Equivalence Between Representations for Samplable Stochastic Processes and its Relationship With Riesz Bases

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***Abstract*—We characterize random signals which can be linearly determined by their samples. This problem is related to the question of the representation of random variables by means of a countable Riesz basis. We study different representations for processes which are linearly determined by a countable Riesz basis. This concerns the representation of continuous time processes by means of discrete samples.**

***Index Terms*—Finite variance random processes, KL-expansions, reproducing kernel Hilbert space, Riesz bases, sampling.**

I. Introduction

Motivated by the Kramer sampling theorem [1] for $L^2(I)$ signals, we study similar conditions for random signals. The random signals or processes considered here are of finite variance but not necessarily stationary. On the other hand, as in Kramer's original result, the samples do not need to be considered uniformly taken. Kramer's result is strongly related to orthonormal bases, but as noted in [2] and [3], what is really needed is a stability condition, and Riesz bases provide an appropriate framework for this. Recalling the definition of a Hilbert space representation given by Parzen [5] of a finite variance and real valued stochastic process, we will study different equivalences between several representations for samplable processes. In this context, a samplable process, will mean a continuous time, or spatial process, which can be completely linearly determined by a series expansion, using a set of countable samples or measurements of the original process. This is related to the problem of reconstructing a signal from its samples. As an example, one of the most known results, related to this problem, is the Whittaker–Shannon–Kotelnikov (WSK) sampling theorem, which also has its stochastic version for wide sense stationary (w.s.s.) random processes.

Theorem 1.1 [6]: Let $\mathfrak{X} = \{X_t\}_{t\in\mathbb{R}}$ be a w.s.s. random process defined over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that its spectral measure is concentrated in a finite interval $(-B, B)$, then

$$X_t = \sum_{n\in\mathbb{Z}} \frac{\sin(Bt - \pi n)}{Bt - \pi n} X_{\frac{\pi n}{B}}, \tag{1}$$

where the convergence is in the $L^2(\Omega, \mathcal{F}, \mathbb{P})$-norm.

This result admits some generalizations for related processes. In particular, note that, (1) implies that the process is completely linearly determined by its samples, i.e., $\overline{\text{span}}\{X_k\}_{k\in\mathbb{Z}} = \overline{\text{span}}\{X_t\}_{t\in\mathbb{R}}$. Lloyd [7] gave necessary and sufficient conditions, in terms of the spectral measure, for a w.s.s. process to be completely linearly determined by its samples. This result can be extended for some nonstationary processes [8]. However, the condition, for a process, of being linearly determined by its samples is weaker than the condition of the samples forming a basis. The study of conditions for a w.s.s. process to have a basis or minimal system goes back to Kolmogorov [6], [9]. However, all these references, as in the case of the WSK theorem, deal with equidistant samples, and are mostly stated for w.s.s. processes. The stochastic version of the WSK theorem, under additional conditions, gives an orthogonal set or a Riesz basis of samples which spans the Hilbert space spanned by the whole process [10]. The representation of signals using Riesz basis has many practical applications [11], in particular, this gives a robust representation of the process under additive noise. A classical generalization of the (deterministic) WSK theorem was given by Kramer [1]. This result allows us to treat the case of nonuniform samples. In [3], a converse of this result is given, stated as conditions on the interpolating functions. Here, by means of the reproducing kernel Hilbert space [12] associated with the process, we will give an analogous to Kramer's result, and its converse, for random processes, which are the (stochastic) integral transform of an appropriate kernel function. W.S.S. processes are particular cases of this. Under additional conditions, we prove that this representation is obtained in a very similar manner to a Karhunen–Loève (KL) expansion. Related representations are studied in [13], where these results proved to be useful for encoding. Similar substitutes are useful in applications such as signal processing and simulation, where it could be necessary to convert the problem of analyzing a continuous time process to that of analyzing a random sequence, and where the classical KL result may be not applied. For example, in [14], the particular case of representing stationary Gaussian random processes by uniformly convergent (in probability) wavelet expansions is treated. To study conditions for uniform convergence is of practical importance. Here, we shall see that

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under suitable conditions on the process, which in this case may be not Gaussian nor stationary either, we also have uniform convergence in probability.

II. General Conditions for the Equivalence Between Representations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, if X is an integrable random variable, we denote $\mathbb{E}(X) = \int_\Omega X d\mathbb{P}$. Let $\mathfrak{T}$ be a set of indexes, which in our case of interest, is considered uncountable. In this paper, we will assume that $\mathfrak{X} = \{X_t\}_{t\in\mathfrak{T}}$ is a finite variance, real valued stochastic process with correlation function $R(t,t') = \mathbb{E}(X_t\, X_{t'})$. This positive definite function defines a reproducing kernel Hilbert space (RKHS) [5], [12], which we will denote as $H(R)$. We are interested in some Riesz basis for $H(\mathfrak{X}) = \overline{\text{span}}\mathfrak{X}$, the closed linear span of $\mathfrak{X}$ in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, and some of their properties.

Definition 1: A Riesz basis for H, a Hilbert space, is a family of vectors $\{v_n\}_n \subset H$, such that $v_n = Ue_n$, where $\{e_n\}_n$ is an orthonormal basis of H, and $U : H \longrightarrow H$ is a bounded bijective operator.

A very useful characterization of Riesz basis is the following well-known theorem.

Theorem 2.1 [11]: Let H be a Hilbert space, $\{v_n\}_n$ is a Riesz basis for $H \iff \{v_n\}_n$ is complete in H, and there exists constants $0 < A < B < \infty$, such that

$$A\sum_k |c_k|^2 \leq \left\|\sum_k c_k v_k\right\|_H^2 \leq B\sum_k |c_k|^2,$$

for all finite scalar sequences $\{c_k\}_k$.

A. Condition for the Existence of a Countable Riesz Basis of $H(\mathfrak{X})$, an Stochastic Kramer Like Theorem and its Converse

The Kramer sampling theorem [1] gives a method for obtaining orthogonal sampling formulas, for functions–signals which are in the range of an appropriate integral operator. The WSK theorem for band limited functions is a particular case of this. In the random case, we can see briefly that something similar happens if we consider processes which are the integral transform of a suitable random measure. Here, $\mathfrak{T}$ denotes a set of indexes, which we assume to be, in general, uncountable, as this is the case of interest in sampling problems.

Theorem 2.2: Let M be a random orthogonal measure over a measurable space $(U, \mathcal{A})$, and let $\mathfrak{X} = \{X_t\}_{t\in\mathfrak{T}} \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$ be an stochastic process defined by $X_t = \int_U k(t,s)dM(s)$, where $k(t,s) = \sum_{n\in\mathbb{Z}} S_n(t)f_n(s)$; $\{f_n\}_n$ is a Riesz basis of $L^2(U, \mathcal{A}, \mu)$, with the control measure $\mu(\,.\,) = \mathbb{E}|M(\,.\,)|^2$, $(S_n(t))_n \in l^2$ and $\{k(t,\,.\,)\}_{t\in\mathfrak{T}}$ is complete in $L^2(\mu)$. Then,

i) $S_n(t)$ belongs to the RKHS generated by $R(t,t') = \mathbb{E}(X_tX_{t'})$ and if $(\alpha_n)_n \in l^2$ is such that $\sum_{n\in\mathbb{Z}} \alpha_n S_n(t) = 0\ \forall\ t$, then $\alpha_n = 0\ \forall\, n$.

ii) There exists $\{Z_n\}_n$, a Riesz basis of $H(\mathfrak{X})$ such that $X_t = \sum_{n\in\mathbb{Z}} S_n(t)Z_n$.

Proof: If $\{f_n\}_n$ is a Riesz basis, then there exists a biorthogonal basis $\{f'_n\}_n$. Recalling that since $\{k(t,\,.\,)\}_{t\in\mathfrak{T}}$ is complete in $L^2(\mu)$ we have an isometry between $H(\mathfrak{X})$ and $L^2(\mu)$. Thus, we define $Z'_n = \int_U f'_n dM$ and $Z_n = \int_U f_n dM$. Then, $\{Z_n\}_n$ is a Riesz basis in $H(\mathfrak{X})$, with a dual basis given by $\{Z'_n\}_n$. From this, $S_n(t) = \mathbb{E}(X_tZ'_n) = \langle k(t,\,.\,), f'_n\rangle_{L^2(\mu)}$. We have

$$\mathbb{E}\left|X_t - \sum_{|n|\leq N} S_n(t)Z_n\right|^2 = \left\|k(t,\,.\,) - \sum_{|n|\leq N} S_n(t)f_n\right\|_{L^2(\mu)}^2 \underset{N\to\infty}{\longrightarrow} 0\,.$$

Finally, suppose that $(\alpha_n)_n \in l^2$ is such that $\sum_{n\in\mathbb{Z}} \alpha_n S_n(t) = 0$ for all t. This is equivalent to $\mathbb{E}\left((\sum_{n\in\mathbb{Z}} \alpha_n Z'_n)X_t\right) = 0$ for all t, then $\sum_{n\in\mathbb{Z}} \alpha_n Z'_n = 0$ and thus $\alpha_n = 0$ for all n, since Z'_n is a Riesz basis. ■

Note that in Theorem 2.2, $\{k(t,\,.\,)\}_{t\in\mathfrak{T}}$ and $L^2(\mu)$ are both *representations* of $\mathfrak{X}$, in the sense given by Parzen in [5].

Definition 2: A Hilbert space H is a representation of a random process $\mathfrak{X} = \{X_t\}_{t\in\mathfrak{T}}$ if H is congruent to $H(\mathfrak{X})$ (i.e., there exists an isomorphism which preserves inner products).

A related notion is the following.

Definition 3: A family of vectors $\{v_t\}_{t\in\mathfrak{T}}$ in a Hilbert space H is a representation of a random process $\mathfrak{X} = \{X_t\}_{t\in\mathfrak{T}}$ if for every $s, t \in \mathfrak{T}$: $\langle v_t, v_s\rangle_H = \mathbb{E}(X_tX_s)$.

It is immediate that $H(R)$ is a representation of $H(\mathfrak{X})$ [5]. The importance of these is that some (mean square) problems may be more easy to solve in another metrically isomorphic Hilbert space. As in [3] it is possible to give a converse of theorem 2.2 (see Theorem 2.3). Giving appropriate conditions on the sampling functions, it is possible to obtain a Riesz basis of the whole space $H(\mathfrak{X})$. In particular, the random process is linearly determined by its samples. In contrast to Garcia's result [3], the hypothesis on the signal, in this random case, of being the image of an integral transform can be dropped. So, in principle, one may conjecture that there exists a larger class of processes with this property. However, it is rather easy to see (see Theorem 2.5) that if there exists a Riesz basis of $H(\mathfrak{X})$ then the process is the integral transform of an appropriate kernel with respect to a random measure.

Theorem 2.3: Let $\mathfrak{T}$ be a set of indexes, generally noncountable, and let $\mathfrak{X} = \{X_t\}_{t\in\mathfrak{T}} \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$ be a stochastic process. Let $H(\mathfrak{X})$ be the closed subspace spanned by $\mathfrak{X}$. Given sequences $\{Z_n\}_{n\in\mathbb{Z}} \subset H(\mathfrak{X})$ and $\{S_n\}_{n\in\mathbb{Z}} \subset H(R)$ such that $X_t = \sum_{n\in\mathbb{Z}} S_n(t)Z_n$, where convergence is in the L^2 norm. The following assertions are equivalent:

i) $\{Z_n\}_{n\in\mathbb{Z}}$ and $\{S_n\}_{n\in\mathbb{Z}}$ verifies

a.1) $(\langle S_n, f\rangle_{H(R)})_n \in l^2$ for all $f \in H(R)$, and if $(a_n)_n \in l^2$ verifies $\sum_{n\in\mathbb{Z}} a_nS_n(t) = 0$ for all $t \in \mathfrak{T}$, then $a_n = 0$, for all $n \in \mathbb{Z}$.

a.2) $(\mathbb{E}(YZ_n))_n \in l^2$ for all $Y \in H(\mathfrak{X})$.

ii) $\{Z_n\}_{n\in\mathbb{Z}}$ and $\{S_n\}_{n\in\mathbb{Z}}$ are Riesz basis of $H(\mathfrak{X})$ and $H(R)$, respectively.

Proof:

$((ii) \implies (i))$*:* If $\{Z_n\}_n$ is a Riesz basis of $H(\mathfrak{X})$, then there exists a biorthogonal basis $\{Z'_n\}_{n\in\mathbb{Z}}$, such that $S_n(t) = \mathbb{E}(Z'_n X_t)$ and $X_t = \sum_{n\in\mathbb{Z}} S_n(t) Z_n$. These $S_n(t)$ are unique, since Z_n is a Riesz basis. It is immediate that $(\mathbb{E}(Z_n Y))_n \in l^2$, for all $Y \in H(\mathfrak{X})$ and $S_n \in H(R)$ from the definition. On the other hand, recalling the theory of reproducing kernels, for every $f \in H(R)$, there exists $Y \in H(\mathfrak{X})$ such that $f(t) = \mathbb{E}(X_t Y)$. This can be written as $f(t) = \tilde{J}(Y)(t)$, where

$$\underset{Y\longmapsto \tilde{J}(Y)=\mathbb{E}(X_t Y)}{\tilde{J} : H(\mathfrak{X}) \longrightarrow H(R),} \tag{2}$$

and where the reproducing kernel Hilbert space $H(R) = Ran(\tilde{J})$, is equipped with the norm [12]: $\|v\|_{H(R)} = \inf\left\{\|w\|_{H(\mathfrak{X})} : \tilde{J}(w) = v\right\}$. In this way, an isometry is defined and thus $\langle S_n, f\rangle_{H(R)} = \mathbb{E}(Z'_n Y)$, but $(\mathbb{E}(Z'_n Y))_n \in l^2$ because $\{Z'_n\}_n$ is also a Riesz basis. In particular, for all $t \in \mathfrak{T} : \sum_{n\in\mathbb{Z}} |S_n(t)|^2 < \infty$. Finally, let $(\alpha_n)_n \in l^2$ be such that $\sum_{n\in\mathbb{Z}} \alpha_n S_n(t) = 0 \; \forall\, t \in \mathfrak{T}$. Hence,

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \alpha_n S_n(t) &= \sum_{n\in\mathbb{Z}} \alpha_n \mathbb{E}(Z'_n X_t)\\ &= \mathbb{E}\left(X_t(\sum_{n\in\mathbb{Z}} \alpha_n Z'_n)\right) = 0 \quad \forall t \in \mathfrak{T}.\end{aligned}$$

Then, $\sum_{n\in\mathbb{Z}} \alpha_n Z'_n \in H(\mathfrak{X})^{\perp} = \{0\}$, thus $\sum_{n\in\mathbb{Z}} \alpha_n Z'_n = 0$ but $\{Z'_n\}_n$ is a basis, then $\alpha_n = 0$, for all n.

$((i) \implies (ii))$*:* We shall see that $\{Z_n\}_n$ of (i) is, indeed, a Riesz basis.

Step I: Let T_k and T be defined as

$$\underset{Y\longmapsto \left(\mathbb{E}(Z'_n Y)\mathbf{1}_{[-k,k]}(n)\right)_n}{T_k : H(\mathfrak{X}) \longrightarrow l^2} \qquad \text{and} \qquad \underset{Y\longmapsto (\mathbb{E}(Z'_n Y))_n}{T : H(\mathfrak{X}) \longrightarrow l^2,}$$

where the Z'_ns are such that $\mathbb{E}(Z'_n X_t) = S_n(t)$, thus,

$$\|T_k(Y) - T(Y)\|^2_{l^2} = \sum_{|n|>k} |\mathbb{E}(Z'_n Y)|^2 \underset{k\longrightarrow\infty}{\longrightarrow} 0,$$

and then pointwise convergence follows from this. On the other hand,

$$\|T_k(Y)\|^2_{l^2} \le \left(\sum_{|n|\le k} \mathbb{E}|Z'_n|^2\right) \mathbb{E}|Y|^2,$$

then, by the Banach–Steinhaus theorem, T is a bounded operator, so there exists B such that

$$\sum_{n\in\mathbb{Z}} |\mathbb{E}(Z'_n Y)|^2 \le B\mathbb{E}|Y|^2 .$$

Then, by Lemma 3.1.6 of [11], we have that $\{Z'_n\}_n$ is Besselian

$$\left\|\sum_{n\in\mathbb{Z}} a_n Z'_n\right\|^2_{H(\mathfrak{X})} \le B \sum_{n\in\mathbb{Z}} |a_n|^2 . \tag{3}$$

For all $(a_n)_n \in l^2$.

Step II: Let $H(S)$ be the reproducing kernel Hilbert space induced by the linear operator

$$\underset{(\alpha_k)_k\longmapsto J(\alpha)=\sum_k \alpha_k S_k(t),}{J : l^2 \longrightarrow H(S)}$$

note that J is well defined, and $H(S)$ is the range of J: $Ran(J)$ equipped with the norm $\|v\|_{H(S)} = \inf\{\|w\|_{l^2} : J(w) = v\}$.

Let us see that $H(S) = H(R)$, in the sense of set inclusions. Let $v \in H(S)$, then $v(t) = \sum_{n\in\mathbb{Z}} \alpha_n S_n(t)$, taking in account that $S_n(t) = \mathbb{E}(Z'_n X_t) = \tilde{J}(Z'_n)(t)$ and that $\sum_{n\in\mathbb{Z}} \alpha_n Z'_n$ converges by (Step I) (3), then by (2) $v(t) = \tilde{J}\left(\sum_{n\in\mathbb{Z}} \alpha_n Z'_n\right)(t)$, and thus $v \in Ran(\tilde{J}) = H(R)$. On the other hand, if $v \in H(R)$, then $v = \tilde{J}(Y)$, for some $Y \in H(\mathfrak{X})$, but if $X_t = \sum_{n\in\mathbb{Z}} Z_n S_n(t)$ and recalling again (2): $v(t) = \sum_{n\in\mathbb{Z}} \mathbb{E}(Z_n Y) S_n(t) = J(\beta)(t)$, with $\beta_n = \mathbb{E}(Z_n Y)$, $\beta \in l^2$, then $v \in Ran(J) = H(S)$.

Step III: Now, let us see that their norms are equivalent. In fact, consider the inclusion map $H(R) \overset{i}{\hookrightarrow} H(S)$ and $v_n = i(v_n) \underset{n\to\infty}{\longrightarrow} w$ in the $H(S)$-norm and such that $v_n \underset{n\to\infty}{\longrightarrow} 0$ in the $H(R)$-norm. But as $H(R)$ and $H(S)$ are both RKHS, then convergence in norm implies pointwise convergence for each $t \in \mathfrak{T}$, so we have that $v_n(t) = i(v_n)(t) \underset{n\to\infty}{\longrightarrow} w(t) = 0$ for all $t \in \mathfrak{T}$, thus $w = 0$, and then by the closed graph theorem i is continuous, but i is also a bijective map, then there exist constants $0 < A \le B < \infty$ such that

$$A\,\|v\|_{H(R)} \le \|i(v)\|_{H(S)} \le B\,\|v\|_{H(R)} ,$$

but we have seen that $v = J(\beta)$, with $\beta_n = \mathbb{E}(Z_n Y)$, and since these coefficients are unique (condition a.1) $\|v\|_{H(S)} = \sum_{n\in\mathbb{Z}} |\mathbb{E}(Z_n Y)|^2$. On the other hand, $v = \tilde{J}(Y)$, thus

$$A\,\|Y\|_{H(\mathfrak{X})} \le \left(\sum_{n\in\mathbb{Z}} |\mathbb{E}(Z_n Y)|^2\right)^{\frac{1}{2}} \le B\,\|Y\|_{H(\mathfrak{X})} ,$$

and then $\{Z_n\}_n$ is a frame [11].

Step IV: Now, we shall see that $\{Z_n\}_n$ is a Riesz basis of $H(\mathfrak{X})$. Indeed $\{Z_n, Z'_n\}_n$ is a biorthogonal system [11], with Z'_n as in (Step I), indeed $X_t = \sum_{n\in\mathbb{Z}} Z_n S_n(t)$ and then $S_m(t) = \sum_{n\in\mathbb{Z}} \mathbb{E}(Z'_m Z_n) S_n(t)$, thus $\mathbb{E}(Z'_m Z_n) = \delta_{n\,m}$, moreover, $\{Z'_n\}_n$ is also a Riesz basis of $H(\mathfrak{X})$ and then the same holds for $S_n = \tilde{J}(Z'_n)$ in $H(R)$. Finally, let us see that $\overline{\text{span}}\{Z_k\}_k = H(\mathfrak{X})$. For this, take $Y \in H(\mathfrak{X})$ such that $\mathbb{E}(Z_k Y) = 0 \; \forall\, k \in \mathbb{Z}$, but this implies $\mathbb{E}(Y X_t) = 0$ for all $t \in \mathfrak{T}$, and then $Y = 0$ a.s. ■

The fact that every samplable process admits a representation as an stochastic integral of a certain type of kernel, in this case, is a consequence of the fact that $H(\mathfrak{X})$ is separable and that all separable Hilbert spaces are isometrically isomorphic. We will give a complete proof of this, in order to make the development of the work self-contained. First, we need the following theorem.

Theorem 2.4 [15, p. 242]: Let the covariance function of a random process $\mathfrak{X} = \{X_t\}_{t\in\mathfrak{T}}$ admit the following representation: $R(t,t') = \int_U k(t,\,.)\overline{k(t',\,.)}d\mu$, where μ is a positive measure over $(U,\mathcal{A})$, and $\{k(t,\,.)\}_{t\in\mathfrak{T}} \subset L^2(\mu)$ is complete. Then, $\mathfrak{X} = \{X_t\}_{t\in\mathfrak{T}}$ admits the following representation: $X_t = \int_U k(t,\,.)dM \;\; a.s.$, where $\{M(A),\;\; A \in \mathcal{A}\} \subset H(\mathfrak{X})$ is an orthogonal random measure, such that $\mathbb{E}|M(\,.\,)|^2 = \mu(\,.\,)$.

With this result, and assuming that $H(\mathfrak{X})$is in the following infinite dimensional, we can prove:

Theorem 2.5: If $\{Z_n\}_n$ is a Riesz basis of $H(\mathfrak{X})$, given $(U,\mathcal{A},\mu)$ a measure space, such that $L^2(\mu)$ is separable and infinite dimensional, then:

There exists an orthogonal random measure M over $(U,\mathcal{A})$, with control measure μ, i.e., $\mu(\,.\,) = \mathbb{E}|M(\,.\,)|^2$, such that there exists a Riesz basis $\{f_n\}_n$ of $L^2(\mu)$, and $X_t = \int_U k(t,s)dM(s)$, with $k(t,s) = \sum_{n\in\mathbb{Z}} S_n(t)f_n(s)$ and $\{S_n\}_n \subset H(R)$ as in a.1) of theorem 2.3, and $\{k(t,.)\}_{t\in\mathfrak{T}}$ a complete system.

Proof: If $\{Z_n\}_{n\in\mathbb{Z}}$ is a Riesz basis of $H(\mathfrak{X})$, then $H(\mathfrak{X})$ is separable as it is $L^2(\mu)$, so there exists an isometric isomorphism $J : H(\mathfrak{X}) \longrightarrow L^2(\mu)$, and taking $f_n = J(Z_n)$, then $\{f_n\}_{n\in\mathbb{Z}}$ is a Riesz basis of $L^2(\mu)$. So, we take,

$$k(t,\,.\,) = \sum_{n\in\mathbb{Z}} \mathbb{E}(X_t Z'_n)f_n = \sum_{n\in\mathbb{Z}} S_n(t)f_n\,.$$

The coefficients are the same, unique, $S_n(t)$s of the previous result, so (a.1) of Theorem 2.3 holds. On the other hand, $X_t = \sum_{n\in\mathbb{Z}} \mathbb{E}(X_t Z'_n)Z_n$, thus $J(X_t) = k(t,\,.\,)$ and then

$$\mathbb{E}(X_t X_{t'}) = \int_U J(X_t)J(X_{t'})d\mu = \int_U k(t,\,.\,)k(t',\,.\,)d\mu,$$

and $\{k(t,\,.\,)\}_{t\in\mathfrak{T}}$ is complete, indeed, take $h \in L^2(\mu)$ such that $\langle k(t,\,.\,),h\rangle_{L^2(\mu)} = 0$, for all $t \in \mathfrak{T}$, and since $\langle f_n,h\rangle_{L^2(\mu)} = \mathbb{E}(Z_n, J^{-1}(h))$ and from the biorthogonality of $\{Z_n\}_{n\in\mathbb{Z}}$ and $\{Z'_n\}_{n\in\mathbb{Z}}$, we have

$$\begin{aligned} 0 &= \sum_{n\in\mathbb{Z}} \mathbb{E}(X_t Z'_n)\langle f_n,h\rangle_{L^2(\mu)} \\ &= \sum_{n\in\mathbb{Z}} \mathbb{E}(X_t Z'_n)\mathbb{E}(Z_n J^{-1}(h)) = \mathbb{E}(X_t J^{-1}(h))\,. \end{aligned}$$

As this holds, for every $t \in \mathfrak{T}$, then $J^{-1}(h) = 0$ a.s. and thus $h = 0$ a.e. $[\mu]$. Finally, from the representation Theorem 2.4, it follows that there exists a random measure M, such that $X_t = \int_U k(t,\,.\,)dM$, and $\mathbb{E}|M(\,.\,)|^2 = \mu(\,.\,)$. ■

Remark: Alternatively, one may construct the random measure M, and the Riesz basis $\{f_n\}_{n\in\mathbb{Z}}$ in the following way: as $H(\mathfrak{X})$ and $L^2(\mu)$ are both separable, take any pair of orthonormal basis $\{Y_n\}_{n\in\mathbb{Z}}$ and $\{g_n\}_{n\in\mathbb{Z}}$ of $H(\mathfrak{X})$ and $L^2(\mu)$, respectively, and define over the algebra of $\mathcal{A}$-measurable subsets with finite μ measure

$$M(A) = \sum_{n\in\mathbb{Z}} \langle \mathbf{1}_A, g_n\rangle_{L^2(\mu)} Y_n, \text{and } f_n = \sum_{m\in\mathbb{Z}} \mathbb{E}(Z_n Y_m)g_m\,.$$

Then, one can verify that if $A \bigcap B = \emptyset$ then $M(A \bigcup B) = M(A) + M(B)$, and

$$\mathbb{E}|M(A)|^2 = \sum_{n\in\mathbb{Z}} |\langle g_n, \mathbf{1}_A\rangle_{L^2(\mu)}|^2 = \int_U |\mathbf{1}_A|^2 d\mu = \mu(A).$$

From this, the measure M extends as usual. Now, given M, the stochastic integral $\int_U f dM$, for $f \in L^2(\mu)$ is constructed in the standard way, defining an isometry.

B. Application: A Sampling Theorem

The following corollary shows how the previous results may be applied to the problem of characterizing process which are linearly determined by its samples, and which also form a Riesz basis.

Corollary 2.1: Let $\{X_t\}_{t\in\mathfrak{T}}$ be a finite variance stochastic process. For sequences $\{X_{t_n}\}_{n\in\mathbb{Z}} \subset H(\mathfrak{X})$, $\{S_n\}_{n\in\mathbb{Z}} \subset H(R)$, $\{t_n\}_{n\in\mathbb{Z}} \subset \mathfrak{T}$, such that $X_t = \sum_{n\in\mathbb{Z}} X_{t_n}S_n(t)$, the following are equivalent:

i)
   (b.1.) $\sum_{n\in\mathbb{Z}} |S_n(t)|^2 < \infty \;\forall\; t \in \mathfrak{T}$, $S_n(t_r) = \delta_{n\,r}$ and $(\langle S_n, f\rangle_{H(R)})_n \in l^2$ for all $f \in H(R)$.
   (b.2.) $(\mathbb{E}(YX_{t_n}))_n \in l^2 \;\forall Y \in H(\mathfrak{X})$.

ii) The sequences $\{X_{t_n}\}_n$ and $\{S_n\}_n$ are Riesz basis of $H(\mathfrak{X})$ and $H(R)$, respectively.

iii) Given $(U,\mathcal{A},\mu)$ a measure space, such that $L^2(\mu)$ is separable and infinite dimensional, there exists M an orthogonal random measure over $(U,\mathcal{A})$, and $k(t,\,.\,) \in L^2(\mu)$ such that $\{k(t_n,\,.\,)\}_{n\in\mathbb{Z}}$ is a Riesz basis of $L^2(\mu)$, with $\mathbb{E}|M(\,.\,)|^2 = \mu(\,.\,)$, $\{S_n\}_n$ is a Riesz basis of $H(R)$ such that $k(t,\,.\,) = \sum_{n\in\mathbb{Z}} k(t_n,\,.\,)S_n(t)$ and $X_t = \int_U k(t,\,.\,)dM$.

C. Examples

1) The WSK Theorem With a Riesz Basis: Let $\mathfrak{X} = \{X_t\}_{t\in\mathbb{R}}$, be defined by

$$X_t = \int_{\mathbb{R}} e^{it\lambda}dM(\lambda),$$

where $M(\,.\,)$ is an orthogonal random measure, such that $\mu(A) = \mathbb{E}|M(A)|^2 = \int_A \phi(\lambda)d\lambda$.[1] For some nonnegative $\phi \in L^1(\mathbb{R})$, is a standard result such that M, exists [15]. Moreover we can take ϕ, such that $A \le \phi \le B$, a.e. on $[-\pi,\pi]$, for some $A, B > 0$, and $\phi = 0$ a.e. on $[-\pi,\pi]^c$. The resulting process is w.s.s. and it is easy to verify that $\{\frac{e^{in\lambda}}{\sqrt{2\pi}}\}_{n\in\mathbb{Z}}$ is a Riesz basis of $L^2(\mathbb{R},d\mu)$ and that, $\{X_n\}_{n\in\mathbb{Z}}$ is also a Riesz basis of $H(\mathfrak{X})$. Moreover, the dual basis is given by $\{\frac{e^{in\lambda}(\phi(\lambda))^{-1}}{\sqrt{2\pi}}\}_{n\in\mathbb{Z}}$, and then $S_n(t) = \mathbb{E}(X_t Z'_n) = \frac{sin(\pi(t-n))}{\pi(t-n)}$, and obviously, $k(t,\lambda) = e^{it\lambda}\mathbf{1}_{[-\pi,\pi]}(\lambda)$.

2) Bessel–Hankel Transforms: Consider $\mathfrak{X} = \{X_t\}_{t\ge 0}$, a process defined in the following way: take $\{W_t\}_{t\in[0,1]}$ a Wiener process, and the orthogonal basis $\{\sqrt{x}J_r(x\lambda_n)\}_{n\in\mathbb{N}} \subset$

[1]In this case, the condition on μ of being absolutely continuous with respect to the Lebesgue measure, is necessary, as it was proved in [10].

$L^2[0,1]$, where λ_n is the nth positive zero of the Bessel function J_r, $r > -1$. Thus, if

$$X_t = \int_{[0,1]} \sqrt{xt} J_r(xt) dW(x),$$

then

$$X_t = \sum_{n\in\mathbb{N}} X_{\lambda_n} \frac{2\sqrt{t\lambda_n} J_r(t)}{J_r'(\lambda_n)(t^2-\lambda_n^2)},$$

where convergence is in the m.s. sense, this follows from the $L^2[0,1]$ convergent, classic formula $\sqrt{xt}J_r(xt) = \sum_{n\in\mathbb{N}} \sqrt{x}J_r(x\lambda_n) \frac{2\sqrt{t\lambda_n}J_r(t)}{J_r'(\lambda_n)(t^2-\lambda_n^2)}$.

III. Karhunen–Loève-Type Expansions

In the previous section, we have seen that the random variables Z_n and the functions S_n are related. Indeed if $\{Z_n'\}_n$ is the associated dual basis we know that $S_n(t) = \mathbb{E}(Z_n' X_t)$. We would like to give some additional condition under which the Z_ns are obtained in a similar fashion to "random Fourier coefficients" as in the original KL expansion. This point of view may be of more practical use in some applications. However, in this case, we shall confine to the case when $\mathfrak{T}$ is an open subset of $\mathbb{R}^d$. First we begin with a result.

Proposition 3.1: Let $\mathfrak{T} \subset \mathbb{R}^d$ be an open subset, let $\{X_t\}_{t\in\mathfrak{T}}$ be a m.s. continuous random process and let ν be a finite Borel measure, such that ν is equivalent to the Lebesgue measure ($\nu \equiv \lambda$) and $\mathbb{E}|X_t|^2 = R(t,t) \in L^1(\nu)$. If there exists $\{Z_n\}_{n\in\mathbb{Z}}$ a Riesz basis of $H(\mathfrak{X})$ and if $\{S_n\}_{n\in\mathbb{Z}}$ is as in the previous result, then $\{S_n\}_n$ is a Bessel sequence (with respect to $\|\,.\,\|_{L^2(\nu)}$), and $T : l^2 \longrightarrow L^2(\nu)$, $(c_n)_n \longmapsto \sum_{n\in\mathbb{Z}} c_n S_n$ defines a linear, bounded, and injective operator.

Remark: Note that under these conditions there exists an stochastically equivalent measurable version of $\{X_t\}_{t\in\mathfrak{T}}$ [15], i.e., a version which is $\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{F}$- measurable, as a function of $(t,\omega) \in \mathfrak{T}\times\Omega$. Indeed, we shall work with that measurable version.

Proof:

$$\left\|\sum_{k=1}^N c_k S_k\right\|^2_{L^2(\nu)} = \int_{\mathfrak{T}} \left|\sum_{k=1}^N c_k S_k(t)\right|^2 d\nu(t) = \int_{\mathfrak{T}} \left|\sum_{k=1}^N c_k \mathbb{E}(Z_n' X_t)\right|^2 d\nu(t).$$

This integral can be bounded, using the Cauchy–Schwartz inequality and the Bessel condition on the Z_ns:

$$\int_{\mathfrak{T}} \left|\mathbb{E}\left(X_t\left(\sum_{k=1}^N c_k Z_k'\right)\right)\right|^2 d\nu(t) \le \int_{\mathfrak{T}} R(t,t)\mathbb{E}\left|\sum_{k=1}^N c_k Z_k'\right|^2 d\nu(t) \le B\left(\int_{\mathfrak{T}} R(t,t)d\nu(t)\right)\sum_{k=1}^N |c_k|^2.$$

Now, if K is a compact subset, $(c_n)_n \in l^2$, and since X_t is m.s. continuous then $R(t,t)$ is continuous and bounded. In a similar manner as in the previous bound, by the Cauchy–Schwarz inequality, we have

$$\left|\sum_{k=M}^N c_k S_k(t)\right| \le \sup_{t\in K} (R(t,t))^{\frac12}\left(\sum_{k=N}^M |c_k|^2\right)^{\frac12} \xrightarrow[N,M\to\infty]{} 0.$$

Then, $\sum_{k\in\mathbb{Z}} c_k S_k$ converges uniformly over compact sets. On the other hand, $|S_n(t) - S_n(s)|^2 \le \mathbb{E}|Z_n'|^2\mathbb{E}|X_t - X_s|^2$, so the S_n are continuous over K, and the same holds for $\sum_{k\in\mathbb{Z}} c_k S_k$ from the uniform convergence. Now, if $(c_n)_n$ is such that $\left\|\sum_k c_k S_k\right\|_{L^2(\nu)} = 0$, thus $\sum_{|k|\le N} c_k S_k \xrightarrow[N\to\infty]{\nu} 0$, and then we have $N_1 < \cdots < N_{k-1} < N_k \nearrow \infty$, a subsequence, such that $\sum_{|k|\le N_r} c_k S_k \xrightarrow[r\to\infty]{} 0$ a.e. $[\nu]$, and since $\nu \equiv \lambda$, then $\sum_{k\in\mathbb{Z}} c_k S_k(t) = 0$, for all t. But the condition (a.1) from Theorem 2.2, implies $c_n = 0$, for all n. Thus,

$$T : l^2 \longrightarrow L^2(\nu), \quad (c_n)_n \longmapsto \sum_{n\in\mathbb{Z}} c_n S_n$$

is a well defined, linear, and bounded operator, which is injective. ■

Now, if $(\mathfrak{T},\mathfrak{F},\nu)$ is a measure space, given the process X_t, and if $\{S_n\}_n$ is a basic sequence, we can find the random variables Z_n, calculating an integral over $\mathfrak{T}$.

Theorem 3.1: Let $\mathfrak{X} = \{X_t\}_{t\in\mathfrak{T}}$ be measurable (i.e., $X(t,\omega)$ is $\mathfrak{F}\otimes\mathcal{F}$-measurable), $\{Z_n\}_n$ a Riesz basis of $H(\mathfrak{X})$, and let ν be a σ-finite measure, such that $\mathbb{E}|X_t|^2 = R(t,t) \in L^1(\nu)$. If $\{S_n\}_{n\in\mathbb{Z}}$ is a basis of $A = \overline{\text{span}}\{S_n\}_n \subset L^2(\nu)$, such that $X_t = \sum_{n\in\mathbb{Z}} Z_n S_n(t)$, where convergence in the m.s. sense, then

i)
$$Tf(\omega) = \int_{\mathfrak{T}} X(t,\omega) f(t) d\nu(t)\,, \tag{4}$$
defines a bounded linear operator from $L^2(\nu)$ to $H(\mathfrak{X})$, and $T(S_n') = Z_n$, where $\{S_n'\}_n$ is the dual basis of $\{S_n\}_n$ with respect to the $L^2(\nu)$-norm.

ii)
$$T'f(t) = \int_{\mathfrak{T}} R(t,t') f(t') d\nu(t'), \tag{5}$$
defines a bounded linear operator from $L^2(\nu)$ to $H(R)$, and $T'(S_n') = S_n^*$, where $\{S_n^*\}_n$ is the dual basis of $\{S_n\}_n$ with respect to the $H(R)$-norm.

Proof:

i) Let us prove first that $T : L^2(\nu) \longrightarrow L^2(\mathbb{P})$ is well defined and bounded. Indeed, if $X(t,\omega)$ is $\mathfrak{F}\otimes\mathcal{F}$-measurable, and as $R(t,t) \in L^1(\nu)$ then $X(t,\omega) \in L^2(\nu)$ for almost all ω $[\mathbb{P}]$. Thus, as $f \in L^2(\nu)$, the integral 4 is well defined. On the other hand, by Minkowski's and Cauchy–Schwarz's inequalities, respectively,

$$\mathbb{E}|Tf|^2 = \mathbb{E}\left|\int_{\mathfrak{X}} X(t,\,.\,)f(t)d\nu(t)\right|^2$$
$$\leq \left(\int_{\mathfrak{X}} (\mathbb{E}|X_t|^2)^{\frac{1}{2}}|f(t)|d\nu\right)^2 \leq \int_{\mathfrak{X}} R(t,t)d\nu(t)\,\|f\|^2_{L^2(\nu)}\,. \tag{6}$$

Now, let us prove that $Ran(T) \subset H(\mathfrak{X})$, indeed, for each $N \in \mathbb{N}$, define $T_N(f) = \sum_{|n|\leq N} Z_n\langle f, S_n\rangle_{L^2(\nu)}$. It is easy to verify that these T_Ns are bounded linear operators from $L^2(\nu)$ to $H(\mathfrak{X})$, and $T_N \underset{N\longrightarrow\infty}{\longrightarrow} T$ strongly, since in a similar way to (6), we can obtain

$$\mathbb{E}|T_Nf - Tf|^2$$
$$\leq \left(\int_{\mathfrak{X}} \mathbb{E}\left|X_t - \sum_{|n|\leq N} Z_nS_n(t)\right|^2 d\nu(t)\right)\|f\|^2_{L^2(\nu)}\,.$$

If $\{Z_n\}_n$ is Riesz basis, there exist $0 < A < B < \infty$ basis constants, so that

$$\mathbb{E}\left|X_t - \sum_{|n|\leq N} Z_nS_n(t)\right|^2 = \mathbb{E}\left|\sum_{|n|>N} Z_nS_n(t)\right|^2$$
$$\leq B\sum_{n\in\mathbb{Z}}|S_n(t)|^2 \leq \frac{B}{A}\mathbb{E}\left|\sum_{n\in\mathbb{Z}} Z_nS_n(t)\right|^2 = \frac{B}{A}R(t,t)\in L^1(\nu)\,.$$

But $\mathbb{E}\left|X_t - \sum_{|n|\leq N} Z_nS_n(t)\right|^2 \underset{N\longrightarrow\infty}{\longrightarrow} 0$, so by Lebesgue's theorem

$$\sup_{\|f\|_{L^2(\nu)}=1} \mathbb{E}\left|T_Nf - Tf\right|^2 \underset{N\longrightarrow\infty}{\longrightarrow} 0\,,$$

from this $Tf \in H(\mathfrak{X})$, since $H(\mathfrak{X})$ is closed. Finally, as $T_N(S'_m) = Z_m$ if $N \geq m$, then $T(S'_m) = Z_m$.

ii) If $\{X_t\}_t$ is such that $X(t,\omega)$ is $\mathfrak{F}\otimes\mathcal{F}$-measurable, then $\mathbb{E}(X_tX_{t'}) = R(t,t')$ is $\mathfrak{F}\otimes\mathfrak{F}$-measurable. On the other hand, by the Cauchy–Schwarz inequality

$$|T'f(t)| \leq \sqrt{R(t,t)}\int_{\mathfrak{X}} \sqrt{R(t',t')}|f(t')|d\nu(t')$$
$$\leq \sqrt{R(t,t)}\left(\int_{\mathfrak{X}} R(t',t')d\nu(t')\right)^{\frac{1}{2}}\|f\|_{L^2(\nu)} < \infty\,.$$

So $T'f$ is well defined and, by Fubini's theorem

$$T'f(t) = \int_{\mathfrak{X}} \mathbb{E}(X_tX_{t'})f(t')d\nu(t') = \mathbb{E}\left(X_tTf\right) = \tilde{J}(Tf)(t)$$

where T is defined as in (4), and $\tilde{J}$ as in (2). Thus, $T' = \tilde{J}\circ T$ defines a bounded operator. ■

Final Remarks: Note that from the proof of Theorem 3.1 we get that $A\sum_n|\langle S_n,f\rangle|^2 \leq \mathbb{E}|Tf|^2$, so if $Tf = 0$ a.s. then $\langle S_n, f\rangle = 0$ for all n, thus $f \in A^{\perp}$. In a similar manner, if $f \in A^{\perp}$, then $Tf = 0$ and so $Ker(T) = A^{\perp}$. In particular, if A is the whole space $L^2(\nu)$, then T is injective. A similar analysis holds for T'.

In these last results, $\{S_n\}_n$ cannot be an unconditional (Riesz) basis with respect the $L^2(\nu)$-norm, indeed, if this would be the case, as the $\{Z_n\}_n$ are already a Riesz basis, we should have that $A\sum_n|S_n(t)|^2 \leq R(t,t) \leq B\sum_n|S_n(t)|^2$, and integrating, we obtain $\sum_n\|S_n\|^2_{L^2(\nu)} < \infty$, which contradicts the fact that $\inf_n\|S_n\| > 0$.

A. About Uniform Convergence in Probability

Let us prove that if the process, in addition, verifies some regularity conditions we have that the previous expansions converge uniformly in probability over any compact interval $J \subset \mathbb{R}$. That is, if $X_{N,t} = \sum_{|n|\leq N} Z_nS_n(t)$, then for every $\epsilon > 0$:

$$\mathbb{P}\left(\sup_{t\in J}|X_{N,t} - X_t| > \epsilon\right) \longrightarrow 0$$

as $N \longrightarrow \infty$. The proof of this theorem is an standard argument, previously we recall Kolmogorov's condition.

Lemma 3.1 [15, p. 191]: Let $\{X_t\}_{t\in J}$ be a separable random process satisfying the following condition: there exists a nonnegative monotonically nondecreasing function $g(t)$ and a function $q(t,x), t\geq 0$ such that $\mathbb{P}(|X_{t_0+t} - X_{t_0}| > xg(t)) \leq q(x,t)$ and $G = \sum_{n\geq 0} g(2^{-n}T) < \infty$ and $Q(x) = \sum_{n\geq 1} 2^nq(x,2^{-n}T) < \infty$. Then,

$$\mathbb{P}\left(\sup_{0\leq t'<t''\leq T}|X_{t'} - X_{t''}| > \lambda\right) \leq Q\left(\frac{\lambda}{2G}\right)\,.$$

By a direct application of this, one can prove

Theorem 3.2: If $\{X_t\}_{t\in\mathbb{R}}$ is such that there exists constants $\alpha \in (1,2]$ and $K > 0$ such that for every $t,s \in \mathbb{R}$: $\mathbb{E}|X_t - X_s|^2 \leq K|t-s|^{\alpha}$ then for every interval $J = [0,T]$:

$$\mathbb{P}\left(\sup_{t\in J}|X_{N,t} - X_t| > \epsilon\right) \longrightarrow 0,$$

as $N \longrightarrow \infty$.

Proof: Writing $X'_{N,t} = X_t - X_{N,t}$. Suppose without loss of generality that $0 \in J$, then $|X_t - X_{N,t}| \leq |X'_{N,t} - X'_{N,0}| + |X'_{N,0}|$. From this,

$$\mathbb{P}\left(\sup_{t\in J}|X_{N,t} - X_t| > \epsilon\right) \tag{7}$$
$$\leq \mathbb{P}\left(\sup_{t\in J\setminus\{0\}}|X'_{N,t} - X'_{N,0}| > \frac{\epsilon}{2}\right) + \mathbb{P}\left(|X'_{N,0}| > \frac{\epsilon}{2}\right)\,.$$

Let us bound the first term of the sum on the right side of this inequality by means of the Kolmogorov condition. For small ϵ,

$0 < \epsilon < \alpha - 1$, and if we denote $\beta = \frac{\alpha-1-\epsilon}{2}$, $\gamma = \alpha - \frac{\epsilon}{2}$, then, by Tchevichev's inequality and the frame bounds

$$\mathbb{P}(|X'_{N,t} - X'_{N,0}| > t^{\beta}x) \leq \frac{\mathbb{E}|X'_{N,t} - X'_{N,0}|^2}{x^2 t^{2\beta}}$$
$$\leq \frac{B}{x^2 t^{2\beta}} \sum_{|n|>N} |\mathbb{E}(Z'_n(X_t - X_0))|^2 \leq B\frac{t^{\gamma-2\beta}}{x^2} T_N,$$

where $T_N = \sup_{t\in J\setminus\{0\}} \frac{1}{t^{\gamma}} \sum_{|n|>N} |\mathbb{E}(Z'_n(X_t - X_0))|^2 = \sup_{t\in J\setminus\{0\}} R_N(t)$, B is the upper frame bound for the sequence $\{Z_n\}_n$, and $\{Z'_n\}_n$ is its dual basis. We would like to show that $T_N \longrightarrow 0$. Indeed, $R_N(t) \searrow 0$ as $N \longrightarrow \infty$ for every $t > 0$. On the other hand, for each fixed $N > 0$, $\sum_{|n|>N} |\mathbb{E}(Z'_n(X_t - X_0))|^2$ is a continuous function on the variable t, since these nonnegative series converges pointwise, each term $|\mathbb{E}(Z'_n(X_t - X_0))|^2$ is continuous by the continuity condition on the process, and by Dini's criterion, then these series converges uniformly on $[0, T]$. Then, $R_N(t)$ is continuous in $(0, T]$, defining $R_N(0) = 0$, let us see that R_N is continuous at 0. If $t > 0$: $R_N(t) \leq \frac{\mathbb{E}|X_t - X_0|^2}{At^{\gamma}} \leq \frac{K}{A} t^{\alpha-\gamma} \longrightarrow 0$, whenever $t \longrightarrow 0$. Then, by Dini's criterion $T_N \longrightarrow 0$. Recalling Kolmogorov's condition of Lemma 3.1, with $G = \sum_{n\geq 0} (2^{-n}T)^{\beta} = T^{\beta} \frac{1}{1-2^{-\beta}}$. Then,

$$Q_N(x) = \frac{BT_N}{x^2} \sum_{n>0} 2^{n(2\beta-\gamma+1)} T^{\gamma-2\beta} = \frac{BT_N}{x^2} \frac{2^{2\beta-\gamma+1}}{1 - 2^{2\beta-\gamma+1}},$$

is such that

$$\mathbb{P}\left(\sup_{t\in J\setminus\{0\}} |X'_{N,t} - X'_{N,0}| > \frac{\epsilon}{2}\right)$$
$$\leq Q_N\left(\frac{\epsilon}{4G}\right) = C(\epsilon, \alpha) T_N \underset{N\longrightarrow\infty}{\longrightarrow} 0\,.$$

Finally, as $\mathbb{P}\left(|X'_{N,0}| > \frac{\epsilon}{2}\right) \leq \frac{4\mathbb{E}|X'_{N,0}|^2}{\epsilon^2} \underset{N\longrightarrow\infty}{\longrightarrow} 0$, recalling (7), the proof is complete. ■

IV. Conclusion

We gave conditions for a finite variance random process to be reconstructed from its discrete samples. The conditions are analogous to the conditions for $L^2(I)$ signals, studied by Kramer in [1] and Garcia [3]. Finally, we relate this expansions to KL-like expansions and study some detailed convergence facts of them, such as uniform convergence.

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