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Self-dual soliton solutions in a Chern–Simons-CP(1) model with a nonstandard kinetic term

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A generalization of the Chern–Simons-CP(1) model is considered by introducing a nonstandard kinetic term. For a particular case, of this nonstandard kinetic term, we show that the model support self-dual Bogomol'nyi equations. The Bogomol'nyi–Prasad–Sommerfield (BPS) energy has a bound proportional to the sum of the magnetic flux and the CP(1) topological charge. The self-dual equations are solved analytically and verified numerically.

Keywords: Chern–Simons gauge theory; topological solitons; CP(1) nonlinear sigma model.

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1. Introduction

The CP(n) sigma model has been investigated in detail since the early '70s mainly as toy models to explore the strong coupling effects of QCD and as effective models of some condensed matter systems.[1–3]

An important issue related to this type of models concern the existence of soliton type solutions. For the simplest CP(1) model topological solutions have been shown to exist.[4] Nevertheless, the solutions are of arbitrary size due to scale invariance. As argued originally by Dzyaloshinsky, Polyakov and Wiegmann[5] a Chern–Simons term can naturally arise in this type of models and the presence of a dimensional parameter could play some role stabilizing the soliton solutions. A first detailed consideration of this problem was done in Ref. 6 where a perturbative analysis

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around the scale invariant solutions (i.e. no Chern–Simons coupling $\kappa = 0$) showed that the solutions were pushed to infinite size. The problem of the stabilizing topological soliton of the pure CP(1) model by including a Chern–Simons and Maxwell term for the gauge field, was done by non-perturbative analysis in Refs. 7–12. Although, these works showed the possibility of stabilizing the CP(1) sigma model solitons, they do not present self-duality Bogomol'nyi equations. This lack of self-duality can present itself as a disadvantage, at least technically. One reason is that self-dual vortices, such as, for example, those of the Abelian Higgs model[13] do not interact by virtue of the stress tensor vanishing identically, and hence multivortex configurations arbitrarily distributed on the plane can be studied systematically.

In the recent years, theories with nonstandard kinetic term, named k-field models, have received much attention. The k-field models are mainly in connection with effective cosmological models[14–20] as well as the tachyon matter,[21] the ghost condensates[22–26] and dark matter.[27] The addition of nonlinear terms to the kinetic part of the Lagrangian has interesting consequences for topological defects, making it possible for defects to arise without a symmetry-breaking potential term.[28] In this context several studies have been conducted, showing that the k-theories can support topological soliton solutions both in models of matter as in gauged models.[29–40] These solitons have certain features such as their characteristic size, which are not necessarily those of the Standard Models.[40–42]

In this paper, we investigate a Chern–Simons-CP(1) model with a nonstandard kinetic term. We will show that introducing a particular nonstandard dynamics in a Chern–Simons-CP(1) model, via a function ω depending on the CP(1) field, we can obtain self-duality Bogomol'nyi equations by minimizing the energy functional of the model. Finally, we will be able to solve the Bogomol'nyi equations and obtain novel analytic expressions for the soliton solutions. This analysis is completed by showing explicitly the principal features of the soliton profiles.

2. The Theoretical Framework

We begin by considering the following $(2+1)$-dimensional Chern–Simons model, coupled to a complex CP(1) field $n(x)$, subject to the constraint $n^\dagger n = 1$

$$S = S_{\rm cs} + \int d^3x(|D_\mu n|^2 - V(n, n^\dagger))\,, \tag{1}$$

where $S_{\rm cs}$ is the Chern–Simons action,

$$S_{\rm cs} = \int d^3x \frac{\kappa}{4}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}\,. \tag{2}$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, is electromagnetic strength-tensor, $D_\mu = \partial_\mu + iA_\mu$ is the covariant derivative. The metric tensor is defined as $g^{\mu\nu} = (1, -1, -1)$. The potential $V(n, n^\dagger)$ is a function of the field $n(x)$ and its complex conjugate. The constraint can be introduced in the variational process via a Lagrange multiplier.

Then, we extremize the following action

$$S = S_{\rm cs} + \int d^3x(|D_\mu n|^2 - V(n, n^\dagger) + \lambda(n^\dagger n - 1))\,. \tag{3}$$

The variation of this action yields the field equations,

$$\frac{1}{2}\kappa\epsilon_{\mu\nu\rho}F^{\nu\rho} + J_\mu = 0\,, \tag{4}$$

$$D_\mu D^\mu n + \frac{\partial V}{\partial n^\dagger} - \lambda n = 0\,, \tag{5}$$

where $J_\mu = i[(D_\mu n)^\dagger n - n^\dagger D_\mu n]$ is the conserved current density and $\lambda = n^\dagger\left(D_\mu D^\mu n + \frac{\partial V}{\partial n^\dagger}\right)$, so that

$$D_\mu D^\mu n + \frac{\partial V}{\partial n^\dagger} = \left[n^\dagger\left(D_\mu D^\mu n + \frac{\partial V}{\partial n^\dagger}\right)\right] n\,. \tag{6}$$

The time component of Eq. (4)

$$\kappa F_{12} = -J_0 \tag{7}$$

is Gauss's law of Chern–Simons dynamics. Integrating it over the entire plane one obtains the important consequence that any object with charge $Q = \int d^2x\, J_0$ also carries magnetic flux $\Phi = \int B\, d^2x$:[43–45]

$$\Phi = -\frac{1}{\kappa}Q\,, \tag{8}$$

where in the expression of magnetic flux we renamed F_{12} as B.

The expression of the energy functional for the static field configuration is

$$E = \int d^2x\left(\frac{\kappa^2}{4}B^2 + |D_i n|^2 + V(n, n^\dagger)\right). \tag{9}$$

As mentioned in Refs. 7–9, the model (1) does not support Bogomol'nyi equations. In fact, as was shown in Ref. 7, it can be established that the energy functional (9) is bounded below by a multiple of the winding number, which is guaranteed to be a nonvanishing energy for nontrivial field configuration. Despite this, it is not possible to saturate the topological bound. This is because the nature of the CP(1) field, which prevents rewriting the expression (9) as sum of square terms plus a topological term. We will consider, here, a generalization of the model (1), which consist on a modification of the model (1) by introducing a nonstandard kinetic term. Specifically, we consider the following $(2+1)$-dimensional model with Chern–Simons-CP(1) Lagrangian

$$S = \int d^3x\left(\frac{\kappa}{4}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} + \omega(n, n^\dagger)|D_0 n|^2 - |D_i n|^2 + V(n, n^\dagger)\right). \tag{10}$$

Here, the function $\omega((n, n^\dagger))$ is, in principle, an arbitrary function of the CP(1) field.

Since, we are interested in time-independent soliton solutions that ensure the finiteness of the action (10), we are looking for stationary points of the energy for which the static field configuration reads

$$E = \int d^2x(-\kappa A_0 B - \omega(n, n^\dagger)A_0^2 + |D_i n|^2 + V(n, n^\dagger))\,. \tag{11}$$

The Gauss law (7) for this system takes the form

$$A_0 = -\frac{\kappa}{2}\frac{B}{\omega(n, n^\dagger)}\,. \tag{12}$$

Substitution of (12) into (11) leads to

$$E = \int d^2x\left(\frac{\kappa^2}{4}\frac{B^2}{\omega(n, n^\dagger)} + |D_i n|^2 + V(n, n^\dagger)\right). \tag{13}$$

In order to implement the Bogomol'nyi–Prasad–Sommerfield (BPS) formalism, we first introduce the usual ansatz for describe CP(1) solutions

$$n = \begin{pmatrix} e^{iN\phi}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}, \qquad \theta = \theta(r)\,, \qquad A_\phi(r) = a(r)\,, \tag{14}$$

where N is the winding number.

In this ansatz, the magnetic field reads as

$$B = \frac{(ra)'}{r}\,, \tag{15}$$

the quantity $|D_i n|^2$ is expressed as

$$|D_i n|^2 = \left(\frac{\theta'}{2}\right)^2 + \left(a + \frac{N}{r}\right)^2 \cos^2\frac{\theta}{2} + a^2\sin^2\frac{\theta}{2}\,.$$

Under this prescription, the energy (13) is expressed as

$$E = \int d^2x\left[\frac{\kappa^2 B^2}{4\omega(\theta)} + V(\theta) + \left(\frac{\theta'}{2}\right)^2 + \left(a + \frac{N}{r}\right)^2\cos^2\frac{\theta}{2} + a^2\sin^2\frac{\theta}{2}\right]. \tag{16}$$

The Ampere's law reads

$$\frac{\kappa^2}{4}\left(\frac{B}{\omega}\right)' - \left[a + \frac{N}{2r}(1+\cos\theta)\right] = 0 \tag{17}$$

and the θ- field equation is

$$\theta'' + \frac{\theta'}{r} + \frac{\kappa^2}{2}\frac{B^2}{\omega^2}\frac{\partial\omega}{\partial\theta} - 2\frac{\partial V}{\partial\theta} + \left(2a + \frac{N}{r}\right)\frac{N}{r}\sin\theta = 0\,. \tag{18}$$

In the following we apply the BPS formalism, so after some manipulations the energy can be expressed in the following quadratic form

$$E = \int d^2x \left\{ \frac{\kappa^2}{4\omega}\left(B \pm \frac{2}{\kappa}\sqrt{\omega V}\right)^2 \mp \kappa B\sqrt{\frac{V}{\omega}} + \left[\frac{\theta'}{2}\sin\frac{\theta}{2} \pm \left(a + \frac{N}{r}\right)\cos\frac{\theta}{2}\right]^2 \right.$$
$$\left. + \left(\frac{\theta'}{2}\cos\frac{\theta}{2} \mp a\sin\frac{\theta}{2}\right)^2 \mp \frac{1}{2}\frac{N\sin\theta}{r}\theta' \right\}, \tag{19}$$

it is minimized by imposing

$$B = \mp\frac{2}{\kappa}\sqrt{\omega V}\,, \tag{20}$$

$$\frac{\theta'}{2}\sin\frac{\theta}{2} = \mp\left(a + \frac{N}{r}\right)\cos\frac{\theta}{2}\,, \tag{21}$$

$$\frac{\theta'}{2}\cos\frac{\theta}{2} = \pm a\sin\frac{\theta}{2}\,, \tag{22}$$

where the upper (lower) signal corresponds to $N > 0$ ($N < 0$).

The last two equations can be reduced to the one given by

$$\theta' = \mp\frac{N}{r}\sin\theta\,, \tag{23}$$

it will be the first BPS equation.

In order to establish a lower bound for the energy of the topological solutions, we choose the function $\omega(\theta)$ as

$$\omega(\theta) = \kappa^2 V(\theta)\,, \tag{24}$$

so the BPS equation (20) of our CP(1) model becomes

$$B = \mp 2V(\theta)\,. \tag{25}$$

In addition, it is interesting to note that the relations (12), (24) and (25), imply

$$A_0 = \pm\frac{1}{\kappa} \tag{26}$$

which lead us to a soliton solution without electric charge.

In this way, the BPS energy becomes

$$E_{\rm BPS} = \int d^2x\left(\mp B \pm \frac{N}{2}\frac{(\cos\theta)'}{r}\right), \tag{27}$$

where the first integral is the magnetic flux and the second is the CP(1) topological charge.[10] The requirement of well-behaved fields at origin and infinity, provide the following boundary conditions for the fields $\theta(r)$ and $a(r)$:

$$\theta(0) = \pi\,, \qquad \theta(\infty) = 0\,, \tag{28}$$

$$a(0) = 0\,, \qquad (ra)(\infty) = -C_N \tag{29}$$

with $C_N > 0 (C_N < 0)$ for $N > 0$ $(N < 0)$.

With such boundary conditions allowing to compute the magnetic flux

$$\Phi = \int d^2x\, B = -2\pi C_N\,, \tag{30}$$

whereas the BPS energy (27) becomes

$$E_{\rm BPS} = 2\pi|C_N| + 2\pi|N|\,. \tag{31}$$

By using BPS equations the energy (19) can be written as

$$E_{\rm BPS} = \int d^2x\, \varepsilon_{\rm BPS}\,, \tag{32}$$

where $\varepsilon_{\rm BPS}$ is the BPS energy density,

$$\varepsilon_{\rm BPS} = 2V(\theta) + \frac{N^2\sin^2\theta}{2r^2}\,, \tag{33}$$

it is a positive quantity.

Since the solutions of self-dual equations (23) and (25) should also be solutions of the second-order equations of motion (17) and (18), we require that the field $a(r)$ obeys

$$a(r) = -\frac{N}{2r}[1+\cos\theta(r)]\,. \tag{34}$$

This equation allows to compute the magnetic field

$$B = \frac{(ar)'}{r} = \mp\frac{N^2}{2r^2}\sin^2\theta\,, \tag{35}$$

which implies an equation for the potential,

$$2V(\theta) = \frac{N^2}{2r^2}\sin^2\theta\,. \tag{36}$$

This is similar to the "superpotential equation" found in Refs. 46 and 47, which relates the potential with topological terms. In the following, we solve the BPS equations for $N > 0$. The first BPS equation (23),

$$\theta' = -\frac{N}{r}\sin\theta\,, \tag{37}$$

is solved explicitly and its solution compatible with the boundary conditions (28) is given by

$$\theta(r) = \arctan\left(\frac{2\left(\frac{r}{r_0}\right)^N}{\left(\frac{r}{r_0}\right)^{2N} - 1}\right)\,, \tag{38}$$

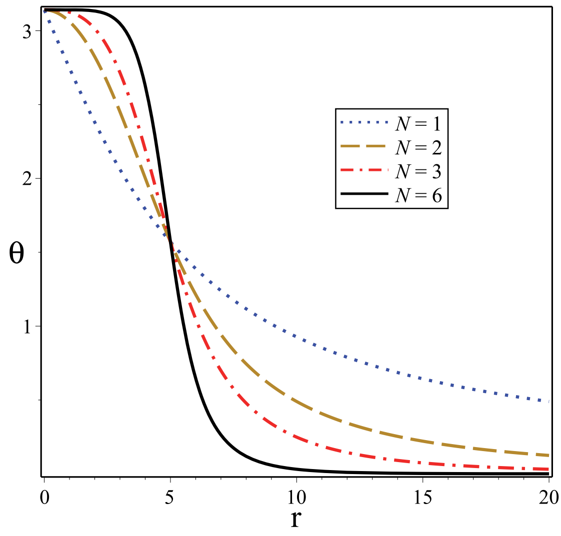

Fig. 1. $\theta(r)$ field profiles.

where r_0 is a parameter characterizing the effective radius of the topological defect. Then, Eq. (34) reads

$$a(r) = -\frac{N}{r}\frac{\left(\frac{r}{r_0}\right)^{2N}}{\left(\frac{r}{r_0}\right)^{2N}+1}\,. \tag{39}$$

It provides the following behavior at $r = 0$,

$$a(r) = -\frac{N}{(r_0)^{2N}}r^{2N-1} + \cdots \tag{40}$$

and for $r \to \infty$, we get

$$a(r) = -\frac{N}{r} + N(r_0)^{2N}\frac{1}{r^{2N+1}} + \cdots\,, \tag{41}$$

it allows to determine asymptotic constant C_N,

$$C_N = N\,, \tag{42}$$

This implies the magnetic flux and BPS energy density are proportional to N.

We have shown the profiles of the BPS solutions for $r_0 = 5$, and some values of the winding number N.

Figure 1 shows the profiles of the θ field. It is clear that for $N > 1$, the asymptotic values is attained rapidly. For larger N, the profile is a rectangle with height π and width r_0.

Figure 2 depicts the profiles of the gauge field, the minimum is located at $r = r_0(2N-1)^{1/2N}$ such that when N increases its position close to the value r_0. Figure 3 shows its asymptotic behavior as explicitly given in Eq. (41).

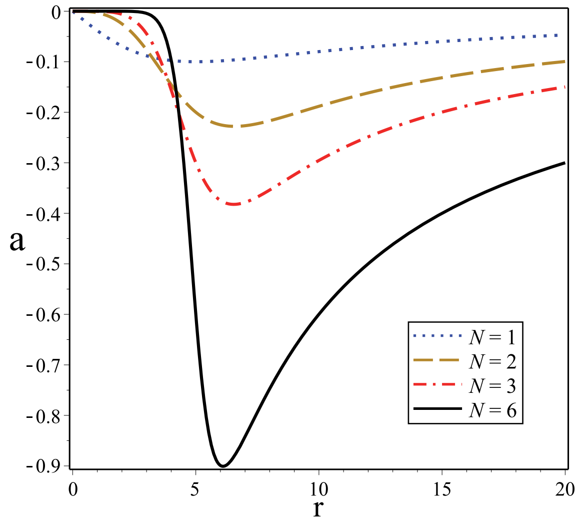

Fig. 2. Gauge field $a(r)$.

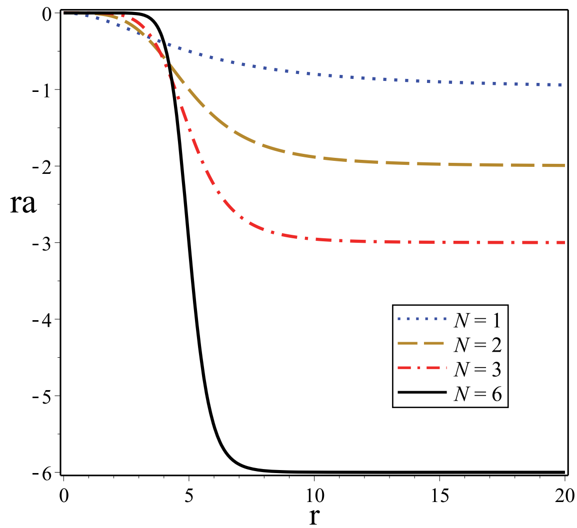

Fig. 3. $ra(r)$ profiles.

Figure 4 depicts the profiles of the magnetic field. For $N = 1$ it is a lump centered at origin, but for $N > 1$ its maximum is located in $r = r_0 \left(\frac{N-1}{N+1}\right)^{1/2N}$. For large values of N, it is located very close to r_0 and its amplitude goes as $\frac{N^2}{2r_0^2}$.

Figure 5 shows the profiles of the BPS energy density which have a similar behavior as the magnetic field.

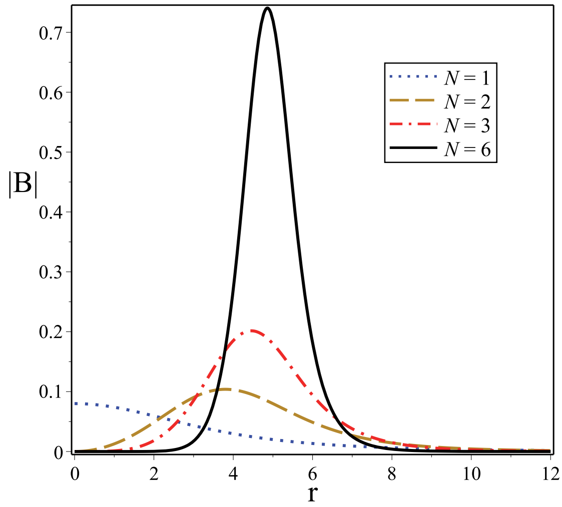

Fig. 4. Magnetic field $|B(r)|$.

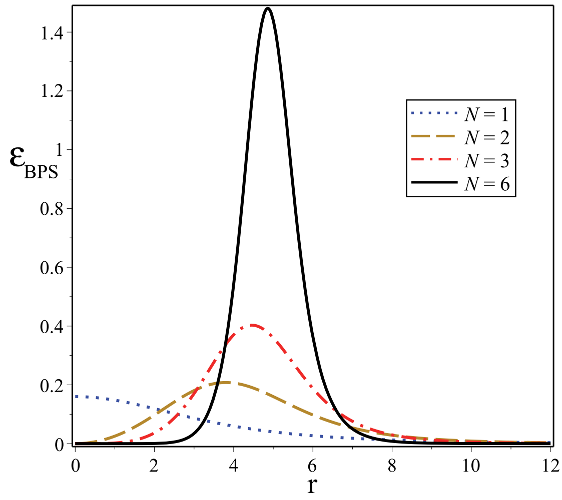

Fig. 5. BPS energy density $\varepsilon_{\mathrm{BPS}}(r)$.

3. Remarks and Conclusions

We have analyzed a Chern–Simons-CP(1) model with generalized kinetic term. Such a generalization allows to obtain self-duality equations whose analytical solutions minimize the energy density. We have obtained a lower bound for the BPS energy given by a sum of two contributions, the first one is due to the magnetic flux and the second one is related to CP(1) topological charge characterizing the BPS solutions. Because our self-dual solutions provide quantized magnetic flux proportional to

N, the CP(1) topological charge (see Eqs. (30) and (42)), the BPS energy result proportional to the CP(1) topological charge.

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