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On Estimates for the Period of Solutions of Equations Involving the ϕ-Laplace Operator

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Abstract. In this paper we give new bounds for the period of solutions to certain Hamiltonian system involving a function ϕ. We also obtain upper and lower bounds which are uniform with respect to the function ϕ. Furthermore, the optimality of this lower bound is established.

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1 Introduction and main results

The main aim of this paper is to give estimates for the period of solutions of the quasilinear ODE

$$\frac{d}{dt}(\phi(x')) + \lambda\phi(x) = 0. \tag{1.1}$$

Throughout this article we consider $\phi : \mathbb{R} \to \mathbb{R}$ an increasing odd homeomorphism of $\mathbb{R}$, Φ the primitive of ϕ with $\Phi(0) = 0$, and x a real function depending on the variable t. Henceforth we denote by $\mathcal{F}$ the set of all functions Φ satisfying previous conditions.

As usual we call *ϕ-Laplace operator* the differential operator $x \mapsto \frac{d}{dt}(\phi(x'))$. This is named *p-Laplace operator* or more briefly *p-Laplacian* in the particular case that $\Phi(x) = |x|^p/p$, $1 < p < \infty$.

Boundary values problems containing ϕ-Laplace operator have been extensively studied (see e.g [4, 7, 10, 11, 12, 13, 14] and the references therein). A large part of the associated literature is devoted to the question of existence of solutions.

The problem of estimating the period of solutions is closely related to the eigenvalue problem on some interval (a,b) of $\mathbb{R}$:

$$\begin{cases} \frac{d}{dt}(\phi(x')) + \lambda\phi(x) = 0 \\ x(a) = x(b) = 0. \end{cases} \tag{1.2}$$

The number λ is an eigenvalue if and only if $2(b-a)$ is an integer multiple of the period of some solution $x(t)$ of equation (1.1) (see [4]).

For certain functions ϕ there exists $T > 0$ such that all solutions of (1.1) have period T. In this case, T depends on λ but does not depend on the initial conditions satisfied by x then, following [2], we say that the equation (1.1) is isochronus. As a consequence the set of eigenvalues is a sequence going to infinity. A well known case of isochrony, although it is not in the form (1.1), is the equation defining tautochrone curve.

The equation (1.1) is the Lagrange equation with respect to the Lagrangian $\mathcal{L}(x,x') = \Phi(x') - \lambda\Phi(x)$. The associated Hamiltonian is the function

$$H(\rho, x) = \Psi(\rho) + \lambda\Phi(x).$$

The variable $\rho = \phi(x')$ is the generalized momentum and Ψ is the complementary function (Legendre transform) of Φ defined by

$$\Psi(x) = \sup_{y\in\mathbb{R}}\{xy - \Phi(y)\}.$$

The lowercase symbol ψ denotes the derivative function of Ψ. The function ψ becomes the inverse of ϕ and therefore $\Psi \in \mathcal{F}$ (see [8]).

As it is known, the Hamiltonian is a conserved quantity along solutions. In this paper we call the quantities H and λ energy and frequency respectively. Since x is one-dimensional, two solutions corresponding to same energy differ in a time translation. Therefore, instead of talking about solutions associated to initial conditions, we will use energy H to indicate solutions of (1.1). Positions x and generalized momentums ρ are solutions of the Hamiltonian system

$$\begin{cases} x'(t) = \frac{\partial H}{\partial \rho} = \psi(\rho) \\ \rho'(t) = -\frac{\partial H}{\partial x} = -\lambda\phi(x) \end{cases} \tag{1.3}$$

(see [1]).

For convenience, we will consider $1/4$ times the period of solutions of the equation (1.1) and we will denote it by $T_\Phi(H,\lambda)$ (we note that the period depends only on energy H and frequency λ). In [5] it was obtained the following explicit formula:

$$T_\Phi(H,\lambda) = \int_0^{\Phi^{-1}\left(\frac{H}{\lambda}\right)} \frac{du}{\psi(\Psi^{-1}(H-\lambda\Phi(u)))}. \tag{1.4}$$

If we consider the change of variable $v = H-\lambda\Phi(u)$ in the integral (1.4), then we obtain

$$T_\Phi(H,\lambda) = \frac{1}{\lambda}\int_0^{H} \frac{dv}{\psi(\Psi^{-1}(v))\,\phi\left(\Phi^{-1}\left(\frac{H-v}{\lambda}\right)\right)}. \tag{1.5}$$

This symmetric convolution formula shows the following relation between periods and frequencies of complementary functions.

Lemma 1.1. *For every $H > 0$ and $\lambda > 0$*

$$T_\Phi(H,\lambda) = \frac{1}{\lambda} T_\Psi\left(\frac{H}{\lambda}, \frac{1}{\lambda}\right). \tag{1.6}$$

Alternatively, (1.6) can be deduced observing that changes of variables $(x,\rho,t) \to (\rho, x, -\lambda^{-1}t)$ transform solutions of (1.3) into solutions of its dual system, i.e the system obtained from (1.3) by means of the substitutions $\phi \leftrightarrow \psi$ and $(H,\lambda) \to (H/\lambda, 1/\lambda)$. Note that $a \leftrightarrow b$ means exchange a and b.

Let us take a moment to show as the classical theory on Hamiltonian system allows us to get (1.4). First, we point out that energy levels of the Hamiltonian function are closed trajectories; hence the solutions are periodic. The solutions with energy $H > 0$ intersect the positive coordinate semi-axis at the points $P := (0, \Psi^{-1}(H))$ and $Q := (\Phi^{-1}(H/\lambda), 0)$. Second, we note that equations (1.3) are invariant with respect to the changes of variables $(x,\rho,t) \to (x,-\rho,-t), (-x,\rho,-t)$, therefore trajectories are symmetric with respect to coordinate axis. These facts imply that the period is four times as long as to go from P to Q. Last, we invoke action-angles variables [1, Section 50] and we consider the generating function

$$W(x,I) = \int^x \Psi^{-1}(H-\lambda\Phi(u))du.$$

Here we assume that I is the action variable and H is function of I. For the angle variable we have that

$$\omega = \frac{\partial W}{\partial I} = \int^x \frac{du}{\psi(\Psi^{-1}(H-\lambda\Phi(u)))}\frac{\partial H}{\partial I}.$$

Now, from [1, p. 280] we know that $\omega = (\partial H/\partial I)t$, therefore

$$t = \int^x \frac{du}{\psi(\Psi^{-1}(H-\lambda\Phi(u)))}$$

and integrating from 0 to $\Phi^{-1}(H/\lambda)$ we get formula (1.4).

For the p-Laplace operator the problem (1.1) is isochronus and, in this case, the formula (1.4) reduces to

$$T_p(\lambda) := T_\Phi(H,\lambda) = \frac{\mathrm{B}\left(\frac{1}{p}, -\frac{1}{p}+2\right)}{(p-1)^{\frac{1}{q}}\lambda^{\frac{1}{p}}} = \frac{\pi(p-1)^{\frac{1}{p}}}{p\sin\left(\frac{\pi}{p}\right)\lambda^{\frac{1}{p}}}, \tag{1.7}$$

where B denotes the beta function. As a consequence, the spectrum of the one-dimensional p-Laplace operator is discrete. It is a remarkable open problem whether the multidimensional p-Laplacian is discrete or continuous.

In [5], M. García-Huidobro, R. Manásevich and F. Zanolin were interested in estimating the spectrum of the ϕ-Laplace operator with the purpose of obtaining, what they called, non resonance intervals, i.e. intervals without eigenvalues. Clearly a sharp bound is better for this goal. In [5] it was obtained the estimate

$$\frac{1}{1+\lambda} \le T_\Phi(H,\lambda) \le \frac{2(\lambda+1)}{\lambda}. \tag{1.8}$$

In this paper we wish to improve these estimates and discuss the possible optimality of the new bounds. That is, we would like to characterize the quantities:

$$U(\lambda) := \sup_{H>0,\Phi\in\mathcal{F}} T_\Phi(H,\lambda)$$
$$L(\lambda) := \inf_{H>0,\Phi\in\mathcal{F}} T_\Phi(H,\lambda).$$

Remark 1.2. It is easy to show that

$$T_\Phi(H,\lambda) = T_{\Phi_{\alpha\beta}}(\beta H,\lambda),$$

where $\Phi_{\alpha\beta}(x) = \beta\Phi(\alpha x)$. Therefore, taking $\beta = H^{-1}$ we get

$$U(\lambda) = \sup_{\Phi\in\mathcal{F}} T_\Phi(1,\lambda) \quad \text{and} \quad L(\lambda) = \inf_{\Phi\in\mathcal{F}} T_\Phi(1,\lambda).$$

We note that we can use the parameter α in order to introduce an extra condition on functions Φ, for example that $\Phi(1) = 1$.

Let $A_\Phi(H,\lambda)$ and $C_\Phi(H,\lambda)$ be defined by

$$A_\Phi(H,\lambda) := \frac{\Psi^{-1}\left(\frac{H}{2}\right)}{\lambda\phi(\Phi^{-1}\left(\frac{H}{2\lambda}\right))} + \frac{\Phi^{-1}\left(\frac{H}{2\lambda}\right)}{\psi\left(\Psi^{-1}(\frac{H}{2})\right)}, \tag{1.9}$$

$$C_\Phi(H,\lambda) := \max\left\{\frac{\Phi^{-1}\left(\frac{H}{\lambda}\right)}{\psi(\Psi^{-1}(H))}, \frac{\Psi^{-1}(H)}{\lambda\phi\left(\Phi^{-1}(\frac{H}{\lambda})\right)}\right\}. \tag{1.10}$$

The following theorem is our starting point.

Theorem 1.3. *If $\Phi \in \mathcal{F}$ then*

$$C_\Phi(H,\lambda) \le T_\Phi(H,\lambda) \le A_\Phi(H,\lambda). \tag{1.11}$$

Throughout this article, we denote by K a positive constant that may depend on Φ and on an arbitrary positive parameter ϵ, and we assume that the value that K represents may change in different occurrences in the same chain of inequalities.

We recall that a nondecreasing function φ is a Δ_2-function when there exists a constant K such that

$$\varphi(2x) \le K\varphi(x), \qquad x \ge 0.$$

We remark that if Φ, Ψ are Δ_2-functions, we get from the previous theorem an estimate of the period by powers of λ.

Corollary 1.4. *If Φ, Ψ are Δ_2-functions then for every $\epsilon > 0$ there exist a constant K such that*

$$K^{-1}\min\left\{\frac{1}{\lambda^{\frac{1}{\beta_\Phi}-\epsilon}}, \frac{1}{\lambda^{\frac{1}{\alpha_\Phi}+\epsilon}}\right\} \le T_\Phi(H,\lambda) \le K\max\left\{\frac{1}{\lambda^{\frac{1}{\beta_\Phi}-\epsilon}}, \frac{1}{\lambda^{\frac{1}{\alpha_\Phi}+\epsilon}}\right\} \tag{1.12}$$

where α_Φ and β_Φ are the Matuszewska-Orlicz indices *(see Section 2 for definitions).*

The next proposition gives better estimates than (1.8) and it also establishes the optimality of the lower bound.

Proposition 1.5. *For any* $\lambda > 0$*, we have that*

$$\min\left\{1, \frac{1}{\lambda}\right\} \le L(\lambda) \le U(\lambda) \le \max\left\{\frac{\lambda+2}{\lambda}, \frac{2\lambda+1}{\lambda}\right\}.$$

Moreover,

$$L(\lambda) = \inf_{p>1} T_p(\lambda) = \min\left\{1, \frac{1}{\lambda}\right\} \le \max\left\{1, \frac{1}{\lambda}\right\} \le \sup_{p>1} T_p(\lambda) \le U(\lambda).$$

We can show that the quantity $\max\{\frac{\lambda+2}{\lambda}, \frac{2\lambda+1}{\lambda}\}$ is optimal with respect to $A_\Phi(H,\lambda)$. More precisely,

Proposition 1.6. *For any* $\lambda > 0$*, we have that*

$$\sup_{H>0, \Phi\in\mathcal{F}} A_\Phi(H,\lambda) = \max\left\{\frac{\lambda+2}{\lambda}, \frac{2\lambda+1}{\lambda}\right\}.$$

The article continues as follows. In Section 2 we present the proofs of the above results. In Section 3 we discuss improvements in the upper bounds of the period.

2 Proofs

Proof. ***Theorem 1.3***. Let (x,ρ) be any non-trivial solution of (1.3) and let

$$H = \Psi(\rho(t)) + \lambda\Phi(x(t))$$

be the energy constant. As we have mentioned above the solutions with energy $H > 0$ intersect the positive coordinate semi-axis at the points $P := (0, \Psi^{-1}(H))$ and $Q := (\Phi^{-1}(\frac{H}{\lambda}), 0)$. Now, we take $R = \left(\Phi^{-1}(\frac{H}{2\lambda}), \Psi^{-1}(\frac{H}{2})\right)$ (see Figure 1).

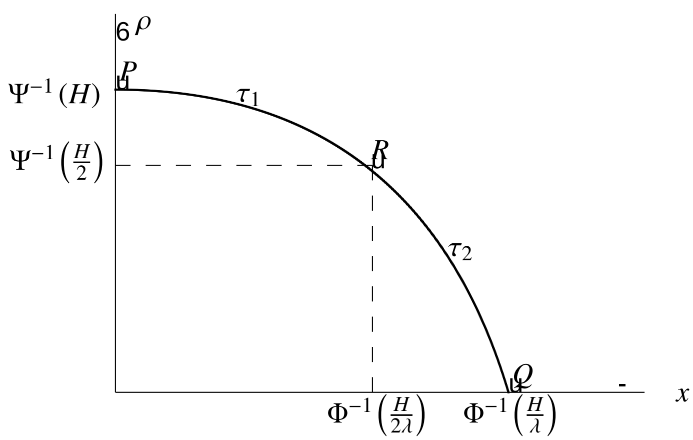

Figure 1. A quarter of a trajectory

If τ_1 is the time taken by a solution to go from P to R in the first quadrant of the $x\rho$-plane and τ_2 is the respective time from R to Q, then

$$T_\Phi(H,\lambda) = \tau_1 + \tau_2. \tag{2.1}$$

We note that $x(t),\rho(t) \geq 0$ for $t \in [0,\tau_1+\tau_2]$.

From (1.3) we have $\rho'(t) = -\lambda\phi(x(t))$. Therefore, since $x(t) \geq 0$ and ϕ is an increasing odd homeomorphism, then ρ is a non increasing function. This fact and the relation $\rho(\tau_1) = \Psi^{-1}(H/2)$ imply that $\rho(t) \geq \Psi^{-1}(H/2)$ for $t \in [0,\tau_1]$. Hence, using the first equation of (1.3) and taking into account the monotonicity of ψ, we obtain

$$x'(t) \geq \psi\left(\Psi^{-1}\left(\frac{H}{2}\right)\right)$$

and integrating between 0 and τ_1 we have

$$\Phi^{-1}\left(\frac{H}{2\lambda}\right) = \int_0^{\tau_1} x'(t)dt \geq \psi\left(\Psi^{-1}\left(\frac{H}{2}\right)\right)\tau_1,$$

thus

$$\tau_1 \leq \frac{\Phi^{-1}(\frac{H}{2\lambda})}{\psi\left(\Psi^{-1}\left(\frac{H}{2}\right)\right)}. \tag{2.2}$$

With the same procedure for $t \in [\tau_1,\tau_1+\tau_2]$, taking into account the second equation of (1.3) and the inequality $x(t) \geq \Phi^{-1}(\frac{H}{2\lambda})$, we obtain

$$\tau_2 \leq \frac{\Psi^{-1}(\frac{H}{2})}{\lambda\phi(\Phi^{-1}\left(\frac{H}{2\lambda}\right))}. \tag{2.3}$$

From (2.1), (2.2) and (2.3) we have the second inequality in (1.11).

In order to prove the first inequality, we note that $\rho(t) \leq \Psi^{-1}(H)$ for $t \in [0,\tau_1+\tau_2]$, thus

$$x'(t) = \psi(\rho(t)) \leq \psi\left(\Psi^{-1}(H)\right),$$

integrating from 0 to $\tau_1+\tau_2$, we get

$$T_\Phi(H,\lambda) \geq \frac{\Phi^{-1}\left(\frac{H}{\lambda}\right)}{\psi(\Psi^{-1}(H))}. \tag{2.4}$$

Analogously, since $x(t) \leq \Phi^{-1}\left(\frac{H}{\lambda}\right)$ and $\rho'(t) = -\lambda\phi(x(t))$, we obtain

$$T_\Phi(H,\lambda) \geq \frac{\Psi^{-1}(H)}{\lambda\phi\left(\Phi^{-1}(\frac{H}{\lambda})\right)}. \tag{2.5}$$

□

With the purpose of establishing Corollary 1.4 we recall some definitions and results from the theory of convex functions. We suggest [3, 6, 8, 9, 15] for definitions, proofs and additional details.

We denote by α_φ and β_φ the so called *Matuszewska-Orlicz indices* of the function φ, which are defined next. Given an increasing, unbounded, continuous function $\varphi : [0,+\infty) \to [0,+\infty)$ such that $\varphi(0) = 0$ we define

$$\alpha_\varphi := \lim_{t\to 0^+} \frac{\log\left(\sup\limits_{u>0} \frac{\varphi(tu)}{\varphi(u)}\right)}{\log(t)}, \quad \beta_\varphi := \lim_{t\to+\infty} \frac{\log\left(\sup\limits_{u>0} \frac{\varphi(tu)}{\varphi(u)}\right)}{\log(t)}. \tag{2.6}$$

It is known that the previous limits exist and $0 \le \alpha_\varphi \le \beta_\varphi \le +\infty$ (see [9, p.84]). The relation $\beta_\varphi < +\infty$ holds true if and only if φ is a Δ_2-function ([9, Theorem 11.7]). If φ is a homeomorphism, by [9, Theorem 11.5], we have that

$$\alpha_{\varphi^{-1}} = \frac{1}{\beta_\varphi}. \tag{2.7}$$

Moreover $\varphi \in \mathcal{F}$ implies $\alpha_\varphi \ge 1$ ([9, Corollary 11.6]). As a consequence, φ^{-1} is a Δ_2-function.

If φ is an increasing Δ_2-function then φ is controlled by above and below by power functions ([6, Section 1], [3, Equations 2.3-2.4] and [9, Theorem 11.13]). More concretely, for every $\epsilon > 0$ there exists a constant $K = K(\varphi, \epsilon)$ such that, for every $t, u \ge 0$,

$$K^{-1} \min\{t^{\beta_\varphi+\epsilon}, t^{\alpha_\varphi-\epsilon}\}\varphi(u) \le \varphi(tu) \le K \max\{t^{\beta_\varphi+\epsilon}, t^{\alpha_\varphi-\epsilon}\}\varphi(u). \tag{2.8}$$

We recall the very well known Young's equality ([8, Equations 2.7-2.8]), for a pair (Φ, Ψ) of complementary functions in $\mathcal{F} \times \mathcal{F}$

$$x\phi(x) = \Phi(x) + \Psi(\phi(x)). \tag{2.9}$$

If Φ and Ψ are Δ_2-functions then the three terms in this formula become balanced. That means

$$x\phi(x) \sim \Phi(x) \sim \Psi(\phi(x)), \quad x > 0 \tag{2.10}$$

where the notation $f \sim g$ means that the ratio f/g remains bounded from above and below by positive constants for positive x. In fact, the relation $x\phi(x) \sim \Phi(x)$ follows from (2.9), the Δ_2-condition for Φ and [15, Theorem 3-1(ii), p.23]. The relation $x\phi(x) \sim \Psi(\phi(x))$ is consequence of the Δ_2-condition for Ψ, because in this case we have $y\psi(y) \sim \Psi(y)$ and the desidered relation is obtained by the substitution $y = \phi(x)$.

From (2.10) we have that there exists $0 < c \le 1 \le C < +\infty$ such that

$$c\Psi(\phi(y)) \le \Phi(y) \le C\Psi(\phi(y)), \quad y > 0.$$

If we replace y by $\psi(\Psi^{-1}(x))$ and we apply Φ^{-1} to all members in the chain of inequalities we obtain

$$\Phi^{-1}(cx) \le \psi(\Psi^{-1}(x)) \le \Phi^{-1}(Cx).$$

As Φ^{-1} is a concave function, $\Phi^{-1}(0) = 0$ and $0 < c \le 1$ we have $\Phi^{-1}(cx) \ge c\Phi^{-1}(x)$. In addition, Φ^{-1} is a Δ_2-function, then there exists a positive constant K such that $\Phi^{-1}(Cx) \le K\Phi^{-1}(x)$ (see [8, p.23]). Finally, we get

$$\Phi^{-1}(x) \sim \psi(\Psi^{-1}(x)), \quad x > 0. \tag{2.11}$$

Proof. ***Corollary 1.4***. By virtue of Theorem 1.3 it is sufficient to prove that the following inequalities

$$\begin{aligned} K^{-1} \min\left\{\frac{1}{\lambda^{\frac{1}{\beta_\Phi}-\epsilon}}, \frac{1}{\lambda^{\frac{1}{\alpha_\Phi}+\epsilon}}\right\} &\le \frac{\Phi^{-1}\left(\frac{H}{\lambda}\right)}{\psi(\Psi^{-1}(H))}, \frac{\Psi^{-1}(H)}{\lambda\phi(\Phi^{-1}(\left(\frac{H}{\lambda}\right)))} \\ &\le K \max\left\{\frac{1}{\lambda^{\frac{1}{\beta_\Phi}-\epsilon}}, \frac{1}{\lambda^{\frac{1}{\alpha_\Phi}+\epsilon}}\right\} \end{aligned} \tag{2.12}$$

hold true for every pair (Φ, Ψ) of complementary Δ_2-functions in $\mathcal{F} \times \mathcal{F}$. Taking account of (2.11) it is possible to substitute $\psi(\Psi^{-1}(H))$ for $\Phi^{-1}(H)$ in (2.12). Now using (2.7) and (2.8) with $\varphi = \Phi^{-1}$ we have that

$$\frac{\Phi^{-1}\left(\frac{H}{\lambda}\right)}{\psi(\Psi^{-1}(H))} \le K\frac{\Phi^{-1}\left(\frac{H}{\lambda}\right)}{\Phi^{-1}(H)} \le K \max\left\{\frac{1}{\lambda^{\frac{1}{\beta_\Phi}-\epsilon}}, \frac{1}{\lambda^{\frac{1}{\alpha_\Phi}+\epsilon}}\right\}.$$

The lower bound is obtained by similar arguments. The other inequalities are obtained by replacing $\Phi \leftrightarrow \Psi$ and $\lambda \leftrightarrow 1/\lambda$. □

Proof. ***Proposition 1.5***

Let $A := \Phi^{-1}\left(\frac{H}{\lambda}\right)$, $B := \Phi^{-1}\left(\frac{H}{2\lambda}\right)$, $C := \Psi^{-1}(H)$, $D := \Psi^{-1}(\frac{H}{2})$ (we note that $\lambda = \Psi(D)/\Phi(B)$), then from Theorem 1.3

$$\max\left\{\frac{A}{\psi(C)}, \frac{C}{\lambda\phi(A)}\right\} \le T_\Phi(H,\lambda) \le \frac{B}{\psi(D)} + \frac{D}{\lambda\phi(B)}.$$

Firstly, dealing with the lower estimate, we have two possibilities $\phi(A) \le C$ or $\psi(C) \le A$, therefore

$$\max\left\{\frac{A}{\psi(C)}, \frac{C}{\lambda\phi(A)}\right\} \ge \min\left\{1, \frac{1}{\lambda}\right\}. \tag{2.13}$$

Secondly, we work on the upper estimate as follows. If $D \le \phi(B)$, using the inequality $\Psi(D) \le \psi(D)D$ and the Young's inequality

$$BD \le \Phi(B) + \Psi(D)$$

we have

$$\frac{B}{\psi(D)} + \frac{D}{\lambda\phi(B)} = \frac{BD}{\psi(D)D} + \frac{D}{\lambda\phi(B)} \le \frac{\lambda+2}{\lambda}.$$

In this manner, we have seen that

$$D \le \phi(B) \Rightarrow T_\Phi(H,\lambda) \le \frac{\lambda+2}{\lambda}. \tag{2.14}$$

Now, exchanging $\Phi \leftrightarrow \Psi$, $B \leftrightarrow D$ (consequently $\lambda \leftrightarrow 1/\lambda$ and $H \leftrightarrow H/\lambda$) and using Lemma 1.1 we obtain

$$\phi(B) \le D \Rightarrow T_\Phi(H,\lambda) = \frac{1}{\lambda} T_\Psi\left(\frac{H}{\lambda}, \frac{1}{\lambda}\right) \le \frac{2\lambda+1}{\lambda}. \tag{2.15}$$

The upper bound of $T_\Phi(H,\lambda)$ follows from (2.14)-(2.15). This concludes the proof of the first part of Proposition 1.5.

Now, we will prove the optimality of the lower bound considering power functions $\Phi(x) = |x|^p$. By elementary limit arguments and performing some calculations we obtain

$$\lim_{p\to 1} T_p(\lambda) = \frac{1}{\lambda} \quad \text{and} \quad \lim_{p\to\infty} T_p(\lambda) = 1.$$

Therefore

$$L(\lambda) \le \min\left\{\lim_{p\to 1} T_p(\lambda), \lim_{p\to\infty} T_p(\lambda)\right\} = \min\left\{1, \frac{1}{\lambda}\right\}.$$

and

$$U(\lambda) \ge \max\left\{\lim_{p\to 1} T_p(\lambda), \lim_{p\to\infty} T_p(\lambda)\right\} = \max\left\{1, \frac{1}{\lambda}\right\}.$$

From these inequalities and (2.13) we obtain the desired result. □

Proof. ***Proposition 1.6***. The inequality

$$A_\Phi(H,\lambda) \le \max\left\{\frac{\lambda+2}{\lambda}, \frac{2\lambda+1}{\lambda}\right\} \tag{2.16}$$

was already proved in the previous proof.

For $a > 0$ we consider the odd functions satisfying for $x \geq 0$

$$\phi_a(x) = \begin{cases} x^a & 0 \leq x \leq 1 \\ \frac{1}{a}x + \frac{a-1}{a} & x > 1. \end{cases}$$

As usual, we denote $\Phi_a(x) = \int_0^x \phi_a(t)\,dt$. It is easy to check that $\Phi_a \in \mathcal{F}$ and $(\phi_a)^{-1}(x) = \phi_{1/a}(x)$. Consequently the complementary function of Φ_a is $\Phi_{1/a}$.

Computing the integral $\int_0^x \phi_a(t)\,dt$ for $x \leq 1$ and $x > 1$ we get

$$\Phi_a(x) = \begin{cases} \frac{x^{a+1}}{a+1} & 0 \leq x \leq 1 \\ \frac{x^2}{2a} + \frac{a-1}{a}x - \frac{(a-1)(2a+1)}{2a(a+1)} & x > 1 \end{cases}$$

Therefore the inverse function $\Phi_a^{-1}(x)$ is equal to $((a+1)x)^{\frac{-a}{a+1}}$ when $0 \leq x \leq 1/(a+1)$. In order to compute $\Phi_a^{-1}(x)$ for $x > 1/(a+1)$, we have to solve the quadratic equation

$$y = \frac{x^2}{2a} + \frac{a-1}{a}x - \frac{(a-1)(2a+1)}{2a(a+1)}$$

for x. Of the two solutions of this equation we are only interested in the largest one. After some elementary calculations we conclude that

$$\Phi_a^{-1}(x) = \begin{cases} ((a+1)x)^{\frac{1}{a+1}} & 0 \leq x \leq 1/(a+1) \\ -a+1+\sqrt{\frac{a}{a+1}}\sqrt{2x(a+1)+(a+2)(a-1)} & x > 1/(a+1) \end{cases}$$

and

$$\frac{d\Phi_a^{-1}}{dx}(x) = \begin{cases} ((a+1)x)^{\frac{-a}{a+1}} & 0 \leq x \leq 1/(a+1) \\ \frac{\sqrt{a+1}\sqrt{a}}{\sqrt{2(a+1)x+(a+2)(a-1)}} & x > 1/(a+1). \end{cases}$$

It is easy to show that

$$\lim_{a\to 0^+} \Phi_a^{-1}(x) = \min(1, x) \quad \text{and} \quad \lim_{a\to+\infty} \frac{d\Phi_a^{-1}}{dx} \equiv 1. \tag{2.17}$$

Now, we compute $\lim\limits_{a\to+\infty} \Phi_a^{-1}(x)$ for $x > 0$. As we can assume $x > 1/(a+1)$, we have

$$\begin{aligned} \lim_{a\to+\infty} \Phi_a^{-1}(x) &= 1 + \lim_{a\to+\infty}\left\{\sqrt{2xa + \frac{a(a+2)(a-1)}{a+1}} - a\right\} \\ &= 1 + \lim_{a\to+\infty} \frac{2xa + \frac{a(a+2)(a-1)}{a+1} - a^2}{\sqrt{2xa + \frac{a(a+2)(a-1)}{a+1}} + a} \\ &= 1 + \lim_{a\to+\infty} \frac{2x - \frac{2}{a+1}}{\sqrt{\frac{2x}{a} + \frac{(a+2)(a-1)}{a(a+1)}} + 1} \\ &= 1 + x \end{aligned} \tag{2.18}$$

We also need to calculate $\lim\limits_{a\to 0^+} \frac{d\Phi_a^{-1}}{dx}$. Firstly, we assume $0 < x < 1$. Since $a \to 0^+$ we can suppose $x < 1/(a+1)$, therefore

$$\lim_{a\to 0^+} \frac{d\Phi_a^{-1}}{dx} = \lim_{a\to 0^+} ((a+1)x)^{\frac{-a}{a+1}} = 1.$$

Secondly, if $x > 1$ then $x > 1/(a+1)$ and consequently

$$\begin{aligned}\lim_{a\to 0^+}\frac{d\Phi_a^{-1}}{dx} &= \lim_{a\to 0^+}\frac{\sqrt{a+1}\,\sqrt{a}}{\sqrt{2\,(a+1)x+(a+2)(a-1)}}\\ &= \lim_{a\to 0^+}\frac{\sqrt{a+1}\,\sqrt{a}}{\sqrt{a^2+2ax+a+2(x-1)}}\\ &= 0.\end{aligned}$$

Finally, if $x = 1$ then $\lim_{a\to 0^+}\frac{d\Phi_a^{-1}}{dx} = 1/\sqrt{3}$. Hence,

$$\lim_{a\to 0^+}\frac{d\Phi_a^{-1}}{dx} = \chi_{(0,1)}, \quad \text{for } x > 0, x \neq 1. \tag{2.19}$$

Let $H > 0$ and $H \neq 2$. Using the formula $d\Phi_a^{-1}/dx = 1/(\phi_a \circ \Phi_a^{-1})$ and taking account of (2.17), (2.18) and (2.19) we have

$$\begin{aligned}\lim_{a\to+\infty} A_{\Phi_a}(H,\lambda) &= \lim_{a\to+\infty}\left[\frac{\Phi_{1/a}^{-1}\left(\frac{H}{2}\right)}{\lambda\phi_a\left(\Phi_a^{-1}\left(\frac{H}{2\lambda}\right)\right)} + \frac{\Phi_a^{-1}\left(\frac{H}{2\lambda}\right)}{\phi_{1/a}\left(\Phi_{1/a}^{-1}\left(\frac{H}{2}\right)\right)}\right]\\ &= \lim_{a\to+\infty}\left[\frac{1}{\lambda}\Phi_{1/a}^{-1}\left(\frac{H}{2}\right)\frac{d\Phi_a^{-1}}{dx}\bigg|_{x=\frac{H}{2\lambda}} + \Phi_a^{-1}\left(\frac{H}{2\lambda}\right)\frac{d\Phi_{1/a}^{-1}}{dx}\bigg|_{x=\frac{H}{2}}\right]\\ &= \frac{1}{\lambda}\min\left(1,\frac{H}{2}\right)+\left(\frac{H}{2\lambda}+1\right)\chi_{(0,1)}\left(\frac{H}{2}\right)\\ &= \begin{cases}\frac{H}{\lambda}+1 & \text{if } H < 2\\ \frac{1}{\lambda} & \text{if } H > 2.\end{cases}\end{aligned}$$

In a similar way

$$\lim_{a\to 0^+} A_{\Phi_a}(H,\lambda) = \begin{cases}\frac{H+1}{\lambda} & \text{if } H < 2\lambda\\ 1 & \text{if } H > 2\lambda.\end{cases}$$

Hence

$$\begin{aligned}\sup_{H>0,\Phi\in\mathcal{F}} A_\Phi(H,\lambda) &\geq \max\left\{\sup_{0<H\neq 2}\lim_{a\to+\infty} A_{\Phi_a}(H,\lambda),\ \sup_{0<H\neq 2\lambda}\lim_{a\to 0^+} A_{\Phi_a}(H,\lambda)\right\}\\ &= \max\left\{\frac{\lambda+2}{\lambda},\frac{2\lambda+1}{\lambda}\right\}.\end{aligned}$$

The result follows taking account of (2.16).

□

3 Additional results

We can improve the upper bound obtained in Theorem 1.3 with a similar argument to that used in its demonstration employing piecewise linear functions instead of piecewise constant ones to bound trajectories.

We consider P, Q, R, τ_1 and τ_2 as in the proof of Theorem 1.3. Let $\tilde{\rho} = ax+b$ and $\tilde{x} = m\rho+n$ be the equations of the straight lines connecting the points P with R and R with Q respectively, then

$a = \frac{\Psi^{-1}(H/2)-\Psi^{-1}(H)}{\Phi^{-1}(H/2\lambda)}$, $b = \Psi^{-1}(H)$, $m = \frac{\Phi^{-1}(H/2\lambda)-\Phi^{-1}(H/\lambda)}{\Psi^{-1}(H/2)}$ and $n = \Psi^{-1}(H/\lambda)$. Due to the concavity of the function $\rho = \Psi^{-1}(H-\lambda\Phi(x))$, for $x \geq 0$ and $\rho \geq 0$, we have that the trajectory $(x(t), \rho(t))$ satisfies $\rho(t) \geq \tilde{\rho}(t)$ for $t \in [0, \tau_1]$ and $x(t) \geq \tilde{x}(t)$ for $t \in [\tau_1, \tau_1 + \tau_2]$.

Taking into account (1.3), we get $x'(t) \geq \psi(\tilde{\rho}(t))$ for $t \in [0, \tau_1]$ and $\rho'(t) \leq -\lambda\phi(\tilde{x}(t))$ for $t \in [\tau_1, \tau_1 + \tau_2]$. Integrating from 0 to τ_1, we obtain

$$\tau_1 \leq \int_0^{\tau_1} \frac{x'(t)dt}{\psi(ax(t)+b)} = \frac{1}{a}\int_{\Psi^{-1}(H)}^{\Psi^{-1}(\frac{H}{2})} \frac{du}{\psi(u)} = \Phi^{-1}\left(\frac{H}{2\lambda}\right) ⨍_{\Psi^{-1}(\frac{H}{2})}^{\Psi^{-1}(H)} \frac{du}{\psi(u)},$$

where $⨍$ denotes the averaged integral. In a similar way, integrating over the interval $[\tau_1, \tau_1+\tau_2]$ the inequality $1 \leq -\lambda^{-1}\rho'/\phi(\tilde{x})$, we have

$$\begin{aligned}\tau_2 &\leq -\lambda^{-1}\int_{\tau_1}^{\tau_1+\tau_2} \frac{\rho'(t)dt}{\phi(\tilde{x}(t))}\\ &= -\lambda^{-1}\int_{\tau_1}^{\tau_1+\tau_2} \frac{\rho'(t)dt}{\phi(m\rho+n)}\\ &\leq \Psi^{-1}\left(\frac{H}{2}\right)\lambda^{-1} ⨍_{\Phi^{-1}(\frac{H}{2\lambda})}^{\Phi^{-1}(\frac{H}{\lambda})} \frac{du}{\phi(u)}.\end{aligned}$$

If we define

$$B_\Phi(H,\lambda) := \Phi^{-1}\left(\frac{H}{2\lambda}\right) ⨍_{\Psi^{-1}(\frac{H}{2})}^{\Psi^{-1}(H)} \frac{du}{\psi(u)} + \Psi^{-1}\left(\frac{H}{2}\right)\lambda^{-1} ⨍_{\Phi^{-1}(\frac{H}{2\lambda})}^{\Phi^{-1}(\frac{H}{\lambda})} \frac{du}{\phi(u)} \tag{3.1}$$

then, according to our previous discussion, we have the following result.

Proposition 3.1. *For any $H > 0$*

$$T_\Phi(H,\lambda) \leq B_\Phi(H,\lambda).$$

Recalling the definition of $A_\Phi(H,\lambda)$ given in (1.9) and bounding the functions $1/\phi$ and $1/\psi$ by their maximum values over the corresponding integration intervals in (3.1), we obtain

$$B_\Phi(H,\lambda) \leq A_\Phi(H,\lambda).$$

However, as it is shown in the following result, the optimal upper bound for $B_\Phi(H,\lambda)$ is the same that for $A_\Phi(H,\lambda)$.

Theorem 3.2. *For any $\lambda > 0$*

$$\sup_{H>0, \Phi\in\mathcal{F}} B_\Phi(H,\lambda) = \max\left\{\frac{\lambda+2}{\lambda}, \frac{2\lambda+1}{\lambda}\right\}.$$

Proof. We consider the functions Φ_a defined in the proof of Proposition 1.6. By performing the change of variables $u = \Psi^{-1}(v)$ and $u = \Phi^{-1}(v)$ in the integrals (3.1) we obtain

$$\begin{aligned}B_{\Phi_a}(H,\lambda) &= \Phi_a^{-1}\left(\frac{H}{2\lambda}\right)\frac{\int_{\frac{H}{2}}^{H}\left|\frac{d\Phi_{1/a}^{-1}}{dv}\right|^2 dv}{\int_{\frac{H}{2}}^{H}\frac{d\Phi_{1/a}^{-1}}{dv}dv} + \frac{1}{\lambda}\Phi_{1/a}^{-1}\left(\frac{H}{2}\right)\frac{\int_{\frac{H}{2\lambda}}^{\frac{H}{\lambda}}\left|\frac{d\Phi_a^{-1}}{dv}\right|^2 dv}{\int_{\frac{H}{2\lambda}}^{\frac{H}{\lambda}}\frac{d\Phi_a^{-1}}{dv}dv}\\ &=: \Phi_a^{-1}\left(\frac{H}{2\lambda}\right)I_1 + \frac{1}{\lambda}\Phi_{1/a}^{-1}\left(\frac{H}{2}\right)I_2.\end{aligned}$$

We note that the functions $\frac{d\Phi_a^{-1}}{dv}$ are decreasing and verify the second equality in (2.17) and (2.19). Therefore they are uniformly bounded on closed intervals excluding 0. Hence, by the Lebesgue dominated convergence theorem, we can see that $\lim_{a\to+\infty} I_2 = 1$. To evaluate $\lim_{a\to+\infty} I_1$ we consider several cases for H. If $H < 2$ then, using (2.19), we obtain $\lim_{a\to+\infty} I_1 = 1$. If $H \geq 2$, taking account of Hölder inequality and the monotonicity of $\frac{d\Phi_{1/a}^{-1}}{dv}$, we get

$$I_1 \leq \left.\frac{d\Phi_{1/a}^{-1}}{dv}\right|_{v=\frac{H}{2}}.$$

Therefore, by elementary calculations we have

$$\limsup_{a\to+\infty} I_1 \leq 1 \quad \text{if } H = 2 \quad \text{and } \lim_{a\to+\infty} I_1 = 0 \quad \text{if } H > 2.$$

Then, from (2.17) and (2.18)

$$\lim_{a\to+\infty} B_{\Phi_a}(H,\lambda) = \left(1 + \frac{H}{2\lambda}\right)\lim_{a\to+\infty} I_1 + \frac{1}{\lambda}\min\left\{1, \frac{H}{2}\right\} = \begin{cases} 1 + \frac{H}{\lambda} & \text{if } H < 2 \\ \frac{1}{\lambda} & \text{if } H > 2 \end{cases}$$

and

$$\limsup_{a\to+\infty} B_{\Phi_a}(2,\lambda) \leq 1 + \frac{2}{\lambda}.$$

Finally, using that B_Φ satifies the anolougous duality formula that T_Φ, we have that $B_{\Phi_a}(H,\lambda) = \frac{1}{\lambda} B_{\Phi_{1/a}}(\frac{H}{\lambda}, \frac{1}{\lambda})$ and cosequently we conclude

$$\begin{aligned}\sup_{H>0,\Phi\in\mathcal{F}} B_\Phi(H,\lambda) &\geq \sup_{H>0}\max\left\{\lim_{a\to0^+} B_{\Phi_a}(H,\lambda), \lim_{a\to+\infty} B_{\Phi_a}(H,\lambda)\right\} \\ &= \max\left\{\frac{\lambda+2}{\lambda}, \frac{2\lambda+1}{\lambda}\right\}.\end{aligned}$$

□

For the functions $\Phi(x) = |x|^p/p$ with $p > 1$, the upper bound obtained in Proposition 1.5 can be improved.

Proposition 3.3. *For any $\lambda > 0$ we have that*

$$\sup_{p>1} T_p(\lambda) \leq \frac{\lambda+1}{\lambda}. \tag{3.2}$$

Proof. From the formula (1.7) for the period T_p, proving (3.2) is equivalent to show that for all $p > 1$

$$(p-1)\left(\frac{\pi}{p\sin\left(\frac{\pi}{p}\right)}\right)^p \leq \lambda\left(1 + \frac{1}{\lambda}\right)^p. \tag{3.3}$$

As the function in the right hand side of (3.3) attains its minimum at $(p-1)$, inequality (3.3) is implied by

$$1 - \frac{x}{\pi} \leq \frac{\sin x}{x}, \tag{3.4}$$

where $x = \pi/p$. Now, since (3.4) is a well known inequality (see inequality (29) on page 47 of [16]), the proof is complete. □

In view of the previous result, we can hypothesize that the inequality $T_\Phi(H,\lambda) \le \frac{\lambda+1}{\lambda}$ holds for every $\Phi \in \mathcal{F}$ and $H > 0$. We could not prove this inequality, however we performed various numerical experiments that support our hypothesis, generating functions randomly in the class $\mathcal{F}$ and computing the period numerically by means of a recursive adaptive Simpson quadrature. The result is consistent with the hypothesis. Nevertheless, likewise in the case of the bounds of A_Φ and B_Φ, the functions Φ_a seem to be approximately extremals, at least among those we have checked. In the Figure 2, we show the hypothetical bound and the graph of $\sup_{H>0} T_{\Phi_a}(H,\lambda)$ for several values of a. In order to compute the supreme, we consider energies $H > 0$ in an equally spaced grid with extremals 0.1 and 11.

We point out that the function U satisfies the inequalities in Proposition 1.5 and $U(\lambda) = \frac{1}{\lambda}U(\frac{1}{\lambda})$. If we suppose that U is meromorphic with an unique pole in 0 then the only option would be that $U(\lambda) = \kappa\frac{1+\lambda}{\lambda}$ with $1 \le \kappa \le 1.5$.

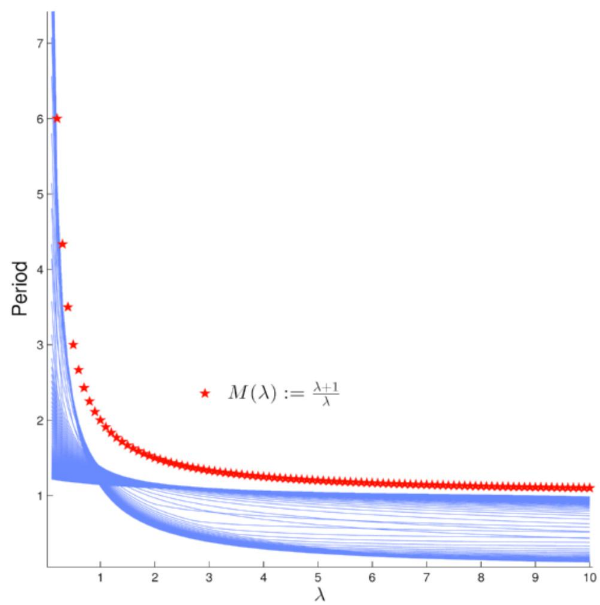

Figure 2. $\sup_{H>0} T_{\Phi_a}(H,\lambda)$ for $a = \frac{1}{200}, \frac{1}{198}, \ldots, 1, 3, \ldots, 70$.

Finally, we would like to pose an interesting open question. In addition to the power functions, is there another function ϕ for which the equation (1.1) is isochronous? We conjecture that the answer is no.

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