

# Perihelion advances for the orbits of Mercury, Earth and Pluto from Extended Theory of General Relativity (ETGR).

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## Abstract

We explore the geodesic movement on an effective four-dimensional hypersurface that is embedded in a five-dimensional Ricci-flat manifold described by a canonical metric, in order to applying to planetary orbits in our solar system. Some important solutions are given, which provide the standard solutions of general relativity without any extra force component. We study the perihelion advances of Mercury, the Earth and Pluto using the extended theory of general relativity (ETGR). Our results are in very good agreement with observations and show how the foliation is determinant to the value of the perihelion's advances. Possible applications are not limited to these kinds of orbits.

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## I. INTRODUCTION, BASIC EQUATIONS, AND MOTIVATION

The advance of the perihelion in the orbit of Mercury is a relativistic effect[1]. Together with the observation of the deflection of light by Dyson, Eddington and Davidson in 1919[2], this result was crucial in the final breakthrough of general relativity. Mercury is the innermost of the four terrestrial planets in the Solar system, moving with high velocity in the gravitational field produced by the Sun. Because of this, Mercury offers unique possibilities for testing general relativity and exploring the limits of alternative theories of gravitation with enough accuracy to be of interest[3]. A compact calculation of the perihelion precession of Mercury in general relativity taking into account a nonzero cosmological Constant  $\Lambda$ , was considered some years ago[4]. The same problem was examined from five-dimensional physics, but with zero cosmological constant[5].

Lately, extensions or modifications to the standard four-dimensional theory of general relativity have a great and increasing impact in top original research in gravitation and cosmology. The spectrum of these proposals includes: theories with compact and noncompact extra dimensions[6], scalar-tensor theories, gravity from non-Riemannian geometries; and  $f(R)$ [7],  $f(R, G)$  and  $f(T)$  theories (e. g. ref them[8]).

In 2009 an extended version of general relativity[9] was introduced from a 5D Ricci-flat space-time, where the extra space-like coordinate is noncompact. After making a static foliation on the extra coordinate, we obtained an effective 4D Schwarzschild-de Sitter space-time in which matter is considered with an equation of state  $\omega = p_m/\rho_m = -1$  4D vacuum state, such that the pressure on the effective 4D manifold is  $P = -3c^4/(8\pi G\psi_0^2)$  and  $\psi_0 = c/H_0$  is the Hubble radius. The resulting effective 4D metric is static, exterior and describes spherically symmetric matter (ordinary matter, dark matter and dark energy) on scales  $r_0 < r_{Sch} < c/H_0$  for black holes or  $r_{Sch} < r < c/H_0$  for ordinary stars with radius  $r_0$ . The radius  $r_{ga}$  is very important because it delimitates distances for which dark energy and ordinary matter (dark matter and ordinary matter) are dominant:  $r > r_{ga}$  ( $r < r_{ga}$ ). We have suggested that ordinary matter, dark matter and dark energy can be considered matter subject to a generalized gravitational field which is attractive on scales  $r < r_{ga}$  and repulsive on scales  $r > r_{ga}$ .

In this work we shall study the effective 4D orbits of some planets (or pseudo-planets in the case of Pluto) of our solar system. In particular we are interested in the calculation of

the perihelion advances of Mercury, Earth and Pluto. In Sect. 2 we review the formalism to calculate the orbits of massive test particles from the extended theory of general relativity (ETGR).

## II. ETGR

In a previous work[9] a 5D extension of general relativity was considered such that the effective 4D gravitational dynamics had a vacuum-dominated,  $\omega = -1$ , equation of state. In this section we shall examine some formal aspects of this theory.

### A. 5D massive test particles dynamics

We consider the extended Schwarzschild-de Sitter 5D Ricci-flat metric  $g_{ab}$ [9]

$$dS^2 = \left(\frac{\psi}{\psi_0}\right)^2 \left[ c^2 f(r) dt^2 - \frac{dr^2}{f(r)} - r^2 (d\theta^2 + \sin^2(\theta) d\phi^2) \right] - d\psi^2, \quad (1)$$

where  $f(r) = 1 - (2G\zeta\psi_0/(rc^2))[1 + c^2r^3/(2G\zeta\psi_0^3)]$  is a dimensionless function. Here,  $\psi$  is the noncompact extra dimension. The space-like coordinates  $\psi$  and  $r$  have length units,  $\theta$  and  $\phi$  are angular coordinates and  $t$  is a time-like coordinate. We denote  $c$  the speed of light. We shall consider that  $\psi_0$  is an arbitrary constant with length units and the constant parameter  $\zeta$  has units of  $(mass)(length)^{-1}$ .

For a massive free test particle outside of a spherically symmetric compact object, the 5D Lagrangian is

$${}^{(5)}L = \frac{1}{2}g_{ab}U^aU^b = \frac{1}{2}\left(\frac{\psi}{\psi_0}\right)^2 \left[ c^2 f(r) (U^t)^2 - \frac{(U^r)^2}{f(r)} - r^2 (U^\theta)^2 - r^2 \sin^2\theta (U^\phi)^2 \right] - \frac{1}{2} (U^\psi)^2. \quad (2)$$

We shall take  $\theta = \pi/2$ . Because  $t$  and  $\phi$  are cyclic coordinates, their associated constants of motion  $p_t$  and  $p_\phi$ , are constants of motion. Using the five-velocity condition  $g_{ab}U^aU^b = 1$ , we obtain the equation of energy for a test particle that moves on space-time (1)

$$\frac{1}{2}(U^r)^2 + \frac{1}{2}\left(\frac{\psi_0}{\psi}\right)^2 (U^\psi)^2 + V_{eff}(r) = E. \quad (3)$$

If we identify the energy,  $E$ , as

$$E = \frac{1}{2}\left(\frac{\psi_0}{\psi}\right)^4 (p_t^2 c^{-2} + p_\phi^2 \psi_0^{-2}) - \frac{1}{2}\left(\frac{\psi_0}{\psi}\right)^2, \quad (4)$$

the effective 5D potential,  $V_{eff}(r)$ , is

$$V_{eff}(r) = - \left( \frac{\psi_0}{\psi} \right)^2 \frac{G\zeta\psi_0}{r} + \left( \frac{\psi_0}{\psi} \right)^4 \left[ \frac{p_\phi^2}{2r^2} - \frac{G\zeta\psi_0 p_\phi^2}{c^2 r^3} \right] - \frac{1}{2} \left( \frac{\psi_0}{\psi} \right)^2 \left[ (U^\psi)^2 \left( \frac{2G\zeta\psi_0}{c^2 r} - \frac{r^2}{\psi_0^2} \right) - \left( \frac{r}{\psi_0} \right)^2 \right]. \quad (5)$$

However, we are interested in the study of this potential for massive test particles on static foliations  $\psi = \psi_0 = c/H_0$ , such that the dynamics evolves on an effective 4D manifold  $\Sigma_0$ . From the point of view of a relativistic observer, this implies that  $U^\psi = 0$ .

## B. Geodesics equations for 5D canonical metrics

We consider a 5D line element  $dS^2 = g_{ab}(x^c)dx^a dx^b$ . We are interested in studying the geodesics equations on a 5D canonical metric

$$dS^2 = \left( \frac{\psi}{\psi_0} \right)^2 ds^2 - d\psi^2, \quad (6)$$

where  $ds^2 = h_{\alpha\beta}(x^\mu)dx^\alpha dx^\beta$ , such that in the absence of external forces the 5D geodesic equation is

$$\frac{d^2 x^a}{dS^2} + \Gamma_{bc}^a \frac{dx^b}{dS} \frac{dx^c}{dS} = 0. \quad (7)$$

For a test particle in a time-like geodesic we must require

$$g_{ab}U^a U^b = 1, \quad (8)$$

such that the velocity components are  $U^c = \frac{dx^c}{dS}$ <sup>1</sup>. To study the effective 4D geodesic equations on a hypersurface obtained after making a constant foliation  $\psi = \psi_0$ , we shall decompose (7) in the geodesic equations

$$\frac{d^2 x^\mu}{ds^2} + \bar{\Gamma}_{\alpha\beta}^\mu \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = - \frac{d^2 s}{dS^2} \left( \frac{ds}{dS} \right)^{-2} \frac{dx^\mu}{ds} - 2 \frac{1}{\psi_0} \delta_\nu^\mu \frac{dx^\nu}{ds} \frac{d\psi}{ds}, \quad (9)$$

$$\frac{d^2 \psi}{ds^2} + \Gamma_{\alpha\beta}^4 \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = - \frac{d^2 s}{dS^2} \left( \frac{ds}{dS} \right)^{-2} \frac{d\psi}{ds}, \quad (10)$$

where

$$\frac{ds}{dS} = \left[ \left( \frac{\psi}{\psi_0} \right)^2 - \left( \frac{d\psi}{ds} \right)^2 \right]^{-1/2}. \quad (11)$$

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<sup>1</sup> The case of 5D null geodesics have been studied in earlier works[10].

Deriving the last expression with respect to  $S$ , we obtain

$$\frac{d^2 s}{dS^2} \left( \frac{ds}{dS} \right)^{-2} = - \left( \frac{ds}{dS} \right)^2 \frac{d\psi}{ds} \left[ \frac{\psi}{\psi_0^2} - \frac{d^2\psi}{ds^2} \right]. \quad (12)$$

Using (8) and (12) in (9) and (10), we obtain

$$\frac{d^2 x^\mu}{ds^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = \frac{dx^\mu}{ds} \frac{d\psi}{ds} \left( \frac{ds}{dS} \right)^2 \left[ \frac{\psi}{\psi_0^2} - \frac{d^2\psi}{ds^2} \right], \quad (13)$$

$$\frac{d^2\psi}{ds^2} + \frac{\psi}{\psi_0^2} = \left( \frac{ds}{dS} \right)^2 \frac{d\psi}{ds} \left[ \frac{\psi}{\psi_0^2} - \frac{d^2\psi}{ds^2} \right]. \quad (14)$$

Using (11) and (12) in (13) we obtain that the right-hand side of (13) becomes null, so that the system (13)-(14) finally becomes

$$\frac{d^2 x^\mu}{ds^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0, \quad (15)$$

$$\frac{d^2\psi}{ds^2} + \frac{\psi}{\psi_0^2} = \frac{2}{\psi} \left( \frac{d\psi}{ds} \right)^2. \quad (16)$$

The solution of this set of equations is

$$\psi(s) = - \frac{2e^{-s/\psi_0}}{\psi_0 [C_1 e^{-2s/\psi_0} + C_2]}. \quad (17)$$

We are interested in studying the induced dynamics of observers who moves on the hypersurface  $\Sigma_0$ , resulting from setting a constant foliation  $\psi(s) = \psi_0$ . In the next section we shall consider this case which will be relevant to the study of planetary dynamics on an effective 4D Schwarzschild-de Sitter space-time.

### III. PHYSICS ON THE 4D MANIFOLD $\Sigma_0$ IN THE SOLAR SYSTEM

Now we consider the static foliation  $\{\Sigma_0 : \psi = \psi_0\}$  on (1). In this case we obtain the effective 4D line element

$$dS_{ind}^2 = c^2 f(r) dt^2 - \frac{dr^2}{f(r)} - r^2 [d\theta^2 + \sin^2(\theta) d\phi^2], \quad (18)$$

which is known as the Schwarzschild-de Sitter metric. From the relativistic point of view, observers that are on  $\Sigma_0$  move with  $U^\psi = 0$ . We assume that the induced matter on  $\Sigma_0$  can be globally described by a 4D energy momentum tensor of a perfect fluid  $T_{\alpha\beta} = (\rho c^2 + P)U_\alpha U_\beta - P g_{\alpha\beta}$ , where  $\rho(t, r)$  and  $P(t, r)$  are respectively the energy density and pressure of the induced matter, such that

$$P = -\rho c^2 = -\frac{3c^4}{8\pi G} \frac{1}{\psi_0^2}, \quad (19)$$

which corresponds to a vacuum equation of state. The energy density of induced matter is denoted  $\rho$ . Because we are interested in studying the orbits of some planets of our solar system, we shall consider that the main gravitational source is the Solar mass  $M_\odot \equiv \zeta\psi_0$  and radius  $r_0$ . We shall assume that we live on the 4D hypersurface  $\Sigma_{H_0} : \psi_0 = cH_0^{-1}$ ,  $H_0$  and  $G\zeta \leq 1/(2\sqrt{27}) \simeq 0.096225$ , being  $H_0 = 73 \frac{km}{sec} Mpc^{-1}$  the present day Hubble constant.

When one takes  $U^\psi = 0$ , the induced potential  $V_{ind}(r)$  on the hypersurface  $\Sigma_0$  is given by

$$V_{ind}(r) = -\frac{GM_\odot}{r} + \frac{p_\phi^2}{2r^2} - \frac{GM_\odot}{c^2} \frac{p_\phi^2}{r^3} - \frac{1}{2} \left( \frac{r}{\psi_0} \right)^2. \quad (20)$$

The first two terms on the right hand side of (20) correspond to the classical potential, the third term is the usual relativistic contribution and the last term is a new contribution coming from 5D metric solution (1). The acceleration associated with the induced potential (20) reads

$$a = -\frac{GM_\odot}{r^2} + \frac{p_\phi^2}{r^3} - \frac{3GM_\odot}{c^2} \frac{p_\phi^2}{r^4} + \frac{r}{\psi_0^2}. \quad (21)$$

By expressing (3) as a function of the angular coordinate,  $\phi$  (indeed assuming  $1/u = r = r(\phi)$ ), we obtain

$$\left( \frac{du}{d\phi} \right)^2 + \left( 1 - \frac{2GM_\odot}{c^2} u \right) (p_\phi^{-2} + u^2) - p_\phi^{-2} (u\psi_0)^{-2} = c^{-2} p_t^2 p_\phi^{-2} + \psi_0^{-2}. \quad (22)$$

This equation of the orbit is almost the same that the one usually obtained in the 4D general theory of relativity for a Schwarzschild-de Sitter metric. However, notice that here the cosmological constant is well determined by the constant  $\psi_0^{-2} = H_0^2/c^2$ , and not any constant of arbitrary signature (as in 4D general relativity). In other words, in our formalism the cosmological constant is determined geometrically by the foliation.

### A. Effective geodesics equations on the 4D hypersurface

If we require that  $S(s) = s$ , we must place (17) in (11). Hence, after taking a constant foliation  $\psi = \psi_0$ , the solution for  $S(s)$  is

$$S(s) = s = -\frac{\psi_0}{2} \ln \left( -\frac{C_2}{C_1} \right). \quad (23)$$

In this case both (15) and (16) evaluated on the foliation  $\psi = \psi_0$  are free of sources

$$\frac{d^2 x^\mu}{ds^2} + \bar{\Gamma}_{\alpha\beta}^\mu \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0, \quad (24)$$

$$\frac{d^2 \psi}{ds^2} + \frac{1}{\psi_0} = 0, \quad (25)$$

where  $\bar{\Gamma}_{\alpha\beta}^{\mu} = \Gamma_{\alpha\beta}^{\mu}|_{\psi=\psi_0}$ . Finally, we must additionally require that  $C_2 = -1/(\psi_0^4 C_1)$  in order to obtain  $\psi(s) = \psi_0$  in (17). Notice that there are no extra force components in (24) and (25).

#### IV. CALCULATION OF THE PERIHELION ADVANCE

In order to study the advances of perihelions for massive test particles in the solar system we consider (22). After making  $u(\phi) = 4/r_s v(\phi)$  and  $M = r_s/256p_\phi^2\psi_0^2$ , we obtain

$$v^2 \left( \frac{dv}{d\phi} \right)^2 = 4v^5 - v^4 + \frac{v^3 r_s^2}{4p_\phi^2} - v^2 \left[ \frac{r_s^2}{16p_\phi^2} - \frac{r_s^2}{16} [p_t^2 p_\phi^{-2} + \psi_0^{-2}] \right] + M,$$

where

$$P_5(v) = 4v^5 - v^4 + \frac{v^3 r_s^2}{4p_\phi^2} - v^2 \left[ \frac{r_s^2}{16p_\phi^2} - \frac{r_s^2}{16} [p_t^2 p_\phi^{-2} + \psi_0^{-2}] \right] + M.$$

The half-period of the orbit will be

$$\phi + \phi_0 = \int_{e_1}^{e_2} \frac{v}{\sqrt{P_5(v)}} dv, \quad (26)$$

where  $e_1$  and  $e_2$  are the real and positive roots of  $P_5(v)$ .

The advance of the perihelion for the orbits will be given by two times the difference between  $\pi$  and the angle described by the orbit in (26)

$$\Delta_\phi^{MI} = 2\pi - 2 \int_{e_1}^{e_2} \frac{v}{\sqrt{P_5(v)}} dv. \quad (27)$$

It must be noted that  $M \ll 1$ . In order to calculate the integral in (27), we shall make the following expansion of  $v/\sqrt{P_5(v)}$

$$\frac{v}{\sqrt{P_5(v)}}|_{M \ll 1} \simeq \frac{1}{\sqrt{P_3(v)}} - \frac{M}{2\sqrt{[P_3(v)]^3}} + \frac{3M^3}{8\sqrt{[P_3(v)]^5}} + \dots, \quad (28)$$

with  $P_5(v) = v^2 P_3(v) + M$ , and

$$P_3(v) = 4v^3 - v^2 + \frac{r_s^2 v}{4p_\phi^2} - \frac{r_s^2}{16p_\phi^2} - \frac{r_s^2}{16} [p_t^2 p_\phi^{-2} + \psi_0^{-2}]. \quad (29)$$

Notice that all the terms in the series are integrable. Finally, if we make the substitution  $v(\phi) = w(\phi) + 1/12$ , we obtain the result

$$\Delta_\phi^{MI} = \phi_1 + \phi_2 = \int_{\epsilon_1}^{\infty} \frac{dw}{\sqrt{4w^3 - g_2 w - g_3}} - \frac{M}{2} \int_{\epsilon_1}^{\infty} \frac{dw}{(w + \frac{1}{2})^2 \sqrt{4w^3 - g_2 w - g_3}} + \dots, \quad (30)$$

with

$$\phi_1 = \int_{\epsilon_1}^{\infty} \frac{dw}{\sqrt{4w^3 - g_2w - g_3}}, \quad (31)$$

$$\phi_2 = -\frac{M}{2} \int_{\epsilon_1}^{\infty} \frac{dw}{\left(w + \frac{1}{2}\right)^2 \sqrt{4w^3 - g_2w - g_3}}, \quad (32)$$

where  $g_2$  and  $g_3$  are the invariants of Weierstrass

$$g_2 = \frac{1}{12} - \frac{r_s^2}{4p_\phi^2}, \quad (33)$$

$$g_3 = \frac{1}{216} + \frac{r_s^2}{16} \left[ \frac{1}{p_\phi^2} \left( 1 - \frac{p_t^2}{c^2} \right) + \frac{1}{\psi_0^2} \right] - \frac{r_s^2}{48p_\phi^2}, \quad (34)$$

and  $e_1 = \epsilon_1 + 1/12$ , such that  $P_3(w = \epsilon_1) = 0$ . The constants  $p_t$  and  $p_\phi$  are the two free parameters of the theory and they are related to the energy by mass unit,  $E = cp_t$ , and the angular momentum by mass unit,  $L_M = cp_\phi$ , such that the invariants of Weierstrass hold

$$g_2 = \frac{1}{12} - \frac{r_s^2 c^2}{4L_M^2}, \quad (35)$$

$$g_3 = -\frac{r_s^2 c^2}{48L_M^2} + \frac{1}{216} + \frac{r_s^2 c^2}{16L_M^2} - \frac{r_s^2}{16} \left[ \frac{E^2}{c^2 L_M^2} + \psi_0^{-2} \right]. \quad (36)$$

Furthermore, because  $0 < r < \infty$ , the range of validity of  $w(\phi)$  is:  $-1/12 < w < \infty$ .

#### A. Limit case with $\psi_0 \rightarrow \infty$

Because  $\psi_0 = c/H$ , the case with zero cosmological constant corresponds to the limit case  $\psi_0 \rightarrow \infty$ . Notice that  $H$  is the Hubble parameter which is experimentally determined so that the foliation  $\psi = \psi_0$  is given physical parameters. If we take this limit in (35) and (36) we obtain exactly the same solution as (30), but the invariant of Weierstrass  $\hat{g}_2$  and  $\hat{g}_3$

$$\hat{g}_2 = \frac{1}{12} - \frac{r_s^2 c^2}{4L_M^2}, \quad (37)$$

$$\hat{g}_3 = -\frac{r_s^2 c^2}{48L_M^2} + \frac{1}{216} + \frac{r_s^2 c^2}{16L_M^2} - \frac{r_s^2}{16} \left[ \frac{E^2}{c^2 L_M^2} \right]. \quad (38)$$

These expressions are in agreement with the results obtained when we use the standard 4D formalism for general relativity.



## V. NUMERICAL RESULTS

With the aim to illustrate the formalism we shall calculate the advance for the perihelions of Mercury, the Earth and Pluto. We shall use for our calculations the respective values for the Schwarzschild radius ( $r_s$ ), the speed of light ( $c$ ) and the Hubble radius ( $c/H$ ):  $r_s = 2.95325008 \times 10^5$  cm,  $c = 2.9979245800 \times 10^{10}$  cm/seg and  $c/H = 1.2701000000 \times 10^{28}$  cm. In all cases we shall consider that the angular momentum by mass unit is given by  $L_M = v_p r_p$ , such that  $v_p$  and  $r_p$  are the velocity and distance, respectively, of the planet at the perihelion.

### A. Mercury

The orbital period of Mercury is 87.9695 Earth days. Its angular moment by mass unit is  $L_M = 2.71308044481 \times 10^{19}$   $cm^2/seg$  and the energy by mass unit being given by  $E = 2.99792454178 \times 10^{10}$   $cm/seg$ . The only finite real root on the physical domain is  $\epsilon_1 = 0.166666640044$ . Using (26), we can calculate the half-period:  $\phi = 3.14159290450$ . It is very important to notice that the result of the second integral in (32) is negligible:  $\phi_2 = -9.71527962041 \times 10^{-52}$ , so that the advance of the perihelion results given totally by the first integral (31):  $\Delta_\phi^{MI} = 42.9773350296$  arcseg/century. This value is in very good agreement with observations:  $\Delta_\phi^{MI}|_{exp} = 42.98 \pm 0.04$  and with predictions of general relativity[4].

### B. Earth

The Earth is densest and fifth-largest of the eight planets in the Solar System. Its angular moment per mass units is  $L_M = 4.52332500000 \times 10^{19}$   $cm^2/seg$  and its energy per mass units is  $E = 2.99792457200 \times 10^{10}$   $cm/seg$ . The only finite real root on the physical domain is  $\epsilon_1 = 0.166666657089$ . Therefore, for an orbital period of 365 days, the half-period can be calculated from eq. (26):  $\phi = 3.14159274386$ . The advance of the perihelion of the Earth can be calculated from the first integral (31):  $\Delta_\phi^{MI} = 3.72390481198$  arcseg/century. This value agree with the experimentally observed value. Because in the case of Mercury the second integral (32) is very small:  $\phi_2 = -3.49514238656 \times 10^{-52}$ .

### C. Pluto

It is well known that Pluto it is not a true planet. It is the second most massive known dwarf planet, after Eris. In this case the angular moment by mass unit is  $L_M = 2.7025100000 \times 10^{19} \text{ cm}^2/\text{seg}$  and the energy by mass unit that we use is  $E = 2.9979245418 \times 10^{10} \text{ cm}/\text{seg}$ , so that the root in the physical domain takes the value  $\epsilon_1 = 0.16666666639$ . Pluto has an orbital period of 247.08 terrestrial years so that the half-period is  $\phi = 3.1415926561$ . This value being given by the first integral (31), because the second one (32) is two orders of magnitude smaller than the other two cases:  $\phi_2 = -9.791421275 \times 10^{-54}$ . With these values we can calculate the advance of the perihelion, which takes the value  $\Delta_\phi^{MI} = 0.000417 \text{ arcseg/century}$ .

## VI. FINAL COMMENTS

Induced matter theory[11–15], has been of much interest in recent years and the exploration of the geodesic equations from a 5D vacuum is an important topic of this theory[16]. In this paper we have re-examined this topic to apply to possible applications of ETGR to orbits like planetary orbits in our solar system. ETGR has been proposed some years ago[9] and has been studied in the framework of astrophysical[17] and cosmological[18, 19] applications. However, the possible applications are not limited to these kinds of orbits. A very important result is the particular solution with  $S(s) = s$  described in Sect. IIIa, for which there are no extra force components due to the foliations on the extra dimension [see (24) and (25)].

We have studied analytically the advances for the perihelions for Mercury, the Earth and Pluto. This work was the first to use ETGR, where the cosmological constant is determined by the foliation  $\psi = \psi_0 = c/H$ , so once the Hubble constant, is experimentally determined, we have the cosmological constant:  $\Lambda = 3/\psi_0^2 = 3H^2$ . In our calculations we have not considered the quadrupolar moment of the Sun, which may be important for the orbit of Mercury[20].

This method can be used to calculate other orbits of comets with large period that come from the Oort cloud. Some of these comets, as for example, the Ison comet, pass very close to the Sun and therefore are subject to an intense gravitational field[21].

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