theta_{s+1})\big) \\ &\quad + \mathcal{O}\big((\theta_{-1})^{n-s-2}\big), \end{aligned} \tag{24}$$

where $\mathcal{O}((\theta_{-1})^{n-s-2})$ represents a sum of terms of degree at most $n-s-2$ in θ_{-1}. On the other hand, by (9) we have

$$\big[\overline{\pi_i(W_i')}\big] = \deg_X m_i'\theta_{-1} + \deg_{\Lambda_0} m_i'\theta_0 + \cdots + \deg_{\Lambda_{s+1}} m_i'\theta_{s+1},$$

where $m_i' \in \mathbb{F}_q[X_{i_0}, \ldots, X_{i_s}, \boldsymbol{\Lambda}, X_i]$ is a polynomial of minimal degree defining $\overline{\pi_i(W_i')}$. Let $\jmath : \mathcal{A}^*(\mathbb{P}^{s+1} \times (\mathbb{P}^n)^{s+2}) \hookrightarrow \mathcal{A}^*((\mathbb{P}^n)^{s+3})$ be the injective $\mathbb{Z}$-map $P \mapsto (\theta_{-1})^{n-s-1}P$ induced by π_i. Since $\overline{\pi_i(W_i')}$ is of pure dimension $t+s = \dim W_i'$, (11) shows that $\jmath([\overline{\pi_i(W_i')}]) \leq [W_i']$, and by definition $[W_i'] \leq [W']$, that is,

$$\jmath\big(\big[\overline{\pi_i(W_i')}\big]\big) = \deg_X m_i'(\theta_{-1})^{n-s} + \sum_{j=0}^{s+1} \deg_{\Lambda_j} m_i'(\theta_{-1})^{n-s-1}\theta_j \leq \big[W'\big],$$

where inequalities are understood in a coefficient-wise sense. By (24) we deduce that $\deg_{\Lambda_j} m_i' \leq (r-s)D^{r-s-1}\delta$ for $0 \leq j \leq s+1$.

Let $m_i \in \overline{\mathbb{F}}_q[X_{i_0}, \ldots, X_{i_s}, \boldsymbol{\Lambda}, X_i]$ be a polynomial of minimal degree defining W_i. Observe that $D_i := \deg_{X_i} m_i > 0$. Let $A_i \in \overline{\mathbb{F}}_q[X_{i_0}, \ldots, X_{i_s}, \boldsymbol{\Lambda}]$ be the (nonzero) polynomial occurring as the coefficient of $X_i^{D_i}$ in m_i, considered as an element of the polynomial ring $\overline{\mathbb{F}}_q[X_{i_0}, \ldots, X_{i_s}, \boldsymbol{\Lambda}][X_i]$. Further, let $A_i^* \in \overline{\mathbb{F}}_q[\boldsymbol{\Lambda}]$ be a nonzero coefficient of A_i, considering A_i as an element of $\overline{\mathbb{F}}_q[\boldsymbol{\Lambda}][X_{i_0}, \ldots, X_{i_s}]$. Finally, let $A_0 \in \overline{\mathbb{F}}_q[\boldsymbol{\Lambda}]$ denote a maximal minor of the generic matrix $(\Lambda_{i,j})_{0 \leq i \leq s+1, 0 \leq j \leq n}$ and set $A := A_0 \cdot \prod_{i \in \Gamma} A_i^* \in \overline{\mathbb{F}}_q[\boldsymbol{\Lambda}]$. We claim that the hypersurface $\mathcal{H}_1 \subset (\mathbb{P}^n)^{s+2}$ defined by the zero locus of A satisfies the requirements of the theorem.

To show this claim, let $\boldsymbol{\lambda} := (\lambda_0, \ldots, \lambda_{s+1}) \in (\mathbb{P}^n)^{s+2} \setminus \mathcal{H}_1$ and denote $m_i^{\boldsymbol{\lambda}} := m_i(X_{i_0}, \ldots, X_{i_s}, \boldsymbol{\lambda}, X_i)$. Since $A_0(\boldsymbol{\lambda}) \neq 0$, we have $\boldsymbol{\lambda} \in \mathcal{U}$. Furthermore, $A_i^*(\boldsymbol{\lambda}) \neq 0$, which implies that $A_i(X_{i_0}, \ldots, X_{i_s}, \boldsymbol{\lambda})$ is a nonzero polynomial of $\overline{\mathbb{F}}_q[X_{i_0}, \ldots, X_{i_s}]$. This polynomial appears as the coefficient of $X_i^{D_i}$ in $m_i^{\boldsymbol{\lambda}}$, considering $m_i^{\boldsymbol{\lambda}}$ as an element of $\overline{\mathbb{F}}_q[X_{i_0}, \ldots, X_{i_s}][X_i]$. We conclude that $m_i^{\boldsymbol{\lambda}}$ is a nonzero polynomial of $\overline{\mathbb{F}}_q[X_{i_0}, \ldots, X_{i_s}, X_i]$ with $\deg_{X_i} m_i^{\boldsymbol{\lambda}} > 0$ vanishing on $\mathsf{M}(L)$ for any $i \in \Gamma$, where L is the linear variety associated with $\boldsymbol{\lambda}$. Then the coordinate function of $\mathsf{M}(L)$ defined by X_i satisfies a nontrivial algebraic equation over $\overline{\mathbb{F}}_q(X_{i_0}, \ldots, X_{i_s})$ for any $i \in \Gamma$. As a consequence, $\mathsf{M}(L)$ has dimension at most s.

Since A_i^* is a multihomogeneous polynomial of $\overline{\mathbb{F}}_q[\boldsymbol{\Lambda}]$ with $\deg_{\Lambda_i} A_i^* \leq (r-s)D^{r-s-1}\delta$ and $|\Gamma| = n - s$, we find that $\deg_{\Lambda_i} A \leq (n-s)(r-s)D^{r-s-1}\delta + 1$. This finishes the proof of the theorem. □

5. On the existence of nonsingular linear sections

In this section we establish a Bertini-type theorem, namely we show the existence of nonsingular linear sections of a singular complete intersection. Combining the main result of this section and Theorem 4.5 we shall be able to obtain an effective Bertini smoothness theorem.

A version of the Bertini theorem asserts that a generic hyperplane section of a nonsingular variety V is nonsingular. A more precise variant asserts that, if $V \subset \mathbb{P}^n$ is a projective variety with singular locus of dimension at most s, then a section of V defined by a generic linear space of $\mathbb{P}^n$ of codimension at least $s+1$ is nonsingular (see, e.g., [16, Proposition 1.3]). In this section we consider the existence of nonsingular linear sections of codimension $s+2$ of a complete intersection having a singular locus of dimension at most s. Identifying each section of this type with a point in the multiprojective space $(\mathbb{P}^n)^{s+2}$, we show the existence of a hypersurface of $(\mathbb{P}^n)^{s+2}$ containing all the linear subvarieties of codimension $s+2$ of $(\mathbb{P}^n)^{s+2}$ which yield singular sections of V. We also estimate the multidegree of this hypersurface.

Let $V \subset \mathbb{P}^n$ be a complete intersection defined by homogeneous polynomials $F_1, \ldots, F_{n-r} \in \mathbb{F}_q[X_0, \ldots, X_n]$ of degrees $d_1 \geq \cdots \geq d_{n-r} \geq 2$ respectively. Let $\Sigma := \operatorname{Sing} V$ and suppose that it has dimension at most $s \leq r-2$. This implies that V is normal, and therefore absolutely irreducible (Theorem 2.1).

As before, given a point $\boldsymbol{\lambda} := (\lambda_0, \ldots, \lambda_{s+1}) \in (\mathbb{P}^n)^{s+2}$ where $\lambda_0, \ldots, \lambda_{s+1}$ are $\overline{\mathbb{F}}_q$-linearly independent, we consider the linear variety

$$L := \{x \in \mathbb{P}^n : \lambda_0 \cdot x = \cdots = \lambda_{s+1} \cdot x\}$$

of (14). Further, if $Y_i := \lambda_i \cdot X$ for $0 \leq i \leq s+1$, we consider the mapping

$$\begin{aligned} \pi : V &\dashrightarrow \mathbb{P}^{s+1} \\ x &\mapsto (\lambda_0 \cdot x : \cdots : \lambda_{s+1} \cdot x) \end{aligned}$$

as defined in (15). Finally, we recall the notations

$$\delta := \deg V = \prod_{i=1}^{n-r} d_i, \qquad D := \sum_{i=1}^{n-r} (d_i - 1), \qquad t := n(s+2).$$

As asserted above, in this section we obtain a condition on $\boldsymbol{\lambda} \in (\mathbb{P}^n)^{s+2}$ which implies that the linear section $V \cap L$ is nonsingular of pure dimension $r - s - 2$.

First we obtain a condition on $\boldsymbol{\lambda}$ which implies that every fiber and the exceptional locus of π have the expected dimension. Recall that the exceptional locus E of π is $V \cap L$.

Lemma 5.1. *There exists a hypersurface $\mathcal{H}_2' \subset (\mathbb{P}^n)^{s+2}$, defined by a multihomogeneous polynomial of $\overline{\mathbb{F}}_q[\boldsymbol{\Lambda}]$ of multidegree at most δ in each group of variables Λ_i, with the following property: let $\boldsymbol{\lambda} \in (\mathbb{P}^n)^{s+2} \setminus \mathcal{H}_2'$ and let $\pi : V \dashrightarrow \mathbb{P}^{s+1}$ be the linear mapping of (15). Then the Zariski closure V_y of every fiber $\pi^{-1}(y)$ is of pure dimension $r - s - 1$ and the exceptional locus E of π is of pure dimension $r - s - 2$.*

Proof. Let $U_0, \dots, U_r$ be groups of $n + 1$ indeterminates over $\overline{\mathbb{F}}_q[X_0, \dots, X_n]$, where $U_i := (U_{i,0}, \dots, U_{i,n})$, and let $\boldsymbol{U} := (U_0, \dots, U_r)$. Denote by $\mathcal{F}_V \in \mathbb{F}_q[\boldsymbol{U}]$ the Chow form of V (see, e.g., [21, Chapter X, §6] or [30, Chapter I, §9]). This is an irreducible polynomial of $\overline{\mathbb{F}}_q[\boldsymbol{U}]$ which characterizes the set of overdetermined linear systems over V, i.e., $\mathcal{F}_V(u_0, \dots, u_r) = 0$ if and only if $V \cap \{u_0 \cdot X = \cdots = u_r \cdot X = 0\}$ is not empty. Furthermore, $\mathcal{F}_V$ is homogeneous in each group of variables U_i and $\deg_{U_i} \mathcal{F}_V = \delta$ for $0 \le i \le r$.

Consider $\mathcal{F}_V$ as a polynomial of $\mathbb{F}_q[U_0, \dots, U_{s+1}][U_{s+2}, \dots, U_r]$ and fix $\boldsymbol{u} := (u_{s+2}, \dots, u_r) \in (\mathbb{P}^n)^{r-s-1}$ such that the multihomogeneous polynomial $B := \mathcal{F}_V(U_0, \dots, U_{s+1}, u_{s+2}, \dots, u_r)$ does not vanish. We claim that any $\boldsymbol{\lambda} := (\lambda_0, \dots, \lambda_{s+1}) \in (\mathbb{P}^n)^{s+2}$ with $B(\boldsymbol{\lambda}) \neq 0$ satisfies the statement of the lemma.

Indeed, by the definition of $\boldsymbol{\lambda}$ and $\boldsymbol{u}$ we have $\mathcal{F}_V(\boldsymbol{\lambda}, \boldsymbol{u}) \neq 0$. This implies

$$V \cap \{\lambda_0 \cdot X = \cdots = \lambda_{s+1} \cdot X = 0, u_{s+2} \cdot X = \cdots = u_r \cdot X = 0\} = \emptyset. \tag{25}$$

Then the mapping $\pi_r : V \to \mathbb{P}^r$ defined by the linear forms $\lambda_0 \cdot X, \dots, \lambda_{s+1} \cdot X, u_{s+2} \cdot X, \dots, u_r \cdot X$ is a finite morphism (see, e.g., [33, §I.5.3, Theorem 8]).

Let $\pi : V \dashrightarrow \mathbb{P}^{s+1}$ be the mapping defined by $\lambda_0 \cdot X, \dots, \lambda_{s+1} \cdot X$. Observe that $\pi = \pi_{r,s} \circ \pi_r$, where $\pi_{r,s} : \mathbb{P}^r \dashrightarrow \mathbb{P}^{s+1}$ is the mapping defined by $(x_0 : \cdots : x_r) \mapsto (x_0 : \cdots : x_{s+1})$. As $\pi_{r,s}$ is surjective, the Zariski closure $L_y \subset \mathbb{P}^r$ of the preimage $\pi_{r,s}^{-1}(y)$ of any point $y \in \mathbb{P}^{s+1}$ is a linear variety of dimension $r - s - 1$. Then the Zariski closure V_y of any fiber $\pi^{-1}(y)$ agrees with the inverse image by π_r of the linear variety $L_y \subset \mathbb{P}^r$, and hence is of pure dimension $r - s - 1$. On the other hand, from (25) we easily conclude that $\mathsf{E} := V \cap \{\lambda_0 \cdot X = \cdots = \lambda_{s+1} \cdot X = 0\}$ is of pure dimension $r - s - 2$. Indeed, every irreducible component of E has dimension at least $r - s - 2$ by, e.g., [33, §I.6.2, Corollary 2]. Furthermore, if there were an irreducible component $\mathcal{C}$

of E of dimension at least $r-s-1$, then $\mathcal{C} \cap \{u_{s+2} \cdot X = \cdots = u_r \cdot X = 0\}$ would be nonempty, contradicting (25).

As a consequence, defining $\mathcal{H}_2' \subset (\mathbb{P}^n)^{s+2}$ as the zero locus of the polynomial $B \in \overline{\mathbb{F}}_q[U_0, \ldots, U_{s+1}]$ finishes the proof of the lemma. $\square$

Next we consider the set of elements $\boldsymbol{\lambda} \in (\mathbb{P}^n)^{s+2}$ such that the corresponding linear variety L does not meet the singular locus Σ.

Lemma 5.2. *There exists a hypersurface $\mathcal{H}_2'' \subset (\mathbb{P}^n)^{s+2}$, defined by a multihomogeneous polynomial of $\overline{\mathbb{F}}_q[\boldsymbol{\Lambda}]$ of multidegree at most $D^{r-s-1}\delta$ in each group of variables Λ_i, with the following property: if $\boldsymbol{\lambda} \in (\mathbb{P}^n)^{s+2} \setminus \mathcal{H}_2''$, then $\Sigma \cap L$ is empty.*

Proof. According to (17), the tangent space T_xV at any point $x \in V$ is the linear variety orthogonal to $\nabla F_1(x), \ldots, \nabla F_{n-r}(x)$. Hence, a point $x \in V$ is singular if and only if $\nabla F_1(x), \ldots, \nabla F_{n-r}(x)$ are linearly dependent, or equivalently, if and only if the Jacobian matrix of $F_1, \ldots, F_{n-r}$ at x has not maximal rank. Let $\Delta_1', \ldots, \Delta_M'$ be the maximal minors of the Jacobian matrix of $F_1, \ldots, F_{n-r}$. Then

$$\Sigma = \big\{x \in V : \Delta_1' = \cdots = \Delta_M' = 0\big\}.$$

All the polynomials Δ_j' are homogeneous of degree D and $\Sigma \subset V$ has dimension at most $s < s+1$. Then Lemma 4.4 shows that there exist linear combinations $H^1, \ldots, H^{r-s-1}$ of $\Delta_1', \ldots, \Delta_M'$ such that the projective variety $Z := V \cap \{H^1 = \cdots = H^{r-s-1} = 0\}$ is of pure dimension $s+1$ with $\Sigma \subset Z$. By the Bézout inequality (5), the degree of Z is at most $D^{r-s-1}\delta$.

Let $\mathcal{F}_Z \in \mathbb{F}_q[\boldsymbol{\Lambda}]$ be the Chow form of Z. Recall that $\mathcal{F}_Z$ is homogeneous in each group of variables Λ_i and $\deg_{\Lambda_i} \mathcal{F}_Z = \deg Z \leq D^{r-s-1}\delta$ for $0 \leq i \leq s+1$.

Let $\boldsymbol{\lambda} \in (\mathbb{P}^n)^{s+2}$ be such that $\mathcal{F}_Z(\boldsymbol{\lambda}) \neq 0$ and let $L \subset \mathbb{P}^n$ be the linear variety $L := \{\lambda_0 \cdot X = \cdots = \lambda_{s+1} \cdot X = 0\}$. Then $Z \cap L$ is empty by the definition of $\mathcal{F}_Z$, and thus so is $\Sigma \cap L$. Therefore, defining $\mathcal{H}_2'' \subset (\mathbb{P}^n)^{s+2}$ as the zero locus of $\mathcal{F}_Z$ finishes the proof of the lemma. $\square$

5.1. An incidence variety for the singular linear sections of codimension $s+2$

Similarly to Section 4.1, we consider the following incidence variety:

$$\begin{aligned} W_s := (V_{\mathrm{sm}} \times \mathcal{U}) \cap \big\{&\Lambda_0 \cdot X = 0, \ldots, \Lambda_{s+1} \cdot X = 0, \\ &\Delta_1(\boldsymbol{\Lambda}, X) = 0, \ldots, \Delta_N(\boldsymbol{\Lambda}, X) = 0\big\}, \end{aligned} \tag{26}$$

where $\mathcal{U} \subset (\mathbb{P}^n)^{s+2}$ is the Zariski open subset of $(s+2) \times (n+1)$-matrices of full rank and $\Delta_1, \ldots, \Delta_N$ are the maximal minors of the generic version $\mathsf{M}(X, \boldsymbol{\Lambda})$ of the matrix of (18). Denote by $\pi_2 : W_s \to \mathcal{U}$ the projection on the second argument. Then each

$\boldsymbol{\lambda} \in \pi_2(W_s)$ corresponds to a linear variety $L \subset \mathbb{P}^n$ of codimension $s+2$ such that $V \cap L$ is singular.

Our first result asserts that W_s is irreducible of dimension $t-1$. At first sight this might be seen as contradicting Proposition 4.3, which shows that the generic polar variety W of (20) has dimension $t+s$, since W_s is the intersection of W with $s+2$ bilinear forms and has codimension $s+1$ in W. Nevertheless, for a generic $\boldsymbol{\lambda} \in (\mathbb{P}^n)^{s+2}$ the corresponding equations in (26) describe the singular locus of $V \cap L$, which is likely to be empty.

Proposition 5.3. *W_s is an irreducible subvariety of $V_{\mathrm{sm}} \times \mathcal{U}$ of dimension $t-1$.*

Proof. As the arguments are similar to those of the proof of Proposition 4.3, we shall omit some details.

Let $\pi_1 : W_s \to V_{\mathrm{sm}}$ be the projection $\pi_1(x, \boldsymbol{\lambda}) := x$. Fix $x \in V_{\mathrm{sm}}$ and consider the fiber $\pi_1^{-1}(x)$. We have $\pi_1^{-1}(x) = \{x\} \times \mathcal{L}$, where $\mathcal{L} \subset \mathcal{U}$ denotes the set of points $\boldsymbol{\lambda} := (\lambda_0, \ldots, \lambda_{s+1})$ such that $\lambda_0 \cdot x = \cdots = \lambda_{s+1} \cdot x = 0$ and the matrix $\mathsf{M}(x, \boldsymbol{\lambda})$ is not of full rank. The latter is equivalent to

$$\langle \lambda_0, \ldots, \lambda_{s+1} \rangle \cap \big\langle \nabla F_1(x), \ldots, \nabla F_{n-r}(x) \big\rangle \neq \{\mathbf{0}\}, \tag{27}$$

where $\langle v_0, \ldots, v_m \rangle \subset \mathbb{A}^{n+1}$ is the linear variety spanned by $v_0, \ldots, v_m$ in $\mathbb{A}^{n+1}$. Let $\mathbb{V} := \{v \in \mathbb{A}^{n+1} : v \cdot x = 0\}$. Observe that $\nabla F_j(x) \in \mathbb{V}$ for $1 \le j \le n-r$. Then (27) holds if and only if $\lambda_0, \ldots, \lambda_{s+1}$ are linearly dependent in the quotient $\overline{\mathbb{F}}_q$-vector space

$$\mathbb{W} := \mathbb{V} / \big\langle \nabla F_1(x), \ldots, \nabla F_{n-r}(x) \big\rangle.$$

This shows that $\mathcal{L}$ is, modulo $\langle \nabla F_1(x), \ldots, \nabla F_{n-r}(x) \rangle$, isomorphic to the Zariski open set $L'_{s+1}(\mathbb{A}^{s+2}, \mathbb{W}) \cap \Phi(\mathcal{U}_{\mathrm{aff}})$ of $L'_{s+1}(\mathbb{A}^{s+2}, \mathbb{W})$, where

$$L'_{s+1}\big(\mathbb{A}^{s+2}, \mathbb{W}\big) := \big\{ f \in \mathrm{Hom}_{\overline{\mathbb{F}}_q}\big(\mathbb{A}^{s+2}, \mathbb{V}\big) : \mathrm{rank}(f) \le s+1 \big\},$$

$\mathcal{U}_{\mathrm{aff}} \subset (\mathbb{A}^{n+1})^{s+2}$ is the affine cone of $\mathcal{U}$ and $\Phi : \mathrm{Hom}_{\overline{\mathbb{F}}_q}(\mathbb{A}^{s+2}, \mathbb{A}^{n+1}) \to \mathrm{Hom}_{\overline{\mathbb{F}}_q}(\mathbb{A}^{s+2}, \mathbb{W})$ is the surjective map induced by the quotient map $\mathbb{A}^{n+1} \to \mathbb{W}$.

According to [7, Proposition 1.1], $L'_{s+1}(\mathbb{A}^{s+2}, \mathbb{W})$ is an irreducible variety of dimension $(s+1)(r+1)$. Since we are considering subspaces of $\mathbb{V}$ of dimension $s+2$ modulo $\langle \nabla F_1(x), \ldots, \nabla F_{n-r}(x) \rangle$, which has dimension $n-r$, it follows that the affine cone of $\pi_1^{-1}(x) = \{x\} \times \mathcal{L}$ is an open dense subset of an irreducible subvariety of $V_{\mathrm{sm}} \times \mathcal{U}_{\mathrm{aff}}$ of dimension $(s+1)(r+1) + (n-r)(s+2) = (n+1)(s+2) - r - 1$. This implies that $\pi_1^{-1}(x) = \{x\} \times \mathcal{L}$ is an irreducible subvariety of $V_{\mathrm{sm}} \times \mathcal{U}$ of dimension $t-r-1$.

As in the proof of Proposition 4.3, the projection $V_{\mathrm{sm}} \times \mathcal{U} \to V_{\mathrm{sm}}$ on the second argument is closed. Let $W_s = \bigcup_j \mathcal{C}_j$ be the decomposition of W_s into irreducible components. Our previous arguments show that $\pi_1 : W_s \to V_{\mathrm{sm}}$ is surjective. Then $\pi_1(W_s) = V_{\mathrm{sm}} = \bigcup_j \pi_1(\mathcal{C}_j)$ and each $\pi_1(\mathcal{C}_j)$ is a closed subset of V_{sm}. Recall that V is a normal complete intersection, and thus irreducible (Theorem 2.1). Then V_{sm} is irreducible and there exists j with $V_{\mathrm{sm}} = \pi_1(\mathcal{C}_j)$. Now we repeat *mutatis mutandis* the second and third paragraph of the proof of [33, §I.6.3, Theorem 8] and deduce that W_s is an irreducible subvariety of $V_{\mathrm{sm}} \times \mathcal{U}$.

Finally, by the theorem on the dimension of fibers (see, e.g., [33, §I.6.3, Theorem 7]), for any $x \in V_{\mathrm{sm}}$ we have

$$t - r - 1 = \dim \pi_1^{-1}(x) = \dim W_s - \dim V_{\mathrm{sm}} = \dim W_s - r.$$

This shows that $\dim W_s = t - 1$ and finishes the proof of the proposition. □

An immediate consequence of Proposition 5.3 is that the Zariski closure of the image of the projection $\pi_2 : W_s \to \mathcal{U}$ is an irreducible variety of dimension at most $t - 1$. Our next result strengthens somewhat this conclusion and provides further quantitative information.

Theorem 5.4. *Let $\mathcal{H}_s \subset (\mathbb{P}^n)^{s+2}$ be the Zariski closure of the image of $\pi_2 : W_s \to \mathcal{U}$. Then $\mathcal{H}_s$ is a hypersurface of $(\mathbb{P}^n)^{s+2}$, defined by a multihomogeneous polynomial of $\overline{\mathbb{F}}_q[\boldsymbol{\Lambda}]$ of degree at most $\delta D^{r-s-2}(D + r - s - 1)$ in each group of variables Λ_i.*

Proof. We first prove that $\mathcal{H}_s$ is a hypersurface. For this purpose, it suffices to show that there exists a zero dimensional fiber $\pi_2^{-1}(\boldsymbol{\lambda})$. Indeed, assuming that such a fiber exists, by the theorem on the dimension of fibers it follows that

$$0 = \dim \pi_2^{-1}(\boldsymbol{\lambda}) \geq \dim W_s - \dim \pi_2(W_s).$$

We conclude that $\dim \pi_2(W_s) \geq \dim W_s = t - 1$. On the other hand, it is clear that $\dim \pi_2(W_s) \leq t - 1$, which proves that $\dim \pi_2(W_s) = t - 1$. Being the Zariski closure $\mathcal{H}_s$ of $\pi_2(W_s)$ irreducible and of dimension $t - 1$, we conclude that it is a hypersurface.

Now we prove the existence of a zero-dimensional fiber of π_2. Fix generic linear forms $\lambda_0 \cdot X, \ldots, \lambda_s \cdot X$. By the Bertini theorem in the form of [16, Proposition 1.3] we have that $V \cap \{\lambda_0 \cdot X = \cdots = \lambda_s \cdot X = 0\}$ is nonsingular of pure dimension $r - s - 1$. We claim that there exists $\lambda_{s+1} \in \mathbb{P}^n$ such that $\boldsymbol{\lambda} := (\lambda_0, \ldots, \lambda_{s+1}) \in \mathcal{U}$ and $V \cap \{\lambda_0 \cdot X = \cdots = \lambda_{s+1} \cdot X = 0\}$ is singular. Indeed, let $L' := \{\lambda_0 \cdot X = \cdots = \lambda_s \cdot X = 0\}$ and let $x \in V \cap L'$. The vectors $\nabla F_1(x), \ldots, \nabla F_{n-r}(x), \lambda_0, \ldots, \lambda_s$ are linearly independent. Then the choice $\lambda_{s+1} := \nabla F_1(x)$ fulfills all our requirements, because $L := L' \cap \{\lambda_{s+1} \cdot X = 0\}$ has dimension $n - s - 2$ and $V \cap L$ has x as a singular point.

From [22, Appendix, Theorem 2] it follows that the singular locus of $V \cap L$ has dimension zero. Since such a singular locus is isomorphic to the fiber $\pi_2^{-1}(\boldsymbol{\lambda})$, we deduce the existence of a zero-dimensional fiber of π_2, which completes the proof of the first assertion.

Next we show the existence of a variety $W_s' \subset (\mathbb{P}^n)^{s+3}$ of pure dimension $t - 1$ and “low” degree containing W_s.

Claim. *There exist linear combinations* $\Delta^1, \ldots, \Delta^{r-s-1}$ *of the polynomials* $\Delta_1(\boldsymbol{\Lambda}, X), \ldots, \Delta_N(\boldsymbol{\Lambda}, X)$ *such that the variety* $W_s' \subset (\mathbb{P}^n)^{s+3}$ *defined by the set of common solutions of*

$$\begin{aligned} F_1 = 0, \ldots, F_{n-r} = 0, \Lambda_0 \cdot X = 0, \ldots, \Lambda_{s+1} \cdot X = 0, \\ \Delta^1(\boldsymbol{\Lambda}, X) = 0, \ldots, \Delta^{r-s-1}(\boldsymbol{\Lambda}, X) = 0, \end{aligned} \tag{28}$$

is of pure dimension $t - 1$.

Proof. Let $L_\Lambda := \{\Lambda_0 \cdot X = 0, \ldots, \Lambda_{s+1} \cdot X = 0\} \subset (\mathbb{P}^n)^{s+3}$ and let $W_s'' \subset (\mathbb{P}^n)^{s+3}$ be the following variety:

$$W_s'' := W_s \cup \big(\big(\Sigma \times (\mathbb{P}^n)^{s+2}\big) \cap L_\Lambda\big) \cup \big(\big(V \times \big((\mathbb{P}^n)^{s+2} \setminus \mathcal{U}\big)\big) \cap L_\Lambda\big).$$

By the definition of W and W_s in (20) and (26) we easily conclude that $W_s = W \cap L_\Lambda$. It follows that $W_s'' = W'' \cap L_\Lambda$, where W'' is the variety of (22). Therefore, by intersecting both sides of (23) with L_Λ we find that

$$W_s'' = \big(\big(V \times (\mathbb{P}^n)^{s+2}\big) \cap L_\Lambda\big) \cap \big\{\Delta_1(\boldsymbol{\Lambda}, X) = 0, \ldots, \Delta_N(\boldsymbol{\Lambda}, X) = 0\big\}.$$

Next we determine the dimension of W_s''. First we observe that $\Sigma \times (\mathbb{P}^n)^{s+2}$ is a cylinder which is well intersected by the equations $\Lambda_0 \cdot X = 0, \ldots, \Lambda_{s+1} \cdot X = 0$. We conclude that $(\Sigma \times (\mathbb{P}^n)^{s+2}) \cap L_\Lambda$ has dimension at most $s + t - (s + 2) < t - 1$.

In the second claim of the proof of Theorem 4.5 we prove that $V \times ((\mathbb{P}^n)^{s+2} \setminus \mathcal{U})$ has dimension $t+r-n+s$. Consider the projection $\pi_2 : (V \times ((\mathbb{P}^n)^{s+2} \setminus \mathcal{U})) \cap L_\Lambda \to (\mathbb{P}^n)^{s+2} \setminus \mathcal{U}$ on the second argument. A generic linear variety of $\mathbb{P}^n$ of codimension $s + 1$ intersects V in a variety of pure dimension $r - s - 1$. Therefore, a generic fiber $\pi_2^{-1}(\boldsymbol{\lambda})$ has dimension $r - s - 1$. Then the theorem on the dimension of fibers shows that

$$r - s - 1 = \dim \pi_2^{-1}(\boldsymbol{\lambda}) \geq \dim\big(V \times \big((\mathbb{P}^n)^{s+2} \setminus \mathcal{U}\big)\big) \cap L_\Lambda - (t - n + s).$$

We deduce that $(V \times ((\mathbb{P}^n)^{s+2} \setminus \mathcal{U})) \cap L_\Lambda$ has dimension at most $t - n + r - 1 < t - 1$. Combining these facts with Proposition 5.3 we conclude that W_s'' has dimension $t - 1$.

Now we apply Lemma 4.4 to $\mathcal{W} := (V \times (\mathbb{P}^n)^{s+2}) \cap L_\Lambda$ and $\mathcal{W}_1 := W_s''$. From Lemma 4.4 we readily deduce the claim. □

The projection $(\mathbb{P}^n)^{s+3} \to (\mathbb{P}^n)^{s+2}$ on the second argument is closed (see, e.g., [33, §I.5.2, Theorem 3]). Let $\mathcal{H}_s' \subset (\mathbb{P}^n)^{s+2}$ be the union of the components of $\pi_2(W_s')$ of dimension $t - 1$. Then $\mathcal{H}_s'$ is a hypersurface containing $\mathcal{H}_s$.

Finally, we estimate the multidegree of $\mathcal{H}_s'$. For this purpose, we consider the class $[W_s']$ of W_s' in the Chow ring $\mathcal{A}^*((\mathbb{P}^n)^{s+3})$ of $(\mathbb{P}^n)^{s+3}$. Denote by θ_{j-2} the class of the inverse image of a hyperplane of $\mathbb{P}^n$ under the jth canonical projection $(\mathbb{P}^n)^{s+3} \to \mathbb{P}^n$ for $1 \leq j \leq s + 2$. According to the definition (28) of W_s', by the multihomogeneous Bézout theorem (10) we deduce that

$$\begin{aligned}[W_s'] &\le \prod_{i=1}^{n-r}(d_i\theta_{-1})\prod_{j=0}^{s+1}(\theta_{-1}+\theta_j)\prod_{k=1}^{r-s-1}(D\theta_{-1}+\theta_0+\cdots+\theta_{s+1}) \\ &= \delta D^{r-s-2}(D+r-s-1)(\theta_{-1})^n(\theta_0+\cdots+\theta_{s+1}) \\ &\quad + \text{terms of lower degree in } \theta_{-1}.\end{aligned}$$

On the other hand, $[\mathcal{H}_s'] = \deg_X H_s'\theta_{-1} + \deg_{\Lambda_0} H_s'\theta_0 + \cdots + \deg_{\Lambda_{s+1}} H_s'\theta_{s+1}$, where $H_s' \in \overline{\mathbb{F}}_q[\Lambda]$ is a polynomial of minimal degree defining $\mathcal{H}_s'$. Let $\jmath : \mathcal{A}^*((\mathbb{P}^n)^{s+2}) \hookrightarrow \mathcal{A}^*((\mathbb{P}^n)^{s+3})$ be the injective $\mathbb{Z}$-map $P \mapsto (\theta_{-1})^n P$ induced by π_2. Then by definition $[\mathcal{H}_s] \le [\mathcal{H}_s']$, and (11) shows that $\jmath(\mathcal{H}_s') \le [W_s']$, that is,

$$\jmath(\mathcal{H}_s') = \deg_X H_s'(\theta_{-1})^{n+1} + \sum_{j=0}^{s+1}\deg_{\Lambda_j} H_s'(\theta_{-1})^n\theta_j \le [W_s'],$$

where inequalities are understood in a coefficient-wise sense. This implies $\deg_{\Lambda_j} H_s' \le \delta D^{r-s-2}(D+r-s-1)$ for $0 \le j \le s+1$ and finishes the proof of the theorem. □

Combining Lemmas 5.1 and 5.2 and Theorem 5.4 we obtain the main result of this section.

Corollary 5.5. *There exists a hypersurface $\mathcal{H}_2 \subset (\mathbb{P}^n)^{s+2}$, defined by a multihomogeneous polynomial of degree at most $(D^{r-s-2}(2D+r-s-1)+1)\delta$ in each group of variables Λ_i, with the following property: if $\boldsymbol{\lambda} \in (\mathbb{P}^n)^{s+2} \setminus \mathcal{H}_2$, and $L \subset \mathbb{P}^n$ and $\pi : V \dashrightarrow \mathbb{P}^{s+1}$ are defined as in (14) and (15), then the following conditions are satisfied:*

1. *$V \cap L$ is nonsingular of pure dimension $r-s-2$;*
2. *$\Sigma \cap L$ is empty;*
3. *the Zariski closure V_y of every fiber $\pi^{-1}(y)$ is of pure dimension $r-s-1$.*

Proof. Let $\mathcal{H}_2 := \mathcal{H}_2' \cup \mathcal{H}_2'' \cup \mathcal{H}_s$, where $\mathcal{H}_2'$, $\mathcal{H}_2''$ and $\mathcal{H}_s$ are the hypersurfaces of $(\mathbb{P}^n)^{s+2}$ of Lemmas 5.1 and 5.2 and Theorem 5.4 respectively. We claim that $\mathcal{H}_2$ satisfies the statement of the corollary.

Indeed, let $\boldsymbol{\lambda} \in (\mathbb{P}^n)^{s+2} \setminus \mathcal{H}_2$. Then Lemma 5.1 shows that 3 is satisfied and $V \cap L$ is of pure dimension $r-s-2$. In particular, $L \subset \mathbb{P}^n$ has codimension $s+2$, namely $\boldsymbol{\lambda} \in \mathcal{U}$. On the other hand, Lemma 5.2 proves 2, which implies $V \cap L = V_{\mathrm{sm}} \cap L$. Finally, since $\boldsymbol{\lambda} \notin \mathcal{H}_s$, we see that $\boldsymbol{\lambda} \notin \pi_2(W_s)$, where W_s is the incidence variety of (26). This means that

$$V \cap L \cap \{\Delta_1(\boldsymbol{\lambda}, X) = 0, \ldots, \Delta_N(\boldsymbol{\lambda}, X) = 0\} = \emptyset.$$

We conclude that $V \cap L$ is nonsingular, because $\Delta_1(\boldsymbol{\lambda}, X), \ldots, \Delta_N(\boldsymbol{\lambda}, X)$ are the maximal minors of the Jacobian matrix of the polynomials defining $V \cap L$.

The degree bound of the statement is an immediate consequence of those of Lemmas 5.1 and 5.2 and Theorem 5.4. □

6. An effective Bertini theorem

This section is devoted to establish an effective version of the Bertini smoothness theorem. The Bertini smoothness theorem (see, e.g., [33, II.6.2, Theorem 2]) asserts that, if $f : V_1 \to V_2$ is a dominant morphism of irreducible varieties defined over a field of characteristic zero with V_1 nonsingular, then there exists a Zariski dense open subset U of V_2 such that the fiber $f^{-1}(y)$ is nonsingular for every $y \in U$. An effective version of this result provides an upper bound of the degree of a proper subvariety of V_2 containing the points defining singular fibers. Our effective version holds for complete intersections without any restriction on the characteristic of the ground field, and generalizes significantly [9, Theorem 5.3].

We remark that an effective version of a weak form of a Bertini theorem is obtained in [1]. Nevertheless, the bound given in [1] is exponentially higher than ours and therefore not suitable for our purposes.

Let $V \subset \mathbb{P}^n$ be a complete intersection defined by homogeneous polynomials $F_1, \dots, F_{n-r} \in \mathbb{F}_q[X_0, \dots, X_n]$ of degrees $d_1 \geq \dots \geq d_{n-r} \geq 2$ respectively. Assume that the singular locus Σ of V has dimension at most $s \leq r - 2$. We use the notations $\delta := \deg V = d_1 \cdots d_{n-r}$ and $D := \sum_{i=1}^{n-r}(d_i - 1)$.

Let $\mathcal{H} := \mathcal{H}_1 \cup \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are the hypersurfaces of Theorem 4.5 and Corollary 5.5 respectively. Let $\boldsymbol{\lambda} := (\lambda_0, \dots, \lambda_{s+1}) \in (\mathbb{P}^n)^{s+2} \setminus \mathcal{H}$, let $Y_j := \lambda_j \cdot X$ for $0 \leq j \leq s+1$ and let $\pi : V \dashrightarrow \mathbb{P}^{s+1}$ be the linear mapping defined by $Y_0, \dots, Y_{s+1}$ as in (15). As before, we denote $L := \{Y_0 = \dots = Y_{s+1} = 0\}$. Recall that the exceptional locus E of π is equal to $V \cap L$.

Remark 6.1. With assumptions and notations as above, $\Sigma \cap L$ is empty, and $\mathsf{M}(L) \cap L = \mathrm{Sing}(V \cap L)$ is also empty.

Proof. By Corollary 5.5, as $\boldsymbol{\lambda} \notin \mathcal{H}_2$, $\Sigma \cap L$ is empty and $V \cap L$ is nonsingular. As a consequence, $\mathrm{Sing}(V \cap L)$ is also empty. From Lemma 4.1 we deduce that $\mathsf{M}(L) \cap L = \mathrm{Sing}(V \cap L)$, which readily implies the remark. □

We shall prove that there exists a nonempty open subset U of $\mathbb{P}^{s+1}$ such that the Zariski closure V_y of $\pi^{-1}(y)$ is nonsingular for every $y \in U$. Furthermore, we shall estimate the degree of the variety $\mathbb{P}^{s+1} \setminus U$ yielding nonsingular fibers.

As a first step, we obtain a sufficient condition for the nonsingularity of the linear section V_y of V defined by a point $y \in \mathbb{P}^{s+1}$. Fix $y := (y_0 : \dots : y_s) \in \mathbb{P}^{s+1}$ and assume without loss of generality that $y_0 \neq 0$. Then

$$V_y = \big\{x \in V : y_j Y_0(x) - y_0 Y_j(x) = 0 \ (1 \leq j \leq s+1)\big\}. \tag{29}$$

In particular, $V \cap L \subset V_y$. Since $\boldsymbol{\lambda} \notin \mathcal{H}$, Corollary 5.5 asserts that $V \cap L$ is nonsingular. As we shall see, this implies that any point of $V \cap L$ is a nonsingular point of V_y.

Now we can state and prove a sufficient condition for the nonsingularity of the linear section V_y. To this end, for a given $x \in V \setminus \mathsf{E}$ we consider as in (16) the linear mapping $\pi_x : T_xV \dashrightarrow \mathbb{P}^{s+1}$ defined by $Y_0, \dots, Y_{s+1}$, i.e., $\pi_x(v) := (\lambda_0 \cdot v : \cdots : \lambda_{s+1} \cdot v)$. We denote by E_x the set of exceptional points of π_x, namely $T_xV \cap L$.

Lemma 6.2. *Let $y \in \mathbb{P}^{s+1}$ be such that for every $x \in \pi^{-1}(y)$ the following conditions are satisfied:*

1. *x is a regular point of V,*
2. *the set E_x has dimension at most $r - s - 2$.*

Then V_y is a nonsingular variety.

Proof. Since $\boldsymbol{\lambda} \notin \mathcal{H}$, by Corollary 5.5 we have that V_y is of pure dimension $r - s - 1$. Then it suffices to prove that for every $x \in V_y$ the tangent space T_xV_y has dimension at most $r - s - 1$. Fix $x \in \pi^{-1}(y)$. Condition 1 implies that T_xV has dimension r. Consider the linear mapping

$$\begin{aligned} \pi_x|_{T_xV_y} : T_xV_y &\dashrightarrow \mathbb{P}^{s+1} \\ v &\mapsto \big(Y_0(v) : \cdots : Y_{s+1}(v)\big). \end{aligned}$$

It is clear that the set $\mathsf{E}_{x,y}$ of exceptional points of $\pi_x|_{T_xV_y}$ is contained in E_x. Since the restriction $\pi|_{V_y} : V_y \dashrightarrow \mathbb{P}^{s+1}$ maps V_y to the point y, the dimension of $\pi_x(T_xV_y)$ is equal to 0. By the Dimension theorem of linear algebra (see, e.g., [20, Chapter 8, Section 1]) we have

$$\dim T_xV_y = \dim \mathsf{E}_{x,y} + \dim \pi_x(T_xV_y) + 1.$$

From this and condition 2 we deduce that

$$r - s - 1 \leq \dim T_xV_y \leq \dim \mathsf{E}_x + 1 \leq r - s - 1.$$

We conclude that $\dim T_xV_y = \dim \mathsf{E}_x + 1 = r - s - 1$ and therefore x is a regular point of V_y.

Finally, let $x \in V_y \setminus \pi^{-1}(y)$. Then x is a regular point of $V \cap L$. As $F_1, \dots, F_{n-r}$, $Y_0, \dots, Y_{s+1}$ define the radical ideal of $V \cap L$, we deduce that $\nabla F_1(x), \dots, \nabla F_{n-r}(x)$, $\lambda_0, \dots, \lambda_{s+1}$ are linearly independent. Assume without loss of generality that $y_0 \neq 0$ and recall that V_y is defined as in (29). Furthermore, $\nabla F_1(x), \dots, \nabla F_{n-r}(x), y_0\lambda_1 - y_1\lambda_0$, $\dots, y_0\lambda_{s+1} - y_{s+1}\lambda_0$ are linearly independent, which implies that x is a regular point of V_y. $\square$

Lemma 6.2 shows that a critical point is the analysis of the set of points $x \in V_{\mathrm{sm}}$ for which E_x has dimension at least $r-s-1$. Lemma 4.1 asserts that this set is the polar variety $\mathsf{M}(L)$. Therefore, linear sections V_y defined by points $y \in \mathbb{P}^{s+1}$ such that $\pi^{-1}(y)$ does not meet $\Sigma \cup \mathsf{M}(L)$ are nonsingular. Our next result shows that $\Sigma \cup \mathsf{M}(L)$ is contained in a subvariety of V of pure dimension s and low degree.

Lemma 6.3. *For $\boldsymbol{\lambda} \notin \mathcal{H}$, there exists a subvariety $Z(L) \subset V$ of pure dimension s and degree at most $D^{r-s}\delta$ with $\mathsf{M}(L) \cup \Sigma \subset Z(L)$.*

Proof. For $x \in V$, we have $x \in \Sigma$ if and only if $\dim T_xV > r$. By (17), this condition holds if and only if $\nabla F_1(x), \ldots, \nabla F_{n-r}(x)$ are linearly dependent. This implies that the matrix $\mathsf{M}(x, \boldsymbol{\lambda})$ of (18) is not of full rank and thus

$$\Sigma \subset \big\{x \in V : \Delta_1(x, \boldsymbol{\lambda}) = \cdots = \Delta_N(x, \boldsymbol{\lambda}) = 0\big\}.$$

On the other hand, for $x \in V_{\mathrm{sm}}$ we have $x \in \mathsf{M}(L)$ if and only if $\Delta_1(x, \boldsymbol{\lambda}) = \cdots = \Delta_N(x, \boldsymbol{\lambda}) = 0$. We conclude that

$$\mathsf{M}(L) \cup \Sigma = \big\{x \in V : \Delta_1(x, \boldsymbol{\lambda}) = \cdots = \Delta_N(x, \boldsymbol{\lambda}) = 0\big\}. \tag{30}$$

Now we apply Lemma 4.4 to $\mathcal{W} := V$ and $\mathcal{W}_1 := \mathsf{M}(L) \cup \Sigma$. Since $\mathsf{M}(L) \cup \Sigma$ has dimension at most s and V has pure dimension r, by Lemma 4.4 there exist linear combinations $\Delta^1(X, \boldsymbol{\lambda}), \ldots, \Delta^{r-s}(X, \boldsymbol{\lambda})$ of the homogeneous polynomials $\Delta_1(X, \boldsymbol{\lambda}), \ldots, \Delta_N(X, \boldsymbol{\lambda})$ of degree D such that the variety $Z(L) := V \cap \{\Delta^1(X, \boldsymbol{\lambda}) = 0, \ldots, \Delta^{r-s}(X, \boldsymbol{\lambda}) = 0\}$ is of pure dimension s and contains $\mathsf{M}(L) \cup \Sigma$. Furthermore, the Bézout inequality (5) shows that $\deg Z(L) \leq D^{r-s} \deg V = D^{r-s}\delta$. This completes the proof of the lemma. □

Now we are ready to state our effective version of the Bertini smoothness theorem.

Theorem 6.4. *Let $\mathcal{H} := \mathcal{H}_1 \cup \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are the hypersurfaces of $(\mathbb{P}^n)^{s+2}$ of Theorem 4.5 and Corollary 5.5 respectively. Let $\boldsymbol{\lambda} := (\lambda_0, \ldots, \lambda_{s+1}) \in (\mathbb{P}^n)^{s+2} \setminus \mathcal{H}$, set $Y_j := \lambda_j \cdot X$ for $0 \leq j \leq s+1$, let $L := \{Y_0 = \cdots = Y_{s+1} = 0\}$ and let $\pi : V \dashrightarrow \mathbb{P}^{s+1}$ be the linear mapping defined by $Y_0, \ldots, Y_{s+1}$. Then there exists a closed set $W(L) \subset \mathbb{P}^{s+1}$ of dimension at most s and degree at most $D^{r-s}\delta$ such that for every $y \in \mathbb{P}^{s+1} \setminus W(L)$, the linear section V_y of V is nonsingular of pure dimension $r-s-1$.*

Proof. Since $\boldsymbol{\lambda} \notin \mathcal{H}$, by Theorem 4.5 it follows that the polar variety $\mathsf{M}(L)$ has dimension at most s. Then Lemma 6.3 proves that there exists a subvariety $Z(L) \subset V$ of dimension s and degree at most $D^{r-s}\delta$ with $\mathsf{M}(L) \cup \Sigma \subset Z(L)$.

Define $W(L) := \overline{\pi(Z(L))}$. Observe that $W(L) \subset \mathbb{P}^{s+1}$ has dimension at most s. Since $\deg Z(L) \leq D^{r-s}\delta$, by (6) we conclude that $\deg W(L) \leq D^{r-s}\delta$.

Let $y \in \mathbb{P}^{s+1} \setminus W(L)$. By Corollary 5.5 we see that V_y is of pure dimension $r-s-1$. Furthermore, by the definition of $W(L)$ we have $\pi^{-1}(y) \cap (\Sigma \cup \mathsf{M}(L)) = \emptyset$. This implies that the conditions of Lemma 6.2 are satisfied, and hence V_y is nonsingular. This finishes the proof of the theorem. □

Remark 6.5. Under the assumptions of Theorem 6.4, for $\boldsymbol{\lambda} \in (\mathbb{P}^n)^{s+2} \setminus \mathcal{H}$ and $y \in \mathbb{P}^{s+1} \setminus W(L)$, the linear section V_y is contained in V_{sm}. Indeed, by the choice of y, any point $x \in \pi^{-1}(y)$ is a regular point of V. On the other hand, if $x \in V_y \setminus \pi^{-1}(y)$, then $x \in V \cap L$, and $V \cap L \subset V_{\mathrm{sm}}$ because $\Sigma \cap L = \emptyset$.

Since each linear section V_y with $\boldsymbol{\lambda} \in (\mathbb{P}^n)^{s+2} \setminus \mathcal{H}$ and $y \in \mathbb{P}^{s+1} \setminus W(L)$ is a nonsingular projective complete intersection, by Theorem 2.1 we conclude that it is absolutely irreducible.

In what follows, we shall frequently use the notation

$$B_{\boldsymbol{d},s} := D^{r-s-2}\delta\big(\big((n-s)(r-s)+2\big)D+r-s-1\big)+\delta+1,$$

where $\delta := d_1 \cdots d_{n-r}$ and $D := \sum_{i=1}^{n-r}(d_i-1)$.

Corollary 6.6. *For $q > \max\{B_{\boldsymbol{d},s}, D^{r-s}\delta\}$, there exists $y \in \mathbb{P}^{s+1}(\mathbb{F}_q)$ such that the linear section V_y is a nonsingular $\mathbb{F}_q$-variety of pure dimension $r-s-1$ with $V_y \subset V_{\mathrm{sm}}$. In other words, V has a nonsingular linear section of pure dimension $r-s-1$ defined over $\mathbb{F}_q$ and contained in V_{sm}.*

Proof. Let $\mathcal{H} := \mathcal{H}_1 \cup \mathcal{H}_2$ be the hypersurface of $(\mathbb{P}^n)^{s+2}$ of Theorem 6.4, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are the hypersurfaces of Theorem 4.5 and Corollary 5.5 respectively. Since $\mathcal{H}_1$ and $\mathcal{H}_2$ are defined by multihomogeneous polynomials of $\overline{\mathbb{F}}_q[\boldsymbol{\Lambda}]$ of degree at most $(n-s)(r-s)D^{r-s-1}\delta+1$ and $\delta(D^{r-s-2}(2D+r-s-1)+1)$ in each group of variables Λ_i respectively, it follows that $\mathcal{H}$ is defined by a multihomogeneous polynomial of $\overline{\mathbb{F}}_q[\boldsymbol{\Lambda}]$ of degree at most $B_{\boldsymbol{d},s}$ in each group of variables Λ_i.

As $q > B_{\boldsymbol{d},s}$, by Corollary 3.3 there exists $\boldsymbol{\lambda} \in (\mathbb{P}^n(\mathbb{F}_q))^{s+2} \setminus \mathcal{H}$. Define $Y_j := \lambda_j \cdot X \in \mathbb{F}_q[X]$ for $0 \leq j \leq s+1$, let $L := \{Y_0 = \cdots = Y_{s+1} = 0\}$ and let $\pi : V \dashrightarrow \mathbb{P}^{s+1}$ be the linear mapping defined by $Y_0, \ldots, Y_{s+1}$.

Then Theorem 6.4 shows that there exists a closed set $W(L) \subset \mathbb{P}^{s+1}$ of dimension at most s and degree at most $D^{r-s}\delta$ such that, for $y \in \mathbb{P}^{s+1} \setminus W(L)$, the linear section V_y is nonsingular. Furthermore, Remark 6.5 asserts that $V_y \subset V_{\mathrm{sm}}$ for any such y.

Since $q > D^{r-s}\delta$, there exists $y \in \mathbb{P}^{s+1}(\mathbb{F}_q) \setminus W(L)$. Then the linear section V_y is defined over $\mathbb{F}_q$ and satisfies the statement of the corollary. □

7. Existence of smooth $\mathbb{F}_q$-rational points

In this section we obtain sufficient conditions for the existence of smooth $\mathbb{F}_q$-rational points of a complete intersection $V \subset \mathbb{P}^n$ defined over $\mathbb{F}_q$, of dimension r, degree δ, multidegree $\boldsymbol{d} := (d_1, \ldots, d_{n-r})$ with $d_1 \geq \cdots \geq d_{n-r} \geq 2$ and singular locus Σ of dimension at most s. More precisely, we establish conditions on q which imply that $V_{\mathrm{sm}}(\mathbb{F}_q)$ is not empty.

The usual approach to this kind of results relies on a combination of estimates on the number of $\mathbb{F}_q$-rational points and upper bounds for the number of singular $\mathbb{F}_q$-rational points. Instead of doing this, we use the effective version of the Bertini smoothness theorem of Section 6 to prove the existence of a nonsingular linear section of V defined over $\mathbb{F}_q$ and contained in V_{sm}. We combine this result with the following well-known estimate on the number of $\mathbb{F}_q$-rational points of a nonsingular complete intersection $W \subset \mathbb{P}^n$ defined over $\mathbb{F}_q$, of dimension r and multidegree $\boldsymbol{d}$, due to P. Deligne [13]:

$$\big||W(\mathbb{F}_q)| - p_r\big| \leq b_r'(n, \boldsymbol{d}) q^{r/2}, \tag{31}$$

where $b_r'(n, \boldsymbol{d})$ denotes the rth primitive Betti number of any nonsingular complete intersection of $\mathbb{P}^n$ of dimension r and multidegree $\boldsymbol{d}$.

We shall frequently use the following explicit expressions for $b_r'(n, \boldsymbol{d})$ with $r \in \{1, 2\}$ (see, e.g., [16, Theorem 4.1]):

$$\begin{aligned} b_1'(n, \boldsymbol{d}) &= (d_1 \cdots d_{n-1})(d_1 + \cdots + d_{n-1} - n - 1) + 2, \\ b_2'(n, \boldsymbol{d}) &= (d_1 \cdots d_{n-2})\left(\binom{n+1}{2} - (n+1) \sum_{1 \leq i \leq n-2} d_i + \sum_{1 \leq i \leq j \leq n-2} d_i d_j\right) - 3. \end{aligned}$$

Remark 7.1. Let $V \subset \mathbb{P}^n$ be a nonsingular complete intersection defined over $\mathbb{F}_q$, of dimension 2 and multidegree $\boldsymbol{d} := (d_1, \ldots, d_{n-2})$. Let $D := \sum_{i=1}^{n-2}(d_i - 1)$. Then $\deg V = d_1 \cdots d_{n-2}$ and we have

$$b_2'(n, \boldsymbol{d}) \leq (n-1) D^2 \deg V. \tag{32}$$

Indeed,

$$-(n+1) \sum_{1 \leq i \leq n-2} d_i + \sum_{1 \leq i \leq j \leq n-2} d_i d_j \leq \sum_{i=1}^{n-2} d_i \left(\sum_{i=1}^{n-2} d_i - n - 1\right) = \sum_{i=1}^{n-2} d_i (D - 3).$$

Using the inequality $\sum_{i=1}^{n-2} d_i \leq (n-1)D$, we obtain

$$b_2'(n, \boldsymbol{d}) \leq \deg V \left(\binom{n+1}{2} + (n-1)D(D-3)\right) \leq (n-1)D^2 \deg V.$$

This shows (32).

Let $B_{\boldsymbol{d},s} := D^{r-s-2}\delta(((n-s)(r-s)+2)D + r - s - 1) + \delta + 1$, where $\delta := d_1 \cdots d_{n-r}$ and $D := \sum_{i=1}^{n-r}(d_i - 1)$. According to Corollary 6.6, if $q > \max\{B_{\boldsymbol{d},s}, D^{r-s}\delta\}$, then there exists a nonsingular linear section $S \subset V_{\mathrm{sm}}$ defined over $\mathbb{F}_q$ of pure dimension $r - s - 1$.

We are going to prove that the number of $\mathbb{F}_q$-rational points in S is strictly positive, showing thus that V has smooth $\mathbb{F}_q$-rational points. We have the following result.

Theorem 7.2. *Let $V \subset \mathbb{P}^n$ be a complete intersection defined over $\mathbb{F}_q$, of dimension $r \geq 2$, degree δ, multidegree $\boldsymbol{d}$ and singular locus Σ of dimension at most $s \leq r-2$. Let $b' := b'_{r-s-1}(n-s-1, \boldsymbol{d})$. If $q > \max\{B_{\boldsymbol{d},s}, D^{r-s}\delta, b'^{\frac{2}{r-s-1}}\}$, then V has a smooth $\mathbb{F}_q$-rational point.*

Proof. Let $S \subset V_{\mathrm{sm}}$ be the nonsingular linear section of V whose existence is shown in Corollary 6.6. Since S is a nonsingular complete intersection defined over $\mathbb{F}_q$ of dimension $r-s-1$, by (31) we have

$$\left|S(\mathbb{F}_q)\right| \geq p_{r-s-1} - b' q^{\frac{r-s-1}{2}} > q^{\frac{r-s-1}{2}}\left(q^{\frac{r-s-1}{2}} - b'\right).$$

The condition on q implies that the right-hand side is positive, finishing the proof of the theorem. □

Next we discuss two particular instances of this result.

Corollary 7.3. *With notations and assumptions as in Theorem 7.2, if*

$$q > \begin{cases} (\delta(D-2)+2)^2, & \text{for } D \geq 5 \text{ or } D=4 \text{ and } n-r>1, \\ (2(n-r+3)D+2)\delta+1, & \text{otherwise}, \end{cases}$$

then V has a smooth $\mathbb{F}_q$-rational point.

Proof. Observe that $b'_1(n-r+1, \boldsymbol{d}) = \delta(D-2)+2$. Therefore, applying Theorem 7.2 with $s = r-2$, we conclude that, if

$$q > \max\left\{\left(2(n-r+3)D+2\right)\delta+1, D^2\delta, \left(\delta(D-2)+2\right)^2\right\}, \tag{33}$$

then V has a smooth $\mathbb{F}_q$-rational point. For $D \leq 2$ we have $D^2\delta \leq (2(n-r+3)D+2)\delta+1$, while $D^2\delta \leq (\delta(D-2)+2)^2$ for $D \geq 3$. As a consequence, (33) is equivalent to the following condition:

$$q > \max\left\{\left(2(n-r+3)D+2\right)\delta+1, \left(\delta(D-2)+2\right)^2\right\}. \tag{34}$$

If $D \geq 6$, then

$$\left(\delta(D-2)+2\right)^2 \geq \left(2(D+3)D+2\right)\delta+1 \geq \left(2(n-r+3)D+2\right)\delta+1.$$

Combining this inequality with (34) and elementary calculations we deduce the statement of the corollary. □

Corollary 7.4. *Let notations and assumptions be as in Theorem 7.2. Suppose further that the singular locus of V has dimension at most $r-3 \geq 0$. If $q > 3D(D+2)^2\delta$, then V has a smooth $\mathbb{F}_q$-rational point.*

Proof. We apply Theorem 7.2 with $s = r-3$. According to Remark 7.1, we have $b_2'(n-r+2, \boldsymbol{d}) \leq (n-r+1)D^2\delta$. Therefore, Theorem 7.2 shows that a sufficient condition for the existence of a smooth $\mathbb{F}_q$-rational point of V is

$$q > \max\bigl\{D^3\delta, D\delta\bigl(\bigl(3(n-r+3)+2\bigr)D+2\bigr)+\delta+1\bigr\}. \tag{35}$$

Using the inequality $n-r \leq D$, we deduce that

$$D\delta\bigl(\bigl(3(n-r+3)+2\bigr)D+2\bigr)+\delta+1 \leq 3D(D+2)^2\delta,$$

which immediately implies the statement of the corollary. $\square$

8. Estimates on the number of $\mathbb{F}_q$-rational points

In this section we estimate $|V(\mathbb{F}_q)|$ for a complete intersection $V \subset \mathbb{P}^n$ defined over $\mathbb{F}_q$, of dimension r and multidegree $\boldsymbol{d} := (d_1, \ldots, d_{n-r})$ with $d_1 \geq \cdots \geq d_{n-r} \geq 2$, having a singular locus of codimension at least 2 or 3. We denote $\delta := \deg V = d_1 \cdots d_{n-r}$ and $D := \sum_{i=1}^{n-r}(d_i - 1)$ as before.

Fix $s \in \{r-2, r-3\}$. Then Theorem 4.5 and Corollary 5.5 show that there exists a hypersurface $\mathcal{H} := \mathcal{H}_1 \cup \mathcal{H}_2 \subset (\mathbb{P}^n)^{s+2}$, defined by a multihomogeneous polynomial of $\overline{\mathbb{F}}_q[\boldsymbol{\Lambda}]$ of degree at most

$$B_{\boldsymbol{d},s} := D^{r-s-2}\delta\bigl(\bigl((n-s)(r-s)+2\bigr)D+r-s-1\bigr)+\delta+1$$

in each group of variables Λ_i, with the following property: for any $\boldsymbol{\lambda} := (\lambda_0, \ldots, \lambda_{s+1}) \in (\mathbb{P}^n)^{s+2} \setminus \mathcal{H}$, let $Y_j := \lambda_j \cdot X$ for $0 \leq j \leq s+1$, let $\pi : V \dashrightarrow \mathbb{P}^{s+1}$ be the linear mapping defined by $Y_0, \ldots, Y_{s+1}$ and let $L := \{Y_0 = \cdots = Y_{s+1} = 0\} \subset \mathbb{P}^n$. Denote by $\mathsf{E} = V \cap L$ the exceptional locus of π. Then the following conditions hold:

1. the polar variety $\mathsf{M}(L)$ has dimension at most s,
2. the Zariski closure V_y of every fiber $\pi^{-1}(y)$ is of pure dimension $r-s-1$,
3. $\mathsf{E} = V \cap L$ is nonsingular of pure dimension $r-s-2$.

For any such matrix $\boldsymbol{\lambda}$, our effective version of the Bertini smoothness theorem (Theorem 6.4) asserts that there exists a variety $W_L := W(L) \subset \mathbb{P}^{s+1}$ of dimension at most s and degree at most $D^{r-s}\delta$ such that for every $y \in \mathbb{P}^{s+1} \setminus W_L$, the linear section V_y is a nonsingular complete intersection. As V_y is defined over $\mathbb{F}_q$ for every $y \in \mathbb{P}^{s+1}(\mathbb{F}_q)$, we can estimate the number $N_y := |V_y(\mathbb{F}_q)|$ for $y \in \mathbb{P}^{s+1}(\mathbb{F}_q) \setminus W_L$ using Deligne's estimate (31). On the other hand, fibers of points in $W_L(\mathbb{F}_q)$ do not make a significant contribution to the asymptotic behavior of $|V(\mathbb{F}_q)|$. We have the following result.

Theorem 8.1. *Let $V \subset \mathbb{P}^n$ be a complete intersection defined over $\mathbb{F}_q$, of dimension $r \geq 2$, multidegree $\boldsymbol{d}$ and singular locus of dimension at most $s \in \{r-2, r-3\}$. Then*

$$\big||V(\mathbb{F}_q)| - p_r\big| \leq b'_{r-s-1} q^{\frac{r+s+1}{2}} + A(n,s,\boldsymbol{d}) q^{r-1},$$

where $A(n,s,\boldsymbol{d}) := 2b'_{r-s-1} + 2(7D^{r-s}\delta + 1)(\delta - 1)$ and $b'_{r-s-1} := b'_{r-s-1}(n-s-1,\boldsymbol{d})$ is the $(r-s-1)$th primitive Betti number of any nonsingular complete intersection of $\mathbb{P}^{n-s-1}$ of dimension $r-s-1$ and multidegree $\boldsymbol{d}$.

Proof. First we observe that, if $D = 1$, then V is a quadric, and the theorem follows from results on the number of $\mathbb{F}_q$-rational points of quadrics (see, e.g., [31, Theorem 2E] or [27, Section 6.2]).

Next we claim that we may assume $q > B_{\boldsymbol{d},s}$. Indeed, suppose that $q \leq B_{\boldsymbol{d},s}$ holds. Since $n - r \leq D$ and $n - s \leq D + r - s$, we have

$$B_{\boldsymbol{d},s} \leq D^{r-s-2}\delta\big(\big((D+r-s)(r-s)+2\big)D + r - s - 1\big) + \delta + 1. \tag{36}$$

For $D = 2$ we have that V is either a cubic hypersurface or an intersection of two quadrics. In both cases, $|V(\mathbb{F}_q)| \leq \delta q^r + p_{r-1}$ (see [32] and [14]), which implies $||V(\mathbb{F}_q)| - p_r| \leq (\delta - 1)q^r \leq B_{\boldsymbol{d},s}(\delta - 1)q^{r-1}$. By (36) we conclude that $B_{\boldsymbol{d},s} \leq 10 \cdot 2^{r-s} \cdot \delta + 1$, which completes the proof in this case.

On the other hand, for $D \geq 3$, according to (7) we have $|V(\mathbb{F}_q)| \leq \delta p_r$, and therefore $||V(\mathbb{F}_q)| - p_r| \leq (\delta - 1)p_r \leq 2B_{\boldsymbol{d},s}(\delta - 1)q^{r-1}$. As a consequence, from (36) we deduce that $B_{\boldsymbol{d},s} \leq 7D^{r-s}\delta + 1$, which implies the theorem in this case. This finishes the proof of the claim.

Now we assume that $q > B_{\boldsymbol{d},s}$. By Corollary 6.6 there exists $\boldsymbol{\lambda} \in (\mathbb{P}^n(\mathbb{F}_q))^{s+2}$ such that conditions 1–3 above are satisfied.

Let V_y be the linear section defined by a point $y \in \mathbb{P}^{s+1}$ and let $N_y := |V_y(\mathbb{F}_q)|$ for $y \in \mathbb{P}^{s+1}(\mathbb{F}_q)$. Then

$$|V(\mathbb{F}_q)| = \sum_{y \in \mathbb{P}^{s+1}(\mathbb{F}_q)} (N_y - e) + e = \sum_{y \in \mathbb{P}^{s+1}(\mathbb{F}_q)} N_y - (p_{s+1} - 1)e, \tag{37}$$

where $e := |(V \cap L)(\mathbb{F}_q)|$. Since $V \cap L$ has dimension $r-s-2$, $e \leq \delta p_{r-s-2}$, and thus $|e - p_{r-s-2}| \leq (\delta - 1)p_{r-s-2}$.

Subtracting p_r at both sides of (37) and taking into account the identity $p_r = p_{s+1}p_{r-s-1} - (p_{s+1} - 1)p_{r-s-2}$, we obtain

$$\begin{aligned}
\big||V(\mathbb{F}_q)| - p_r\big| &\le \sum_{y\in\mathbb{P}^{s+1}(\mathbb{F}_q)} |N_y - p_{r-s-1}| + (p_{s+1}-1)(\delta-1)p_{r-s-2} \\
&\le \sum_{y\in\mathbb{P}^{s+1}(\mathbb{F}_q)} |N_y - p_{r-s-1}| + 2(\delta-1)q^{r-1}. \qquad (38)
\end{aligned}$$

Let $W_L := W(L) \subset \mathbb{P}^{s+1}$ be the variety of the statement of Theorem 6.4. We can decompose the first term of the right-hand side of (38) as

$$\sum_{y\in\mathbb{P}^{s+1}(\mathbb{F}_q)} |N_y - p_{r-s-1}| = \sum_{y\notin W_L(\mathbb{F}_q)} |N_y - p_{r-s-1}| + \sum_{y\in W_L(\mathbb{F}_q)} |N_y - p_{r-s-1}|.$$

In order to estimate the first term in the right-hand side, Theorem 6.4 asserts that, if $y \notin W_L(\mathbb{F}_q)$, then V_y is a nonsingular complete intersection of $\mathbb{P}^{n-s-1}$ defined over $\mathbb{F}_q$, of pure dimension $r-s-1$, degree δ and multidegree $\boldsymbol{d}$. By (31) we deduce that

$$\begin{aligned}
\sum_{y\notin W_L(\mathbb{F}_q)} |N_y - p_{r-s-1}| &\le b'_{r-s-1} q^{\frac{r-s-1}{2}} p_{s+1} \\
&\le b'_{r-s-1} q^{\frac{r+s+1}{2}} + 2b'_{r-s-1} q^{r-1}. \qquad (39)
\end{aligned}$$

On the other hand, for $y \in W_L(\mathbb{F}_q)$ we have $N_y \le \delta p_{r-s-1}$. Since $\delta \ge 2$, we obtain $|N_y - p_{r-s-1}| \le (\delta-1)p_{r-s-1}$. From (7) it follows that $|W_L(\mathbb{F}_q)| \le \deg W_L \cdot p_s$ and thus

$$\begin{aligned}
\sum_{y\in W_L(\mathbb{F}_q)} |N_y - p_{r-s-1}| &\le (\delta-1)p_{r-s-1} \cdot \deg W_L \cdot p_s \\
&\le 4(\delta-1)\deg W_L \cdot q^{r-1}. \qquad (40)
\end{aligned}$$

Combining (38), (39), (40), we conclude that

$$\big||V(\mathbb{F}_q)| - p_r\big| \le b'_{r-s-1} q^{\frac{r+s+1}{2}} + 2\big(b'_{r-s-1} + \big(2D^{r-s}\delta + 1\big)(\delta-1)\big)q^{r-1}.$$

From this estimate we easily deduce the statement of the theorem. □

Next we estimate the number of smooth $\mathbb{F}_q$-rational points of a singular complete intersection as above.

Theorem 8.2. *Let notations and assumptions be as in Theorem 8.1. Then*

$$\big||V_{\mathrm{sm}}(\mathbb{F}_q)| - p_r\big| \le b'_{r-s-1} q^{\frac{r+s+1}{2}} + B(n,s,\boldsymbol{d})q^{r-1},$$

where $B(n,s,\boldsymbol{d}) := 2b'_{r-s-1} + 2(2D^{r-s}\delta+1)(\delta-1) + 2(s+2)(\delta-1)B_{\boldsymbol{d},s}$.

Proof. Let $\mathcal{H} \subset (\mathbb{P}^n)^{s+2}$ be the hypersurface of Theorem 6.4. Recall that $\mathcal{H}$ is defined by a multihomogeneous polynomial of $\overline{\mathbb{F}}_q[\boldsymbol{\Lambda}]$ of degree at most $B_{\boldsymbol{d},s}$ in each group of variables Λ_i. We have

$$\begin{aligned}\big||V_{\mathrm{sm}}(\mathbb{F}_q)| - p_r\big| &= \frac{1}{p_n^{s+2}}\Bigg(\sum_{\boldsymbol{\lambda}\in((\mathbb{P}^n)^{s+2}\setminus\mathcal{H})(\mathbb{F}_q)} \big||V_{\mathrm{sm}}(\mathbb{F}_q)| - p_r\big| + \sum_{\boldsymbol{\lambda}\in\mathcal{H}(\mathbb{F}_q)} \big||V_{\mathrm{sm}}(\mathbb{F}_q)| - p_r\big|\Bigg) \\ &\le \frac{1}{p_n^{s+2}}\Bigg(\sum_{\boldsymbol{\lambda}\in((\mathbb{P}^n)^{s+2}\setminus\mathcal{H})(\mathbb{F}_q)} \big||V_{\mathrm{sm}}(\mathbb{F}_q)| - p_r\big| + \big|\mathcal{H}(\mathbb{F}_q)\big|(\delta-1)p_r\Bigg).\end{aligned}$$

By (13) we have $|\mathcal{H}(\mathbb{F}_q)| \le p_n^{s+2} - (q^n - \min\{q, B_{\mathbf{d},s}\}q^{n-1})^{s+2}$. Hence,

$$\frac{|\mathcal{H}(\mathbb{F}_q)|}{(p_n)^{s+2}}(\delta-1)p_r \le 2(s+2)(\delta-1)B_{\mathbf{d},s}q^{r-1}.$$

For each $\boldsymbol{\lambda} \in ((\mathbb{P}^n)^{s+2} \setminus \mathcal{H})(\mathbb{F}_q)$, Theorem 6.4 shows that there exists a variety $W_L \subset \mathbb{P}^{s+1}$ of dimension at most s and degree at most $D^{r-s}\delta$ such that for every $y \in \mathbb{P}^{s+1}\setminus W_L$, the Zariski closure V_y of the fiber $\pi^{-1}(y)$ is a nonsingular complete intersection contained in V_{sm}. Then, arguing as in the proof of Theorem 8.1, we obtain

$$\frac{1}{p_n^{s+2}}\sum_{\boldsymbol{\lambda}\notin\mathcal{H}(\mathbb{F}_q)} \big||V_{\mathrm{sm}}(\mathbb{F}_q)| - p_r\big| \le b'_{r-s-1}q^{\frac{r+s+1}{2}} + 2\big(b'_{r-s-1} + \big(2D^{r-s}\delta+1\big)(\delta-1)\big)q^{r-1}.$$

From this inequality we easily deduce the statement of the theorem. □

8.1. Normal complete intersections

In this section we consider the case $s := r-2$ of Theorems 8.1 and 8.2.

Corollary 8.3. *Let $V \subset \mathbb{P}^n$ be a normal complete intersection defined over $\mathbb{F}_q$, of dimension $r \ge 2$, degree δ and multidegree $\boldsymbol{d}$. Then we have*

$$\big||V(\mathbb{F}_q)| - p_r\big| \le \big(\delta(D-2)+2\big)q^{r-1/2} + 14D^2\delta^2q^{r-1}, \tag{41}$$

$$\big||V_{\mathrm{sm}}(\mathbb{F}_q)| - p_r\big| \le \big(\delta(D-2)+2\big)q^{r-1/2} + 11(r+1)D^2\delta^2q^{r-1}. \tag{42}$$

Proof. Applying Theorems 8.1 and 8.2 with $s = r-2$, we obtain

$$\big||V(\mathbb{F}_q)| - p_r\big| \le b'_1q^{r-1/2} + A(n, r-2, \boldsymbol{d})q^{r-1},$$
$$\big||V_{\mathrm{sm}}(\mathbb{F}_q)| - p_r\big| \le b'_1q^{r-1/2} + B(n, r-2, \boldsymbol{d})q^{r-1},$$

where $b'_1 := b'_1(n-r+1, \boldsymbol{d})$,

$$A(n, r-2, \boldsymbol{d}) := 2b_1' + 2\big(7D^2\delta + 1\big)(\delta - 1),$$
$$B(n, r-2, \boldsymbol{d}) := 2b_1' + 2\big(2D^2\delta + 1\big)(\delta - 1) + 2r(\delta - 1)B_{\mathbf{d}, r-2}.$$

Since $b_1' = \delta(D-2) + 2$, we easily deduce (41). On the other hand, using the inequality $n - r \leq D$ we readily obtain (42). $\square$

For a normal complete intersection V as in Corollary 8.3, we have the following estimate (see [16, Corollary 6.2]):

$$\big||V(\mathbb{F}_q)| - p_r\big| \leq \big(\delta(D-2) + 2\big)q^{r-1/2} + 9 \cdot 2^{n-r}\big((n-r)d + 3\big)^{n+1} q^{r-1}, \tag{43}$$

where $d := \max_{1 \leq i \leq n-r} d_i$. On the other hand, if $q > 2(n-r)d\delta + 1$, then we have the following estimate (see [9, Corollary 6.2]):

$$\big||V(\mathbb{F}_q)| - p_r\big| \leq \big(\delta(D-2) + 2\big)q^{r-1/2} + 2\big((n-r)d\delta\big)^2 q^{r-1}. \tag{44}$$

These are the most accurate estimates for normal complete intersections to the best of our knowledge.

The right-hand sides of (41), (43) and (44) have the same first term and different second terms. For the sake of comparison, we observe that

$$\begin{aligned} 2^{n-r}\big((n-r)d + 3\big)^{n+1} &\geq \big(2(n-r)\big)^{n-r}\left(\sum_{i=1}^{n-r} \frac{d_i}{n-r}\right)^{n-r}\left(\sum_{i=1}^{n-r} d_i\right)^{r+1} \\ &\geq \big(2(n-r)\big)^{n-r}\prod_{i=1}^{n-r} d_i \left(\sum_{i=1}^{n-r} d_i\right)^{r+1} \\ &\geq \big(2(n-r)\big)^{n-r} D^2\delta \left(\sum_{i=1}^{n-r} d_i\right)^{r-1}, \end{aligned}$$

where the mid inequality is due to the AM–GM inequality. This allows us to draw several conclusions. First, for varieties of high dimension, say $r \geq (n+1)/2$, (41) and (44) are clearly preferable to (43). In particular, for hypersurfaces the second term in the right-hand side of both (41) and (44) is roughly quartic in δ, while the one (43) contains an exponential term δ^{n+1}. On the other hand, for varieties of low dimension the second term in the right-hand side of (43) might be preferable to (41) and (44). In particular, for curves the former is roughly linear in δ while the latter is quadratic in δ. In this sense, we may say that (41)–(44) somewhat complement (43). Finally, the right-hand side of (44) is slightly lower than that of (41) but holds only for $q > 2(n-r)d\delta + 1$, while (44) holds without any restriction on q.

8.2. Complete intersections which are regular in codimension 2

Next we consider complete intersections which are regular in codimension 2, namely $s \leq r-3$. We have the following result.

Corollary 8.4. *Let $V \subset \mathbb{P}^n$ be a complete intersection defined over $\mathbb{F}_q$, of dimension $r \geq 3$, degree δ and multidegree $\boldsymbol{d}$, with a singular locus of dimension at most $r-3$. Then*

$$\big||V(\mathbb{F}_q)| - p_r\big| \leq 14D^3\delta^2 q^{r-1}, \tag{45}$$

$$\big||V_{\mathrm{sm}}(\mathbb{F}_q)| - p_r\big| \leq (34r-20)D^3\delta^2 q^{r-1}. \tag{46}$$

Proof. By Theorems 8.1 and 8.2 it follows that

$$\big||V(\mathbb{F}_q)| - p_r\big| \leq A(n, r-3, \boldsymbol{d})q^{r-1},$$
$$\big||V_{\mathrm{sm}}(\mathbb{F}_q)| - p_r\big| \leq B(n, r-3, \boldsymbol{d})q^{r-1},$$

where

$$A(n, r-3, \boldsymbol{d}) := 3b_2' + 2\big(7D^3\delta + 1\big)(\delta - 1),$$
$$B(n, r-3, \boldsymbol{d}) := 3b_2' + 2\big(2D^3\delta + 1\big)(\delta - 1) + 2(r-1)(\delta-1)B_{\mathbf{d},r-3},$$

and $b_2' := b_2'(n-r+2, \boldsymbol{d})$. According to Remark 7.1, $b_2' \leq (n-r+1)D^2\delta \leq (D+1)D^2\delta$. Then a simple calculation proves the corollary. $\square$

Under the hypotheses of Corollary 8.4, we have [16, Theorem 6.1]:

$$\big||V(\mathbb{F}_q)| - p_r\big| \leq b_2'(n-r+2, \boldsymbol{d})q^{r-1} + 9 \cdot 2^{n-r} \cdot \big((n-r)d+3\big)^{n+1} q^{r-3/2}. \tag{47}$$

In the comparison of (45) and (47) similar remarks can be made as in the case of normal complete intersections: for high-dimensional varieties (45) may be more accurate than (47), while for low-dimensional varieties (47) may be preferable. Nevertheless, the exponentials in n in the second term of the right-hand side of (47) may hamper its application, even for low-dimensional varieties. In fact, in [10] and [28] we use (41) and (45) to estimate the average cardinality of the value set of polynomials with prescribed coefficients, with a significant gain over what is obtained applying (43) and (47).

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