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A further study on the postbuckling of extensible elastic rods

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Abstract

The postbuckling of extensible elastic rods is studied using non-linear geometric models. Accordingly the kinematics and equilibrium are stated. Nine different strain–stress relationships are analyzed. The classical Strength of Materials approach is compared and discussed with other eight constitutive laws stated with *Lagrangian* and *Eulerian* descriptions. The well-known Cauchy and Green methods in Continuum Mechanics are alternatively employed. Four of the approaches are worked out until an explicit solution of the secondary equilibrium path is obtained. The analysis is applicable to small strain problems. The linearized problem is presented for all the laws together with numerical results for rods with various values of the extensibility parameter. The secondary equilibrium paths are numerically evaluated to illustrate the degree of discrepancy. A specific example that displays unexpected unstable behavior is shown. Both critical loads and postbuckling curves are coincident when the theoretical problem of an inextensible rod is solved. It is shown that even when small strains are addressed, the extensibility influence gives rise to disagreement of the postbuckling response when using the different alternatives. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

This paper deals with the influence of the constitutive relation on the postbuckling of an extensible rod under axial load. This is a classical problem in the theory of stability; however it will be shown that different constitutive models lead to significant changes in the postbuckling behavior. The classical problem of the *elastica* (i.e. the equilibrium configurations of inextensible bars under axial compression) was studied by the pioneers James and Daniel Bernoulli, Euler and Lagrange [1,2]. A geometrically exact analysis was already found to exist in the governing equations reported by Love [3] for a specific constitutive law. Although Euler's solution includes a complete analysis in the postbuckling region, its focus was on the stability limit. The postcritical behavior was of secondary interest and the analytical complexity of the governing non-linear equations was not easy to overcome at that time. After the foundations of the general of elastic stability by Poincaré published on 1885, many researchers have contributed to the development of this branch of Applied Mechanics. The asymptotic approach of Koiter [4] has been a key development in the analysis of the initial postbuckling of structures. In the early 1960s two groups continued the

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study of the buckling theories, at Harvard University in the United States and University College London in England, including accurate experimental tests on small-scale models by Roorda [5], and theoretical developments by Thompson and Hunt [6], Huseyin [7], Croll and Walker [8] and El Naschie [9].

An important discussion on constitutive models for rods may be found in a paper by Koenig and Bolle [10] which employs a 3D modeling of a beam assuming large displacements and rotations yet remaining in the elastic regime. Though the kinematics and equilibrium are valid for general rods (i.e. large strains and/or plastic behavior) the constitutive law that they propose is limited to small strains. Pai et al. [11] investigate the appropriate formulation for the geometrically non-linear analysis. Gummadi and Palazotto [12] deal with large strain problems in beams and arches which give rise to some comments in the present work. Regarding the extensibility of the rod, Sampaio and Almeida [13,14] extend the classic static bifurcation problem of the theoretical case of inextensibility to include the influence of axial deformation, accepting the same constitutive equations used earlier by Euler.

In this work the problem of the elastic bar with an axial compressive load is addressed by assuming simultaneous axial and flexional deformations. Large displacements and rotations are admitted though the strains are assumed to remain small. The Navier–Bernoulli (or Kirchhoff) hypothesis (i.e. the cross-sections remain plane after deformation) is undertaken. Such an assumption is commonly adopted, see for instance [11]. The other restriction is the constant thickness.

The paper includes two introductory sections (Sections 2 and 3) to review the kinematics and equilibrium conditions of the problem. The aim is to arrive at the necessary equations so as to achieve a self-contained paper.

Three main procedures lead to eight alternative constitutive formulations. The Cauchy method (CM) (the statement of a linear isotropic stress–strain relation) as well as the Green method (GM) (derivation of the constitutive law from a suitable potential) are well known within the field of Continuum Mechanics. In the Green method one may choose to derive the stress tensor from the infinite series expression (exact) of the energy, then to linearize it and finally to apply it to the particular problem under study (GM1). An alternative is to adopt a truncated energy expression and then to derive the stress tensor (GM2). A third option is to start from the statement of the energy for the post-buckling of the rod leading to the constitutive law (GM3). Thus with the use of the Cauchy method (CM) and the three versions of the Green method (GM1, GM2, GM3) the following eight constitutive laws are considered:

- Relations among tensors with *Lagrangian* description (material form) using the second Piola–Kirchhoff stress tensor and the Green–St. Venant strain tensor. Truesdell [15] and Fung [16] accept this approach. CM, GM1 and GM2 lead to the same law (it will be named LA). GM3 gives a slightly different one (named LB).
- From the relations among tensors with *Eulerian* description (spatial form) six formulations arise. These will be stated and developed in Section 4.2. Three of these involve the Cauchy and the Almansi–Hammel tensors (they will be named EUA, EUB, EUC). The other three (named EUD, EUE and EUF) deal with relations between the Cauchy and the Hencky tensors [15].
- A linear relationship is accepted between the Cauchy tensor and a measure of deformation named ε (specific axial strain). This is a well-known Strength of Materials approach and the results are compared with those of the other eight models.

As will be shown, such constitutive models lead to different secondary equilibrium paths in each formulation. Furthermore when each solution is linearized, as is done in Section 6, neither one is coincident for the critical loads.

It should be mentioned that after the *Lagrangian* and *Eulerian* stress–strain relations are introduced in the equilibrium equations they are worked out without disregarding any term or introducing additional simplifying hypotheses. If one would attempt to equate the *Lagrangian*, *Eulerian* and Strength of Materials' solutions strong restrictions should have to be accepted.

When dealing with the theoretical problem of an inextensible rod, the results for both the critical loads and the secondary equilibrium paths are the same in all models. This is not the case with the more accurate model of an extensible rod. In the authors' opinion the extensibility influence justifies the study of alternative constitutive laws for the postbuckling of an elastic, highly flexible rod, as done in this work.

2. Kinematics of deformation

This section and the next one contain the necessary theory to arrive at the equilibrium equations.

In Fig. 1 the system under study is schematized, i.e. the axis of a compressed strip of length L and thickness h is referred to a Cartesian, orthogonal system. An equilibrium state in plane XY is now imposed which is not the (trivial) straight configuration. The same Cartesian orthogonal reference will be used to position the points at the material and deformed configuration (XYZ and xyz coordinates, respectively)

In Fig. 2, how a portion of length ΔX in the reference configuration (I) is generically located in configuration (II) is shown. In this paper, we use Navier–Bernoulli (or Kirchhoff) hypothesis and assume that the thickness remains constant during deformation. As observed in Fig. 2 the following notation is used:

$$x = X + u,$$

$$y = Y + v,$$

$$z = Z. \tag{1}$$

Also $P = P(X, Y)$, $P_0 \equiv P(X, 0)$, $\mathbf{d} = u\breve{\imath} + v\breve{\jmath} + w\breve{k} = \mathbf{d}(X, Y)$ is the displacement vector, $\breve{\imath}$, $\breve{\jmath}$ and $\breve{k}$ are the unit vectors in correspondence with the reference axes system (XYZ). In particular, in this problem, $w = \hat{w}(X, Y) \equiv 0$. Furthermore, we denote the axis displacements as $u_0 = \hat{u}(X, 0) = u_0(X)$ and $v_0 = \hat{v}(X, 0) = v_0(X)$. It is easy to find, under the assumed hypotheses, that

$$u = u_0 - Y s_\theta, \tag{2a}$$

$$v = v_0 - Y(1 - c_\theta), \tag{2b}$$

where $\theta = \theta(X)$ is the axis rotation and $s_\theta \equiv \sin\theta$, $c_\theta \equiv \cos\theta$. For simplicity, the following notation will be used for the derivatives: $(\cdot)' \equiv \partial(\cdot)/\partial X$, $(\cdot)'' \equiv \partial^2(\cdot)/\partial X^2$, etc., $\overline{(\cdot)} \equiv \partial(\cdot)/\partial Y$, $\overline{\overline{(\cdot)}} \equiv \partial^2(\cdot)/\partial Y^2$, etc.

The Green–St. Venant strain tensors are

$$E_{11}(X, Y) = u' + \tfrac{1}{2}[u'^2 + v'^2], \tag{3a}$$

$$E_{22}(X, Y) = \bar{v} + \tfrac{1}{2}[\bar{u}^2 + \bar{v}^2], \tag{3b}$$

$$2E_{12}(X, Y) = \bar{u} + v' + u'\bar{u} + v'\bar{v} = 2E_{21}(X, Y), \tag{3c}$$

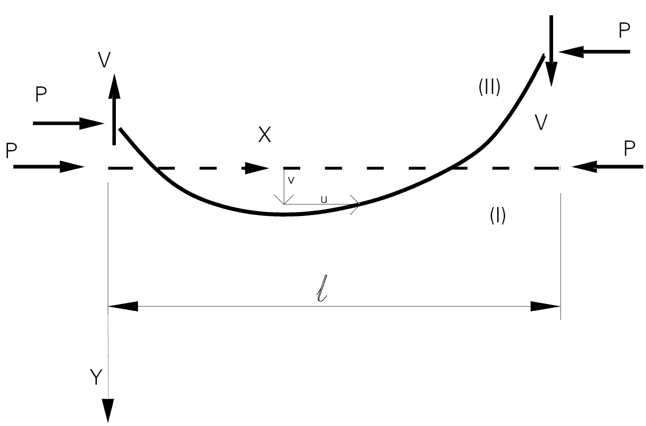

Fig. 1. System under study; (I) reference configuration, (II) equilibrium configuration (for exploration).

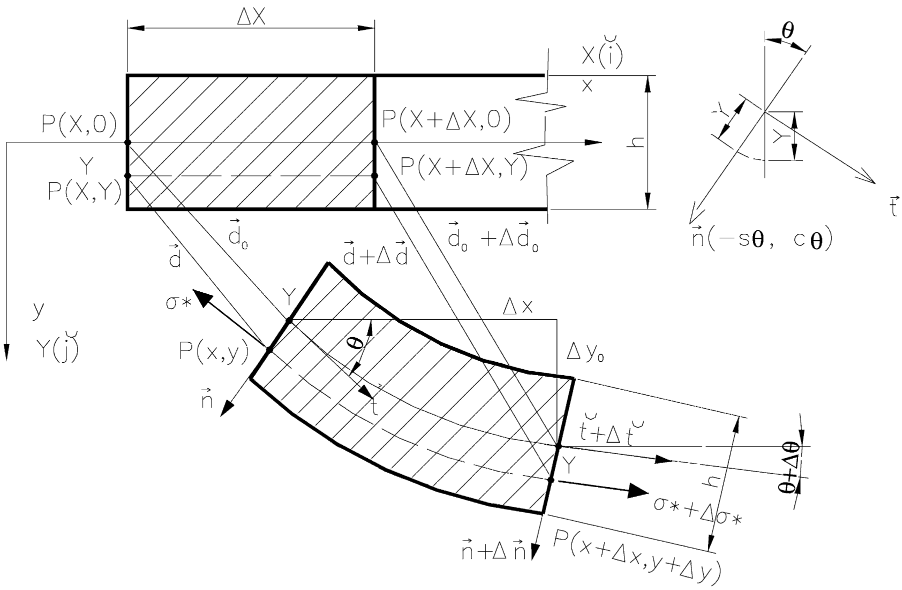

Fig. 2. Scheme of the beam deformation.

which when referred to the rod axes are written as

$$E_{11}(X,Y) = E_{11}(X,0) - \theta' Y[(1+u_0')c_\theta + v_0' s_\theta] + \frac{(\theta' Y)^2}{2}, \tag{4}$$

$$E_{22}(X,Y) = 0, \tag{5}$$

$$2E_{12}(X,Y) = -(1+u_0')s_\theta + v_0' c_\theta. \tag{6}$$

We now introduce an additional notation: $\lambda = \lambda(X,Y)$ as the *stretching* (a length relationship); in particular $\lambda_0 \equiv \lambda(X,0)$ and $\varepsilon = \varepsilon(X,Y)$ the specific axial deformation of a fiber parallel to the rod axis. As is well known

$$\varepsilon = \varepsilon(X,Y) = \lambda - 1 = \sqrt{1 + 2E_{11}(X,Y)} - 1, \tag{7a}$$

$$\varepsilon_0 = \varepsilon(X,0) = \lambda_0 - 1 = \sqrt{1 + 2E_{11}(X,0)} - 1. \tag{7b}$$

Furthermore, let us introduce the strain gradient tensor $\mathbf{F} = \mathbf{F}(X,Y)$ of components F_{ij} which, using index notation and summation convention (in Eq. (1), $x_1 = x$; $x_2 = y$; $x_3 = z$ and $X_1 = X$; $X_2 = Y$; $X_3 = Z$), is written as $F_{ij} = \partial x_i / \partial X_j$.

In matrix notation one obtains

$$\mathbf{F}(F_{ij}) = \begin{pmatrix} (1+u') & \bar{u} & 0 \\ v' & (1+\bar{v}) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} (1+u') & -s_\theta & 0 \\ v' & c_\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{8}$$

It is known that if $\breve{M}(M_i)$ and $\breve{m}(m_i)$ are the directions of an arbitrary filament before and after the deformation, respectively, and λ_M being the stretching of such fiber, it is verified that (see for instance [17])

$$\lambda_M m_i = F_{ij} M_j. \tag{9}$$

Let us apply Eq. (9) to the axis fiber for which

$$M_1 = 1, \quad M_2 = 0, \quad m_1 = c_\theta, \quad m_2 = s_\theta \tag{10}$$

since in this case $\breve{m} \equiv \breve{t}$ is imposed. Then (also $\lambda_M \equiv \lambda_0$)

$$\lambda_0 c_\theta = (1 + u_0'),$$

$$\lambda_0 s_\theta = v_0'. \tag{11}$$

The immediate consequence is that from Eqs. (4) and (6) together with Eq. (11) the following result holds:

$$E_{11}(X, Y) = E_{11_0} - \lambda_0 \theta' Y + \frac{\theta'^2 Y^2}{2}, \tag{12a}$$

$$E_{12}(X, Y) = E_{21}(X, Y) = 0, \tag{12b}$$

where

$$E_{11_0} \equiv E_{11}(X, 0) = \frac{\lambda_0^2 - 1}{2}. \tag{13}$$

These expressions (together with Eq. (5)) are logical consequences of the previously accepted hypothesis: neither the specific strain ε_Y in the thickness direction nor the angular distortion γ_{XY} between orthogonal fibers originally parallel to XY exist.

So the St. Venant–Green tensor is written in matrix notation as

$$\mathbf{E} = \begin{pmatrix} E_{11}(X, Y) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{14}$$

The main invariants of $\mathbf{E}$ are $I_E = E_{11}$; $II_E = III_E = 0$.

The determinant of the matrix (8) may be found using Eqs. (11) and (12a),

$$|\mathbf{F}| = \lambda. \tag{15}$$

Due to the mass conservation principle, if $\mathrm{d}v$ and $\mathrm{d}V$ are the volume elements of the deformed and the reference configuration, respectively and ρ and ρ_0 are the corresponding densities, the following expression is verified:

$$\mathrm{d}v = \frac{\rho_0}{\rho}\,\mathrm{d}V = |\mathbf{F}|\,\mathrm{d}V = \lambda\,\mathrm{d}V. \tag{16}$$

At this stage a verification should be made: the fibers originally parallel to axis X (for any Y) and once deformed, should remain parallel to the axis with rotation θ. For this purpose we should use Eq. (9) in which $\breve{M} = \breve{\imath}$ $(\forall Y)$. To find the direction $\breve{m}$ of these fibers the stretching of which is called λ (see, for instance, [17]), one writes $\lambda m_i = F_{ij} M_j = F_{i1}$. Using this as well as Eqs. (8) and (11) one can draw two main conclusions for our purposes: all the fibers originally parallel to axis X are located, in the equilibrated configuration, parallel to the axis on the same cross section since $\forall Y$: $m_1 = c_\theta$; $m_2 = s_\theta$. Such fibers have the following length relationship and specific axial deformation:

$$\lambda \equiv \lambda(X, Y) = \lambda_0 - \theta' Y, \tag{17a}$$

$$\varepsilon = \varepsilon(X, Y) = \lambda - 1 = \varepsilon_0 - \theta' Y. \tag{17b}$$

Eqs. (17a) and (17b) may be also derived from expressions (7a), (7b) and (12a). Additionally, it is possible to write $F_{11} = \lambda c_\theta$; $F_{12} = -s_\theta$; $F_{21} = \lambda s_\theta$; $F_{22} = c_\theta$.

Let us now state the Almansi–Hammel (*Eulerian*) tensor $\mathbf{e}$. Having taken an orthogonal Cartesian system as a general reference for both the original and the deformed configurations we obtain $\mathbf{e}$ from a tensor transformation of $\mathbf{E}$, which is written as

$$\mathbf{e} = \frac{E_{11}}{\lambda^2}\begin{pmatrix} c_\theta^2 & s_\theta c_\theta & 0 \\ s_\theta c_\theta & s_\theta^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{18}$$

Its main invariants are $I_e = E_{11}/\lambda^2$; $II_e = III_e = 0$. In this problem it is convenient to calculate the tensor $\mathbf{e}^*$ referred to the deformed local reference axes — $(\breve{t}\breve{n}\breve{k})$ ($\breve{k}$ is the unit vector corresponding to axis $z = Z$) — for any value of coordinate Y. The tensor $\mathbf{e}^*$ may be found by means of a tensorial transformation at a point, i.e. $e^*_{pq} = e_{rs} a_{rp} a_{sq}$ in which a_k denotes the $\breve{t}$ and/or $\breve{n}$ components with respect to (x, y, z). Furthermore, one arrives at

$$\mathbf{e}^* = \frac{E_{11}}{\lambda^2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{19}$$

The main invariants are $I_{e^*} = E_{11}/\lambda^2$; $II_{e^*} = III_{e^*} = 0$. On the other hand, $\mathbf{e}^*$ may be simply calculated knowing Eq. (7a) and also that $\lambda = 1 + \varepsilon = 1/\sqrt{1 - 2e^*_{11}}$ since one deals with λ (or ε) of the fibers parallel to the axis before and after the deformation; after equating these expressions, $e^*_{11} = E_{11}/\lambda^2$ is verified. Additionally, one may observe that

$$E_{11} = E_{11}(X, Y) = \frac{\lambda^2 - 1}{2}, \tag{20}$$

$$e^*_{11} = e^*_{11}(X, Y) = \frac{\lambda^2 - 1}{2\lambda^2}. \tag{21}$$

It can also be shown that $e^*_{22} = e^*_{12} = 0$, thus verifying tensor (19). Expressions (20) and (21) are for this study of major importance.

Finally, let us introduce the Hencky tensor [15] defined as

$$\mathbf{h} = -\tfrac{1}{2}\log(\mathbf{I} - 2\mathbf{e}). \tag{22}$$

When a reference system $xyz(\breve{i}\breve{j}\breve{k})$ is used, an infinite series in the $e_{\breve{i}\breve{j}}$'s is obtained for each component. In the system $(\breve{t}\breve{n}\breve{k})$, $\mathbf{h}^*$ is

$$\mathbf{h}^* = -\tfrac{1}{2}\log(\mathbf{I} - 2\mathbf{e}^*) = \mathbf{e}^* + \mathbf{e}^{*2} + \tfrac{4}{3}\mathbf{e}^{*3} + \cdots. \tag{23}$$

Fortunately when using the reference $(\breve{t}\breve{n}\breve{k})$, from Eq. (19) and according to Eq. (23)

$$h_{11} = \frac{E_{11}}{\lambda^2} + \frac{E_{11}^2}{\lambda^4} + \frac{4}{3}\frac{E_{11}^3}{\lambda^6} + \cdots$$

$$= -\frac{1}{2}\log\left(1 - 2\frac{E_{11}}{\lambda^2}\right) \tag{24}$$

others being $h_{ij} = 0$. Then from Eqs. (20) and (24), $h_{11} = \log\lambda$. In general, $h_{ij} = \delta_{1i}\delta_{1j}\log\lambda$ and

$$\mathbf{h}^* = \log\lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{25}$$

whose invariants are $I_{h^*} = \log\lambda = \log|\mathbf{F}| = \log(\rho_0/\rho)$; $II_{h^*} = III_{h^*} = 0$.

3. Equilibrium

In Fig. 3 the internal forces reduced to the axis of the deformed beam portion are indicated in the same element depicted in Fig. 2. As stated before, we are dealing with a static problem in which body forces are not considered and only an axial load

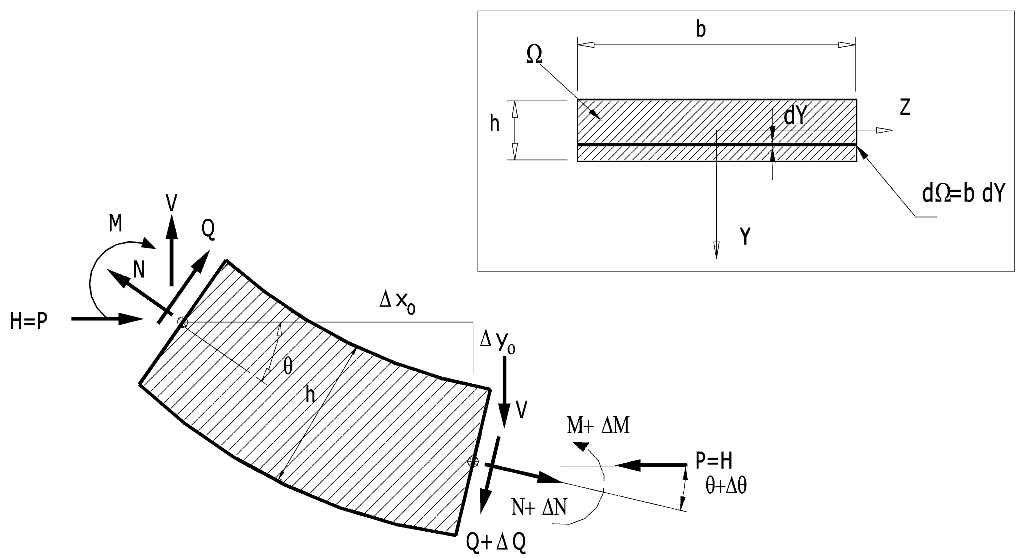

Fig. 3. Reduced internal forces in a generic portion of the deformed beam.

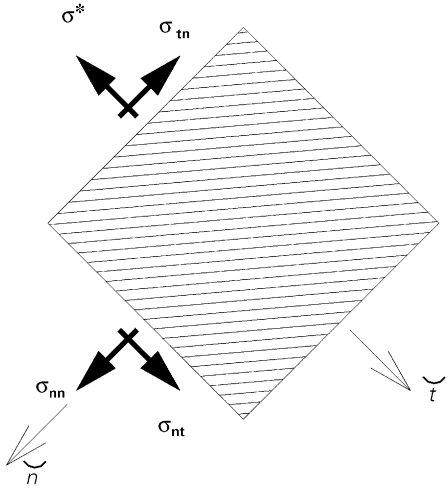

Fig. 4. Scheme of normal and tangential stresses.

P is included in the study. The equilibrium equations may be stated in the following way: two equations of force equilibrium along directions X and Y and one of moment equilibrium with respect to any point belonging to the (XY) plane, all of them among characteristic internal forces.

The element's cross section area being $\mathrm{d}\Omega = b\,\mathrm{d}Y$, the following expressions are obtained:

$$N = N(X) = \int_{\Omega} \sigma^*(X, Y)\,\mathrm{d}\Omega$$

$$= b\int_{-h/2}^{h/2} \sigma^*(X, Y)\,\mathrm{d}Y, \tag{26}$$

$$M = M(X) = \int_{\Omega} \sigma^*(X, Y) Y\,\mathrm{d}\Omega$$

$$= b\int_{-h/2}^{h/2} \sigma^*(X, Y) Y\,\mathrm{d}Y \tag{27}$$

(see σ^* in Figs. 3 and 4). Due to Navier–Bernoulli hypothesis the shear forces should be considered but they are derived from integration of the tangential stresses. They should be taken into account only with a static character and not as coming from deformation. One is able to calculate (constitutively) only σ^*.

After working out the equilibrium expressions (using Taylor expansions, mean value theorem, limits for $\Delta X \to \infty$) and defining

$$H \equiv H(X) = -Nc_\theta + Qs_\theta, \tag{28}$$

$$V \equiv V(X) = Ns_\theta + Qc_\theta \tag{29}$$

or alternatively

$$N = -(Hc_\theta - Vs_\theta), \tag{30}$$

$$Q = Hs_\theta + Vc_\theta. \tag{31}$$

Finally, the moment equilibrium may be written as

$$M' - [Pv_0' + V(1 + u_0')] = 0 \tag{32}$$

which in turn may be expressed after using Eq. (11) as

$$M' - \lambda_0(Ps_\theta + Vc_\theta) = 0, \tag{33}$$

where, as mentioned before, P and V are constants. Now observing expression (31), the equilibrium condition (33) is further reduced to

$$M' - \lambda_0 Q = 0. \tag{34}$$

Also after observing Eq. (32) and integrating, one obtains

$$M - [Pv_0 + V(X + u_0)] = \text{constant}. \tag{35}$$

This condition — given by either Eq. (32), (33) or (34) — is the only differential equation to be solved. The other two are identically satisfied with $P = V = \text{constant}$.

4. Constitutive relationships

As is known, two fundamental methods allow for a rational statement of constitutive laws within the scope of the so-called elastic isotropic solids. They will be used in this work:

1. Cauchy method, and
2. Green method.

Table 1
Summary of the nine constitutive laws studied in this work

Basic strain tensor	Short name	Constitutive law	Elastic constant	Derived from method
E	LA	$\sigma^* = C_A^{\mathrm{L}} \dfrac{\lambda(\lambda^2-1)}{2}$	$C_A^{\mathrm{L}} \equiv 2G_A^{\mathrm{L}};\ \Lambda_A^{\mathrm{L}} = 0$	CM = GM1 = GM2
	LB	$\sigma^* = C_B^{\mathrm{L}} \dfrac{\lambda(\lambda^2-1)}{2}$	$C_B^{\mathrm{L}} \equiv \Lambda_B^{\mathrm{L}} + 2G_B^{\mathrm{L}}$	GM3
	EUA	$\sigma^* = C_A^{\mathrm{E}} \dfrac{(\lambda^2-1)}{2\lambda^2}$	$C_A^{\mathrm{E}} \equiv 2G_A^{\mathrm{E}};\ \Lambda_A^{\mathrm{E}} = 0$	CM = GM1
e	EUB	$\sigma^* = C_B^{\mathrm{E}} \dfrac{(\lambda^2-1)}{2\lambda^5}$	$C_B^{\mathrm{E}} \equiv 2G_B^{\mathrm{E}};\ \Lambda_B^{\mathrm{E}} = 0$	GM2
	EUC	$\sigma^* = C_C^{\mathrm{E}} \dfrac{(\lambda^2-1)}{2\lambda^5}$	$C_C^{\mathrm{E}} \equiv \Lambda_C^{\mathrm{E}} + 2G_C^{\mathrm{E}}$	GM3
	EUD	$\sigma^* = C_D^{\mathrm{E}} \log\lambda$	$C_D^{\mathrm{E}} \equiv 2G_D^{\mathrm{E}};\ \Lambda_D^{\mathrm{E}} = 0$	CM = GM1
h	EUE	$\sigma^* = C_E^{\mathrm{E}} \dfrac{\log\lambda}{\lambda}$	$C_E^{\mathrm{E}} \equiv 2G_E^{\mathrm{E}};\ \Lambda_E^{\mathrm{E}} = 0$	GM2
	EUF	$\sigma^* = C_F^{\mathrm{E}} \dfrac{\log\lambda}{\lambda}$	$C_F^{\mathrm{E}} \equiv \Lambda_F^{\mathrm{E}} + 2G_F^{\mathrm{E}}$	GM3
–	S	$\sigma^* = C^{\mathrm{S}}(\lambda-1)$	$C^{\mathrm{S}} \equiv 2G^{\mathrm{S}} = E;\ \Lambda^{\mathrm{S}} = 0$	CM[a]

[a] A Cauchy method but not between tensors; E is the Young's modulus.

The first one starts from accepting the most general *linear law* which entails strain and stress tensors. It will be referred to as CM.

Instead the second method, for hyperelastic materials, accepts a potential for the stresses, say a strain energy, which is an analytic function of the strain tensors. In this work the authors study the following alternatives to the Green method:

GM1: The linearization of the stress tensor from the general expression of the energy (infinite series). When dealing with applications for systems with one dimension small with respect to the others (as here with the rod) the strain is infinitesimal although displacement gradients or rotations may be large and the displacements–load relationship no longer linear. Then, it is acceptable to linearize the general expression of the Cauchy tensor though not the strain tensor [15]. After this particular linearization the stress–strain law is obtained.

GM2: the use of an energy expression up to the second order in the strain tensors and the posterior derivation of the stress tensor.

GM3: the statement of the energy for the particular problem herein analyzed and the posterior derivation of the stress tensor.

Furthermore, the *Lagrangian* and *Eulerian* descriptions are explored with all the methods. Also the Hencky tensor is included in the study. One would obtain 12 constitutive relations from the application of the above-mentioned alternatives. Some of them, as shown below in detail, are coincident. So we deal with eight distinct constitutive laws and additionally with the classical Strength of Materials approach.

In this work the stress to be calculated is $\sigma^* = \sigma^*(X, Y)$ acting normally on an area element of the plane cross section of the deformed beam (see

Table 2
Green method's stresses

Description	Strain tensor	Green method's stresses
Lagrangian	$\mathbf{E}$	$S_{ij} = \dfrac{\partial \Sigma(\mathbf{E})}{\partial E_{ij}}$
Eulerian	$\mathbf{e}$	$\sigma_{ij} = \dfrac{\rho}{\rho_0}(\delta_{kj} - 2e_{kj})\dfrac{\partial \Sigma(\mathbf{e})}{\partial e_{ki}}$
	$\mathbf{h}$	$\sigma_{ij} = \dfrac{\rho}{\rho_0}\dfrac{\partial \Sigma(\mathbf{h})}{\partial h_{ij}}$

Fig. 4). Now, it is evident that σ^* is a component of the Cauchy tensor referred to the axes $(\breve{t},\breve{n},\breve{k})$. Once σ^* is known as a function of the strain tensors, $N(X)$ and $M(X)$ are found by means of Eqs. (26) and (27).

The resulting equations and notation are summarized in Table 1. The detailed derivation is made in this section. G and Λ are the Lamé constants. Recall from the Introduction that the *Lagrangian* alternatives are named LA and LB and the *Eulerian* ones EUA, EUB, EUC, EUD, EUE and EUF, the last three being related to the Hencky tensor. Note that solutions LB, EUC and EUF differ respectively from LA, EUB and EUE only by a constant definition. This subject will be discussed later on.

Since the energy essentially depends on the Green–St. Venant tensor (or on the right Cauchy–Green one), as may be shown [17], the values of the same stress tensor calculated with the different approaches will not conduce to the same result. In effect, the energy is in general an infinite series in the corresponding strain tensors. Its truncation leads to a discrepancy in the results. A summary of the expression for the energy derivation of the stress tensor is presented in Table 2. A more detailed treatment is given below.

4.1. Lagrangian statements

4.1.1. Lagrangian statement LA

Let $\mathbf{S}$ of components $S_{ij}(=S_{ji})$ be the second Piola–Kirchhoff stress tensor [15] (or only Kirchhoff [16]) linearly related to the Green–St. Venant strain tensor $\mathbf{E}$. The general isotropic linear expression is

$$S^{\mathrm{L}}_{A_{ij}} = \Lambda^{\mathrm{L}}_A \delta_{ij} I_E + 2G^{\mathrm{L}}_A E_{ij} \tag{36}$$

in which δ_{ij} are the second-order Kronecker delta; $I_E \equiv E_{\alpha\alpha} = E_{11} + E_{22} + E_{33}$ the linear invariant of $\mathbf{E}$, and Λ^{L}_A and G^{L}_A the elastic Lamé constants. These are not necessarily the same ones of the infinitesimal strain problem nor the constants to be used in the other constitutive propositions. Note that expression (36) may be obtained indistinctly from any of the following methods: (a) the Cauchy method (CM), (b) the linearization of the second Piola–Kirchoff tensor S_{ij} obtained by means of the Green's method (hyperelastic material) (GM1) and, (c) the acceptance of a strain energy of second order stated in the Green–St. Venant strain tensor followed by an application of Green's method (GM2). The energy expression for the latter approach is

$$\Sigma^{\mathrm{L}}_A = \frac{\Lambda^{\mathrm{L}}_A + 2G^{\mathrm{L}}_A}{2} I_E^2 - 2G^{\mathrm{L}}_A II_E \tag{37}$$

and the stress tensor is obtained from the following derivation:

$$S^{\mathrm{L}}_{A_{ij}} = \frac{\partial \Sigma^{\mathrm{L}}_A}{\partial E_{ij}}. \tag{38}$$

As is known [15,16] the Cauchy stress tensor of components σ_{ij} is applied to Cartesian area elements in the deformed configuration, that is, over surfaces around a generic point, normal to (x, y, z) $(\equiv (X, Y, Z))$, respectively. To obtain it as a function of $\mathbf{S}$ the following expressions are to be used:

$$\sigma^{\mathrm{L}}_{A_{ij}} = \frac{\rho}{\rho_0} F_{ip} F_{jq} S_{pq} = \frac{1}{|\mathbf{F}|} F_{ip} F_{jq} S_{pq}. \tag{39}$$

Making use of the tensor character of the Cauchy tensor it is possible to find any normal stress on an arbitrary plane at the considered point and even more in an arbitrary direction on that plane. In the case under study only the component $E_{11}(X, Y) \neq 0$ because of which only two components of $\mathbf{S}$ may not be null. In effect, from Eq. (36)

$$S_{11} = S_{11}(X, Y) = (\Lambda^{\mathrm{L}}_A + 2G^{\mathrm{L}}_A) E_{11}(X, Y), \tag{40}$$

$$S_{22} = S_{22}(X, Y) = \Lambda^{\mathrm{L}}_A E_{11}(X, Y). \tag{41}$$

For this plane stress problem the *Eulerian* Cauchy tensor has only three non-null components (from

Eq. (39); recall also Eq. (15)):

$$\sigma_{11} = \sigma_{11}(X, Y) = \frac{1}{\lambda}[F_{11}^2 S_{11} + F_{12}^2 S_{22}], \tag{42}$$

$$\sigma_{12} = \sigma_{21} = \sigma_{12}(X, Y) = \frac{1}{\lambda}[F_{11}F_{21}S_{11} + F_{12}F_{22}S_{22}], \tag{43}$$

$$\sigma_{22} = \sigma_{22}(X, Y) = \frac{1}{\lambda}[F_{21}^2 S_{11} + F_{22}^2 S_{22}], \tag{44}$$

where $\lambda = \lambda(X, Y)$ is obtained from Eq. (17a). Then due to Eqs. (40)–(44) and the expressions of F_{ij} given in Section 2, after denoting $r \equiv \Lambda_A^L/(\Lambda_A^L + 2G_A^L)$ the components of the Cauchy tensor result:

$$\sigma_{11} = (\Lambda_A^L + 2G_A^L)E_{11}\left(\lambda c_\theta^2 + r\frac{s_\theta^2}{\lambda}\right), \tag{45}$$

$$\sigma_{12} = \sigma_{21} = (\Lambda_A^L + 2G_A^L)E_{11}\left(\lambda - \frac{r}{\lambda}\right)s_\theta c_\theta, \tag{46}$$

$$\sigma_{22} = (\Lambda_A^L + 2G_A^L)E_{11}\left(\lambda s_\theta^2 + r\frac{c_\theta^2}{\lambda}\right). \tag{47}$$

By means of a plane tensorial transformation in a point of a generic section (Mohr expression for an arbitrary Y),

$$\sigma_{tt} \equiv \sigma^* = \sigma^*(X, Y) = \sigma_{ij}(\breve{t})_i(\breve{t})_j = \sigma_{11}c_\theta^2 + 2\sigma_{12}s_\theta c_\theta + \sigma_{22}s_\theta^2 \tag{48}$$

holds and then recalling Eq. (20) and taking into account Eqs. (45)–(47) it may be found that

$$\sigma^* = (\Lambda_A^L + 2G_A^L)(\lambda^2 - 1)\frac{\lambda}{2}, \tag{49}$$

where, as stated by Eq. (17a), $\lambda(X, Y) = \lambda_0(X) - \theta'(X)Y$. Analogously, let us now calculate stresses σ_{nn} and $\sigma_{tn}(=\tau_{tn})$ (see Fig. 4):

$$\sigma_{nn} = \sigma_{ij}(\breve{n})_i(\breve{n})_j = \sigma_{11}s_\theta^2 - 2\sigma_{12}s_\theta c_\theta + \sigma_{22}c_\theta^2, \tag{50}$$

$$\sigma_{nt} = \sigma_{ij}(\breve{n})_i(\breve{t})_j = \sigma_{tn} = (\sigma_{22} - \sigma_{11})s_\theta c_\theta + \sigma_{12}(c_\theta^2 - s_\theta^2), \tag{51}$$

which, after replacement of Eqs. (45)–(47), yield

$$\sigma_{nn} = \frac{\Lambda_A^L E_{11}}{\lambda}, \tag{52}$$

$$\sigma_{nt} = \sigma_{tn} \equiv 0. \tag{53}$$

Eq. (52) puts forth the same question arising in classical Strength of Materials. If a Poisson effect is considered, a normal stress σ_{nn} in the thickness direction would arise. However, since by hypothesis the single external load in this study is P, in $Y = \pm h/2$,

$$\sigma_{nn}(X, \pm h/2) = 0, \tag{54}$$

should hold: that is, due to Eq. (52)

$$E_{11}(X, \pm h/2) = 0 \tag{55}$$

or

$$\Lambda_A^L = 0. \tag{56}$$

After observing Eqs. (12a) and (12b), Eq. (55) implies two simultaneous conditions

$$E_{11_0} - \lambda_0\theta'\frac{h}{2} + \theta'^2\frac{h^2}{8} = 0, \tag{57}$$

$$E_{11_0} + \lambda_0\theta'\frac{h}{2} + \theta'^2\frac{h^2}{8} = 0. \tag{58}$$

Since λ_0 is never null the following result should hold for $h \neq 0$:

$$\theta'(X) \equiv E_{11_0}(X) \equiv 0 \tag{59}$$

which is absurd since this would conduce to an undeformed (trivial) solution, obviously not the aim of our search. In consequence, Eq. (56) must be satisfied which implies that the only non-null stress component in the reference $(\breve{t}, \breve{n}, \breve{k})$ is

$$\sigma^* = C_A^L\frac{\lambda(\lambda^2 - 1)}{2} = C_A^L\varepsilon\frac{\lambda(\lambda + 1)}{2} = C_A^L\varepsilon\frac{(1 + \varepsilon)(2 + \varepsilon)}{2}, \tag{60}$$

where $C_A^L \equiv 2G_A^L$. If the elastic constants are thought as resulting from a classical linear *hookean* behavior, the expression derived above yields a null Poisson effect as well as $C_A^L = 2G_A^L = E$ (E: Young's modulus). In what follows, C_A^L will be tackled as an elastic *Lagrangian* constant about which no supposition will be made.

Finally, after finding σ^* from Eq. (60) the expression of the internal forces $N(X)$ and $M(X)$ for each deformed section may be calculated. From this and after integrating (Eqs. (26) and (27)) we have the following results:

$$N = N_A^{\mathrm{L}} \equiv C_A^{\mathrm{L}} \Omega \eta_A^{\mathrm{L}}, \tag{61}$$

$$M = M_A^{\mathrm{L}} \equiv C_A^{\mathrm{L}} J \mu_A'^{\mathrm{L}}. \tag{62}$$

The following notation has been introduced:

$$\eta_A^{\mathrm{L}} = \eta_A^{\mathrm{L}}(X) \equiv \frac{\lambda_0}{2}[\lambda_0^2 + \alpha^2 - 1], \tag{63}$$

$$\mu_A'^{\mathrm{L}} = \mu_A'^{\mathrm{L}}(X) \equiv \frac{\theta'}{2}\left[1 - 3\lambda_0^2 - \frac{3\alpha^2}{5}\right], \tag{64}$$

$$\alpha \equiv \frac{\theta' h}{2}, \tag{65}$$

where J is the moment of inertia with respect to Z (centroidal axis of the section) and Ω is the cross-sectional area.

4.1.2. Lagrangian statement LB

Let us first state the energy for the particular problem studied in this work, i.e. the postbuckling of a rod. Given the fact that $I_E = E_{11}$ and $II_E = II_{e^*} = II_e = 0$ the derivation of the tensor $\mathbf{S}$ after modifying Eq. (37) gives rise to

$$\Sigma_B^{\mathrm{L}} = \frac{\Lambda_B^{\mathrm{L}} + 2G_B^{\mathrm{L}}}{2} I_E^2 \tag{66}$$

and from Eq. (38) gives rise to

$$S_{11} = (\Lambda_B^{\mathrm{L}} + 2G_B^{\mathrm{L}})E_{11} \tag{67}$$

and $S_{22} = 0$. Compare these results with Eqs. (40) and (41). Now, following the same steps as in the LA alternative, the stresses in the reference $(\breve{t}\breve{n}\breve{k})$ are

$$\sigma_{tt} = \sigma^* = (\Lambda_B^{\mathrm{L}} + 2G_B^{\mathrm{L}})\left(\frac{\lambda^2 - 1}{2}\right)\lambda, \tag{68}$$

$$\sigma_{nn} = \sigma_{nt} = 0. \tag{69}$$

As may be observed, one arrives at results analogous to the ones of the previous section (see Eq. (49)). The nullity of σ_{nn} is here not imposed but is a natural result. Thus, the difference between the LA and LB alternatives is only with the elastic constant definition, as follows:

$$\sigma^* = C_B^{\mathrm{L}}\left(\frac{\lambda^2 - 1}{2}\right)\lambda \tag{70}$$

with $C_B^{\mathrm{L}} \equiv \Lambda_B^{\mathrm{L}} + 2G_B^{\mathrm{L}}$. With this definition, expressions (61)–(65) are valid for the LB constitutive law.

4.2. Eulerian statements

In this subsection six constitutive statements using relationships between *Eulerian* tensors is developed. They are named EUA, EUB, EUC, EUD, EUE and EUF (the last three related to the Hencky tensor).

Regarding EUA, let us start from the general expression of the energy as a function of the Almansi–Hammel tensor and use the Green method GM1, i.e. the general expression of σ_{ij} as a function of the tensors $\mathbf{e}$ is linearized. The resulting stress–strain relation is

$$\sigma_{ij} = 2G_A^{\mathrm{E}} e_{ij} + \Lambda_A^{\mathrm{E}} \delta_{ij} I_e \equiv \sigma_{A_{ij}}^{\mathrm{E}}. \tag{71}$$

in which $I_e = e_{\alpha\alpha} = e_{11} + e_{22} + e_{33}$ is the strain linear invariant and G_A^{E} and Λ_A^{E} are Lamé elastic constants. This expression may also be seen as the application of the Cauchy method (CM), i.e. accepting a *hookean* constitutive relationship between the *Eulerian* tensors for isotropic materials. Eq. (71) constitute the simplest isotropic relationship [16,18] satisfying the material indifference and invariance requirements.

On the other hand, the following definite positive energy is accepted:

$$\Sigma_B^{\mathrm{E}} = \frac{\Lambda_B^{\mathrm{E}} + 2G_B^{\mathrm{E}}}{2} I_e^2 - 2G_B^{\mathrm{E}} II_e. \tag{72}$$

Such expression retains terms up to order two in the Almansi–Hammel tensor $\mathbf{e}$. From Green's method (GM2) the next stress–strain relationship yields

$$\sigma_{B_{ij}}^{\mathrm{E}} = \frac{\rho}{\rho_0}(\delta_{kj} - 2e_{kj})\frac{\partial \Sigma_B^{\mathrm{E}}}{\partial e_{ki}} \tag{73}$$

which when worked out renders the following:

$$\sigma^{\mathrm{E}}_{B_{ij}} = \frac{\rho}{\rho_0}(\delta_{kj} - 2e_{kj})(\Lambda^{\mathrm{E}}_B \delta_{ki} I_e + 2G^{\mathrm{E}}_B e_{ki}). \tag{74}$$

Contrary to the *Lagrangian* approach, the constitutive statement (71) for the formulation EUA is not coincident with the one obtained following EUB, i.e. Eq. (74). Statement EUC is derived from the energy stated for the particular problem under study, i.e.

$$\Sigma^{\mathrm{E}}_C = \frac{\Lambda^{\mathrm{E}}_C + 2G^{\mathrm{E}}_C}{2} I_e^2. \tag{75}$$

Making use of an expression similar to Eq. (73) one obtains (GM3):

$$\sigma^{\mathrm{E}}_{C_{ij}} = \frac{\rho}{\rho_0}(\delta_{kj} - 2e_{kj})(\Lambda^{\mathrm{E}}_C + 2G^{\mathrm{E}}_C)\delta_{ki} I_e. \tag{76}$$

Finally let us state the alternative EUD (or Hencky). Recall the Hencky tensor expression (22). Methods CM and GM1 yield the following constitutive law:

$$\sigma^{\mathrm{E}}_{D_{ij}} = \Lambda^{\mathrm{E}}_D \delta_{ij} I_h + 2G^{\mathrm{E}}_D h_{ij}. \tag{77}$$

Now, assuming the same formal expression of energy (72) but this time stated with the Hencky tensor

$$\Sigma^{\mathrm{E}}_E = \frac{\Lambda^{\mathrm{E}}_E + 2G^{\mathrm{E}}_E}{2} I_h^2 - 2G^{\mathrm{E}}_E II_h \tag{78}$$

and again by means of the alternative of Green's method GM2, one is able to find the corresponding stress–strain law (EUE),

$$\sigma^{\mathrm{E}}_{E_{ij}} = \frac{\rho}{\rho_0}\frac{\partial \Sigma^{\mathrm{E}}_E}{\partial h_{ij}} \tag{79}$$

or

$$\sigma^{\mathrm{E}}_{E_{ij}} = \frac{\rho}{\rho_0}(\Lambda^{\mathrm{E}}_E \delta_{ij} I_h + 2G^{\mathrm{E}}_E h_{ij}). \tag{80}$$

To state alternative EUF the energy (75) is written in terms of **h** as

$$\Sigma^{\mathrm{E}}_F = \frac{\Lambda^{\mathrm{E}}_F + 2G^{\mathrm{E}}_F}{2} I_h^2 \tag{81}$$

which after the application of an expression similar to Eq. (79) yields

$$\sigma^{\mathrm{E}}_{F_{ij}} = \frac{\rho}{\rho_0}(\Lambda^{\mathrm{E}}_F + 2G^{\mathrm{E}}_F)\delta_{ij} I_h. \tag{82}$$

Now each alternative will be worked out so as to obtain the internal forces N and M.

4.2.1. Eulerian statement EUA

The stress–strain relation is given by expression (71). It is also possible to state the *Eulerian* constitutive equations in the orthogonal local axes $(\breve{t}, \breve{n}, \breve{k})$. Then the relationship between the tensor $\boldsymbol{\sigma}^*$ (of components σ^*_{ij}) and the tensor $\mathbf{e}^*$ (of components e^*_{ij}) from Eq. (19), and since isotropy implies unique value of G^{E}_A and Λ^{E}_A at the point,

$$\sigma^*_{ij} = 2G^{\mathrm{E}}_A e^*_{ij} + \Lambda^{\mathrm{E}}_A \delta_{ij} I^*_e. \tag{83}$$

Obviously $\sigma^*_{11} \equiv \sigma^*$. This magnitude is needed to calculate the characteristic internal forces. Alternatively, σ^* may be found by tensorial transformation of the tensor $\boldsymbol{\sigma}$ (Eq. (71)). Both ways are coincident. Then making use of Eqs. (83) and (19) the following results are obtained:

$$\sigma^*_{11} = \sigma^* = (\Lambda^{\mathrm{E}}_A + 2G^{\mathrm{E}}_A)e^*_{11}, \tag{84}$$

$$\sigma^*_{22} = \Lambda^{\mathrm{E}}_A e^*_{11}, \tag{85}$$

$$\sigma^*_{12} = \sigma^*_{21} = 0. \tag{86}$$

Again, as was done in the *Lagrangian* statement, the nullity of σ^*_{22} in $Y = \pm h/2$, is imposed. Then

$$\Lambda^{\mathrm{E}}_A = 0. \tag{87}$$

Consequently, and in view of Eqs. (21), (84) and (87), one arrives at

$$\sigma^* = C^{\mathrm{E}}_A \frac{(\lambda^2 - 1)}{2\lambda^2} = \frac{C^{\mathrm{E}}_A}{2}\left(1 - \frac{1}{\lambda^2}\right)$$
$$= C^{\mathrm{E}}_A \varepsilon \frac{(1+\lambda)}{2\lambda^2} = C^{\mathrm{E}}_A \varepsilon \frac{(2+\varepsilon)}{2(1+\varepsilon)^2} \tag{88}$$

with $C^{\mathrm{E}}_A = 2G^{\mathrm{E}}_A$.

According to Eqs. (17a), (26), (27) and (88) the constitutive relations may be written as

$$N = N_A^{\mathrm{E}} = C_A^{\mathrm{E}}\Omega\eta_A^{\mathrm{E}}, \tag{89}$$

$$M = M_A^{\mathrm{E}} = C_A^{\mathrm{E}} J\mu_A'^{\mathrm{E}}, \tag{90}$$

where

$$\eta_A^{\mathrm{E}} = \eta_A^{\mathrm{E}}(X) \equiv \frac{1}{2}\left[1 - \frac{1}{(\lambda_0^2 - \alpha^2)}\right], \tag{91}$$

$$\mu_A'^{\mathrm{E}} = \mu_A'^{\mathrm{E}}(X) \equiv \frac{3\theta'}{4\alpha^3}\left[\log\left(\frac{\lambda_0 + \alpha}{\lambda_0 - \alpha}\right) - \frac{2\lambda_0\alpha}{(\lambda_0^2 - \alpha^2)}\right]. \tag{92}$$

4.2.2. Eulerian statement EUB

This approach starts from the stress–strain law (74). Now if the reference $(\breve{t}\breve{n}\breve{k})$ in the deformed rod is considered and recalling $\mathbf{e}^*$ from Eq. (19), it is possible to write

$$I_{e^*} = E_{11}/\lambda^2. \tag{93}$$

Taking $i = j = 1$ in Eq. (74) but referenced to $(\breve{t}\breve{n}\breve{k})$, σ_{11}^* can be written as

$$\sigma_{11}^* = \sigma^* = \frac{\rho E_{11}}{\rho_0\lambda^2}(\Lambda_B^{\mathrm{E}} + 2G_B^{\mathrm{E}})\left(1 - 2\frac{E_{11}}{\lambda^2}\right). \tag{94}$$

On the other hand, recall that

$$\frac{\rho}{\rho_0} = \frac{1}{|\mathbf{F}|} = \frac{1}{\lambda}. \tag{95}$$

Furthermore after stating σ_{22}^* and imposing its nullity at $Y = \pm h/2$, one again concludes that $\Lambda_B^{\mathrm{E}} = 0$. Recalling that $E_{11} = (\lambda^2 - 1)/2$ one has

$$\sigma^* = \frac{G_B^{\mathrm{E}}(\lambda^2 - 1)}{\lambda^5} = \frac{C_B^{\mathrm{E}}(\lambda^2 - 1)}{2\lambda^5} = \frac{C_B^{\mathrm{E}}\varepsilon(2 + \varepsilon)}{2(1 + \varepsilon)^5}, \tag{96}$$

with $C_B^{\mathrm{E}} \equiv 2G_B^{\mathrm{E}}$. Now, making use of Eqs. (26) and (27) the internal forces render the following:

$$N = N_B^{\mathrm{E}} = C_B^{\mathrm{E}}\Omega\eta_B^{\mathrm{E}}, \tag{97}$$

$$M = M_B^{\mathrm{E}} = C_B^{\mathrm{E}} J\mu_B'^{\,\mathrm{E}} \tag{98}$$

with

$$\eta_B^{\mathrm{E}} = \frac{\lambda_0}{2(\lambda_0^2 - \alpha^2)^4}[(\lambda_0^2 - \alpha^2)^2 - (\lambda_0^2 + \alpha^2)], \tag{99}$$

$$\mu_B'^{\mathrm{E}} = \frac{\theta'}{2}\left[\frac{3(\lambda_0^2 - \alpha^2)^2 - 5\lambda_0^2 + \alpha^2}{(\lambda_0^2 - \alpha^2)^4}\right]. \tag{100}$$

4.2.3. Eulerian statement EUC

The derivation of the constitutive law from the energy stated for the particular problem of the title (consider that $I_{e^*} = E_{11}/\lambda^2$) gives rise to the expression of σ_{11}^* similar to the alternative EUB (see Eq. (94) although $\sigma_{22}^* = 0$ a priori). Consequently, one arrives at the constitutive law EUC which differs from EUB by a constant;

$$\sigma^* = \frac{C_C^{\mathrm{E}}(\lambda^2 - 1)}{2\lambda^5} = \frac{C_C^{\mathrm{E}}\varepsilon(2 + \varepsilon)}{2(1 + \varepsilon)^5}, \tag{101}$$

with $C_C^{\mathrm{E}} \equiv \Lambda_C^{\mathrm{E}} + 2G_C^{\mathrm{E}}$. Eqs. (97)–(100) are also valid with this new constant definition.

4.2.4. Eulerian statement EUD (Hencky)

The constitutive law is Eq. (77). Fortunately, as mentioned in Section 2, when using the reference $\breve{t}\breve{n}\breve{k}$ the Hencky tensor may be expressed very simply as $h_{ij} = \delta_{i1}\delta_{j1}\log\lambda$ and in particular $h_{11} = \log\lambda$ the only non-null component. Then

$$\sigma_{11}^* = \sigma^* = (\Lambda_D^{\mathrm{E}} + 2G_D^{\mathrm{E}})\log\lambda. \tag{102}$$

Again, when stating the nullity of σ_{22}^* for $Y = \pm h/2$ one finds that $\Lambda_D^{\mathrm{E}} = 0$. Finally

$$\sigma^* = C_D^{\mathrm{E}}\log\lambda = C_D^{\mathrm{E}}\log(1 + \varepsilon) \tag{103}$$

with $C_D^{\mathrm{E}} \equiv 2G_D^{\mathrm{E}}$. The equilibrium conditions and integration render the following expressions for the internal forces:

$$N = N_D^{\mathrm{E}} = C_D^{\mathrm{E}}\Omega\eta_D^{\mathrm{E}}, \tag{104}$$

$$M = M_D^{\mathrm{E}} = C_D^{\mathrm{E}} J\mu_D'^{\,\mathrm{E}} \tag{105}$$

with

$$\eta_D^{\mathrm{E}} = \frac{1}{2\alpha}\left[-2\alpha + \lambda_0 \log\left(\frac{\lambda_0+\alpha}{\lambda_0-\alpha}\right) + \alpha \log(\lambda_0^2-\alpha^2)\right], \tag{106}$$

$$\mu_D'^{\mathrm{E}} = \frac{3\theta'}{2\alpha^3}\left[\left(\frac{\lambda_0^2-\alpha^2}{2}\right)\log\left(\frac{\lambda_0+\alpha}{\lambda_0-\alpha}\right) - \lambda_0\alpha\right]. \tag{107}$$

4.2.5. Eulerian statement EUE (Hencky)

The corresponding stress–strain relationship is written in Eq. (80). Then

$$\sigma_{11}^{*} = \sigma^* = (\Lambda_E^{\mathrm{E}} + 2G_E^{\mathrm{E}})\frac{\log\lambda}{\lambda}. \tag{108}$$

Again, when stating the nullity of σ_{22}^* for $Y = \pm h/2$ one finds that $\Lambda_E^{\mathrm{E}} = 0$. Finally

$$\sigma^* = C_E^{\mathrm{E}}\frac{\log\lambda}{\lambda} = C_E^{\mathrm{E}}\frac{\log(1+\varepsilon)}{(1+\varepsilon)} \tag{109}$$

with $C_E^{\mathrm{E}} \equiv 2G_E^{\mathrm{E}}$. Integration according to Eqs. (26) and (27) yields the internal forces in this formulation:

$$N = N_E^{\mathrm{E}} = C_E^{\mathrm{E}}\Omega\eta_E^{\mathrm{E}}, \tag{110}$$

$$M = M_E^{\mathrm{E}} = C_E^{\mathrm{E}}J\mu_E'^{\mathrm{E}} \tag{111}$$

with

$$\eta_E^{\mathrm{E}} = \frac{1}{4\alpha}\{[\log(\lambda_0+\alpha)]^2 - [\log(\lambda_0-\alpha)]^2\}, \tag{112}$$

$$\mu_E'^{\mathrm{E}} = \frac{3\theta'\lambda_0}{2\alpha^3}\langle(\lambda_0-\alpha)\log(\lambda_0-\alpha) - (\lambda_0+\alpha)\log(\lambda_0+\alpha) + 2\alpha \tag{113}$$

$$- \frac{\lambda_0}{2}\{[\log(\lambda_0-\alpha)]^2 - [\log(\lambda_0+\alpha)]^2\}\rangle. \tag{114}$$

4.2.6. Eulerian statement EUF (Hencky)

The derivation of the constitutive law from the energy stated for the particular problem of the title (consider that $I_{h^*} = \log\lambda$ and $II_{h^*} = III_{h^*} = 0$) gives rise to an expression of σ_{11}^* analogous to alternative EUE (see Eq. (108) but $\sigma_{22}^* = 0$ a priori). Consequently one arrives at the constitutive law EUF different from EUE only by a constant,

$$\sigma^* = C_F^{\mathrm{E}}\frac{\log\lambda}{\lambda} = C_F^{\mathrm{E}}\frac{\log(1+\varepsilon)}{(1+\varepsilon)} \tag{115}$$

with $C_F^{\mathrm{E}} \equiv \Lambda_F^{\mathrm{E}} + 2G_F^{\mathrm{E}}$. Eqs. (110)–(114) are formally valid with this new constant definition.

4.3. Strength of Materials' statement

For the sake of comparison the already known approach of Strength of Materials [14] is now addressed. In this case one simply assumes that

$$\sigma^* = C^{\mathrm{S}}\varepsilon = C^{\mathrm{S}}(\lambda-1), \tag{116}$$

where $C^{\mathrm{S}} \equiv E$ (modulus of elasticity). According to Eqs. (26) and (27) it is possible to write

$$N(X) = N^{\mathrm{S}} = C^{\mathrm{S}}\Omega\eta^{\mathrm{S}}, \tag{117}$$

$$M(X) = M^{\mathrm{S}} = C^{\mathrm{S}}J\mu'^{\mathrm{S}} \tag{118}$$

in which

$$\eta^{\mathrm{S}} = \varepsilon_0 = \lambda_0 - 1, \tag{119}$$

$$\mu'^{\mathrm{S}} = -\theta'. \tag{120}$$

5. Solution of the problem

Once the general conditions of equilibrium — expressions (30) and (33) — and the constitutive equations — found in the previous section — are available, it is possible to find the differential equation which governs the postbuckling problem. In this paper four solutions will be contrasted:

(a) a *Lagrangian* statement;
(b) an *Eulerian* statement (EUA) and
(c) the classical Strength of Material's approach.

Formulations LA and LB are included in (a) (they differ in the elastic constant definition). Explicit solutions of the other formulations, EUB to EUF, are not included.

The governing non-linear differential equations will be obtained and from them the secondary equilibrium paths. The linearization of the nine approaches will be reported in Section 6. In Section 7 the postbuckling curves are numerically evaluated for a particular example: an extensible simply supported rod. The discrepancy in the results will be graphically depicted. Also an unexpected unstable behavior when using EUA in a rather short rod will be shown.

After non-dimensionalizing the variables, $\hat{X} \equiv X/\ell$ $(0 \leqslant \hat{X} \leqslant 1)$ (ℓ is the original length of the rod), the differential equation of rotational equilibrium, for instance Eq. (33), is written now as

$$\mu'' - \lambda_0(\beta s_\theta + \gamma c_\theta) = 0, \tag{121}$$

where $\mathrm{d}(\cdot)/\mathrm{d}\hat{X} = (\cdot)'$ and the following parameters are introduced:

$$\beta \equiv \frac{P\ell^2}{CJ}, \quad \gamma \equiv \frac{V\ell^2}{CJ}. \tag{122}$$

Also it is true (from the constitutive relationship) that, in general, $N = N(X) = C\Omega\eta(X)$ and from expression (30), $N = -(Hc_\theta - Vs_\theta)$. Then if we introduce

$$i^2 \equiv \frac{J}{\Omega\ell^2} \tag{123}$$

and equate both N expressions we obtain (since $H = P$) the following:

$$\eta(\hat{X}) = -i^2[\beta c_\theta(\hat{X}) - \gamma s_\theta(\hat{X})]. \tag{124}$$

The value of $\lambda_0(\hat{X})$ may be derived from the last equation. Replacing it in $\mu'(\hat{X})$ expression and then from Eq. (121) the definitive differential equation is obtained. The value of λ_0 will be different in each of the approaches, introducing the notation φ^{L}_A, φ^{E}_A or φ^{S}, respectively. For the sake of algebraic simplicity only problems with $V = \gamma = 0$ will be considered. In Ref. [19] the authors have solved the Strength of Material's equation for $\gamma \neq 0$. In what follows the variable $\hat{X}$ will be omitted.

5.1. Lagrangian solution LA ($\equiv$ LB)

For this case and according to Eq. (63), expression (124) renders

$$\frac{\lambda_0}{2}(\lambda_0^2 + \alpha^2 - 1) = -i^2\beta^{\mathrm{L}}_A c_\theta \tag{125}$$

in which, if

$$q \equiv h/\ell \Rightarrow i^2 = q^2/12 \text{ for the rectangular section}$$
$$\text{and now } \alpha = \theta' q/2. \tag{126}$$

In order to find $\varphi^{\mathrm{L}}_A(\equiv \lambda_0)$ we will proceed as follows: first, the cubic equation (125) is solved for λ_0 after introducing the following notation:

$$b \equiv -\beta^{\mathrm{L}}_A i^2 c_\theta, \quad a \equiv \frac{1-\alpha^2}{3}, \quad \cos v \equiv ba^{-3/2}. \tag{127}$$

From the three roots (the three are real if $b^2 - a^3 < 0$) we choose the one whose limit tends to unity when the axial rigidity $\to \infty$ ($\Rightarrow i^2 \to 0$). This requirement leads to

$$\varphi^{\mathrm{L}}_A = 2a^{1/2}\cos(v/3) = \varphi^{\mathrm{L}}_A(\theta, \theta'). \tag{128}$$

At the same time, from Eq. (64) one has

$$\mu'^{\mathrm{L}}_A = \frac{\theta'}{2}\left(1 - 3\varphi^{\mathrm{L}^2}_A - \frac{3\alpha^2}{5}\right) = \mu'^{\mathrm{L}}_A(\theta, \theta') \tag{129}$$

from which the differential equation (121) results for this case,

$$\mu''^{\mathrm{L}}_A - \varphi^{\mathrm{L}}_A\beta^{\mathrm{L}}_A s_\theta = 0. \tag{130}$$

After performing rather cumbersome derivatives the second order, non-linear DE in θ is obtained:

$$\theta'' + \varphi^{\mathrm{L}}_A\beta^{\mathrm{L}}_A s_\theta \Phi^{\mathrm{L}}_A = 0, \tag{131}$$

where

$$\Phi^{\mathrm{L}}_A = \Phi^{\mathrm{L}}_A(\theta, \theta') = \frac{V^*}{U},$$

$$V^* = V^*(\theta, \theta') = 1 + 3\theta' R,$$

$$U = U(\theta, \theta') = \frac{3\varphi^{\mathrm{L}^2}_A - 1}{2} + \frac{9\alpha^2}{10} + 3\varphi^{\mathrm{L}}_A W,$$

$$W = W(\theta, \theta')$$
$$= -\tfrac{2}{3}\alpha^2 a^{-1/2}[\cos v/3 + \beta_A^{L} i^2 (a^3 - b^2)^{-1/2}$$
$$\times c_\theta \sin v/3],$$
$$R = R(\theta, \theta') = \tfrac{2}{3}\theta' i^2 a^{1/2}(a^3 - b^2)^{-1/2} \sin v/3.$$

The particular case of an inextensible member requires the following restrictions: $\varphi_A^{L} = 1$ ($\varepsilon_0 = 0$); $i^2 = q = 0$; $W = \alpha = R = 0$; $U = V^* = 1 \Rightarrow \Phi_A^{L} = 1$. Thus $\mu_A'^{L} = -\theta'$ is obtained, coincident with the Strength of Materials solution (see below).

5.2. Eulerian solution EUA

For this approach, according to Eq. (91), expression (124) is

$$\frac{1}{2}\left[1 - \frac{1}{(\lambda_0^2 - \alpha^2)}\right] = -i^2 \beta_A^{E} c_\theta. \tag{132}$$

Here $\varphi_A^{E}(=\lambda_0)$ is simply found from Eq. (132):

$$\varphi_A^{E} = \left(\alpha^2 + \frac{1}{K}\right)^{1/2} = \varphi_A^{E}(\theta, \theta'), \tag{133}$$

where $K = K(\theta) \equiv 1 + 2\beta_A^{E} i^2 c_\theta$. On the other hand, from Eqs. (92) and (133)

$$\mu_A'^{E} = \frac{3\theta'}{4\alpha^3}\left[\log\left(\frac{\varphi_A^{E} + \alpha}{\varphi_A^{E} - \alpha}\right) - 2K\varphi_A^{E}\alpha\right]$$
$$= \mu_A'^{E}(\theta, \theta') \tag{134}$$
$$= -\frac{3\theta'}{\varphi_A^{E^3}} \sum_{j=0}^{\infty} \frac{(j+1)}{(2j+3)}\left(\frac{\alpha}{\varphi_A^{E}}\right)^{2j}. \tag{135}$$

The differential equation (121) is now written as

$$\mu_A''^{E} - \varphi_A^{E}\beta_A^{E} s_\theta = 0 \tag{136}$$

which after the necessary algebraic steps results in the non-linear DE in θ:

$$\theta'' + \frac{\varphi_A^{E}\beta_A^{E} s_\theta}{\Phi_A^{E}} = 0, \tag{137}$$

where

$$\Phi_A^{E} = \Phi_A^{E}(\theta, \theta') = 3K(K\varphi_A^{E} - 2\Upsilon)$$

using

$$\Upsilon = \Upsilon(\theta, \theta') = \frac{1}{\varphi_A^{E}} \sum_{j=0}^{m} \frac{(j+1)}{(2j+3)}\left(\frac{\alpha}{\varphi_A^{E}}\right)^{2j},$$

with $m \to \infty$ but it may be set at will depending on the desired accuracy. Again considering an inextensible bar ($q = i^2 = 0$; $\varphi_A^{E} = K = 1$ and $\Upsilon = 1/3 \Rightarrow \Phi_A^{E} = 1$) the equation $\mu_A'^{E} = -\theta'$ is obtained.

5.3. Strength of Materials' solution

When equating the internal normal force given by η^{S} of expression (119) with the external one (recall Eq. (124)) the value of λ_0 (with $\gamma = 0$) yields

$$\varphi^{S} = \lambda_0 = 1 - \beta^{S} i^2 c_\theta \tag{138}$$

which substituted in Eq. (121), taking into account Eq. (120), gives rise to

$$\theta'' + \beta^{S}(1 - \beta^{S} i^2 c_\theta) s_\theta = 0 \tag{139}$$

coincident with the one reported in [14]. In this case

$$\beta^{S} = \frac{P\ell^2}{C^{S} J}, \tag{140}$$

where $C^{S} \equiv E$ is the classical modulus of elasticity (Young's modulus).

Next the linearization of all the alternative formulations will be performed. The postbuckling curves are numerically evaluated for a particular example: an extensible simply supported rod in Section 7.

6. Linearization: critical loads

In order to find the bifurcation points, a linearization procedure will be carried out on the differential equations. This is achieved by assuming that

$$|\theta| \ll 1 \Rightarrow \theta^2 = 0, \tag{141}$$

$$|\theta'| \ll 1 \Rightarrow \theta'^2 = 0. \tag{142}$$

Introducing this simplification the differential equations of solutions LA, EUA, EUB, EUD, EUE and S (LB, EUC and EUF differ from LA, EUB and EUE respectively in a constant) are reduced to

$$\theta'' + k^2\theta = 0 \tag{143}$$

in which k^2 stands for the following expressions according to the considered approach:

$$k_A^{L^2} = \frac{2\beta_A^L \varphi_A^L}{(3\varphi_A^{L^2} - 1)}, \tag{144a}$$

with

$$\varphi_A^L = \frac{2}{\sqrt{3}} \cos\left[\frac{\arccos(-3\sqrt{3}\beta_A^L i^2)}{3}\right], \tag{144b}$$

$$k_A^{E^2} = \frac{\beta_A^E}{(1 + 2\beta_A^E i^2)^2}, \tag{145}$$

$$k_B^{E^2} = \frac{2\beta_B^E \lambda_0^7}{(5 - 3\lambda_0^2)}, \tag{146}$$

$$k_D^{E^2} = \beta_D^E \exp(-2\beta_D^E i^2), \tag{147}$$

$$k_E^{E^2} = \beta_E^E \lambda_0^3, \tag{148}$$

$$k^{S^2} = \beta^S(1 - \beta^S i^2). \tag{149}$$

The other equilibrium condition — regarding the normal force N — is written for each of the formulations and after linearization as

$$2\beta_A^L i^2 + \lambda_0^3 - \lambda_0 = 0, \tag{150}$$

$$\lambda_0^2(1 + 2\beta_A^E i^2) - 1 = 0, \tag{151}$$

$$2\beta_B^E i^2 \lambda_0^5 + \lambda_0^2 - 1 = 0, \tag{152}$$

$$\log \lambda_0 + \beta_D^E i^2 = 0, \tag{153}$$

$$\log \lambda_0 + \lambda_0 \beta_E^E i^2 = 0, \tag{154}$$

$$\beta^S i^2 + \lambda_0 - 1 = 0. \tag{155}$$

It should be noted that when dealing with the *Lagrangian*, *Eulerian EUA and EUD* and Strength of Materials solutions it was rather convenient to analytically obtain λ_0 from Eqs. (150), (151), (153) and (155), respectively, and then their replacement in the moment equilibrium equation (143) with the corresponding value of k^2 (Eqs. (144a), (145), (147) and (149)). In the other cases, i.e. the *Eulerian* solutions EUB and EUE, this advantage was not apparent and the equilibrium equations were used interchanging the order.

The solution of Eq. (143) is of the form

$$\theta = \theta(\hat{X}) = A \sin k\hat{X} + B \cos k\hat{X} \quad (0 \leqslant \hat{X} \leqslant 1). \tag{156}$$

A typical eigenproblem arises after stating the two homogeneous boundary conditions in θ and/or θ' in accordance with the type of support. The eigenvalues are the bifurcation points.

In order to fix ideas let us analyze the simply supported bar with an axial compressive load. In this case the boundary conditions are

$$\theta'(0) = \theta'(1) = 0 \tag{157}$$

with which the following result should stand for all the k's:

$$k^2 = (n\pi)^2 \quad (n = 1, 2, 3, \dots). \tag{158}$$

Observe that in the case of an inextensible bar, $i = 0$, one obtains $k_A^{L^2} = \beta_A^L$, $k_A^{E^2} = \beta_A^E$, $k_B^{E^2} = \beta_B^E$, and so on.

6.1. Lagrangian case

The linearized solution is coincident for approaches LA and LB if the results are referred to the non-dimensionalized load parameter (i.e. β). From expressions (144a) and (158) the following expression should be satisfied:

$$\frac{2\beta_A^{L^{(n)}} \varphi_A^{L^{(n)}}}{3\varphi_A^{L^{(n)^2}} - 1} = (n\pi)^2 \tag{159}$$

which when solved for φ_A^L gives

$$\varphi_A^{L^{(n)}} = \frac{\beta_A^{L^{(n)}}}{3(n\pi)^2}\left[1 + \sqrt{1 + 3\frac{(n\pi)^4}{\beta_A^{L^{(n)^2}}}}\right]. \tag{160}$$

The choice of the positive sign is justified since $\varphi_A^L(\equiv \lambda_0)$ is essentially positive.

Table 3
Bifurcation points using the *Lagrangian* approach LA = LB for an inextensible bar ($q = 0$, bifurcation points equal to $(n\pi)^2$) and extensible bars ($q = 0.01, 0.1, 0.35$); $n = 1, \ldots, 5$ and $n = 20$ denote the order of the bifurcation load

q	n					
	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=20$
0	9.8696	39.478	88.826	157.91	246.74	3947.8
0.01	9.8679	39.452	88.695	157.50	245.73	3690.6
0.1	9.7076	36.906	76.031	118.84	156.87	230.53
0.35	7.9625	16.580	18.380	18.707	18.794	18.852

Requiring the positiveness of the different k's (otherwise one would not have an eigenproblem) and $\varphi_A^{\mathrm{L}} > 0$, one obtains

$$\varphi_A^{\mathrm{L}} \geqslant \frac{1}{\sqrt{3}}. \tag{161}$$

On the other hand, from Eq. (144b) and since $\beta > 0$, the cosine argument is between 30 and 60° with which

$$\frac{\sqrt{3}}{3} \leqslant \varphi_A^{\mathrm{L}^{(n)}} \leqslant 1 \tag{162}$$

that includes Eq. (161). By the same argument it is also found that

$$\beta_A^{\mathrm{L}^{(n)}} \leqslant \frac{\sqrt{3}}{(3i)^2}. \tag{163}$$

From Eqs. (160) and (162) another restriction for $\beta_A^{\mathrm{L}^{(n)}}$ yields

$$\beta_A^{\mathrm{L}^{(n)}} \leqslant (n\pi)^2. \tag{164}$$

Eq. (164) should combine with Eq. (163). The novelty is that due to Eq. (164) the *Lagrangian* critical loads are always smaller than those found by Euler for all the values of i.

On the other hand (easy to see from Eq. (159)) if n is very large ($n \to \infty$),

$$\varphi_A^{\mathrm{L}^2} \to \tfrac{1}{3} \quad (n \to \infty) \tag{165}$$

with which the value of $\beta_A^{\mathrm{L}^{(n)}}$ is limited by

$$\beta_A^{\mathrm{L}^{(n)}} \to \frac{\sqrt{3}}{(3i)^2}. \tag{166}$$

The limit (166) is an accumulation point of the *Lagrangian* critical loads. According to Eq. (163) the limit is approached from below.

Numerical results of bifurcation points for several values of n (mode) and q ($= h/\ell$ in the rectangular cross section), based on the *Lagrangian* constitutive statement are listed in Table 3.

6.2. Eulerian case EUA

The following condition derives from Eqs. (145) and (158):

$$\frac{\beta_A^{\mathrm{E}^{(n)}}}{(1 + 2\beta_A^{\mathrm{E}^{(n)}} i^2)^2} = (n\pi)^2 \tag{167}$$

which can be solved for $\beta_A^{\mathrm{E}^{(n)}}$:

$$\beta_A^{\mathrm{E}^{(n)}} = \frac{1 - (2n\pi i)^2 \pm \sqrt{1 - 2(2n\pi i)^2}}{2(2n\pi i^2)^2}. \tag{168}$$

In order for $\beta_A^{\mathrm{E}^{(n)}}$ to be real the next inequality should be true:

$$1 - 2(2n\pi i)^2 \geqslant 0 \tag{169}$$

or in other form

$$n \leqslant \frac{1}{2\sqrt{2}\pi i} \tag{170}$$

from which a relevant as well as an unusual conclusion may be drawn when the *Eulerian* statement EUA is used: *there is a finite number of critical loads.* The same result found in a similar way, but using

Table 4
Same as Table 3 but using the *Eulerian* approach EUA

q	n					
	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=20$
0	9.8696	39.478	88.826	157.91	246.74	3947.8
0.01	9.8728	39.530	89.090	158.75	248.79	4572.4
	3.6×10^8	9.1×10^7	4.0×10^7	2.2×10^7	1.1×10^7	7.8×10^5
0.1	10.208	45.724	132.32	—	—	—
	3.5×10^4	7873.1	2720.5	—	—	—
0.35	19.027	—	—	—	—	—
	126.08	—	—	—	—	—

Table 5
Same as Table 3 but using the *Eulerian* approach EUB = EUC

q	n					
	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=20$
0	9.8696	39.478	88.826	157.91	246.74	3947.8
0.01	9.8777	39.608	89.488	160.02	251.92	5923.8
	1.7×10^{13}	5.4×10^{11}	7.1×10^{10}	1.6×10^{10}	5.4×10^8	3.6×10^6
0.1	10.756	59.238	546.00	—	—	—
	1.6×10^{13}	3.6×10^4	1171.31	—	—	—
0.35	—	—	—	—	—	—
	—	—	—	—	—	—

the Strength of Materials formulation (see Section 6.6 below) has been reported in Ref. [14].

In case $i \neq 0$, the value of the square root in expression (168) is smaller than the first term. The two signs are theoretically possible for the same n and i. This fact gives rise to a double-bifurcation point. In all cases n must satisfy Eq. (170). Table 4 depicts values of bifurcation points for inextensible and extensible bars.

It should be noted that, for a value of n given, there is a $q_{\lim}$ to guarantee the existence of real bifurcation loads, i.e. from Eq. (170)

$$q_{\lim^{\mathrm{E}}_A} = \frac{0.389849}{n}. \tag{171}$$

Also it is possible to show that unlike the *Lagrangian* formulation, the Euler loads are a lower bound for the $\beta_A^{\mathrm{E}^{(n)}}$, i.e.

$$\beta_A^{\mathrm{E}^{(n)}} \geqslant (n\pi)^2. \tag{172}$$

6.3. *Eulerian case EUB*

Again when the non-dimensionalized load is used, the bifurcation points EUB and EUC are the same. The critical loads may be found (from Eqs. (146) and (158)) by means of the expression

$$\beta_B^{\mathrm{E}} = \frac{(n\pi)^2(5-3\lambda_0^2)}{2\lambda_0^7} \tag{173}$$

using the value of λ_0 obtained from the N equilibrium (after linearization) (152),

$$(n\pi)^2 i^2 (5-3\lambda_0^2) + \lambda_0^4 - \lambda_0^2 = 0. \tag{174}$$

The bifurcation loads corresponding to the *Eulerian* approach EUB are listed in Table 5. Both inextensible and extensible bars are considered.

Table 6
Same as Table 3 but using the *Eulerian* approach EUD (Hencky)

q	n					
	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=20$
0	9.8696	39.478	88.826	157.91	246.74	3947.8
0.01	9.8712	39.504	88.958	158.33	247.76	4236.6
	6.6×10^5	5.7×10^5	5.2×10^5	4.8×10^5	4.5×10^5	2.4×10^5
0.1	10.036	42.366	105.98	232.75	—	—
	3527.3	2485.4	1807.9	1233.1	—	—
0.35	12.823	—	—	—	—	—
	123.93	—	—	—	—	—

Table 7
Same as Table 3 but using the *Eulerian* approach EUE = EUF (Hencky)

q	n					
	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=20$
0	9.8696	39.478	88.826	157.91	246.74	3947.8
0.01	9.8720	39.517	89.024	118.54	248.27	4388.9
	1.7×10^8	6.9×10^7	3.9×10^7	2.6×10^7	1.9×10^7	1.9×10^6
0.1	10.120	43.889	115.77	282.56	—	—
	6.6×10^4	1.9×10^4	8156.6	3446.1	—	—
0.35	14.616	—	—	—	—	—
	439.16	—	—	—	—	—

Again there is a range of n and q ($i^2 = q^2/12$ with $q = h/\ell$ for a rectangular cross section) in which no real bifurcation points exist. The same comment is valid for the rest of the *Eulerian* linearized problem.

6.4. Eulerian case EUD (Hencky)

This solution derived from the constitutive alternative found from the Cauchy method and using the Hencky tensor yields the linearized equation

$$(n\pi)^2 \exp(2i^2\beta_D^{\mathrm{E}^{(n)}}) - \beta_D^{\mathrm{E}^{(n)}} = 0 \tag{175}$$

from which the bifurcations loads may be calculated. Table 6 shows the corresponding values of the bifurcation points which like the other *Eulerian* solutions are double for each mode excepting the inextensible rod.

6.5. Eulerian case EUE (Hencky)

This approach using the Hencky tensors renders the following equation to find the critical loads for a simply supported rod. From Eqs. (148) and (158)

$$\beta_E^{\mathrm{E}^{(n)}} = \frac{(n\pi)^2}{\lambda_0^3} \Rightarrow \lambda_0 = \sqrt[3]{\frac{(n\pi)^2}{\beta_E^{\mathrm{E}^{(n)}}}} \tag{176}$$

which replaced in Eq. (154) gives the solving equation

$$\log \sqrt[3]{\frac{(n\pi)^2}{\beta_E^{\mathrm{E}^{(n)}}}} + \beta_E^{\mathrm{E}^{(n)}} \sqrt[3]{\frac{(n\pi)^2}{\beta_E^{\mathrm{E}^{(n)}}}} i^2 = 0. \tag{177}$$

Table 7 shows the corresponding values of the bifurcation points. Again such points are double when $q \neq 0$ (extensible rods). The results are coincident for the linearized approach EUF.

Table 8
Same as Table 3 but using the Strength of Materials approach

q	n					
	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=20$
0	9.8696	39.478	88.826	157.91	246.74	3947.8
0.01	9.8704	39.491	88.892	158.12	247.25	4087.0
	1.199×10^5	1.199×10^5	1.199×10^5	1.198×10^5	1.197×10^5	1.159×10^5
0.1	9.9521	40.870	96.603	187.07	347.19	—
	1190.0	1159.1	1103.4	1012.92	852.80	—
0.35	11.135	—	—	—	—	—
	86.824	—	—	—	—	—

6.6. Strength of Materials case

β^{S} may be found from the following expression (recall Eqs. (149) and (158)):

$$\beta^{\mathrm{S}^{(n)}}(1-\beta^{\mathrm{S}^{(n)}}i^2)=(n\pi)^2 \tag{178}$$

from which

$$\beta^{\mathrm{S}^{(n)}}=\frac{1\pm\sqrt{1-(2n\pi i)^2}}{2i^2}. \tag{179}$$

Both signs make sense but also, as said before, a finite number of bifurcation loads arise in order for $\beta^{\mathrm{S}^{(n)}}$ to be real, i.e.

$$n\leqslant\frac{1}{2\pi i}. \tag{180}$$

Results (179) and (180) were reported in [14]. Similar to the *Eulerian* case, the Strength of Materials approach yields eventual critical loads always larger than the well-known Euler loads ($=(n\pi)^2$). Table 8 depicts critical loads found for inextensible as well as extensible rods according to this formulation.

7. Secondary equilibrium paths: numerical examples

An analysis of the postbuckling secondary equilibrium paths, β vs. displacement, will be performed in this section. As mentioned in Section 5, four solutions are numerically handled, say *Lagrangian* (LA $\equiv$ LB, unless a constant), *Eulerian* EUA and Strength of Materials formulations. The last is solved for the sake of comparison. Firstly a word must be said regarding the elastic constants C_A^{L}, C_B^{L}, C_A^{E} and C^{S}. The representation of the equilibrium curves, due to the non-dimensionalization, is independent of the real value of those constants. The comparison in the results is then only qualitative since they are performed among non-dimensionalized quantities. For instance, it was shown that the *Lagrangian* bifurcation loads are smaller than Euler loads of linear buckling, while *Eulerian* and Strength of Materials' ones are larger; actually it should be taken into account that the Euler loads for a simply supported bar are, respectively,

$$P_{A\ \mathrm{cr}}^{\mathrm{L}}=\frac{\pi^2C_A^{\mathrm{L}}J}{\ell^2}\quad\text{as for LB}, \tag{181}$$

$$P_{A\ \mathrm{cr}}^{\mathrm{E}}=\frac{\pi^2C_A^{\mathrm{E}}J}{\ell^2}, \tag{182}$$

$$P_{\mathrm{cr}}^{\mathrm{S}}=\frac{\pi^2C^{\mathrm{S}}J}{\ell^2}=\frac{\pi^2EJ}{\ell^2}. \tag{183}$$

At the first sight, they are not comparable since a criterion among the constants should be established. Or at least — as happens with the infinitesimal deformations case — a test (similar to a simple tensile test) should be standardized so as to find the mechanical constants. The authors considered that this discussion is beyond the scope of the present work. Thus the issue of elastic constants remains open.

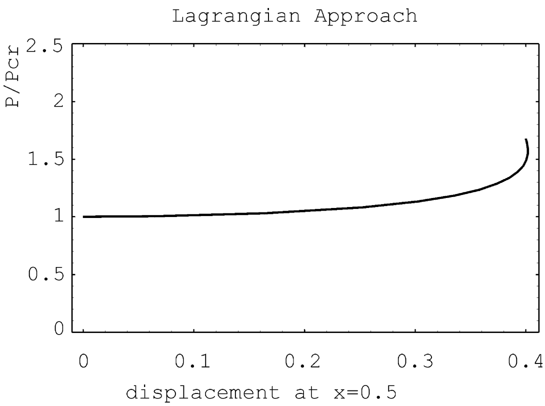

Fig. 5. Secondary equilibrium path corresponding to the first mode. *Lagrangian* approach. Extensible bar with $q = 0.1$. LA $\equiv$ LB.

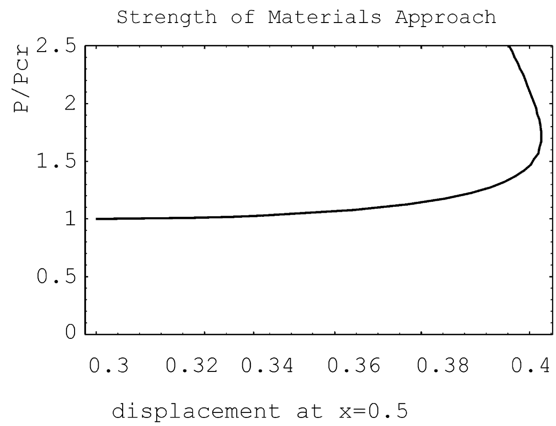

Fig. 7. Same as Fig. 5: Strength of Materials approach.

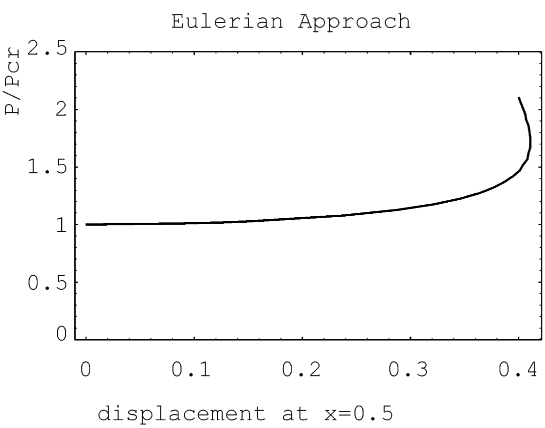

Fig. 6. Same as Fig. 5: *Eulerian* EUA approach.

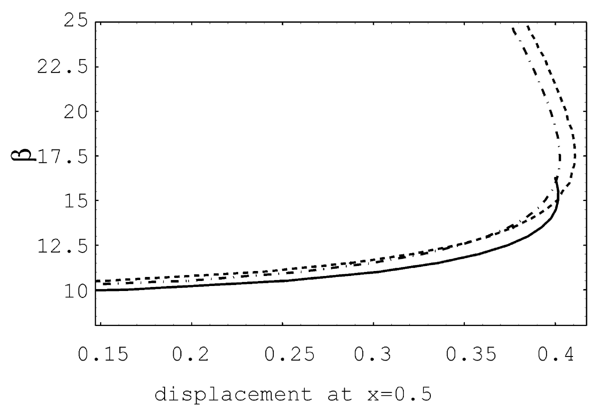

Fig. 8. Same as Fig. 5: Superposition of three solutions: —, *Lagrangian* formulation; - - -, *Eulerian* formulation; - · -, Strength of Materials formulation.

The bifurcation curves for each approach are plotted in Figs. 5–7. In all cases an extensible rod with $q = 0.1$ was assumed and the loads are referred to their respective critical load; $n = 1$.

Fig. 8 shows a superposition of the results from three formulations, i.e. *Lagrangian LA*, *Eulerian EUA* and Strength of Materials. In this graph the parameter β is not referred to the critical one. As said before, no criterion is assumed with respect to the constant. Then the comparison is only qualitative. The *Lagrangian* approach yields real values only in a limited range (see Figs. 5 and 8).

A special case of an unstable behavior using the EUA statement occurs, for instance, when $q = 0.35$ and the equilibrium curve is shown in Fig. 9. A lower limit load β of about 14.35 arises, which is significantly smaller than the first bifurcation load (see Table 4 with $n = 1$). In the range between the limit load and the bifurcation point two *elastica* are obtained for each load value. The corresponding modal shapes are schematically drawn in Fig. 10 for the case of $\beta = 15$. Obviously, this is the mathematical solution of the postbuckling behavior of

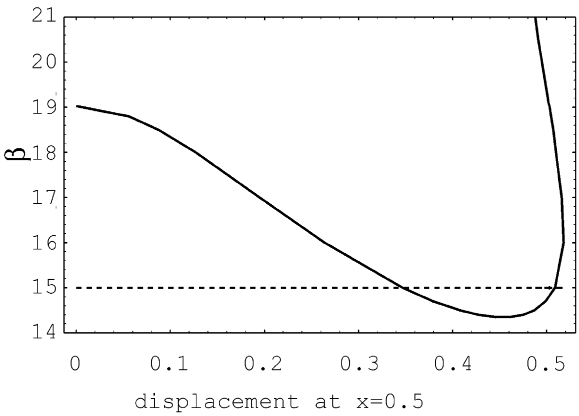

Fig. 9. Example of unstable behavior using the *Eulerian* approach EUA. $q = 0.35$.

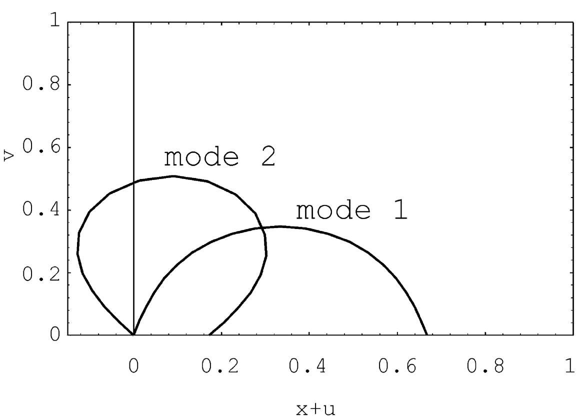

Fig. 10. Modal shapes corresponding to $\beta = 15$ in Fig. 9.

a rather short rod and a comment on this regard is included in the next section.

8. Discussion

The numerical values included in Tables 4–8, serve to illustrate the fact that the linearized Eulerian and the Strength of Materials solutions exhibit double bifurcation points for each mode. Also they have a finite number of real critical loads. These features are not present in the *Lagrangian* formulation. An accumulation point for the critical load when $n \to \infty$ is a characteristic of this approach. Furthermore, the non-dimensionalized *Lagrangian* critical loads are smaller than the Euler's loads while all the *Eulerian*'s loads are higher.

Regarding the postbuckling curves, the rod with extensibility parameter $q = 0.1$ exhibits a stable behavior, as expected, though the *Lagrangian* equilibrium paths are real only in a range of the plot. When dealing with extensible rods the three alternatives yield different results (curves). On the other hand, it was shown analytically that all solutions are coincident when the rod is theoretically inextensible. The unstable behavior found with the *Eulerian* formulation EUA (Fig. 9) for rather short bars is an unexpected result. Neither the *Lagrangian* nor the Strength of Materials solutions exhibited this change when their behavior was studied. One may infer that the constitutive equation (71) would not be exactly derivable from a non-negative strain energy. Truesdell [15] made a similar comment on Seth's work. Fung [16] employed an expression similar to Eq. (71) applying the Cauchy method. This result must be considered as a counterexample to the assumption of an hyperelastic behavior in the stress–strain constitutive relation (71). Instead, since the Eulerian statements EUB and EUC (not solved numerically in this paper) come essentially from positive potentials (strain energy), it could be asserted that this unstable behavior would not arise.

As is known, any rod of real materials is extensible. So the inextensible model is an idealization of the behavior. It should be mentioned when stating the inextensibility condition (i.e. $\lambda_0 = 1$) that the formulations are only compatible with $q = 0$.

Although not dealing with the postbuckling of extensible elastic rods, Gummadi and Palazotto [12] address the issue of the *Lagrangian* and *Eulerian* formulations and comment that both approaches are to be considered equal when small strains are involved. An interesting graph is included showing the range of practical validity of this assertion. However in the present work, it was found that even for small strain there are discrepancies in the response (say critical loads or secondary equilibrium paths) when using the different alternatives proposed above.

8.1. Trivial (primary) equilibrium paths

As was mentioned only the secondary equilibrium paths of the bifurcation problem have been addressed. The trivial solution (pure compression, primary path) arises after imposing $\theta = \theta' = v_0 = 0 \Rightarrow \alpha = 0$, with which the rotational equilibrium is satisfied identically. In this problem when an extensible rod is tackled only the equilibrium equation of the force N should be satisfied, i.e. in general,

$$\eta = -\, i^2 \beta \quad \forall \beta \quad \text{with } c_\theta = 1,\, \alpha = 0. \tag{184}$$

Formally, the equation to be solved to find β_{cr} is identical to Eq. (184). However in the linearization procedure one takes a mathematical limit. Solution (184) is obviously valid if $\lambda_0 > 0$. Note that in the Strength of Materials solution, Eq. (138) with $\theta = 0$ is written as

$$\lambda_0 = 1 - \beta^S i^2. \tag{185}$$

Then one finds

$$\beta^S < \frac{1}{i^2} \tag{186}$$

a bound to the validity of this solution. Also for all the formulations, and dealing with compression, $\beta \geqslant 0$ and $\lambda_0 < 1$.

8.2. Strength of Materials approach

In the traditional (and specially in the Strength of Materials) approach the constitutive equation links the Cauchy tensor (equilibrated deformed configuration) with the axial strain of the corresponding fiber. Thus the stresses would be those required to produce an axial deformation of the fiber considered. The axial (tensile or compression) test in which the axial load deforms the sample is erroneously extended to each body point. This conceptual misunderstanding arises from imagining a micro, local axial test in each point. Now, Hooke's law proposes a relationship between the stress components and the specific strains. Notwithstanding the first of these entities should be associated with area elements oriented in each point while the strains are related to the behavior of fibers at the material point. Recall that the tensor character of the measures of deformation is only valid in the infinitesimal theory. From this one may conclude that in order to fulfill the invariance conditions required by the constitutive relationships, it is necessary to relate tensors. When dealing with non-infinitesimal deformations one should handle strain tensors and not measures of deformation which do not have a tensorial character.

Let us try to justify the constitutive equations of Strength of Materials. Expansion of the expressions of η_A^L and η_A^E up to a quadratic approximation in the strain gradients involved in the problem, i.e. u_0', v_0' and θ' leads to

$$\eta_A^L \sim u_0' + \frac{1}{2}(3u_0'^2 + v_0'^2) + \frac{\alpha^2}{2}, \tag{187}$$

$$\eta_A^E \sim u_0' + \frac{1}{2}(-\,3u_0'^2 + v_0'^2) - \frac{\alpha^2}{2}. \tag{188}$$

In order for these two expressions to be coincident the influences of ${u_0'}^2$ and θ'^2 should be disregarded and, since the quadratic approximation of ε_0 is

$$\varepsilon_0 = \lambda_0 - 1 = \sqrt{1 + 2E_{11_0}} - 1 = u_0' + \frac{v_0'^2}{2}, \tag{189}$$

one arrives at

$$\eta_A^L = \eta_A^E = \eta^S = u_0' + \frac{v_0'^2}{2} = \varepsilon_0. \tag{190}$$

The same reasoning may be carried out with the expressions for $\mu_A'^L$ and $\mu_A'^E$:

$$\mu_A'^L \sim -\,\theta' + 3\theta' u_0', \tag{191}$$

$$\mu_A'^E \sim -\,\theta' - 3\theta' {u'}_0. \tag{192}$$

To make them equal, the term $\theta' u_0'$ should be neglected. Then

$$\mu_A'^L = \mu_A'^E = \mu'^S = -\,\theta'. \tag{193}$$

These strong restrictions should have to be accepted if one would attempt to equate the *Lagrangian*, *Eulerian* and Strength of Materials formulations.

Also, and with the purpose of illustration, it is known that (Frenet–Serret) (see Fig. 2)

$$\chi_0 \breve{n} = \frac{\mathrm{d}\breve{t}}{\mathrm{d}s_0} = \frac{\mathrm{d}\theta}{\mathrm{d}s_0}\breve{n} \tag{194}$$

from which the curvature of the deformed axis is $\chi_0 = \mathrm{d}\theta/\mathrm{d}s_0$. On the other hand,

$$\theta' = \frac{\mathrm{d}\theta}{\mathrm{d}s_0} s_0' = \chi_0 s_0' \tag{195}$$

but also

$$\varepsilon_0 = \lim_{\Delta X \to 0} \frac{\Delta s_0 - \Delta X}{\Delta X} = \lim_{\Delta X \to 0} \frac{\Delta s_0}{\Delta X} - 1$$

$$= s_0' - 1 = \lambda_0 - 1. \tag{196}$$

Finally one can find that

$$\theta' = \lambda_0 \chi_0 = (1 + \varepsilon_0)\chi_0. \tag{197}$$

Consequently it is observed from Eqs. (118) and (120) that the moment is not proportional to χ_0 when the bar is extensible.

9. Conclusions

The postbuckling analysis of extensible bars has been carried out in this work by means of nine alternative constitutive laws which lead to the so-called *Lagrangian* LA and LB, *Eulerian* EUA, EUB, EUC, EUD, EUE and EUF as well as the Strength of Materials, solutions. Some of these constitute — to the authors' knowledge — a step forward in the postbuckling analysis of extensible rods and the last one is included for the sake of comparison.

Only two assumptions, commonly accepted, are made: first, the Navier–Bernoulli condition is imposed to the geometrical formulation; second, the thickness is assumed as constant for all the states. Regarding the constitutive models all the approaches were worked out without further simplifications. Explicit solutions of the non-linear differential equations are given for the *Lagrangian, Eulerian* EUA and Strength of Materials' approaches.

The linearized problem was solved for all the alternatives and the critical loads were obtained for each formulation for various modes and values of the extensibility parameter.

The postbuckling curves were numerically evaluated for an extensible simply supported rod. The secondary equilibrium paths were computed for extensible bars using *Lagrangian* LA (= LB), *Eulerian* EUA and Strength of Materials solutions. The case of $q = 0.1$ is reported in graphs showing a stable behavior. The discrepancy of the results is graphically depicted. Also an unexpected unstable behavior was observed when using EUA in a rather short rod and a comment regarding this feature is included. When dealing with the theoretical problem of an inextensible rod, all the results (both critical loads and secondary equilibrium paths) are coincident but this is not the case for the real extensible rod. In the authors' opinion the influence of extensibility justifies the study of alternative constitutive models for the postbuckling of an elastic, highly flexible rod, as is done in this work. The authors have analyzed the alternative formulations for the constitutive laws to the solution of the postbuckling of extensible rods, which are, at least, as valid as the traditionally known and commonly used Strength of Material formulation.

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