

# Optimal Frame Completions with Prescribed Norms for Majorization

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Received: 20 December 2013 / Published online: 9 July 2014  
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**Abstract** Given a finite sequence of vectors  $\mathcal{F}_0$  in a  $d$ -dimensional complex Hilbert space  $\mathcal{H}$  we characterize in a complete and explicit way the optimal completions of  $\mathcal{F}_0$  obtained by appending a finite sequence of vectors with prescribed norms, where optimality is measured with respect to majorization (of the eigenvalues of the frame operators of the completed sequences). Indeed, we construct (in terms of a fast algorithm) a vector—that depends on the eigenvalues of the frame operator of the initial sequence  $\mathcal{F}_0$  and the sequence of prescribed norms—that is a minimum for majorization among all eigenvalues of frame operators of completions with prescribed norms. Then, using the eigenspaces of the frame operator of the initial sequence  $\mathcal{F}_0$  we describe the frame operators of all optimal completions for majorization. Hence, the concrete optimal completions with prescribed norms can be obtained using recent algorithmic constructions related with the Schur-Horn theorem.

**Keywords** Frames · Frame completions · Majorization · Convex potentials · Schur-Horn theorem

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Communicated by Peter G. Casazza.

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## Mathematics Subject Classification 42C15 · 15A60

### 1 Introduction

A finite sequence of vectors  $\mathcal{F} = \{f_i\}_{i=1}^n$  in a  $d$ -dimensional complex Hilbert space  $\mathcal{H}$  is a frame for  $\mathcal{H}$  if the sequence spans  $\mathcal{H}$ . It is well known that finite frames provide (stable) linear encoding-decoding schemes. As opposed to bases, frames are not subject to linear independence; indeed, it turns out that the redundancy allowed in finite frames can be turned into robustness of the transmission scheme that they induce, which makes frames a useful device for transmission of signals through noisy channels (see [4–6, 11, 21, 24, 25]).

The frame operator of a given a sequence of vectors  $\mathcal{F} = \{f_i\}_{i=1}^n$  in  $\mathcal{H}$  is the positive operator on  $\mathcal{H}$  defined as

$$S_{\mathcal{F}} = \sum_{i=1}^n f_i \otimes f_i,$$

where  $g \otimes f$  is the linear operator in  $\mathcal{H}$  given by:  $g \otimes f(h) = \langle h, f \rangle g$  for every  $h \in \mathcal{H}$ . Thus, a sequence  $\mathcal{F}$  is a frame for  $\mathcal{H}$  if and only if  $S_{\mathcal{F}}$  is an invertible operator. When the frame operator is a multiple of the identity, the frame is called tight. Tight frames allow for redundant linear representations of vectors that are formally analogous to the linear representations given by orthonormal basis; this feature makes tight frames a distinguished class of frames that is of interest for applications. In several applications we would like to consider tight frames that have some other prescribed properties leading to what is known in the literature as frame design problems [1, 7, 8, 13–16, 23]. It turns out that in some cases it is not possible to find a frame fulfilling the previous demands.

An alternative approach to deal with the construction of frames with prescribed parameters and nice associated reconstruction formulas was posed in [2] by Benedetto and Fickus; they defined a functional, called the frame potential, and showed that minimizers of the frame potential (within a convenient set of frames) are the natural substitutes of tight frames with prescribed parameters (see also [10, 20, 22, 27] and [9, 28, 29] for related problems in the context of fusion frames). Moreover, in [27] it is shown that minimizers of the frame potential under suitable restrictions (considered in the literature) are structural minimizers in the sense that they coincide with minimizers of more general *convex potentials*.

Recently, there has been interest in the following optimal frame completion problem: given an initial sequence  $\mathcal{F}_0$  in  $\mathcal{H}$  and a sequence of positive numbers  $\mathbf{a}$  then, compute the sequences  $\mathcal{G}$  in  $\mathcal{H}$  whose elements have norms given by the sequence  $\mathbf{a}$  and such that the completed sequence  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G})$  is such that the eigenvalues of its frame operators are as concentrated as possible: thus, ideally, we would search for completions  $\mathcal{G}$  such that  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G})$  is a tight frame. Unfortunately, it is well known that there might not exist such completions (see [16–18, 26, 30, 31]). In this setting, the initial sequence of vectors can be considered as a checking device for the measurement, and therefore we search for a complementary set of measurements (given by

vectors with prescribed norms) in such a way that the complete set of measurements is optimal in some sense. Following [2, 10] we could measure optimality in terms of the frame potential i.e., we could search for completions with prescribed norms  $\mathcal{G}$  such that  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G})$  minimizes the frame potential among such completions; alternatively, we could measure optimality in terms of the so-called mean squared error (MSE) of the completed sequence (see [18]). More generally, we can consider a natural extension of the previous problems: given a functional defined on the set of frames, compute the frame completions with prescribed norms that minimize this functional. Moreover, this last problem raises the question of whether the completions that minimize these functionals coincide i.e., whether the minimizers are structural in this setting.

A first step towards the solution of the general version of the completion problem was made in [30]. There we showed that under certain hypothesis (feasible cases, see Sect. 2.3), optimal frame completions with prescribed norms are structural (do not depend on the particular choice of functional), as long as we consider convex potentials, that contain the MSE and the frame potential. On the other hand, it is easy to show examples in which the previous result does not apply (non-feasible cases); in these cases the optimal frame completions with prescribed norms were not known even for the MSE nor the frame potential.

In [31] we considered the structure of completions that minimize a fixed convex potential (non feasible case). There, we showed that the eigenvalues of optimal completions with respect to a fixed convex potential are uniquely determined by the solution of an optimization problem in a compact convex subset of  $\mathbb{R}^d$  for a convex objective function that is associated to the convex potential in a natural way. Then, we showed an important geometrical feature of optimal completions  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G})$  for a fixed convex potential, namely that the vectors in the completion  $\mathcal{G}$  are eigenvectors of the frame operator of the completed sequence  $\mathcal{F}$  (see Sect. 2.2 for a detailed exposition of these results). Based on these facts, we developed an algorithm that allowed us to compute the solutions of the completion problem for small dimensions. In this setting we conjectured some properties of the optimal frame completions in the general case, based on common features of the solutions of several examples obtained by this algorithm (see Sect. 3 for a detailed description of these conjectures).

In this paper, building on our previous work [30] and [31], we give a complete and explicit description of the spectral and geometrical structure of optimal completions with prescribed norms with respect to a convex potential induced by a strictly convex function. Our approach is constructive and allows to develop a fast and effective algorithm that computes the spectral structure of optimal completions. As we shall see, given an initial sequence  $\mathcal{F}_0$  in  $\mathcal{H}$  and a sequence of positive numbers  $\mathbf{a}$ , both the spectral and geometrical structure of optimal completions depend only on the frame operator of  $\mathcal{F}_0$  and  $\mathbf{a}$ , but they do not depend on the particular choice of the convex potential. Hence, we show that in the general case the minimizers of convex potentials (induced by strictly convex functions) are structural.

In order to obtain the previous results, we begin by proving the properties of general optimal completions conjectured in [31]. These properties (that are structural, in the sense that they do not depend on the convex potential) are then used to compute several other structural parameters—that involve the notion of feasibility developed

in [30]—that completely describe the spectral structure of optimal completions. As a consequence of this description, we conclude that optimal solutions have the same eigenvalues and hence, the eigenvalues of optimal completions are minimum for the so-called majorization preorder. Moreover, all the parameters involved in the description of the spectral structure of optimal completions can be computed in terms of fast algorithms. With the spectral data and results from [30] we completely describe the set positive matrices that correspond to the frame operators of sequences  $\mathcal{G}$  with norms prescribed by  $\mathbf{a}$  and such that  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G})$  are optimal. Finally, every optimal completion  $\mathcal{G}$  can be computed by using recent results from [7] (see also [14] and [19]).

The paper is organized as follows. In Section 2 we describe the context of our main problem—namely, optimal completions with prescribed norms, where optimality is described in terms of majorization—and give a detailed account of several related results that were developed in our previous works [30] and [31] that we shall need in the sequel, in a way suitable for this note; in particular, we include a new construction of the spectra of optimal completions in the feasible cases. In Sect. 3 we introduce new structural parameters—that can be efficiently computed in terms of explicit algorithms—and show how to give a complete description of the spectra of optimal completions for strictly convex potentials, in terms of these parameters in the general case. This allows us to show that the spectra of such optimal completions do not depend on the choice of strictly convex potential, so that minimizers are then structural. The proofs of the technical results of this section are presented in Sect. 4. In particular, we settle in the affirmative some features of the structure of optimal completions for strictly convex potentials that were conjectured in [31]. As a byproduct we also settle in the affirmative a conjecture on local minimizers of strictly convex potentials with prescribed norms posed in [27].

## 2 Optimal Completions with Prescribed Norms

In this section we give a detailed description of the optimal completion problem and recall some notions and results from our previous work [30,31]. We point out that the exposition of the results in Sect. 2.3 differs from that of [31], since this new presentation is better suited for our present purposes.

By now, finite frame theory is a well established area of intensive research. For a modern introduction to several aspect of this subject see [12]. In what follows we shall use the following notations and terminology:

**Notations and terminology:** let  $\mathcal{F} = \{f_i\}_{i=1}^n$  be a finite sequence in a complex  $d$ -dimensional Hilbert space  $\mathcal{H}$ . Then,

1.  $T_{\mathcal{F}} \in L(\mathbb{C}^n, \mathcal{H})$  denotes the synthesis operator of  $\mathcal{F}$  given by  $T_{\mathcal{F}}((\alpha_i)_{i=1}^n) = \sum_{i=1}^n \alpha_i f_i$ .
2.  $T_{\mathcal{F}}^* \in L(\mathcal{H}, \mathbb{C}^n)$  denotes the analysis operator of  $\mathcal{F}$  and it is given by  $T_{\mathcal{F}}^*(f) = (\langle f, f_i \rangle)_{i=1}^n$ .
3.  $S_{\mathcal{F}} \in L(\mathcal{H})$  denotes the frame operator of  $\mathcal{F}$  and it is given by  $S_{\mathcal{F}} = T_{\mathcal{F}} T_{\mathcal{F}}^*$ . Hence,  $Sf = \sum_{i=1}^n \langle f, f_i \rangle f_i = \sum_{i=1}^n f_i \otimes f_i(f)$  for  $f \in \mathcal{H}$ . Notice that  $S_{\mathcal{F}}$  is positive by construction.

4. We say that  $\mathcal{F}$  is a frame for  $\mathcal{H}$  if  $\mathcal{F}$  spans  $\mathcal{H}$ ; equivalently,  $\mathcal{F}$  is a frame for  $\mathcal{H}$  if  $S_{\mathcal{F}}$  is a positive invertible operator acting on  $\mathcal{H}$ .
5. In order to check whether  $\mathcal{F}$  is a frame, we will inspect the spectrum of  $S_{\mathcal{F}}$ . Thus, given a positive operator  $S \in L(H)$ ,  $\lambda(S) = (\lambda_i)_{i=1}^d \in \mathbb{R}_{\geq 0}^d$  denotes the eigenvalues of  $S_{\mathcal{F}}$ , counting multiplicities and arranged in non-increasing order i.e.  $\lambda_1 \geq \dots \geq \lambda_d \geq 0$ .

### 2.1 Presentation of the Problem

In several applied situations it is desired to construct a sequence  $\mathcal{G}$  in a complex  $d$ -dimensional Hilbert space  $\mathcal{H}$  in such a way that the frame operator of  $\mathcal{G}$  is given by some positive operator  $B$  and the squared norms of the frame elements are prescribed by a sequence of positive numbers  $\mathbf{a} = (a_i)_{i=1}^k$ . That is, given a fixed positive operator  $B$  on  $\mathcal{H}$  and  $\mathbf{a} \in \mathbb{R}_{>0}^k$ , we analyze the existence (and construction) of a sequence  $\mathcal{G} = \{g_i\}_{i=1}^k$  such that  $S_{\mathcal{G}} = B$  and  $\|g_i\|^2 = a_i$ , for  $1 \leq i \leq k$ . This is known as the classical frame design problem. It has been treated by several research groups (see for example [1, 7, 8, 13–16, 23]). In what follows we recall a solution of the classical frame design problem in the finite dimensional setting.

**Proposition 2.1** ([1, 26]) *Let  $B$  be a positive operator on  $\mathcal{H}$  and let  $\lambda(B) = (\lambda_i)_{i=1}^d$ . Consider  $\alpha_1 \geq \dots \geq \alpha_k > 0$ . Then there exists a sequence  $\mathcal{G} = \{g_i\}_{i=1}^k$  in  $\mathcal{H}$  with frame operator  $S_{\mathcal{G}} = B$  such that  $\|g_i\|^2 = \alpha_i$  for every  $1 \leq i \leq k$  if and only if*

$$\sum_{i=1}^j \alpha_i \leq \sum_{i=1}^j \lambda_i, \quad \text{for } 1 \leq j \leq \min\{k, d\} \quad \text{and} \quad \sum_{i=1}^k \alpha_i = \sum_{i=1}^d \lambda_i. \quad (1)$$

□

Recently, researchers have made a step forward in the classical frame design problem and have asked about the structure of *optimal* frames with prescribed parameters. In particular, there has been interest in the following problem: let  $\mathcal{H} \cong \mathbb{C}^d$  and let  $\mathcal{F}_0 = \{f_i\}_{i=1}^{n_0}$  be a fixed (finite) sequence of vectors in  $\mathcal{H}$ . Consider a sequence  $\mathbf{a} = (a_i)_{i=1}^k$  of positive numbers such that  $\text{rk } S_{\mathcal{F}_0} \geq d - k$  and denote by  $n = n_0 + k$ . Then, with this fixed data, the problem is to construct a sequence

$$\mathcal{G} = \{g_i\}_{i=1}^k \quad \text{with} \quad \|g_i\|^2 = a_i \quad \text{for } 1 \leq i \leq k,$$

such that the resulting completed sequence  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G})$  - obtained by appending the sequence  $\mathcal{G}$  to  $\mathcal{F}_0$ —is a frame such that the eigenvalues of the frame operator of  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G})$  are as concentrated as possible: thus, ideally, we would search for completions  $\mathcal{G}$  such that  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G})$  is a tight frame. Unfortunately, it is well known that there might not exist such completions (see [16–18, 26, 30, 31]). In this setting, the initial sequence of vectors can be considered as a checking device for the measurement, and therefore we search for a complementary set of measurements (given by vectors with prescribed norms) in such a way that the complete set of measurements is optimal

in some sense. We could measure optimality in terms of the frame potential i.e., we search for a frame  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G})$ , with  $\|g_i\|^2 = a_i$  for  $1 \leq i \leq k$ , and such that its frame potential  $FP(\mathcal{F}) = \text{tr } S_{\mathcal{F}}^2$  is minimal among all possible such completions (indeed, this problem has been considered before in the particular case in which  $\mathcal{F}_0 = \emptyset$  in [2, 10, 20, 22, 27]); alternatively, we could measure optimality in terms of the so-called mean squared error (MSE) of the completed sequence  $\mathcal{F}$  i.e.  $\text{MSE}(\mathcal{F}) = \text{tr}(S_{\mathcal{F}}^{-1})$  (see [18]). More generally, we can measure robustness of the completed frame  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G})$  in terms of general convex potentials:

**Definition 2.2** Let us denote by

$$\text{Conv}(\mathbb{R}_{\geq 0}) = \{f : [0, \infty) \rightarrow [0, \infty) : f \text{ is a convex function}\}$$

and  $\text{Conv}_s(\mathbb{R}_{\geq 0}) = \{f \in \text{Conv}(\mathbb{R}_{\geq 0}) : f \text{ is strictly convex}\}$ . Following [27] we consider the (generalized) convex potential  $P_f$  associated to any  $f \in \text{Conv}(\mathbb{R}_{\geq 0})$ , given by

$$P_f(\mathcal{F}) = \text{tr } f(S_{\mathcal{F}}) = \sum_{i=1}^d f(\lambda_i(S_{\mathcal{F}})) \quad \text{for } \mathcal{F} = \{f_i\}_{i=1}^n \in \mathcal{H}^n,$$

where the matrix  $f(S_{\mathcal{F}})$  is defined by means of the usual functional calculus. △

In order to describe the main problems we first fix the notation that we shall use throughout the paper.

**Definition 2.3** Let  $\mathcal{H}$  be a complex  $d$ -dimensional Hilbert space, let  $\mathcal{F}_0 = \{f_i\}_{i=1}^{n_0}$  be a sequence of vectors in  $\mathcal{H}$  and  $\mathbf{a} = (a_i)_{i=1}^k$  be a positive nonincreasing sequence such that  $d - \text{rk } S_{\mathcal{F}_0} \leq k$ . Define  $n = n_0 + k$ . Then

1. In what follows we say that  $(\mathcal{F}_0, \mathbf{a})$  are initial data for the completion problem (CP).
2. For these data we consider the set

$$\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0) = \{(\mathcal{F}_0, \mathcal{G}) \in \mathcal{H}^n : \mathcal{G} = \{g_i\}_{i=1}^k \quad \text{and} \quad \|g_i\|^2 = a_i \text{ for } 1 \leq i \leq k\},$$

When the initial data  $(\mathcal{F}_0, \mathbf{a})$  are fixed, we shall use the notations  $S_0 = S_{\mathcal{F}_0}$  and  $\lambda = \lambda(S_0)^\uparrow$  will denote the eigenvalues of  $S_0$  arranged in a non-decreasing order i.e.  $\lambda_1 \leq \dots \leq \lambda_d$ , where  $x^\uparrow \in \mathbb{R}^d$  denotes the vector obtained from  $x \in \mathbb{R}^d$  by re-arranging the coordinates of  $x$  in non-decreasing order.

**Main problems:** (Optimal completions with prescribed norms for majorization) Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP and let  $f \in \text{Conv}_s(\mathbb{R}_{\geq 0})$ .

- P1. Give an explicit description (both spectral and geometrical) of  $\mathcal{F} \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$  that are the minimizers of  $P_f$  in  $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ .
- P2. Construct a fast algorithm that efficiently computes all possible  $\mathcal{F} \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$  that are the minimizers of  $P_f$  in  $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ .
- P3. Verify that the set of  $\mathcal{F} \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$  that are the minimizers of  $P_f$  in  $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$  is the same for every  $f \in \text{Conv}_s(\mathbb{R}_{\geq 0})$ . △

In previous works we have obtained some results related with the problems above. Indeed, in [30] we obtained a partial affirmative answer to P3, while in [31] we obtained some partial results related with P1 and a non-efficient algorithm as in P2 that worked in small examples (see Sects.2.2 and 2.3 below).

In this paper, building on our previous work, we completely solve the three problems above in terms of a constructive (algorithmic) approach.

### 2.2 On the Structure of the Minimizers of $P_f$ on $C_a(\mathcal{F}_0)$

In this section we collect results from [31] that we shall use in this paper. Throughout this section we fix the initial data  $(\mathcal{F}_0, \mathbf{a})$  for the CP. Recall that  $\lambda = (\lambda_i)_{i=1}^d$  are the eigenvalues of  $S_0 = S_{\mathcal{F}_0}$  arranged in a non-decreasing order. Therefore we recast the results from [31] by reversing the ordering used in that work. Also notice that we are assuming that the sequence  $\mathbf{a}$  is arranged in non-increasing order, that is,  $a_1 \geq a_2 \geq \dots \geq a_k > 0$ .

In what follows, the notion of majorization will play a fundamental role: recall that given  $x, y \in \mathbb{R}^d$  we say that  $x$  is **submajorized** by  $y$ , and write  $x \prec_w y$ , if

$$\sum_{i=1}^k x_i^\downarrow \leq \sum_{i=1}^k y_i^\downarrow \quad \text{for every } k \in \mathbb{I}_d,$$

where  $x^\downarrow \in \mathbb{R}^d$  denotes the vector obtained from  $x$  by rearrangement of its entries in non-increasing order. If  $x \prec_w y$  and  $\sum_{i=1}^d x_i = \sum_{i=1}^d y_i$ , then we say that  $x$  is **majorized** by  $y$ , and write  $x \prec y$ . In particular, Prop. 2.1 states that the eigenvalues of  $B$  majorize the sequence of squared norms (to be precise, we must add zeros to one of the two vectors if they have different sizes).

Our analysis of the completed frames  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G})$  depends on  $\mathcal{F}$  through  $S_{\mathcal{F}} = S_0 + S_{\mathcal{G}}$ . Hence, the following result plays a central role in our approach.

**Proposition 2.4** *Let  $(\mathcal{F}_0, \mathbf{a})$  be the initial data for the CP and let  $S$  be a positive operator on  $\mathcal{H}$ . Then  $S$  is the frame operator for some completion  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G}) \in C_a(\mathcal{F}_0)$  if and only if*

$$S \geq S_0 \quad \text{and} \quad \mathbf{a} \prec \lambda(S - S_0) = \lambda(S_{\mathcal{G}}). \tag{2}$$

Let  $\mu = (\mu_i)_{i=1}^d$  be such that  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_d$  (i.e.  $\mu = \mu^\downarrow$ ). By Proposition 2.4 we get the following partition:

$$C_a(\mathcal{F}_0) = \bigsqcup_{\mathbf{a} \prec \mu = \mu^\downarrow} \{ \mathcal{F} = (\mathcal{F}_0, \mathcal{G}) \in C_a(\mathcal{F}_0) : \lambda(S_{\mathcal{G}}) = \mu \} \tag{3}$$

Building on Lidskii’s inequality (see [3, III.4]) we obtained the following result:

**Theorem 2.5** *Let  $\mu = \mu^\downarrow$  with  $\mathbf{a} \prec \mu$ . Recall that  $\lambda = \lambda^\uparrow = \lambda(S_0)^\uparrow$ . Then,*

1. *The set  $\{ \lambda(S_{\mathcal{F}}) : \mathcal{F} = (\mathcal{F}_0, \mathcal{G}) \in C_a(\mathcal{F}_0) : \lambda(S_{\mathcal{G}}) = \mu \}$  is convex.*

2.  $(\lambda + \mu)^\downarrow \prec \lambda(S_{\mathcal{F}})$  for all  $\mathcal{F} \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$  such that  $\lambda(S_{\mathcal{F}} - S_0) = \mu$ .
3. If  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G})$  is such that  $\lambda(S_{\mathcal{G}}) = \mu$  and  $\lambda(S_{\mathcal{F}}) = (\lambda + \mu)^\downarrow$  then  $S_0$  and  $S_{\mathcal{G}}$  commute.

□

*Remark 2.6* Fix  $\mu$  such that  $\mathbf{a} \prec \mu$ . Consider  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G}) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$  such that the spectrum of  $S_{\mathcal{G}}$  is  $\mu$ . Let  $f \in \text{Conv}_s(\mathbb{R}_{\geq 0})$  and let  $P_f$  be the convex potential induced by  $f$ . By the well known interplay between majorization and minimization of convex functions (see [28]) and Theorem 2.5 we see that  $P_f(\mathcal{F}) \leq P_f(\mathcal{F}')$  for every completion  $\mathcal{F}' \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$  such that  $\lambda(S_{\mathcal{F}'} - S_0) = \mu$  if and only if

$$\lambda(S_{\mathcal{F}}) = (\lambda + \mu)^\downarrow = (\lambda^\uparrow + \mu^\downarrow)^\downarrow. \tag{4}$$

That is, if we consider the partition of  $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$  described in Eq. (3), then in each slice defined by  $\mu$  the minimizers of the potential  $P_f$  are characterized by the spectral condition (4). This shows that in order to search for global minimizers of  $P_f$  on  $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$  we can restrict our attention to the set of completions  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G}) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$  such that

$$\lambda(S_{\mathcal{F}}) = (\lambda + \lambda(S_{\mathcal{G}}))^\downarrow. \tag{5}$$

Indeed, Eqs. (3) and (4) show that if  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G})$  is a minimizer of  $P_f$  in  $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$  then the eigenvalues of  $S_{\mathcal{F}}$  are such that  $\lambda(S_{\mathcal{F}}) = (\lambda + \lambda(S_{\mathcal{G}}))^\downarrow$ . Therefore, we analyze the existence and uniqueness of  $\prec$ -minimizers on the set  $\{\lambda + \mu : \mu \in \mathbb{R}_{\geq 0}^d, \mu = \mu^\downarrow \text{ and } \mathbf{a} \prec \mu\}$ .

**Theorem 2.7** *Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP. Denote by  $\lambda = \lambda(S_{\mathcal{F}_0})^\uparrow$ . Then*

1. *The set  $\{\lambda + \mu : \mu \in \mathbb{R}_{\geq 0}^d, \mu = \mu^\downarrow \text{ and } \mathbf{a} \prec \mu\}$  is compact and convex.*
2. *If  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G}) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ , with  $\lambda(S_{\mathcal{G}}) = \mu$  and  $\lambda(S_{\mathcal{F}}) = (\lambda + \mu)^\downarrow$ , then there exists a orthonormal basis of  $\mathcal{H}$ ,  $\{v_i\}_{i=1}^d$  such that*

$$S_{\mathcal{G}} = \sum_{i=1}^d \mu_i \cdot v_i \otimes v_i \quad \text{and} \quad S_{\mathcal{F}} = S_0 + S_{\mathcal{G}} = \sum_{i=1}^d (\lambda_i + \mu_i) \cdot v_i \otimes v_i.$$

□

The following result is a reduction of the computation of the spectral structure of the optimal completions with respect to a fixed convex potential.

**Theorem 2.8** *Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP and let  $f \in \text{Conv}_s(\mathbb{R}_{\geq 0})$ . Denote by  $\lambda = \lambda(S_{\mathcal{F}_0})^\uparrow$ . Then, there exists a vector  $\mu_f(\lambda, \mathbf{a}) = \mu = \mu^\downarrow \in \mathbb{R}_{\geq 0}^d$  such that  $\mathbf{a} \prec \mu$  and:*

1.  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G}) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$  is a global minimizer of  $P_f \iff \lambda(S_{\mathcal{F}}) = (\lambda + \mu)^\downarrow$  and  $\lambda(S_{\mathcal{G}}) = \mu$ .



2.  $\mu_f(\lambda, \mathbf{a})$  is **uniquely** determined by the conditions:  $\mu = \mu^\downarrow \in \mathbb{R}_{\geq 0}^d$ ,  $\mathbf{a} \prec \mu$ , and

$$\sum_{i=1}^d f(\lambda_i + \mu_i) = \min \left\{ \sum_{i=1}^d f(\lambda_i + \gamma_i) : \gamma = \gamma^\downarrow = (\gamma_i)_{i=1}^d \in \mathbb{R}_{\geq 0}^d \text{ and } \mathbf{a} \prec \gamma \right\}. \tag{6}$$

□

Denote by  $\nu_f(\lambda, \mathbf{a}) \stackrel{\text{def}}{=} \lambda + \mu_f(\lambda, \mathbf{a})$ , the optimal spectrum as in item 1 of Theorem 2.8 above.

**Theorem 2.9** *Let  $f \in \text{Conv}_s(\mathbb{R}_{\geq 0})$  and assume that  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G})$  is a global minimizer of  $P_f$  on the set of completions  $\mathcal{F}' = (\mathcal{F}_0, \mathcal{G}') \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$  such that  $\lambda(S_{\mathcal{F}'}) = (\lambda + \lambda(S_{\mathcal{G}'}))^\downarrow$ . Then, there exists a partition  $\{J_i\}_{i=1}^p$  of  $\{1, \dots, k\}$  and  $c_1 > \dots > c_p > 0$  such that*

1. *The subfamilies  $\mathcal{G}_i = \{g_j\}_{j \in J_i}$  (for  $1 \leq i \leq p$ ) are mutually orthogonal, i.e.  $S_{\mathcal{G}} = \bigoplus_{i=1}^p S_{\mathcal{G}_i}$ .*
2. *The frame operators  $S_{\mathcal{G}_i}$  and  $S_{\mathcal{F}_0}$  commute, for every  $1 \leq i \leq p$ .*
3. *We have that  $S_{\mathcal{F}} g_j = c_i g_j$ , for every  $j \in J_i$  and every  $1 \leq i \leq p$ .*

*The statement is still valid if we assume that  $\mathcal{F}$  is just a local minimizer, but if we also assume as a hypothesis that  $\mathcal{F}$  satisfies item 2 (for example if  $S_{\mathcal{F}_0} = 0$ ).* □

### 2.3 The Feasible Case of the CP

In this section we recall the results from [30] that we shall need in the sequel. Throughout this section we keep the notation used previously. That is, given the initial data  $(\mathcal{F}_0, \mathbf{a})$  for the CP, we denote by  $S_0 = S_{\mathcal{F}_0}$ ,  $\lambda = \lambda(S_0)^\uparrow$ . Let  $t = \text{tr } S_0 + \text{tr } \mathbf{a} = \sum_{i=1}^d \lambda_i + \sum_{i=1}^k a_i$ . Let  $L(\mathcal{H})^+$  denote the convex cone of positive (semidefinite) operators acting on  $\mathcal{H}$ . In [30] we introduced the following set

$$U_t(S_0, k) = \{S_0 + B : B \in L(\mathcal{H})^+, \text{rk } B \leq k, \text{tr}(S_0 + B) = t\} \subseteq L(\mathcal{H})^+.$$

In [30, Theorem 3.12] it is shown that there exist  $\prec$ -minimizers in  $U_t(S_0, k)$ . Indeed, there exists  $\mu(\lambda, \mathbf{a}) = \mu(\lambda, \mathbf{a})^\downarrow \in \mathbb{R}_{\geq 0}^d$ —that can be effectively computed by a fast algorithm—such that, if  $\nu(\lambda, \mathbf{a}) \stackrel{\text{def}}{=} \lambda + \mu(\lambda, \mathbf{a})$  then  $\nu(\lambda, \mathbf{a}) \in \mathbb{R}_{> 0}^d$  and  $S \in U_t(S_0, k)$  is a  $\prec$ -minimizer if and only if  $\lambda(S) = \nu(\lambda, \mathbf{a})^\downarrow$ .

Notice that by construction  $\nu(\lambda, \mathbf{a})$  is not necessarily an ordered vector (nor decreasing, nor increasing); yet, in terms of the terminology from [30], we have that  $\nu_{\lambda, m}(t) = \nu(\lambda, \mathbf{a})^\downarrow$ . Thus, we have reversed the order of the vector  $\mu(\lambda, \mathbf{a})$ —accordingly with reversing the order of  $\lambda = \lambda(S_{\mathcal{F}_0})^\uparrow$ —and we have changed the description of the vector  $\nu(\lambda, \mathbf{a})$ —while preserving all of their majorization properties—with respect to [30]. Nevertheless, we point out that the ordering of the entries of the vector  $\nu(\lambda, \mathbf{a})$  presented here plays a crucial role in simplifying the exposition of the results herein, as it guaranties that  $\mu(\lambda, \mathbf{a}) = \nu(\lambda, \mathbf{a}) - \lambda$ .

The following definition and remark show the relevance of the notions introduced above for the computation of the spectral structure of solutions for the optimal completion problem.

**Definition 2.10** With the previous notations, we say that the pair  $(\lambda, \mathbf{a})$  is **feasible** if  $\mu(\lambda, \mathbf{a})$  satisfies that  $\mathbf{a} < \mu(\lambda, \mathbf{a})$ . △

*Remark 2.11* With the previous notations, assume that the pair  $(\lambda, \mathbf{a})$  is feasible and denote  $\mu = \mu(\lambda, \mathbf{a})$ . In this case (see [30]) for any  $S$  which is a  $<$ -minimizer in  $U_t(S_0, k)$  it holds that  $\lambda(S - S_0) = \mu$  and hence, by Proposition 2.4, we conclude that  $S$  is the frame operator for some completion in  $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ . Moreover, Proposition 2.4 also shows that the frame operators of completions in  $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$  are in  $U_t(S_0, k)$ . Then  $S$  is also a  $<$ -minimizer in the set of frame operators of sequences in  $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ . Therefore, since  $x < y$  implies  $\sum_{i=1}^d f(x_i) \leq \sum_{i=1}^d f(y_i)$  for every  $f \in \text{Conv}(\mathbb{R}_{\geq 0})$  (see [28] for a detailed account of these facts), any completion  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G}) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$  such that  $S_{\mathcal{F}} = S$  is a minimizer of  $P_f$  for every  $f \in \text{Conv}(\mathbb{R}_{\geq 0})$ .

On the other hand, as a consequence of the geometrical structure of  $S = S_{\mathcal{F}}$  as above (see [30,31]), we conclude that there exists  $c > 0$  such that  $S_{\mathcal{F}} g_i = c g_i$  for every  $1 \leq i \leq k$ . That is, in this case the structure of the completing sequence  $\mathcal{G}$  given in Theorem 2.9 is trivial: the partition of  $\{1, \dots, k\}$  has only one member and there exists a unique constant  $c = c_1$ . △

It is worth pointing out that it is easy to construct examples of initial data  $(\mathcal{F}_0, \mathbf{a})$  for the CP such that the pair  $(\lambda, \mathbf{a})$  is not feasible (see [30]), so that comments in Remark 2.11 do not apply in these cases.

**Definition 2.12** Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP. For every  $r \in \mathbb{I}_d$  we denote by

$$Q_r \stackrel{\text{def}}{=} \frac{1}{r} \left[ \text{tr } \mathbf{a} + \sum_{i=1}^r \lambda_i \right]$$

△

*Remark 2.13* Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP with  $k \geq d$ , recall the notations  $\lambda = \lambda(S_{\mathcal{F}_0})^\uparrow$  and  $t = \text{tr } \mathbf{a} + \text{tr } \lambda = d Q_d$ . As shown in [30] we can explicitly compute the vector  $v(\lambda, \mathbf{a})$  previously defined according to the following two cases:

1. Since  $\lambda = \lambda^\uparrow$  then  $\lambda_d = \max_{i \in \mathbb{I}_d} \lambda_i$ . If

$$Q_d = \frac{t}{d} = \frac{\text{tr } \mathbf{a} + \text{tr } \lambda}{d} \geq \lambda_d \quad \text{then} \quad v(\lambda, \mathbf{a}) = Q_d \mathbb{1}_d = \frac{t}{d} \mathbb{1}_d, \tag{7}$$

where  $\mathbb{1}_d \in \mathbb{R}^d$  is the vector with all its entries equal to one.

2. If  $\lambda_d > Q_d = \frac{t}{d}$  then there exists  $s \in \mathbb{I}_{d-1}$  such that

$$\lambda_s \leq Q_s < \lambda_{s+1} \quad \text{and} \quad v(\lambda, \mathbf{a}) = (Q_s \mathbb{1}_s, \lambda_{s+1}, \dots, \lambda_d). \tag{8}$$

Moreover,  $\lambda \leq v(\lambda, \mathbf{a}) = v(\lambda, \mathbf{a})^\uparrow$  and  $\text{tr } v(\lambda, \mathbf{a}) \stackrel{2.12}{=} \text{tr } \mathbf{a} + \text{tr } \lambda = t$ .

These properties of the vector  $v(\lambda, \mathbf{a})$  are proved in [30], modulo a reordering. If  $Q_d < \lambda_d$ , an alternative proof of them is to take  $s = \max \{r \in \mathbb{I}_d : Q_r \geq \lambda_r\}$  and define  $v = v(\lambda, \mathbf{a}) = (Q_s \mathbb{1}_s, \lambda_{s+1}, \dots, \lambda_d)$ . Direct computations show that  $\lambda_s \leq Q_s < \lambda_{s+1}$ , so that  $\lambda \leq v = v^\uparrow$  with  $\text{tr } v = t$ . This also implies, using [30, Lemma 5.11], that  $v \prec \lambda(S)$  for every  $S \in U_t(S_0, k)$ .

In what follows we obtain another explicit description of the vector  $v(\lambda, \mathbf{a})$  in case  $d \leq k$  and  $Q_d < \lambda_d$ . That is, we compute the parameter  $s$  of Eq. (8). The way in which it is found is the key for the developments of Sect. 3. Our present techniques differ substantially from those introduced in [30]. We begin by showing that the vector  $v(\lambda, \mathbf{a})$  above is unique. Then, we show that the computation of  $v(\lambda, \mathbf{a})$  for  $k < d$  can be reduced to the case when  $k = d$ . First we need to state a technical result:  $\triangle$

**Lemma 2.14** *Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP with  $k \geq d$  and let  $r \in \mathbb{I}_d$ . Then*

1. *If  $r < d$  and  $Q_r < \lambda_{r+1}$  then  $Q_r < Q_j$ , for every  $j$  such that  $r < j \leq d$ .*
2. *If  $r < d$  and  $Q_r \leq \lambda_{r+1}$  then  $Q_r \leq Q_j$ , for every  $j$  such that  $r < j \leq d$ .*
3. *If  $\lambda_r \leq Q_r$  then  $Q_r \leq Q_j$ , for every  $j$  such that  $1 \leq j < r$ .*

*Proof* Denote by  $c = Q_r$  for a fixed  $r < d$ . Recall that  $\lambda = \lambda^\uparrow$ . If  $j > r$  then

$$c < \lambda_{r+1} \implies Q_j = \frac{1}{j} \left( \text{tr } \mathbf{a} + \sum_{i=1}^r \lambda_i + \sum_{i=r+1}^j \lambda_i \right) > \frac{1}{j} (rc + (j-r)c) = c.$$

The proof of item 2 is identical. On the other side, if  $j < r$  then

$$\lambda_r \leq c \implies Q_j = \frac{1}{j} \left( \text{tr } \mathbf{a} + \sum_{i=1}^r \lambda_i - \sum_{i=j+1}^r \lambda_i \right) \geq \frac{1}{j} (rc - (r-j)c) = c.$$

$\square$

**Proposition 2.15** *Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP with  $k \geq d$  and suppose also that  $Q_d = \frac{1}{d} [\text{tr } \mathbf{a} + \text{tr } \lambda] < \lambda_d$ . Then*

1. *There exists a unique index  $s \in \mathbb{I}_{d-1}$  such that  $\lambda_s \leq Q_s < \lambda_{s+1}$ , and in this case*

$$s = \max \{w \in \mathbb{I}_{d-1} : Q_w = \min_{j \in \mathbb{I}_d} Q_j\} \quad \text{and} \quad v(\lambda, \mathbf{a}) = (Q_s \mathbb{1}_s, \lambda_{s+1}, \dots, \lambda_d) \tag{9}$$

2. *If another index  $r \leq d - 1$  satisfies that  $\lambda_r \leq Q_r \leq \lambda_{r+1}$ , then*
  - (a)  $Q_r = \min_{j \in \mathbb{I}_d} Q_j = Q_s$  and  $r \leq s$ .
  - (b) If  $r < s$ , then  $Q_r = \lambda_{r+1} = \lambda_s$  and also  $v(\lambda, \mathbf{a}) = (Q_r \mathbb{1}_r, \lambda_{r+1}, \dots, \lambda_d)$ .
3. *Given  $\rho = (c \mathbb{1}_r, \lambda_{r+1}, \dots, \lambda_d)$  (or  $\rho = c \mathbb{1}_d$ ) such that  $\lambda \leq \rho = \rho^\uparrow$  and  $\text{tr } \rho = \text{tr } v(\lambda, \mathbf{a})$  then  $\rho = v(\lambda, \mathbf{a})$ .*

*Proof* The existence of an index  $s$  such as in item 1 is guaranteed by the properties of  $v(\lambda, \mathbf{a})$  stated in [30] or in Remark 2.13. The formula given in Eq. (9), which shows the uniqueness of the index  $s$ , is a direct consequence of Lemma 2.14. Assume that  $\lambda_r \leq Q_r \leq \lambda_{r+1}$ . Then  $Q_r = \min_{1 \leq j \leq d} Q_j = Q_s$  and  $r \leq s$  by Lemma 2.14. If  $r < s$ , then  $Q_s = \frac{1}{s} (r Q_r + \sum_{i=r+1}^s \lambda_i) = Q_r$ . This clearly implies all the equalities of item (b). Finally, observe that item 2  $\implies$  item 3.  $\square$

*Remark 2.16* (Reduction of the computation of  $v(\lambda, \mathbf{a})$  to the case  $k \geq d$ ) Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP with  $d > k$ . Then if

$$\tilde{\lambda} = (\lambda_1, \dots, \lambda_k) \in (\mathbb{R}^k)^\uparrow \quad \text{then} \quad v(\lambda, \mathbf{a}) = (v(\tilde{\lambda}, \mathbf{a}), \lambda_{k+1}, \dots, \lambda_d), \quad (10)$$

and  $v(\tilde{\lambda}, \mathbf{a})$  is constructed as in Proposition 2.15 (notice that in this case  $\tilde{\lambda} \in \mathbb{R}^{\tilde{d}}$  with  $\tilde{d} = k$ ).

The proof is direct by observing that, extracting the entries  $\lambda_{k+1}, \dots, \lambda_d$  of the vector  $v(\lambda, \mathbf{a})$  as described in [30, Def. 4.13], the vector that one obtains (with the reverse order) satisfies the conditions of item 3 of Proposition 2.15 relative to the pair  $(\tilde{\lambda}, \mathbf{a})$ .  $\triangle$

The following result is in a sense a converse to Remark 2.11. It establishes that if there exists  $f \in \text{Conv}_s(\mathbb{R}_{\geq 0})$  and a minimizer  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G})$  of  $P_f$  in  $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$  such that the structure of the completing sequence  $\mathcal{G}$  as described in Theorem 2.9 is trivial, then the underlying pair  $(\lambda, \mathbf{a})$  is feasible. Recall the notation  $v_f(\lambda, \mathbf{a})$  given in Theorem 2.8.

**Lemma 2.17** *Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP, with  $k \geq d$  and let  $f \in \text{Conv}_s(\mathbb{R}_{\geq 0})$ . Let  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G})$  be a minimum for  $P_f$  on  $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$  such that  $\lambda(S_{\mathcal{F}}) = (\lambda + \lambda(S_{\mathcal{G}}))^\downarrow$ . Suppose that, for some  $c > 0$ ,*

$$W = R(S_{\mathcal{G}}) \neq \mathcal{H} \quad \text{and} \quad S_{\mathcal{F}}|_W \in L(W) = c I_W.$$

*Let  $\mu = \mu^\downarrow = \lambda(S_{\mathcal{G}})$  and  $s \stackrel{\text{def}}{=} \dim W = \max\{i : \mu_i \neq 0\}$ . Then*

$$\lambda_s < c \leq \lambda_{s+1} \quad \text{so that} \quad (\lambda, \mathbf{a}) \quad \text{is feasible and} \quad v_f(\lambda, \mathbf{a}) = v(\lambda, \mathbf{a}).$$

*The same final conclusion trivially holds if  $s = \dim W = d$  and  $S_{\mathcal{F}} = c I$ .*

*Proof* Suppose that  $s < d$ . By hypothesis  $v_f(\lambda, \mathbf{a}) = \lambda^\uparrow + \mu^\downarrow = (c \mathbb{1}_s, \lambda_{s+1}, \dots, \lambda_d)$  and it satisfies that  $\lambda(S_{\mathcal{F}}) = v_f(\lambda, \mathbf{a})^\downarrow$ . Since  $\mathbf{a} < \mu = \mu^\downarrow$  then  $\text{tr } \mu = \text{tr } \mathbf{a} > \sum_{i=1}^s a_i$ , because  $s < d \leq k$ . Suppose now that  $c > \lambda_{s+1}$ . For small  $t > 0$  consider the vector

$$\gamma(t) = (c \mathbb{1}_{s-1}, (c - t), \lambda_{s+1} + t, \lambda_{s+2}, \dots, \lambda_d) \in \mathbb{R}^d \quad \text{with} \quad \text{tr } \gamma(t) = \text{tr } S_{\mathcal{F}}.$$

Let  $\mu(t) = \gamma(t) - \lambda$ . For every  $t$  we have that  $\text{tr } \mu(t) = \text{tr } \mu$ . On the other hand, if

$$t < \frac{\mu_s}{2} \implies \mu(t) = (\mu_1, \dots, \mu_{s-1}, \mu_s - t, t, 0 \mathbb{1}_{d-s-1}) = \mu(t)^\downarrow \in (\mathbb{R}_{\geq 0}^d)^\downarrow.$$

It is easy to see that if also  $t < \sum_{i=s+1}^k a_i$  then still  $\mathbf{a} < \mu(t)$ . So there exists  $\mathcal{F}' \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$  such that  $\lambda(S_{\mathcal{F}'}) = \gamma(t)^\downarrow$ . Notice that, since  $(c-t, \lambda_{s+1}+t) < (c, \lambda_{s+1})$  strictly, then  $P_f(\mathcal{F}') = \text{tr } f(\gamma(t)) < \text{tr } f(v_f(\lambda, \mathbf{a})) = P_f(\mathcal{F})$ , a contradiction. Hence  $c \leq \lambda_{s+1}$ .

The condition  $\lambda_s < c$  follows from the fact that  $c - \lambda_s = \mu_s > 0$ . These facts show that  $\lambda = \lambda^\uparrow \leq v_f(\lambda, \mathbf{a}) = v_f(\lambda, \mathbf{a})^\uparrow \implies v_f(\lambda, \mathbf{a}) = v(\lambda, \mathbf{a})$  (by item 3 of Proposition 2.15). In particular,  $\mathbf{a} < \lambda(S_{\mathcal{G}}) = \mu = v(\lambda, \mathbf{a}) - \lambda = \mu(\lambda, \mathbf{a})$  so that  $(\lambda, \mathbf{a})$  is feasible.  $\square$

### 3 Uniqueness and Characterization of the Minimum

In this section we shall state the main results of the paper. For the sake of clarity of the exposition, we postpone the more technical proofs until Sect. 4.

**3.1** (Fixed data, notations and terminology) Until Theorem 3.8, we fix  $f \in \text{Conv}_s(\mathbb{R}_{\geq 0})$  and  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G}) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$  a minimizer of  $P_f$  on  $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ .

1. By Theorem 2.8,  $\lambda(S_{\mathcal{F}}) = (\lambda + \lambda(S_{\mathcal{G}}))^\downarrow$  and  $\lambda(S_{\mathcal{G}}) = \mu_f(\lambda, \mathbf{a}) = v_f(\lambda, \mathbf{a}) - \lambda$ . Then, by Theorem 2.7 there exists an ONB  $\{v_i\}_{i=1}^d$  such that

$$S_{\mathcal{G}} = \sum_{i=1}^d \mu_i \cdot v_i \otimes v_i \quad \text{and} \quad S_{\mathcal{F}} = S_{\mathcal{F}_0} + S_{\mathcal{G}} = \sum_{i=1}^d (\lambda_i + \mu_i) v_i \otimes v_i. \quad (11)$$

2. Let  $s_{\mathcal{F}} = \max \{i \leq d : \mu_i \neq 0\} = \text{rk } S_{\mathcal{G}}$ . Denote by  $W = R(S_{\mathcal{G}})$ , which reduces  $S_{\mathcal{F}}$ .
3. Let  $S = S_{\mathcal{F}}|_W \in L(W)$  and  $\sigma(S) = \{c_1, \dots, c_p\}$  (where  $c_1 > c_2 > \dots > c_p > 0$ ).
4. Let  $K_j = \{i \leq s_{\mathcal{F}} : \lambda_i + \mu_i = c_j\}$  and  $J_j = \{i \leq k : S g_i = c_j g_i\}$ . By Theorem 2.9,

$$\{1, \dots, s_{\mathcal{F}}\} = \bigsqcup_{j=1}^p K_j \quad \text{and} \quad \{1, \dots, k\} = \bigsqcup_{j=1}^p J_j.$$

5. Since  $R(S_{\mathcal{G}}) = \text{span}\{g_i : 1 \leq i \leq k\} = W = \bigoplus_{i=1}^p \ker(S - c_i I_W)$  then for every  $1 \leq j \leq p$ ,

$$W_j \stackrel{\text{def}}{=} \text{span}\{g_i : i \in J_j\} = \ker(S - c_j I_W) = \text{span}\{v_i : i \in K_j\}, \quad (12)$$

because  $g_i \in \ker(S - c_j I_W)$  for every  $i \in J_j$ . Note that, by Theorem 2.9, each  $W_j$  reduces both  $S_{\mathcal{F}_0}$  and  $S_{\mathcal{G}}$ .

6. If  $p = 1$  then  $J_1 = \{1, \dots, k\}$  and  $S = c_1 I_W$ . Hence the minimum  $\mathcal{F}$  satisfies the hypothesis of Lemma 2.17, so that the pair  $(\lambda, \mathbf{a})$  is feasible.

7. We denote by  $h_i = \lambda_i + a_i$  for every  $1 \leq i \leq d$ . Given  $j \leq r \leq d$ , let

$$P_{j,r} = \frac{1}{r - j + 1} \sum_{i=j}^r h_i = \frac{1}{r - j + 1} \sum_{i=j}^r \lambda_i + a_i,$$

be the initial averages. We abbreviate  $P_{1,r} = P_r$ .

△

*Remark 3.2* (A reduction procedure) Consider the data, notations and terminology fixed in 3.1. For any  $j \leq p - 1$  denote by

$$\begin{aligned} I_j &= \{1, \dots, d\} \setminus \bigcup_{i \leq j} K_i, L_j = \{1, \dots, k\} \setminus \bigcup_{i \leq j} J_i, \lambda^{L_j} \\ &= (\lambda_i)_{i \in I_j}, \mathcal{G}_j = (g_i)_{i \in L_j}, \mathbf{a}^{L_j} = (a_i)_{i \in L_j} \end{aligned}$$

and take some sequence  $\mathcal{F}_0^{(j)}$  in  $\mathcal{H}_j = [\bigoplus_{i \leq j} W_i]^\perp$  such that  $S_{\mathcal{F}_0^{(j)}} = S_0|_{\mathcal{H}_j}$  (notice that, by construction,  $\mathcal{H}_j$  reduces  $S_0$ ).

Then, it is straightforward to show that  $\mathcal{F}_j = (\mathcal{F}_0^{(j)}, \mathcal{G}_j)$  is a (global) minimizer of  $P_f$  on  $\mathcal{C}_{\mathbf{a}_j}(\mathcal{F}_0^{(j)})$  in  $\mathcal{H}_j$ , i.e. an optimal completion for the reduced problem. Indeed, recall that the minimality is computed in terms of the map  $\text{tr } f(\gamma) = \sum_{i=1}^d f(\gamma_i)$ , for  $\gamma \in \mathbb{R}_{\geq 0}^d$ , which works independently in each entry of  $\lambda(S_{\mathcal{F}}) = v_f(\lambda, \mathbf{a})^\downarrow$ .

The importance of the previous remarks lies in the fact that they provide a powerful reduction method to compute the structure of the sets  $\mathcal{G}_i, K_i$  and  $J_i$  for  $1 \leq i \leq p$  as well as the set of constants  $c_1 > \dots > c_p > 0$ . Indeed, assume that we are able to describe the sets  $\mathcal{G}_1, K_1, J_1$  and the constant  $c_1$  in some structural sense, using the fact that these sets are extremal (e.g. these sets are built on  $c_1 > c_j$  for  $2 \leq j \leq p$ ).

Then, in principle, we could apply these structural arguments to find  $\mathcal{G}_2, K_2, J_2$  and the constant  $c_2$ , using the fact that these are now extremal sets of  $\mathcal{F}_1$ , which is a  $P_f$  minimizer of the reduced CP for  $(\mathcal{F}_0^{(1)}, \mathbf{a}^{L_1})$ . On the other hand, the minimality of the final reduction  $\mathcal{F}_{p-1}$  produces a pair  $(\lambda^{L_{p-1}}, \mathbf{a}^{L_{p-1}})$  which is feasible by item 6 of 3.1, because it has a unique constant  $c_p$  associated to the unique set  $K_p$ . As we shall see, this strategy can be implemented to obtain (inductively) a precise description of the sets above. △

*Remark 3.3* Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP with  $d \leq k$ , fix  $f \in \text{Conv}_s(\mathbb{R}_{\geq 0})$  and let  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G}) \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$  be a global minimum for  $P_f$  on  $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$ . In Sect. 4.1 we shall prove the following properties of the sets  $J_j$  and  $K_j$  defined in item 4. of 3.1 describing  $\mu_f(\lambda, \mathbf{a})$  and  $v_f(\lambda, \mathbf{a})$ :

1. Each set  $J_j$  and  $K_j$  consists of consecutive indices, for  $1 \leq j \leq p$ .
2. The sets  $K_j$  and  $J_j$  have the same number of elements, for  $1 \leq j \leq p - 1$ .
3. Moreover,  $J_1 < \dots < J_p$  (i.e. if  $l \in J_i$  and  $h \in J_j$  with  $i < j \Rightarrow l < h$ ) and  $K_1 < \dots < K_p$ . In particular, by items 1 and 2 above,  $K_j = J_j$  for  $1 \leq j \leq p - 1$ .

△

We state the properties of the sets  $J_j$  and  $K_j$ ,  $1 \leq j \leq p$  described in Remark 3.3 in the following:

**Theorem 3.4** *With the notations of Remark 3.3 (so that, in particular,  $d \leq k$ ) then*

1. *There exist  $0 = s_0 < s_1 < s_2 < \dots < s_{p-1} < s_p = \mathcal{F}$  such that*

$$K_j = J_j = \{s_{j-1} + 1, \dots, s_j\}, \quad \text{for } 1 \leq j \leq p - 1,$$

$$K_p = \{s_{p-1} + 1, \dots, s_p\}, J_p = \{s_{p-1} + 1, \dots, k\}.$$

2. *The vector  $v_f(\lambda, \mathbf{a}) = (c_1 \mathbb{1}_{s_1}, \dots, c_p \mathbb{1}_{s_p - s_{p-1}}, \lambda_{s_p+1}, \dots, \lambda_d)$ , where*

$$c_r = \frac{1}{s_r - s_{r-1}} \sum_{i=s_{r-1}+1}^{s_r} h_i = P_{s_{r-1}+1, s_r} \quad \text{for } 1 \leq r \leq p - 1,$$

*or also  $c_r = \lambda_j + \mu_j$  for every  $j \in K_r = J_r$  for  $1 \leq r \leq p - 1$ .*

3. *The constant  $c_p$  (and the index  $s_p$ ) is determined by the identity*

$$(c_p \mathbb{1}_{s_p - s_{p-1}}, \lambda_{s_p+1}, \dots, \lambda_d) = v(\lambda^{I_{p-1}}, \mathbf{a}^{L_{p-1}}), \tag{13}$$

*where, with the notations in Remark 3.2 (notice that  $I_{p-1} = \{s_{p-1} + 1, \dots, d\}$  and  $L_{p-1} = \{s_{p-1} + 1, \dots, k\}$  according to item 1 above)*

$$\lambda^{I_{p-1}} = (\lambda_i)_{i=s_{p-1}+1}^d \in \mathbb{R}^{d_p}, c\mathbf{a}^{L_{p-1}} = (a_i)_{i=s_{p-1}+1}^k \in \mathbb{R}^{k_p} \text{ (with } d_p \leq k_p)$$

*and  $v(\lambda^{I_{p-1}}, \mathbf{a}^{L_{p-1}})$  is computed as in Remark 2.13.*

*Proof* See Sect. 4.2. □

Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP. Assume that  $v_f(\lambda, \mathbf{a}) = (c_1 \mathbb{1}_{s_1}, \dots, \lambda_{s_1+1}, \dots, \lambda_d)$  i.e. with  $p = 1$ , in the notations of Theorem 3.4. Then, by Lemma 2.17, the pair  $(\lambda, \mathbf{a})$  is feasible and  $v_f(\lambda, \mathbf{a}) = v(\lambda, \mathbf{a})$ .

In what follows we shall need the following notion, that allow us to show feasibility in the more general case in which, in the notations of Theorem 3.4,  $p > 1$ .

**Definition 3.5** Let  $\lambda = \lambda^\uparrow \in \mathbb{R}^d$  and  $\mathbf{a} = \mathbf{a}^\downarrow \in \mathbb{R}_{>0}^k$ , with  $d \leq k$ .

1. Given  $s \leq d - 1$  denote by

$$\lambda^s = (\lambda_{s+1}, \dots, \lambda_d) \in \mathbb{R}^{d-s} \quad \text{and} \quad \mathbf{a}^s = (a_{s+1}, \dots, a_k) \in \mathbb{R}^{k-s},$$

the truncations of the original vectors  $\lambda$  and  $\mathbf{a}$ .

2. We say that the index  $s$  is **feasible** if the pair  $(\lambda^s, \mathbf{a}^s)$  is feasible (see Definition 2.10) i.e. if  $v(\lambda^s, \mathbf{a}^s) - \lambda^s < \mathbf{a}^s$ .

3. If  $s \leq d - 1$  is feasible then, since  $d - s \leq k - s$ , we define

$$v_s \stackrel{\text{def}}{=} v(\lambda^s, \mathbf{a}^s) \stackrel{(9)}{=} (c \mathbb{1}_{r-s}, \lambda_{r+1}, \dots, \lambda_d) \quad \text{where} \quad c = Q_{s,r}$$

for the unique  $r > s$  such that  $\lambda_r \leq c < \lambda_{r+1}$  (or  $v_s = Q_{s,d} \mathbb{1}_{d-s}$  if  $\lambda_d \leq Q_{s,d}$ ). Notice that  $(\lambda^s)_i \leq (v_s)_i$  for  $1 \leq i \leq d - s$  and  $v_s = v_s^\uparrow$ . △

**Proposition 3.6** *Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP. With the notations of Theorem 3.4, the global minimum  $v_f(\lambda, \mathbf{a})$  satisfies that*

1. *The index  $s_{p-1}$  (where the feasible part begins) is determined by*

$$s_{p-1} = \min\{s \leq d : s \text{ is feasible}\},$$

*and the index  $s_p$  is determined by Eq. (13) and Remark 2.13.*

2. *The following recursive method allows to describe the vector  $v_f(\lambda, \mathbf{a})$  as in Theorem 3.4:*

(a) *The index  $s_1 = \max \{j \leq s_{p-1} : P_{1,j} = \max_{i \leq s_{p-1}} P_{1,i}\}$ , and  $c_1 = P_{1,s_1}$ .*

(b) *If the index  $s_j$  is already computed and  $s_j < s_{p-1}$ , then*

$$\begin{aligned} s_{j+1} &= \max \{s_j < r \leq s_{p-1} : P_{s_j+1,j} = \max_{s_j < i \leq s_{p-1}} P_{s_j+1,i}\} \quad \text{and} \quad c_{j+1} \\ &= P_{s_j+1,s_{j+1}}. \end{aligned}$$

*Proof* See Propositions 4.16 and 4.12. □

The following are the main results of the paper. In order to state them, we introduce the spectral picture of the completions with prescribed norms, given by

$$\Lambda(\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)) \stackrel{\text{def}}{=} \{\lambda(S_{\mathcal{F}}) : \mathcal{F} \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)\}.$$

**Theorem 3.7** *Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP with  $d \leq k$ . Then the vector  $v = v_f(\lambda, \mathbf{a})$  is the same for every  $f \in \text{Conv}_s(\mathbb{R}_{\geq 0})$ . Therefore,*

$$v^\downarrow \in \Lambda(\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)) \quad \text{and} \quad v^\downarrow \prec \gamma \quad \text{for every} \quad \gamma \in \Lambda(\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)). \tag{14}$$

*Proof* By Proposition 3.6, the minima  $v = v_f(\lambda, \mathbf{a})$  are completely characterized by the data  $(\lambda, \mathbf{a})$  without interference of the map  $f$ . Therefore, given any  $\gamma \in \Lambda(\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0))$ ,

$$\text{tr } f(v) \leq \text{tr } f(\gamma) \quad \text{for every} \quad f \in \text{Conv}_s(\mathbb{R}_{\geq 0}) \implies v \prec \gamma.$$

□

The following result shows that the structure of optimal completions in  $\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$  in case  $d > k$  can be obtained from the case in which  $k = d$ .



**Theorem 3.8** *Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP with  $d > k$ . If we let*

$$\lambda' = (\lambda_1, \dots, \lambda_k) \in (\mathbb{R}_{\geq 0}^k)^\uparrow \quad \text{then} \quad v_f(\lambda, \mathbf{a}) = (v_f(\lambda', \mathbf{a}), \lambda_{k+1}, \dots, \lambda_d),$$

where  $v_f(\lambda', \mathbf{a})$  is constructed as in Proposition 3.6 (since  $d' = k$ , by construction of  $\lambda' \in (\mathbb{R}_{\geq 0}^{d'})^\uparrow$ ). In this case the vector  $v_f(\lambda, \mathbf{a})$  is the same for every  $f \in \text{Conv}_s(\mathbb{R}_{\geq 0})$  and also satisfies Eq. (14).

*Proof* It is clear that any  $\delta = \delta^\downarrow \in \mathbb{R}_{\geq 0}^d$  such that  $\mathbf{a} < \delta$  must have  $\delta_{k+1} = \dots = \delta_d = 0$ . It is easy to see that this fact implies that

$$\begin{aligned} & \{\lambda^\uparrow + \delta^\downarrow : \delta \in \mathbb{R}_{\geq 0}^d \quad \text{and} \quad \mathbf{a} < \delta\} \\ & = \{(\gamma, \lambda_{k+1}, \dots, \lambda_d) : \gamma_i = \lambda_i + \delta_i^\downarrow, \quad 1 \leq i \leq k\}. \end{aligned} \tag{15}$$

We know that  $v_f(\lambda, \mathbf{a}) - \lambda = \mu = \mu^\downarrow$  and that  $\mathbf{a} < \mu \implies \mu_{k+1} = \dots = \mu_d = 0$ . Therefore

$$v_f(\lambda, \mathbf{a}) = \mu^\downarrow + \lambda^\uparrow \implies v_f(\lambda, \mathbf{a}) = (\rho, \lambda_{k+1}, \dots, \lambda_d), \tag{16}$$

for  $\rho \in \Lambda(\mathcal{C}_{\mathbf{a}}(\mathcal{F}'_0))$  for some  $\mathcal{F}'_0$  such that  $\lambda(S_{\mathcal{F}'_0})^\uparrow = \lambda'$ . Then  $\sum_{i=1}^d f(v_f(\lambda, \mathbf{a})_i) = \sum_{i=1}^d f(\rho_i) + \sum_{i=k+1}^d f(\lambda_i)$ . By Eq. (6) and Eq. (15),

$$\begin{aligned} \sum_{i=1}^d f(v_f(\lambda, \mathbf{a})_i) &= \min \left\{ \sum_{i=1}^d f(\lambda_i + \gamma_i) : \gamma = \gamma^\downarrow \in \mathbb{R}_{\geq 0}^d, \mathbf{a} < \gamma \right\} \\ &= \left[ \min \left\{ \sum_{i=1}^k f(\lambda'_i + \gamma_i) : \gamma = \gamma^\downarrow \in \mathbb{R}_{\geq 0}^k, \mathbf{a} < \gamma \right\} \right] + \sum_{i=k+1}^d f(\lambda_i). \end{aligned}$$

Using Eq. (6) again we deduce that  $\rho = v_f(\lambda', \mathbf{a})$ . Since  $v_f(\lambda', \mathbf{a})$  is constructed as in Proposition 3.6, then it is the same vector for every strictly convex map  $f$ ; hence,  $v_f(\lambda, \mathbf{a})$  is the same vector for every strictly convex map  $f$ , so that  $v_f(\lambda, \mathbf{a})^\downarrow$  is a minimum for majorization on  $\Lambda(\mathcal{C}_{\mathbf{a}}(\mathcal{F}_0))$ .  $\square$

*Remark 3.9* The construction of the minimum  $v_f(\lambda, \mathbf{a})$  given by Proposition 3.6 is algorithmic, as it can be easily implemented in MATLAB. It only depends on an -already available, see [30] - routine for checking feasibility, which is fast and efficient.  $\triangle$

### 4 Proofs of Some Technical Results

In this section we present detailed proofs of several statements in Sect. 3. All these results assume that the initial data  $(\mathcal{F}_0, \mathbf{a})$  for the CP satisfies that  $k \geq d$ . As seen in Theorem 3.8, the general case can be reduced to this situation.

4.1 Description of the Sets  $K_i$  and  $J_i$ .

**4.1** We begin by recalling the notations of 3.1: Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP, with  $k \geq d$ . Fix a convex map  $f \in \text{Conv}_s(\mathbb{R}_{\geq 0})$ . We consider the following objects:

1. Let  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G}) \in \mathcal{C}_a$  be a global minimum for  $P_f$  on  $\mathcal{C}_a(\mathcal{F}_0)$  (or a local minimum if  $\mathcal{F}_0 = \emptyset$ ). Therefore,

$$\mathcal{F} \in \mathcal{C}_a^{\text{op}}(\mathcal{F}_0) \stackrel{\text{def}}{=} \{\mathcal{F} = (\mathcal{F}_0, \mathcal{G}) : \lambda(S_{\mathcal{F}}) = (\lambda + \lambda(S_{\mathcal{G}}))^{\downarrow}\}.$$

2. By Theorem 2.7 there exists an orthonormal basis of  $\mathcal{H} \{v_i\}_{i=1}^d$  such that

$$S_{\mathcal{G}} = \sum_{i=1}^d \mu_i \cdot v_i \otimes v_i \quad \text{and} \quad S_{\mathcal{F}} = S_{\mathcal{F}_0} + S_{\mathcal{G}} = \sum_{i=1}^d (\lambda_i + \mu_i) v_i \otimes v_i.$$

3. Let  $s_{\mathcal{F}} = \max\{1 \leq i \leq d : \mu_i \neq 0\} = \text{rk } S_{\mathcal{G}}$ . Denote by  $W = R(S_{\mathcal{G}})$ , which reduces  $S_{\mathcal{F}}$ .
4. Let  $S = S_{\mathcal{F}}|_W \in L(W)$  and  $\sigma(S) = \{c_1, \dots, c_p\}$  (where  $c_1 > c_2 > \dots > c_p$ ).
5. Let  $K_j = \{1 \leq i \leq s : \lambda_i + \mu_i = c_j\}$  and  $J_j = \{1 \leq i \leq k : S g_i = c_j g_i\}$ . Then

$$\{1, \dots, s_{\mathcal{F}}\} = \bigcup_{1 \leq j \leq p} K_j \quad \text{and} \quad \{1, \dots, k\} = \bigcup_{1 \leq k \leq p} J_k.$$

We remark that, if  $\mathcal{F}_0 = \emptyset$ , these facts are still valid for local minima by Theorem 2.9. △

The next three Propositions give a complete proof of Theorem 3.4. The first of them justifies the convention that  $\lambda = \lambda(S_{\mathcal{F}_0})^{\uparrow}$ .

*Remark 4.2* In what follows we shall need the following elementary property of majorization (see [3]): if  $x_1, y_1 \in \mathbb{R}^r$  and  $x_2, y_2 \in \mathbb{R}^s$  are such that

$$x_i < y_i \quad \text{for } i = 1, 2 \implies x = (x_1, x_2) < y = (y_1, y_2) \quad \text{in } \mathbb{R}^{r+s}. \quad (17)$$

△

**Proposition 4.3** *Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP with  $\lambda = \lambda(S_{\mathcal{F}_0})^{\uparrow}$ , and consider the notations of 4.1. If  $p > 1$ , then*

$$i \in K_1 \implies i < j \quad (\implies \lambda_i \leq \lambda_j) \quad \text{for every } j \in \bigcup_{r>1} K_r = \{1, \dots, s_{\mathcal{F}}\} \setminus K_1.$$

*Inductively, by means of Remark 3.2, we deduce that all sets  $K_j$  consist on consecutive indices, and that  $K_i < K_j$  (in terms of their elements) if  $i < j$ .*

*Proof* Suppose that there are  $i \in K_1$  and  $j \in K_r$  (for some  $r > 1$ ) such that  $j < i$ . Then  $\lambda_j \leq \lambda_i$  and  $\mu_i \leq \mu_j$ . For  $t > 0$  very small, let  $\mu_i(t) = \mu_i - t > 0$  and  $\mu_j(t) = \mu_j + t$ . Consider the vector  $\mu(t)$  obtained by changing in  $\mu$  the entries  $\mu_i$  by  $\mu_i(t)$  and  $\mu_j$  by  $\mu_j(t)$ . Observe that not necessarily  $\mu(t) = \mu(t)^\downarrow$ , but we are indeed sure that  $c_1 > c_r$ .

Nevertheless, by Remark 4.2,  $(\mu_i, \mu_j) \prec (\mu_i(t), \mu_j(t)) \implies \mathbf{a} \prec \mu \prec \mu(t)$ . Therefore there exists  $\mathcal{F}' = (\mathcal{F}_0, \mathcal{G}') \in \mathcal{C}_{\mathbf{a}}(\mathcal{F}_0)$  such that, using the ONB of Eq. (11),

$$S_{\mathcal{G}'} = \sum_{h=1}^d \mu_h(t) \cdot v_h \otimes v_h \quad \text{and} \quad S_{\mathcal{F}'} = S_{\mathcal{F}_0} + S_{\mathcal{G}'} = \sum_{h=1}^d (\lambda_h + \mu_h(t)) v_h \otimes v_h.$$

Denote by  $V = \text{span}\{v_i, v_j\}$ , which reduces both  $S_{\mathcal{F}}$  and  $S_{\mathcal{F}'}$ . Also  $S_{\mathcal{F}'}|_{V^\perp} = S_{\mathcal{F}}|_{V^\perp}$ . Considering the restrictions to  $V$  as operators in  $L(V) \cong \mathcal{M}_2(\mathbb{C})$  we get that

$$\begin{aligned} \lambda(S_{\mathcal{F}'}|_V) &= (\lambda_i + \mu_i(t), \lambda_j + \mu_j(t)) \\ &= (c_1 - t, c_r + t) \prec (c_1, c_r) = \lambda(S_{\mathcal{F}}|_V) \quad \text{strictly,} \end{aligned}$$

for  $t$  small enough in such a way that  $c_1 - t > c_r + t$ , so that  $(c_1 - t, c_r + t) = (c_1 - t, c_r + t)^\downarrow$ . Then

$$\begin{aligned} f(c_1 - t) + f(c_r + t) &< f(c_1) + f(c_r) \implies P_f(\mathcal{F}') \\ &= F(\lambda(S_{\mathcal{F}'})) < F(\lambda(S_{\mathcal{F}})) = P_f(\mathcal{F}), \end{aligned}$$

a contradiction. The inductive argument follows from Remark 3.2. □

**4.4** In the following two statements we assume that, for some  $f \in \text{Conv}_s(\mathbb{R}_{\geq 0})$ , the sequence  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G}) \in \mathcal{C}_{\mathbf{a}}^{\text{op}}(\mathcal{F}_0)$  is a global minimum for  $P_f$ , or it is a local minimum if  $S_{\mathcal{F}_0} = 0$  and  $\lambda = 0$ . In both cases 4.1 applies. △

**Proposition 4.5** *Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP, and let  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G}) \in \mathcal{C}_{\mathbf{a}}^{\text{op}}(\mathcal{F}_0)$  as in 4.1 and 4.4. Suppose that  $p > 1$ . Given  $h \in J_i$  and  $l \in J_r$  then*

$$i < r \implies a_h - a_l \geq c_i - c_r > 0.$$

*In particular, the sets  $J_i$  consist of consecutive indices, and  $J_1 < J_2 < \dots < J_p$  (in terms of their elements).* □

*Proof* Let us assume that  $i < r \leq p$ ,  $h \in J_i$  and  $l \in J_r$ , but  $l < h$  (even less: that  $a_l \geq a_h$ ). Then

$$g_l \otimes g_l \leq S_{\mathcal{G}} \leq S_{\mathcal{F}} \quad \text{and} \quad S_{\mathcal{F}} g_l = c_r g_l \implies a_h = \|g_h\|^2 \leq \|g_l\|^2 = a_l \leq c_r < c_i.$$

We also know that  $\langle g_l, g_h \rangle = 0$ . Denote by  $w_h = \frac{g_h}{\|g_h\|} = a_h^{-1/2} g_h$  and  $w_l = \frac{g_l}{\|g_l\|} = a_l^{-1/2} g_l$ . Let

$$\begin{aligned} g_h(t) &= \cos(t) g_h + \sin(t) \|g_h\| w_l \quad \text{and} \\ g_l(t) &= \cos(\gamma t) g_l + \sin(\gamma t) \|g_l\| w_h \quad \text{for } t \in \mathbb{R} \end{aligned}$$

for some convenient  $\gamma > 0$  that we shall find later. Let  $\mathcal{F}_\gamma(t)$  be the sequence obtained by changing in  $\mathcal{F}$  the vectors  $g_h$  by  $g_h(t)$  and  $g_l$  by  $g_l(t)$ , for every  $t \in \mathbb{R}$ . Notice that  $\|g_h(t)\|^2 = a_h$  and  $\|g_l(t)\|^2 = a_l$  for every  $t \in \mathbb{R}$ , so that all the sequences  $\mathcal{F}_\gamma(t) \in \mathcal{C}_a(\mathcal{F}_0)$ .

Let  $W = \text{span}\{w_h, w_l\}$ , a subspace which reduces  $S_{\mathcal{F}}$  and  $S_{\mathcal{F}_\gamma(t)}$ . Note that  $g_h(t), g_l(t) \in W$ . In the matrix representation with respect to this basis of  $W$  we get that

$$g_h \otimes g_h = \begin{bmatrix} a_h & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} w_h \\ w_l \end{matrix}, g_h(t) \otimes g_h(t) = a_h \begin{bmatrix} \cos^2(t) & \cos(t) \sin(t) \\ \cos(t) \sin(t) & \sin^2(t) \end{bmatrix} \begin{matrix} w_h \\ w_l \end{matrix},$$

$$g_l \otimes g_l = \begin{bmatrix} 0 & 0 \\ 0 & a_l \end{bmatrix} \begin{matrix} w_h \\ w_l \end{matrix} \quad \text{and} \quad g_l(t) \otimes g_l(t) = a_l \begin{bmatrix} \sin^2(\gamma t) & \cos(t) \sin(t) \\ \cos(t) \sin(t) & \cos^2(\gamma t) \end{bmatrix} \begin{matrix} w_h \\ w_l \end{matrix}$$

If we denote by  $S(t) = S_{\mathcal{F}_\gamma(t)}$ , we get that

$$S(t) = S_{\mathcal{F}} - g_h \otimes g_h - g_l \otimes g_l + g_h(t) \otimes g_h(t) + g_l(t) \otimes g_l(t).$$

Therefore  $S(t)|_{W^\perp} = S_{\mathcal{F}}|_{W^\perp}$ . On the other hand,  $S_{\mathcal{F}}|_W = \begin{bmatrix} c_i & 0 \\ 0 & c_r \end{bmatrix}$ . Then

$$S(t)|_W = \begin{bmatrix} c_i + a_h(\cos^2(t) - 1) + a_l \sin^2(\gamma t) & a_h \cos(t) \sin(t) + a_l \cos(\gamma t) \sin(\gamma t) \\ a_h \cos(t) \sin(t) + a_l \cos(\gamma t) \sin(\gamma t) & c_r + a_h \sin^2(t) + a_l^2(\cos^2(\gamma t) - 1) \end{bmatrix} \stackrel{\text{def}}{=} A_\gamma(t).$$

Note that  $\text{tr } A_\gamma(t) = c_i + c_r$  for every  $t \in \mathbb{R}$ . Therefore  $\lambda(A_\gamma(t)) < (c_i, c_r)$  strictly  $\iff \|A_\gamma(t)\|_2^2 < c_i^2 + c_r^2$ . Hence we consider the map  $m_\gamma : \mathbb{R} \rightarrow \mathbb{R}$  given by

$$m_\gamma(t) = \|A_\gamma(t)\|_2^2 = \text{tr}(A_\gamma(t)^2) \quad \text{for every } t \in \mathbb{R}.$$

Note that  $S(0) = S_{\mathcal{F}} \implies m_\gamma(0) = c_i^2 + c_r^2$ . We shall see that, for a convenient choice of  $\gamma$ , it holds that  $m'_\gamma(0) = 0$  but  $m''_\gamma(0) < 0$ . This will contradict the (local) minimality of  $\mathcal{F}$ , because  $m_\gamma$  would have in this case a maximum at  $t = 0$ , so that  $\lambda(A_\gamma(t)) < (c_i, c_r)$  strictly  $\xrightarrow{(19)} \lambda(S_{\mathcal{F}_\gamma(t)}) < \lambda(S_{\mathcal{F}})$  strictly  $\implies P_f(\mathcal{F}_\gamma(t)) < P_f(\mathcal{F})$  for every  $t$  near 0.

Indeed, we first compute the derivatives of the entries  $a_{ij}$  of  $A_\gamma(t)$ :

$$\begin{aligned} a'_{11} &= -a_h \sin(2t) + \gamma a_l \sin(2\gamma t) & a''_{11} &= 2[-a_h \cos(2t) + \gamma^2 a_l \cos(2\gamma t)] \\ a'_{12} &= a_h \cos(2t) + \gamma a_l \cos(2\gamma t) & \text{and} & & a''_{22} &= 2[a_h \cos(2t) - \gamma^2 a_l \cos(2\gamma t)] \\ a'_{22} &= a_h \sin(2t) - \gamma a_l \sin(2\gamma t) \end{aligned}$$

So  $a'_{11}(0) = 0, a'_{22}(0) = 0$  and  $a_{12}(0) = 0$ . Then, for  $i, j \in \{1, 2\}$  we have that

$$(a'_{ij})'(0) = 2 a_{ij}(0) a'_{ij}(0) = 0 \quad \text{and} \quad (a'_{ij})''(0) = 2((a'_{ij})^2(0) + a_{ij}(0) a''_{ij}(0)).$$

Therefore  $(a_{11}^2)''(0) = 4c_i(-a_h + \gamma^2 a_l)$ ,  $(a_{12}^2)''(0) = 2(a_h + \gamma a_l)^2$  and  $(a_{22}^2)''(0) = -4c_r(-a_h + \gamma^2 a_l)$ . We conclude that  $m'_\gamma(0) = 0$  (for every  $\gamma \in \mathbb{R}$ ) and that

$$m''_\gamma(0) = 4 \left[ c_i(-a_h + \gamma^2 a_l) + (a_h + \gamma a_l)^2 - c_r(-a_h + \gamma^2 a_l) \right],$$

which is quadratic polynomial on  $\gamma$  with discriminant (if we drop the factor 4) given by

$$D = a_h a_l \left[ a_h a_l - (a_l + (c_i - c_r))(a_h - (c_i - c_r)) \right].$$

As we are assuming that  $a_l \geq a_h$  then  $D > 0$ , because

$$(a_l + (c_i - c_r))(a_h - (c_i - c_r)) = a_l a_h - (c_i - c_r)(a_l - a_h) - (c_i - c_r)^2 < a_l a_h.$$

Hence there exists  $\gamma \in \mathbb{R}$  such that  $m''_\gamma(0) < 0$ . Observe that as long as  $0 < (c_i - c_r)(a_l - a_h) + (c_i - c_r)^2$  ( $\iff a_h - a_l < c_i - c_r$ ) we arrive at the same contradiction.  $\square$

The following result is inspired on some ideas from [10].

**Proposition 4.6** *Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP, and let  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G}) \in \mathcal{C}_a^{\text{op}}(\mathcal{F}_0)$  as in 4.1 and 4.4. For every  $j < p$ , the subsequence  $\{g_i\}_{i \in J_j}$  of  $\mathcal{G}$  is linearly independent.*

*Proof* Suppose that there exists  $1 \leq j \leq p - 1$  such that  $\{g_i\}_{i \in J_j}$  is linearly dependent. Hence there exists coefficients  $z_l \in \mathbb{C}$ ,  $l \in J_j$  (not all zero) such that  $|z_l| \leq 1/2$  and

$$\sum_{l \in J_j} \bar{z}_l a_l g_l = 0. \tag{18}$$

Let  $I_j \subseteq J_j$  be given by  $I_j = \{l \in J_j : z_l \neq 0\}$  and let  $h \in \mathcal{H}$  such that  $\|h\| = 1$  and  $S_{\mathcal{F}}h = c_p h$ . For  $t \in (-1, 1)$  let  $\mathcal{F}(t) = (\mathcal{F}_0, \mathcal{G}(t))$  where  $\mathcal{G}(t) = \{g_i(t)\}_{i \in \mathbb{I}_k}$  is given by

$$g_l(t) = \begin{cases} (1 - t^2 |z_l|^2)^{1/2} g_l + t z_l a_l h & \text{if } l \in I_j \\ g_l & \text{if } 1 \leq l \leq k \setminus I_j. \end{cases}$$

Fix  $l \in I_j$ . Let  $\text{Re}(A) = \frac{A+A^*}{2}$  denote the real part of each  $A \in L(\mathcal{H})$ . Then

$$g_l(t) \otimes g_l(t) = (1 - t^2 |z_l|^2) g_l \otimes g_l + t^2 |z_l|^2 a_l^2 h \otimes h + 2(1 - t^2 |z_l|^2)^{1/2} t \text{Re}(h \otimes a_l z_l g_l)$$

Let  $S(t)$  denote the frame operator of  $\mathcal{F}(t)$  and notice that  $S(0) = S_{\mathcal{F}}$ . Note that

$$S(t) = S_{\mathcal{F}} + t^2 \sum_{l \in I_j} |z_l|^2 \left( -g_l \otimes g_l + a_l^2 h \otimes h \right) + R(t)$$

where  $R(t) = 2 \sum_{l \in I_j} (1 - t^2 |z_l|^2)^{1/2} t \operatorname{Re}(h \otimes a_l z_l g_l)$ . Then  $R(t)$  is a smooth function such that

$$R(0) = 0, R'(0) = \sum_{l \in I_j} \operatorname{Re}(h \otimes a_l z_l g_l) = \operatorname{Re}(h \otimes \sum_{l \in I_j} a_l z_l g_l) = 0,$$

and such that  $R''(0) = 0$ . Therefore  $\lim_{t \rightarrow 0} t^{-2} R(t) = 0$ . We now consider

$$W = \operatorname{span} (\{g_l : l \in I_j\} \cup \{h\}) = \operatorname{span} \{g_l : l \in I_j\} \perp \mathbb{C} \cdot h.$$

Then  $\dim W = s + 1$ , for  $s = \dim \operatorname{span}\{g_l : l \in I_j\} \geq 1$ . By construction, the subspace  $W$  reduces  $S_{\mathcal{F}}$  and  $S(t)$  for  $t \in \mathbb{R}$ , in such a way that  $S(t)|_{W^\perp} = S_{\mathcal{F}}|_{W^\perp}$  for  $t \in \mathbb{R}$ . On the other hand

$$S(t)|_W = S_{\mathcal{F}}|_W + t^2 \sum_{l \in I_j} |z_l|^2 (-g_l \otimes g_l + a_l^2 h \otimes h) + R(t) = A(t) + R(t) \in L(W), \tag{19}$$

where we use the fact that the ranges of the selfadjoint operators in the second and third term in the formula above clearly lie in  $W$ . Then  $\lambda(S_{\mathcal{F}}|_W) = (c_j \mathbb{1}_s, c_p) \in (\mathbb{R}_{>0}^{s+1})^\downarrow$  and

$$\lambda\left(\sum_{l \in I_j} |z_l|^2 g_l \otimes g_l\right) = (\gamma_1, \dots, \gamma_s, 0) \in (\mathbb{R}_{\geq 0}^{s+1})^\downarrow \quad \text{with } \gamma_s > 0,$$

where we have used the definition of  $s$  and the fact that  $|z_l| > 0$  for  $l \in I_j$ . Hence, for sufficiently small  $t$ , the spectrum of the operator  $A(t) \in L(W)$  defined in (19) is

$$\lambda(A(t)) = (c_j - t^2 \gamma_s, \dots, c_j - t^2 \gamma_1, c_p + t^2 \sum_{l \in I_j} a_l^2 |z_l|^2) \in (\mathbb{R}_{\geq 0}^{s+1})^\downarrow,$$

where we have used the fact that  $\langle g_l, h \rangle = 0$  for every  $l \in I_j$ . Let us now consider

$$\lambda(R(t)) = (\delta_1(t), \dots, \delta_{s+1}(t)) \in (\mathbb{R}_{\geq 0}^{s+1})^\downarrow \quad \text{for } t \in \mathbb{R}.$$

Recall that in this case  $\lim_{t \rightarrow 0} t^{-2} \delta_j(t) = 0$  for  $1 \leq j \leq s + 1$ . Using Weyl’s inequality on Eq. (19), we now see that  $\lambda(S(t)|_W) < \lambda(A(t)) + \lambda(R(t)) \stackrel{\text{def}}{=} \rho(t) \in (\mathbb{R}_{\geq 0}^{s+1})^\downarrow$ . We know that

$$\begin{aligned} \rho(t) &= (c_j - t^2 \gamma_s + \delta_1(t), \dots, c_j - t^2 \gamma_1 + \delta_s(t), c_p + t^2 \sum_{l \in I_j} a_l^2 |z_l|^2 + \delta_{s+1}(t)) \\ &= \left(c_j - t^2 \left(\gamma_s - \frac{\delta_1(t)}{t^2}\right), \dots, c_j - t^2 \left(\gamma_1 - \frac{\delta_s(t)}{t^2}\right), c_p + t^2 \left(\sum_{l \in I_j} a_l^2 |z_l|^2 + \frac{\delta_{s+1}(t)}{t^2}\right)\right). \end{aligned}$$

A direct test shows that, for small  $t$ , this  $\rho(t) < \lambda(S_{\mathcal{F}}|_W) = (c_j \mathbb{1}_s, c_p)$  strictly. Then, since  $f$  is strictly convex, for every sufficiently small  $t$  we have that

$$\begin{aligned} P_f(\mathcal{F}(t)) &\leq \operatorname{tr} f(\lambda(S_{\mathcal{F}}|_{W^\perp})) + \operatorname{tr} f(\rho(t)) \\ &< \operatorname{tr} f(\lambda(S_{\mathcal{F}}|_{W^\perp})) + \operatorname{tr} f(\lambda(S_{\mathcal{F}}|_W)) = P_f(\mathcal{F}). \end{aligned}$$

This last fact contradicts the assumption that  $\mathcal{F}$  is a local minimizer of  $P_f$  in  $C_{\mathbf{a}}^{\text{op}}(\mathcal{F}_0)$ . □

*Remark 4.7* Proposition 4.6 allows to show that in case  $\mathcal{F}_0 = \emptyset$  then local and global minimizers of a convex potential  $P_f$ , induced by  $f \in \text{Conv}_s(\mathbb{R}_{\geq 0})$ , on  $C_{\mathbf{a}}(\mathcal{F}_0)$ —endowed with the product topology—coincide, as conjectured in [27].

Recall that a local minimizer  $\mathcal{F}$  is a juxtaposition of tight frame sequences  $\{\mathcal{F}_i\}_{i=1}^p$  which generate pairwise orthogonal subspaces of  $\mathcal{H}$ . Notice that by [31, Lemma 4.9]  $\mathcal{F}$  is a frame for  $\mathcal{H}$ . Moreover, by Proposition 4.5, it is constructed using a partition of  $\mathbf{a}$  with consecutive indices.

Now by inspection of the proof of Proposition 4.6 we see that only one of such frame sequences can be a linearly dependent set: that with the smallest tight constant  $c_p$ . This forces that the (ordered) spectrum  $\nu$  of a local minimizer must be either  $\nu = c\mathbb{1}_d$  or

$$\nu = (a_1, a_2, \dots, a_r, c, \dots, c), \quad \text{where } a_r > c \geq a_{r+1},$$

and  $c$  is the constant of the unique tight subframe constructed with a linear dependent sequence of vectors with norms given by  $\{a_i\}_{i=r+1}^k$  (notice that this forces  $c \geq a_{r+1}$ ). But it is not difficult to see that this vector can be constructed in a unique way, that is, there is only one  $r$  such that

$$a_{r+1} \leq c = \frac{1}{d} \left( \text{tr}(\mathbf{a}) - \sum_{i=1}^r a_i \right) < a_r.$$

That is, the spectrum of local minimizers is unique and therefore local and global minimizers of  $P_f$  coincide, for every potential  $P_f$  as above. △

### 4.2 Several Proofs

Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP with  $\lambda = \lambda(S_{\mathcal{F}_0})^\uparrow$ ,  $\mathbf{a} = \mathbf{a}^\downarrow$  and  $d \leq k$ . Recall that we denote by  $h_i = \lambda_i + a_i$  for every  $1 \leq i \leq d$  and, given  $j \leq r \leq d$ , we denote by

$$P_{j,r} = \frac{1}{r-j+1} \sum_{i=j}^r h_i = \frac{1}{r-j+1} \sum_{i=j}^r \lambda_i + a_i.$$

We shall abbreviate  $P_{1,r} = P_r$ .

**4.8 (Proof of Theorem 3.4)** We rewrite its statement: Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP with  $d \geq k$ . Let  $\mathcal{F} = (\mathcal{F}_0, \mathcal{G}) \in C_{\mathbf{a}}^{\text{op}}(\mathcal{F}_0)$  be a **global** minimum for  $P_f$  on  $C_{\mathbf{a}}^{\text{op}}(\mathcal{F}_0)$ . Using the notations of 3.1, assume that  $\lambda = \lambda(S_{\mathcal{F}_0})^\uparrow$ ,  $\mu = \mu^\downarrow = \mu_f(\lambda, \mathbf{a})$  and  $\mathbf{a} = \mathbf{a}^\downarrow$ . Then

1. There exist indices  $0 = s_0 < s_1 < \dots < s_{p-1} < s_p = s_{\mathcal{F}} = \max\{1 \leq j \leq d : \mu_j \neq 0\}$  such that

$$\begin{aligned}
 K_j &= J_j = \{s_{j-1} + 1, \dots, s_j\}, \quad \text{for } j \in \mathbb{I}_{p-1}, \\
 K_p &= \{s_{p-1} + 1, \dots, s_p\}, \quad J_p = \{s_{p-1} + 1, \dots, k\}.
 \end{aligned}
 \tag{20}$$

2. The vector  $v_f(\lambda, \mathbf{a}) = (c_1 \mathbb{1}_{s_1}, \dots, c_p \mathbb{1}_{s_p - s_{p-1}}, \lambda_{s_p+1}, \dots, \lambda_d)$ , where

$$c_r = \frac{1}{s_r - s_{r-1}} \sum_{i=s_{r-1}+1}^{s_r} h_i = P_{s_{r-1}+1, s_r} \quad \text{for } r \in \mathbb{I}_{p-1},
 \tag{21}$$

or also  $c_r = \lambda_j + \mu_j$  for every  $j \in K_r = J_r$  for  $r \in \mathbb{I}_{p-1}$ .

3. The constant  $c_p$  (and the index  $s_p$ ) is determined by the identity

$$(c_p \mathbb{1}_{s_p - s_{p-1}}, \lambda_{s_p+1}, \dots, \lambda_d) = v(\lambda^{I_{p-1}}, \mathbf{a}^{L_{p-1}}),$$

where, with the notations in Remark 3.2 (notice that  $I_{p-1} = \{s_{p-1} + 1, \dots, d\}$  and  $L_{p-1} = \{s_{p-1} + 1, \dots, k\}$  according to item 1 above)

$$\lambda^{I_{p-1}} = (\lambda_i)_{i=s_{p-1}+1}^d \in \mathbb{R}^{d_p}, \quad \mathbf{a}^{L_{p-1}} = (a_i)_{i=s_{p-1}+1}^k \in \mathbb{R}^{k_p} \quad (\text{with } d_p \leq k_p)$$

and  $v(\lambda^{I_{p-1}}, \mathbf{a}^{L_{p-1}})$  is computed as in Remark 2.13.

*Proof* Recall from Eq. (12) that for every  $1 \leq j \leq p - 1$

$$W_j \stackrel{\text{def}}{=} \ker(S - c_j I_W) = \text{span}\{v_i : i \in K_j\} = \text{span}\{g_i : i \in J_j\}$$

By Proposition 4.6  $|J_j| = \dim W_j = |K_j|$  for  $j < p$ . Using now Propositions 4.3 and 4.5, we deduce that there exist indices  $0 = s_0 < s_1 < s_2 < \dots < s_{p-1} < s_p = s_{\mathcal{F}} = \max\{1 \leq j \leq d : \mu_j \neq 0\}$  such that the sets  $K_j$  and  $J_j$  satisfy Eq. (20). Using Eq. (12) again,

$$S_{\mathcal{G}}|_{W_j} = \sum_{i \in J_j} g_i \otimes g_i \implies \text{tr } S_{\mathcal{G}}|_{W_j} = \sum_{i \in K_j} \mu_i = \sum_{i \in J_j} a_i.
 \tag{22}$$

Therefore  $(s_j - s_{j-1}) c_j = \text{tr } S|_{W_j} = \text{tr } S_{\mathcal{F}_0}|_{W_j} + \text{tr } S_{\mathcal{G}}|_{W_j} = \sum_{i \in K_j} h_i$ , for every  $j < p$ . Then the vector  $v_f(\lambda, \mathbf{a}) = (c_1 \mathbb{1}_{s_1}, \dots, c_p \mathbb{1}_{s_p - s_{p-1}}, \lambda_{s_p+1}, \dots, \lambda_d)$ , where the constants  $c_r$  are given by Eq. (21) for  $r < p$ . Item 3 follows from Remark 3.2 and Lemma 2.17.  $\square$

**Lemma 4.9** *Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP with  $k \geq d$ . Given  $1 \leq m \leq d$ ,*

$$(a_j)_{j=1}^m < (P_m - \lambda_j)_{j=1}^m \iff P_m \geq P_i \quad \text{for every } 1 \leq i \leq m \iff P_{1,m} = \max_{1 \leq i \leq m} \{P_{1,i}\}.$$

*Proof* Straightforward.  $\square$



*Remark 4.10* Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP with  $k \geq d$  and recall the description of a minimum  $v_f(\lambda, \mathbf{a})$  given in Theorem 3.4. As in Lemma 4.9 (or by an inductive argument using Remark 3.2) we can assure that for every  $r \leq p - 1$ , the constants

$$c_r = P_{s_{r-1}+1, s_r} \geq P_{s_{r-1}+1, j} \quad \text{for every } j \text{ such that } s_{r-1} + 1 \leq j \leq s_r. \quad (23)$$

It uses that  $(a_j)_{j=s_{r-1}+1}^{s_r} < (\mu_j)_{j=s_{r-1}+1}^{s_r} = (c_r - \lambda_j)_{j=s_{r-1}+1}^{s_r}$ , a consequence of Eq. (22). △

**Lemma 4.11** *Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP. With the notations of Theorem 3.4, the global minimum  $v_f(\lambda, \mathbf{a})$ , its constants  $c_j$  and the indices  $s_j$  (for  $1 \leq j \leq p$ ) satisfy the following properties:*

1. Suppose that  $p > 1$ . For every  $1 \leq j \leq p - 1$  such that  $j > 1$ , the constant  $c_j$  satisfies that

$$c_j = P_{s_{j-1}+1, s_j} = \frac{1}{s_j - s_{j-1}} \sum_{i=s_{j-1}+1}^{s_j} h_i < \frac{1}{s_j} \sum_{i=1}^{s_j} h_i = P_{1, s_j}. \quad (24)$$

2. Fix  $1 \leq j \leq p - 1$  such that  $j > 1$ . Then

$$P_{1, t} < P_{1, s_{j-1}} \quad \text{for every } s_{j-1} < t \leq s_{p-1}. \quad (25)$$

3. In particular the averages  $P_{1, s_j} = \frac{1}{s_j} \sum_{i=1}^{s_j} h_i < \frac{1}{s_{j-1}} \sum_{i=1}^{s_{j-1}} h_i = P_{1, s_{j-1}}$  for  $2 \leq j \leq p - 1$ .

*Proof* The inequality of item 1 follows since

$$\begin{aligned} \sum_{i=1}^{s_j} h_i &= \sum_{i=1}^{s_1} h_i + \sum_{i=s_1+1}^{s_2} h_i + \dots + \sum_{i=s_{j-1}+1}^{s_j} h_i \\ &= s_1 c_1 + (s_2 - s_1) c_2 + \dots + (s_j - s_{j-1}) c_j > s_j c_j. \end{aligned}$$

Now we prove the inequality of Eq. (25): Given an index  $t$  such that  $s_{j-1} < t \leq s_j$ ,

$$\begin{aligned} t P_{1, t} &= s_{j-1} P_{1, s_{j-1}} + \sum_{i=s_{j-1}+1}^t h_i \\ &= s_{j-1} P_{1, s_{j-1}} + (t - s_{j-1}) \frac{1}{(t - s_{j-1})} \sum_{i=s_{j-1}+1}^t h_i \\ &\stackrel{(25)}{\leq} s_{j-1} P_{1, s_{j-1}} + (t - s_{j-1}) c_j \\ &< s_{j-1} P_{1, s_{j-1}} + (t - s_{j-1}) c_{j-1} \\ &\leq s_{j-1} P_{1, s_{j-1}} + (t - s_{j-1}) P_{1, s_{j-1}} = t P_{1, s_{j-1}}, \end{aligned}$$

where we used the fact that  $c_{j-1} \leq P_{1, s_{j-1}}$  for  $1 \leq j - 1 \leq p - 1$ , which follows from item 1. In particular we have proved item 3, and this also proves that Eq. (25) holds for  $s_j < t \leq s_{p-1}$ . □

**Proposition 4.12** *With the notations of Theorem 3.4, the global minimum  $v = v_f(\lambda, \mathbf{a})$ , its constants  $c_j$  and the indices  $s_j$  (for  $1 \leq j \leq p$ ) satisfy the following properties: suppose we know the index  $s_{p-1}$ , and that  $p > 1$ . Then we have a recursive method to reconstruct  $v$ :*

1. *The index  $s_1 = \max \{j \leq s_{p-1} : P_{1,j} = \max_{i \leq s_{p-1}} P_{1,i}\}$ , and  $c_1 = P_{1,s_1}$ .*
2. *If we already compute the index  $s_j$  and  $s_j < s_{p-1}$ , then*

$$s_{j+1} = \max \{s_j < r \leq s_{p-1} : P_{s_j+1,r} = \max_{s_j < i \leq s_{p-1}} P_{s_j+1,i}\}$$

and  $c_{j+1} = P_{s_j+1,s_{j+1}}$ .

*Proof* The formula  $P_{1,s_1} = \max_{i \leq s_{p-1}} P_{1,i}$  follows from Lemma 4.9 and Eq. (25) of Lemma 4.11, which also implies that  $s_1$  must be the greater index (before  $s_{p-1}$ ) satisfying this property.

The iterative program works by applying the last fact to the successive truncations of  $v$  which are still minima in their neighborhood, by Remark 3.2. □

**Definition 4.13** Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP. Assume that  $d \leq k$ . We denote by

$$\lambda = \lambda(S_{\mathcal{F}_0})^\uparrow \quad \text{and} \quad h_i = \lambda_i + a_i \quad \text{for every} \quad 1 \leq i \leq d.$$

Given  $j, r \in \{0, 1, \dots, d\}$  such that  $j < r$ , by  $Q_{j,r}$  we denote the final averages:

$$Q_{j,r} = \frac{1}{r-j} \left[ \sum_{i=j+1}^r h_i + \sum_{i=r+1}^k a_i \right] = \frac{1}{r-j} \left[ \sum_{i=j+1}^k a_i + \sum_{i=j+1}^r \lambda_i \right]. \quad (26)$$

Notice that the numbers  $Q_r$  defined in 2.12 satisfy  $Q_r = Q_{0,r}$ . △

Recall the notion of feasible indices given in Definition 3.5: given  $1 \leq s \leq d - 1$  denote by  $\lambda^s = (\lambda_{s+1}, \dots, \lambda_d) \in \mathbb{R}^{d-s}$  and  $\mathbf{a}^s = (a_{s+1}, \dots, a_k)$ , the truncations of the original vectors  $\lambda$  and  $\mathbf{a}$ . Recall that the index  $s$  is feasible if the pair  $(\lambda^s, \mathbf{a}^s)$  is feasible for the CP. In any case we denote by

$$v_s = v(\lambda^s, \mathbf{a}^s) = (c \mathbb{1}_{r-s}, \lambda_{r+1}, \dots, \lambda_d) \quad \text{where} \quad c = Q_{s,r}$$

for the unique  $r > s$  such that  $\lambda_r \leq c < \lambda_{r+1}$ . This means that  $\lambda_s \leq v_s \in (\mathbb{R}_{>0}^{d-s})^\uparrow$  and that  $\text{tr } v_s = \text{tr } \lambda^s + \text{tr } \mathbf{a}^s$ .

**Lemma 4.14** *Fix an index  $0 \leq s \leq d - 1$ . Then*

1. *The index  $r$  associated to  $v_s$  as in the previous notations is given by*

$$r = \max\{1 \leq w \leq d : w > s \quad \text{and} \quad Q_{s,w} = \min_{j>s} Q_{s,j}\}.$$

In other words,  $r$  is the unique index which satisfies: Given  $j > s$ ,

$$Q_{s,r} < Q_{s,j} \text{ if } j > r \text{ and } Q_{s,r} \leq Q_{s,j} \text{ if } j < r. \tag{27}$$

2. Given an index  $1 \leq l \leq d - 1$ ,

$$l > s \text{ and } Q_{s,l} < \lambda_{l+1} \implies l \geq r, \tag{28}$$

where  $r$  is the index associated to  $v_s$  of item 1.

*Proof* Item 1 follows from Proposition 2.15 applied to  $\lambda^s$  and  $\mathbf{a}^s$ .

Item 2 : Assume that  $l < l + 1 \leq r$ . Then  $Q_{s,l} < \lambda_{l+1} \leq \lambda_r \leq Q_{s,r}$ . In this case

$$\begin{aligned} \text{tr } \lambda^s + \text{tr } \mathbf{a}^s &\stackrel{(28)}{=} (l - s) Q_{s,l} + \sum_{i=l+1}^d \lambda_i \\ &= (l - s) Q_{s,l} + \sum_{l+1 \leq i \leq r} \lambda_i + \sum_{i=r+1}^d \lambda_i \\ &< (r - s) Q_{s,r} + \sum_{i=r+1}^d \lambda_i \stackrel{(28)}{=} \text{tr } \lambda^s + \text{tr } \mathbf{a}^s, \end{aligned}$$

a contradiction. Hence  $l \geq r$ . □

**Proposition 4.15** Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP which is not feasible, with  $k \geq d$ . Let

$$s^* = \min\{1 \leq s \leq d : s \text{ is feasible}\}.$$

Let  $v^*$  be constructed using the recursive method of Proposition 4.12, by using  $s^*$  instead of  $s_{p-1}$  (which can always be done). Then if we get the constants  $c_1 > \dots > c_{q-1}$ , and we define  $c_q$  as the feasibility constant of  $\lambda^{s^*}$  and  $\mathbf{a}^{s^*}$ , then  $c_{q-1} > c_q$ .

*Proof* For simplicity of the notations, by working with the pair  $(\lambda^{s_{q-2}}, \mathbf{a}^{s_{q-2}})$ , we can assume that  $q = 2$ . Denote by  $s_1 = s^* < s_2$  and  $c_1, c_2$  the indices and constants given by:

$$c_1 = \frac{1}{s_1} \sum_{i=1}^{s_1} h_i = P_{1,s_1} \quad \text{and} \quad c_2 = Q_{s_1,s_2} = \frac{1}{s_2 - s_1} \left( \sum_{i=s_1+1}^{s_2} h_i + \sum_{i=s_2+1}^k a_i \right), \tag{29}$$

and we must show that  $c_1 > c_2$ . Recall that  $h_i = \lambda_i + a_i$ . We can assume that:

- By Proposition 4.12,  $c_1 \geq \frac{1}{p} \sum_{i=1}^p h_i = P_{1,p}$  for every  $1 \leq p \leq s_1$ .

- $c_2 \geq \frac{1}{p-s_1} \sum_{i=s_1+1}^p h_i = P_{s_1+1,p}$  for every  $s_1 + 1 \leq p \leq s_2$ .
- $\lambda_{s_2} \leq c_2 < \lambda_{s_2+1}$ ,

where the second item follows by the feasibility of  $s^*$  and the last item states that  $c_2$  is the feasible constant for the second block.

Suppose that  $c_1 \leq c_2$  and we will arrive to a contradiction by showing that, in such case, the pair  $(\lambda, \mathbf{a})$  would be feasible (that is,  $s^* = 0$  or  $s_{q-2}$ ). In order to do that, let

$$1 \leq t \leq d \quad \text{and} \quad b \stackrel{\text{def}}{=} Q_t = \frac{1}{t} \left( \sum_{i=1}^t h_i + \sum_{i=t+1}^k a_i \right)$$

be the unique constant such that  $\lambda_t \leq b < \lambda_{t+1}$ , which appears in  $v(\lambda, \mathbf{a})$ . Then

$$c \stackrel{\text{def}}{=} Q_{s_2} = \frac{1}{s_2} \left( \sum_{i=1}^{s_2} h_i + \sum_{i=s_2+1}^k a_i \right) = \frac{1}{s_2} (s_1 c_1 + (s_2 - s_1) c_2) \leq c_2 < \lambda_{s_2+1}.$$

By Eq. (28) we can deduce that  $t \leq s_2$ . Moreover, by item 1 of Lemma 4.14 we know that

$$b = Q_t = \frac{1}{t} \left( \sum_{i=1}^t h_i + \sum_{i=t+1}^k a_i \right) \leq \frac{1}{p} \left( \sum_{i=1}^p h_i + \sum_{i=p+1}^k a_i \right) = Q_p \quad \text{for every } 1 \leq p \leq d. \tag{30}$$

In particular,  $b \leq c \leq c_2$ . On the other side,  $c_1 \leq b$ . Indeed, if  $v = v(\lambda, \mathbf{a})$  then

$$\lambda \leq v^* \quad \text{and} \quad t = \text{tr } v^* = \text{tr } v \implies v < v^* \implies b = v_1 \geq v_1^* = c_1,$$

because  $c_1 \leq c_2 \implies v^* = (v^*)^\uparrow$  and since  $v = v^\uparrow$  is the  $<$ -minimum of the set

$$\{\lambda(S)^\uparrow : S_{\mathcal{F}_0} \leq S \quad \text{and} \quad \text{tr } S = t\} = \{\rho = \rho^\uparrow : \lambda \leq \rho \quad \text{and} \quad \text{tr } \rho = t\},$$

by the remarks at the beginning of Sect. 2.3 and Proposition 2.15.

To show the feasibility, by Lemma 4.9 we must show that  $b \geq P_{1,p}$  for every  $p \in \mathbb{I}_t$ . First, if we are in the case  $t \leq s_1$ , this is clear since  $b \geq c_1 \geq P_{1,p}$  for every  $p \leq s_1$ . Finally, suppose that  $t \geq s_1 + 1$ . As before,  $b \geq c_1$  implies  $b \geq P_{1,p}$  for every  $p \leq s_1$ . On the other hand, if  $s_1 < p \leq t$  then Lemma 4.14 applied to  $v_{s_1}$  (whose “ $r$ ” is  $s_2$ ) assures that

$$c_2 < Q_{s_1,t} \implies (t - s_1) c_2 \leq \sum_{i=s_1+1}^t h_i + \sum_{i=t+1}^k a_i \stackrel{(31)}{=} t b - s_1 c_1.$$

Since  $p \leq t$  and  $b \leq c_2$ , this implies that  $(p - s_1) c_2 \leq p b - s_1 c_1$ . Therefore

$$p P_{1,p} = s_1 c_1 + (p - s_1) P_{s_1+1,p} \leq s_1 c_1 + (p - s_1) c_2 \leq p b.$$

□

**Proposition 4.16** *Let  $(\mathcal{F}_0, \mathbf{a})$  be initial data for the CP. With the notations of Theorem 3.4, the global minimum  $v_f(\lambda, \mathbf{a})$  satisfies that*

$$s_{p-1} = \min\{1 \leq s \leq d : s \text{ is feasible}\}.$$

*Proof* Denote by  $s^*$  the minimum of the statement. Since  $s_{p-1}$  is feasible (recall the remark after Definition 3.5), then  $s^* \leq s_{p-1}$ . On the other hand, let us construct the vector  $v^*$  of Proposition 4.15, using the iterative method of Proposition 4.12 with respect to the index  $s = s^*$ , and the solution for the feasible pair  $(\lambda^{s^*}, \mathbf{a}^{s^*})$  after  $s^*$ . Write  $v^* = (v_1^*, \dots, v_s^*, c \mathbf{1}_{r-s}, \lambda_{r+1}, \dots, \lambda_d)$ , where  $c$  is the constant of the feasible part of  $v^*$ . Observe that Proposition 4.15 assures that  $c < \min\{v_i^* : 1 \leq i \leq s\}$ .

Using this fact and Proposition 4.12 it is easy to see that the vector  $\mu = v^* - \lambda^\uparrow$  satisfies that  $\mu = \mu^\downarrow$ . On the other hand Lemma 4.9 and Remark 4.10 assure that  $\mathbf{a} < \mu$  (using the majorization in each block and the fact that  $\mathbf{a} = \mathbf{a}^\downarrow$ ). Then  $v^* \in \{\lambda + \mu^\downarrow : \mu \in \mathbb{R}_{\geq 0}^d \text{ and } \mathbf{a} < \mu\}$ . Moreover, in each step of the construction of the minimum  $v = v_f(\lambda, \mathbf{a})$  we have to get the same index  $s_j = s_j(v^*)$  of  $v^*$  or there exists a step where the maximum which determines  $s_j$  (for  $v_f(\lambda, \mathbf{a})$ ) satisfies that  $s_j > s^*$  (in the eventual case in which  $s_{p-1} > s^*$ ).

In both cases, we get that  $v_i^* \leq v_i$  for every index  $1 \leq i \leq s^*$ . Consider the subvector of  $v^*$  given by  $\rho = (v_1^*, \dots, v_s^*, \lambda_{r+1}, \dots, \lambda_d) \in \mathbb{R}^{s+d-r}$ , and the respective part of  $v_f(\lambda, \mathbf{a})$  given by  $\xi = (v_1, \dots, v_s, v_{r+1}, \dots, v_d)$ . Since  $\text{tr } v^* = \text{tr } v$ , the previous remarks show that

$$\rho \leq \xi \implies \rho \prec_w \xi \implies (\rho, c \mathbf{1}_{r-s}) \prec (\xi, v_{s+1}, \dots, v_r),$$

where the final majorization follows using Lemma 4.6 of [30], which can be used since the constant  $c < \min\{v_i^* : 1 \leq i \leq s\}$  by Proposition 4.15 (and because  $c < \lambda_{r+1}$ ). Since majorization is invariant under rearrangements, we deduce that  $v^* \prec v$ .

Finally, using Theorem 2.8 we know that  $v = v_f(\lambda, \mathbf{a})$  is the unique minimum for the map  $\text{tr } f(\cdot)$  in the set  $\{\lambda + \mu^\downarrow : \mu \in \mathbb{R}_{\geq 0}^d \text{ and } \mathbf{a} < \mu\}$ . This implies that  $v^* = v$ , and therefore  $s_{p-1} = s^*$ .  $\square$

**Acknowledgments** This work was partially supported by CONICET (PIP 0435/10) and Universidad Nacional de La Plata (UNLP 11 X585).

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