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McVittie solution in $f(T)$ gravity

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Abstract We show that McVittie geometry, which describes a black hole embedded in a FLRW universe, not only solves the Einstein equations but also remains as a non-deformable solution of $f(T)$ gravity. This search for GR solutions that survive in $f(T)$ gravity is facilitated by a null tetrad approach. We also show that flat FLRW geometry is a consistent solution of $f(T)$ dynamical equations not only for $T = -6H^2$ but also for $T = 0$, which could be a manifestation of the additional degrees of freedom involved in $f(T)$ theories.

1 Introduction

It is well known that Riemann–Cartan spacetime, the underlying arena of Einstein–Cartan gravity, is a geometry possessing non-vanishing torsion and curvature [1–4]. Einstein's general relativity (GR) is defined in a spacetime with zero torsion and non-vanishing curvature. But there is another way of simplifying the Riemann–Cartan spacetime: by imposing the vanishing of the curvature tensor and letting the torsion undetermined. In this case, the spacetime geometry is completely described in terms of the vierbein (tetrad) field, since both the metric and the connection depend on it. The connection is called Weitzenböck connection and has torsion but vanishing curvature. For a certain choice of a Lagrangian quadratic in this torsion, the theory reduces to the so-called Teleparallel Equivalent of General Relativity (TEGR) [5–9], whose dynamical equations are equivalent to those of GR [10–16]. A decade ago, a novel approach to modified gravity was proposed by using TEGR as a starting point: the so-called $f(T)$ gravity [17,18], which mimics the proposal of $f(R)$ gravity by extending the Lagrangian through an arbitrary function. From the very beginning, the modified teleparallel gravity approach attracted attention since it successfully describes an inflationary scenario without the introduction of any inflaton field [17,19], together with an explanation for the accelerated expansion of our universe without dark energy [20]. Moreover, the action of $f(T)$ gravity contains only first derivatives of the dynamical variables so that the dynamical equations are of second order, which is an unusual feature in modified gravity. Soon enough, it was also established that the action is not local Lorentz invariant [17,21–23], which relates to the fact that $f(T)$ gravity presents additional degrees of freedom, although their physical nature is not yet well understood [21–25]. A lot of work has been done from then on (e. g. [21–45], among others). Following the steps of $f(T)$ gravity as an extension of TEGR, it is worth mentioning that other alternative theories in teleparallel framework were also developed in Born–Infeld [19,46–49], Kaluza–Klein [50], or Lovelock [51] schemes.

In this work, we study the McVittie solution [52,53], which describes a black hole embedded in a FLRW cosmology, in the context of modified teleparallelism. The McVittie spacetime is regular everywhere on and outside the black hole horizon, and also away from the big-bang singularity, when the cosmology is dominated at late times by a positive cosmological constant. The spherically symmetric geometry is parameterized by a function $a(t)$ and a constant mass parameter m. Of course, it reduces to FLRW cosmology with factor scale $a(t)$ at large radius, and to a black hole with mass m for proper limits. An exhaustive study of McVittie spacetime was developed in [54–56] (see also Ref. [57]). There were pointed out some misleading conceptions about the geometrical interpretation of the solutions that are shown in [58], where the McVittie geometry and its casual structure are thoroughly analyzed. It is fair to say that the McVittie solution does not describe astrophysical black holes because of missing ingredients such as accretion and rotation. However, it is still interesting as a first approach to study compact objects in an expanding universe.

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Due to the lack of local Lorentz invariance, looking for dynamical solutions in $f(T)$ gravity could be an awkward task. The symmetries of the metric does not throw light on the path one must follow. In [41] we have shown that a null tetrad approach seems to be a useful way to find solutions with spherical or axial symmetry. We will apply the same strategy here.

The outline of the paper is the following. In Sect. 2 we briefly introduce teleparallel gravity and the so-called $f(T)$ gravity. In Sect. 3 we present the McVittie spacetime. In Sect. 4 we look for McVittie-like solutions in $f(T)$ gravity by taking advantage of the null tetrad approach, which provides a simple way to find suitable frames. In Sect. 5 we show that FLRW cosmology consistently solves the $f(T)$ equations for two different tetrads not linked by a symmetry transformation of the equations. Finally, we state the conclusions in Sect. 6.

2 Teleparallel framework

Teleparallelism is a framework to describe gravity where the role of the metric tensor $g_{\mu\nu}$ is played by the tetrad or *vierbein* $\{\mathbf{e}_a(x^\mu)\}$, $a = 0, 1, 2, 3$. The tetrad is a set of four vectors at each point of the spacetime which are linked to the metric by the condition of orthonormality $\mathbf{e}_a \cdot \mathbf{e}_b = \eta_{ab}$, where $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski symbol. In coordinate bases, we write the tetrad and its dual co-tetrad $\{\mathbf{e}^a(x^\mu)\}$ as $\mathbf{e}_a = e_a^{\ \mu}\,\partial_\mu$ and $\mathbf{e}^a = e^a_{\ \mu}\,\mathrm{d}x^\mu$, where $e_a^{\ \mu}$ and $e^a_{\ \mu}$ are the components of the tetrad and its inverse which obey duality relationships ($e^a_{\ \mu}\, e_b^{\ \mu} = \delta^a_b$ and $e^a_{\ \mu}\, e_a^{\ \nu} = \delta^\nu_\mu$). By combining duality and orthonormality, one can get the metric tensor of the manifold in terms of a tetrad in the tangent space,

$$g_{\mu\nu} = \eta_{ab}\ e^a_{\ \mu}\, e^b_{\ \nu}. \tag{1}$$

Of course, this relationship is invariant under (local) Lorentz transformations of the tetrad. As seen, we have used Greek indices to label coordinates and tensor components in a coordinate basis; instead Latin indices $a, b, \ldots = 0, 1, 2, 3$ label the vectors taking part in the tetrad, and tensor components with respect to these Lorentzian frames. These indices are lowered and raised by the metric tensor and the Minkowski symbol, respectively.

It is well known that General Relativity can be reformulated by using the tetrad as the dynamical variable. In fact, in the so-called Teleparallel Equivalent of General Relativity (TEGR) [10–13] the action is built from the object of anholonomity $\mathrm{d}\mathbf{e}^a$, which reads $\mathrm{d}\mathbf{e}^a = e^a_{\ \mu}\ T^\mu_{\ \ \rho\nu}\ \mathrm{d}x^\rho \wedge \mathrm{d}x^\nu$ in a coordinate basis, where

$$T^\mu_{\ \ \rho\nu} = e_b^{\ \mu}(\,\partial_\rho e^b_{\ \nu}\ -\ \partial_\nu e^b_{\ \rho}). \tag{2}$$

$\mathrm{d}\mathbf{e}^a$ coincides with the torsion $\mathbf{T}^a = \mathrm{d}\mathbf{e}^a + \boldsymbol{\omega}^a_b \wedge \mathbf{e}^b$ when the spin connection $\boldsymbol{\omega}^a_b$ is chosen to be zero. Such a choice is the Weitzenböck connection [59], which also implies that the curvature $\mathbf{R}^a_b = \mathrm{d}\boldsymbol{\omega}^a_b + \boldsymbol{\omega}^a_c \wedge \boldsymbol{\omega}^c_b$ vanishes. According to Eq. (2) the Christoffel symbols for the Weitzenböck connection are $\overset{\mathrm{w}}{\Gamma}{}^\mu_{\ \rho\nu} = e_a^{\ \mu}\ \partial_\nu e^a_{\ \rho} = -e^a_{\ \rho}\ \partial_\nu e_a^{\ \mu}$. The Weitzenböck connection leads to the vanishing of the covariant derivative of the tetrad, which also means that this connection is metric-compatible (see Eq. (1)). This property defines an absolute parallelism in the spacetime. In fact, $\overset{\mathrm{w}}{\nabla} e_a^{\ \mu} \equiv 0$ means that a parallel transported vector keeps its projections on the tetrad constant, irrespective of the path as a consequence of the zero curvature. This is the reason why the name *teleparallelism* comes up. As can be seen, the Weitzenböck torsion (2) is invariant only under global Lorentz transformations of the tetrad. So, Weitzenböck geometry selects a preferred global frame modulo global Lorentz transformations, in spite of the metric being invariant under local Lorentz transformations.

2.1 The teleparallel equivalent of general relativity

The TEGR action is

$$S_T = \frac{1}{2\kappa}\int \mathrm{d}^4x\ e\ T + \int \mathrm{d}^4x\ e\ \mathcal{L}_{\mathrm{matter}}, \tag{3}$$

where $\kappa = 8\pi G$, $e = \det[e^a_{\ \mu}] = \sqrt{-g}$ and the *torsion scalar* or *Weitzenböck invariant* T is defined by the contraction of the torsion tensor (2) and the *superpotential*, that is,

$$T \equiv S_\rho^{\ \ \mu\nu}\, T^\rho_{\ \ \mu\nu}, \tag{4}$$

where $S_\rho^{\ \ \mu\nu} \equiv \frac{1}{2}(K^{\mu\nu}_{\ \ \ \ \rho} + T_\lambda^{\ \ \lambda\mu}\,\delta^\nu_\rho - T_\lambda^{\ \ \lambda\nu}\delta^\mu_\rho)$, with $K^{\mu\nu}_{\ \ \ \ \rho} \equiv \frac{1}{2}\,(T_\rho^{\ \ \mu\nu} - T^{\mu\nu}_{\ \ \ \ \rho} + T^{\nu\mu}_{\ \ \ \ \rho})$ the contorsion tensor. Then the torsion scalar can be expressed by the following quadratic combination of the components of the torsion tensor:

$$T = \frac{1}{4}T_\rho^{\ \ \mu\nu}\, T^\rho_{\ \ \mu\nu} + \frac{1}{2}T_\rho^{\ \ \mu\nu}\, T^\rho_{\ \ \nu\mu} - T_\rho^{\ \ \mu\rho}\, T^\sigma_{\ \ \sigma\mu}. \tag{5}$$

One can compute the Levi-Civita scalar curvature R in terms of the tetrad by using Eq. (1); then a relation is obtained between T and R which shows that the Einstein–Hilbert Lagrangian only differs from the TEGR Lagrangian by a four-divergence,

$$T = -R[\mathbf{e}_a] + 2\, e^{-1}\,\partial_\rho(e\, T^{\mu\ \ \rho}_{\ \ \mu}). \tag{6}$$

Hence the equations of motion are fully equivalent, showing the equivalence between GR and TEGR pictures. Although the tetrad field has 16 independent components, in contrast with the 10 independent components of the symmetric metric tensor, TEGR and GR have the same number of degrees of freedom, as expected from their equivalence. In fact, Eq. (6) implies that TEGR only governs the dynamics of the metric, which is invariant under local Lorentz transformations of the tetrad. Then the tetrad is only determined modulo this local symmetry group. Therefore, the extra components of the tetrad do not represent new dynamical degrees of freedom. In fact, the local Lorentz invariance of the relation (1) means that the metric is related with an infinite set of tetrad fields, connected by local Lorentz transformations. Note also that teleparallel vacuum solutions do not force T to vanish, as opposite to R in GR. This point is clearly shown in Eq. (6) where $R = 0$ implies that T is a four-divergence.

2.2 $f(T)$ gravity

In analogy with $f(R)$ gravity [60–65], where the GR Lagrangian is extended to an arbitrary function f of the curvature scalar R, $f(T)$ gravity is obtained by replacing the TEGR Lagrangian with an arbitrary function f of the torsion scalar T [17–19],

$$S_{f(T)} = \frac{1}{2\kappa} \int \mathrm{d}^4x\, e\, f(T) + \int \mathrm{d}^4x\, e\, \mathcal{L}_{\text{matter}}. \tag{7}$$

The dynamical equations of $f(T)$ gravity are computed by varying the modified teleparallel action with respect to the tetrad, yielding

$$\begin{aligned} &4\, e_a^{\ \lambda}\, S_\lambda^{\ \mu\nu}\, \partial_\mu T\, f''(T) \\ &\quad + 4 \left[e_a^{\ \lambda}\, T^\rho_{\ \mu\lambda}\, S_\rho^{\ \mu\nu} + e^{-1} \partial_\mu (e\, e_a^{\ \lambda} S_\lambda^{\ \mu\nu}) \right] f'(T) \\ &\quad - e_a^{\ \nu}\, f(T) = -2\kappa\, e_a^{\ \lambda}\, \mathcal{T}_\lambda^{\ \nu}, \end{aligned} \tag{8}$$

where $\mathcal{T}_\lambda^{\ \nu}$ is the energy-momentum tensor. Of course, TEGR dynamics is recovered if $f(T) = T$. Remarkably, it is straightforward to verify that the equations of motion are of second order with respect to the tetrad (just as in GR with respect to the metric), since $T^\rho_{\ \mu\lambda}$ and $S_\rho^{\ \mu\nu}$ are linear in the first derivatives of the tetrad. Instead, other alternative theories of gravity have dynamical equations of higher order.

From the equivalence relation expressed in Eq. (6), it is manifest that $f(T)$ gravity is a non-local Lorentz invariant theory; by extending the TEGR Lagrangian to a function $f(T)$, the four-divergence (non-invariant) term remains encapsulated inside the function f.[1]

While TEGR is a theory for the metric (as GR is), $f(T)$ gravity is a theory for the tetrad. This is so because the modified teleparallel dynamical equations are not invariant under local Lorentz transformations but only under global Lorentz transformations.[2] Thus the dynamical equations possess information exclusively associated with the tetrad field. This means that $f(T)$ theories dynamically endow the spacetime with an absolute parallelism. This also means that, if one is looking for a solution associated with a given metric, then the symmetries of the metric are not sufficient to anticipate the form of the tetrad solving the equations. For instance, it was shown that Schwarzschild geometry [31] and non-flat FLRW spacetimes [32] require non-trivial tetrads in order to consistently solve the equations of motion. Therefore the gravitational field is encoded in a set of preferred reference frames, which should not depend on the function f considered [37]. Along this line, it was proposed that a set of suitable frames would be defined up to *certain* local Lorentz transformations, which are a remnant symmetry group which depends on the specific spacetime considered [42]. In fact, in a mathematical context, it has been known for a long time that the vector fields capable to parallelize a manifold are not unique (see for instance, Ref. [77]).

In the present work, we rely on results displayed in [31,41] where Schwarzschild and Kerr spacetimes proved to remain as vacuum solutions in $f(T)$ gravity, since both geometries admit a tetrad where the torsion scalar vanishes ($T = 0$) or is constant ($T = T_c$). In fact if $\partial_\mu T = 0$, Eq. (8) can be arranged as follows:

$$G_\eta^{\ \nu} - \frac{1}{2}\, \delta_\eta^{\ \nu}\, T_c + \frac{1}{2}\, \delta_\eta^{\ \nu}\, \frac{f(T_c)}{f'(T_c)} = \frac{\kappa}{f'(T_c)}\, \mathcal{T}_\eta^{\ \nu}, \tag{9}$$

where we define $G_\eta^{\ \nu}$ from Eq. (8) by taking $f(T) = T$,

$$\begin{aligned} G_\eta^{\ \nu} \equiv &-2e^a_{\ \eta} \left[e_a^{\ \lambda}\, T^\rho_{\ \mu\lambda}\, S_\rho^{\ \mu\nu} + e^{-1}\, \partial_\mu (e\, e_a^{\ \lambda}\, S_\lambda^{\ \mu\nu}) \right] \\ &+ \frac{1}{2}\, \delta_\eta^{\ \nu}\, T \end{aligned} \tag{10}$$

($G_{(\mu\nu)}$ is the Einstein tensor). Remarkably the dynamical equations (9), when the energy-momentum tensor is symmetric, are nothing but (TEGR) Einstein equations with a scaled Newton constant $\tilde{G} = G/f'(T_c)$ and cosmological constant $\Lambda = (T_c - f(T_c)/f'(T_c))/2$. Therefore, if $f(T = 0) = 0$ then the TEGR solutions having $T = 0$ remain as solutions of $f(T)$ gravity (in the case of non-vacuum solutions we should adjust the Newton constant). Even if $T_c \neq 0$ one could still regard any TEGR solution having $T = T_c$ as a solution to $f(T)$ equations for proper values of cosmological and Newton constants. In the following sections we will take advantage of this property to figure out whether a metric solving Einstein's equations could survive as a solution of $f(T)$ equations or not. Following the strategy displayed in [41], we will exploit the local Lorentz invariance of TEGR to look for solutions having $T = 0$ by starting from a known

[1] The equivalence between GR and TEGR is broken after the generalization procedure; therefore $f(T)$ gravity is not at all equivalent to $f(R)$ gravity.

[2] It should be noted too that the local Lorentz symmetry could be broken at the level of quantum gravity (Planck scale) and many candidate theories of quantum gravity predict at some point the loss of local Lorentz symmetry. In fact, there are many attempts to examine in detail the range in which the Lorentz symmetry is preserved

Footnote 2 continued
[66–71]. Other alternative descriptions of gravity in vogue nowadays such that Hořava–Lifshitz [72–74] and Æther–Einstein [75,76] are not local Lorentz invariant in the time sector.

solution. If we succeed, then we can state that the solution so obtained remains as a solution to $f(T)$ gravity. The null tetrad approach will be very useful for this purpose.

3 McVittie geometry

McVittie geometry [52] describes a cosmological black hole, which means a black hole solution embedded in an expanding FLRW universe. A comprehensive review of McVittie geometry was carried out by Nolan in the late-1990s [54–56]. Nonetheless, Kaloper et al. [58] went one step further by analyzing some misconceptions in the geometrical interpretation of the solution. The original McVittie spacetime assumes vanishing (Levi-Civita) spatial curvature in the asymptotically FLRW region, but also it can easily be generalized so as to include positive or negative spatial curvature [57]. Here we will focus on the spatially flat case. It is assumed that the spatial curvature of the FLRW does not significantly influence the dynamics around the central mass, accounting for that the radius of curvature is (much) greater than the gravitational radius of the mass source. Then the standard McVittie geometry is described by the metric

$$\mathrm{d}s^2 = \left(\frac{1-\mu}{1+\mu}\right)^2 \mathrm{d}t^2 - (1+\mu)^4 a^2(t)\, \mathrm{d}\mathbf{x}^2, \tag{11}$$

where $\mu = m(2a(t)|\mathbf{x}|)^{-1}$, with m/G the mass of the source, $a(t)$ is the asymptotic cosmological scale factor, and the center of the spherical symmetry is at $\mathbf{x} = 0$. This is an exact solution to Einstein's equations for an arbitrary mass m provided that $a(t)$ solves the Friedmann equation. As might be expected, if $m = 0$ the geometry corresponds to the standard flat FLRW spacetime, and if $a(t) = 1$ the line element describes the Schwarzschild solution (in isotropic coordinates). For $a \sim e^{H_0 t}$, the metric reduces to the case of Schwarzschild–de Sitter (by adopting a positive cosmological constant). It can be verified that McVittie solution has a spacelike and an inhomogeneous singularity at $\mu = 1$ (which means that $a(t)|\mathbf{x}| = m/2$) where the surface lies in the causal past of all spacetime events so that it should be properly interpreted as a cosmological big-bang singularity (see Ref. [58] for more detail).

In analogy to the FLRW case for a perfect fluid, the energy density scales with the scale factor $a(t)$ and commands the expansion rate

$$\rho = \frac{3H^2(t)}{\kappa}, \tag{12}$$

where $H(t) = \dot{a}(t)/a(t)$ is the Hubble parameter. Remarkably, the energy density is constant along slices where t is constant, but the pressure on fixed t slices is not homogeneous,

$$p = \frac{1}{\kappa}\left(-3\,H(t)^2 + 2\,\dot{H}^2(t)\left(\frac{m + 2\,|\mathbf{x}|\,a(t)}{m - 2\,|\mathbf{x}|\,a(t)}\right)\right). \tag{13}$$

The inhomogeneous pressure constitutes the necessary non-gravitational balancing force (when the mass is constant and the energy density is spatially homogeneous) to compensate for the gravitational attraction of the central mass.

The coordinates employed in the line element given by Eq. (11) are such that $|\mathbf{x}|$ covers the exterior of the black hole twice, that is, $m/2 < |\mathbf{x}| < \infty$ covers the same exterior region as $0 < |\mathbf{x}| < m/2$. Therefore, Kaloper et al. [58] propose another coordinate choice which in turn imitates better the familiar static form of the Schwarzschild metric. The new radial coordinate is defined by

$$\mathbf{R} = (1+\mu)^2\, a(t)\, \mathbf{x}, \tag{14}$$

where $|\mathbf{R}| = R$ represents the "spherical area" coordinate. Since the relation between $a(t)|\mathbf{x}|$ and R is quadratic, the coordinate transformation (14) actually defines two separate branches,

$$a(t)\,|\mathbf{x}| = \frac{m}{2}\left(\frac{R}{m} - 1 \pm \sqrt{\left(\frac{R}{m} - 1\right)^2 - 1}\right)^{-1}. \tag{15}$$

The physically relevant branch is the one with the $-$ sign, since $R \to \infty$ implies $|\mathbf{x}| \to \infty$, therefore the geometry is asymptotically FLRW-like.

By applying the transformation defined in (14), the McVittie metric becomes

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{R} - H(t)^2 R^2\right)\mathrm{d}t^2 + \frac{2H(t)R}{\sqrt{1-2m/R}}\,\mathrm{d}R\,\mathrm{d}t - \frac{\mathrm{d}R^2}{1-2m/R} - R^2 \mathrm{d}\Omega^2, \tag{16}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$. It is now evident that a constant value of H leads to the Schwarzschild–de Sitter metric in coordinates which are analogous to outgoing Eddington–Finkelstein coordinates.

4 McVittie solution in $f(T)$ gravity

As previously mentioned, $f(T)$ gravity is not locally Lorentz invariant. This means that tetrads connected by local Lorentz transformations, which reproduce the same metric tensor, are not equivalent at the level of the equations of motion of $f(T)$: not all of them will represent a proper parallelizing field of frames (in the sense of being a consistent solution of the dynamical equations). We can state that $f(T)$ gravity selects the parallelization of the spacetime. Finding suitable tetrad solutions is then quite awkward in $f(T)$ theory, since the symmetry of the searched metric is not sufficient to determine the form of the tetrad. However, as explained at the end of

Sect. 2, one can exploit the local Lorentz invariance of TEGR solutions to force the scalar torsion T to be zero. If such a purpose is attainable, we will obtain a tetrad solving the $f(T)$ gravity equations too (whenever $f(T=0)\neq 0$). This also means that the same metric solving TEGR equations is admissible in $f(T)$ gravity. Since the condition $T=0$ is not affected by global linear transformations of the tetrad, it could be easier to look for the condition $T=0$ by using a null tetrad [41]: given an orthonormal tetrad $\{\mathbf{e}^a\}=\{\mathbf{e}^0,\mathbf{e}^1,\mathbf{e}^2,\mathbf{e}^3\}$, a null tetrad can be defined as

$$\{\mathbf{n}^a\}=\{\mathbf{l},\mathbf{n},\mathbf{m},\overline{\mathbf{m}}\} \\ =\left\{\frac{(\mathbf{e}^0-\mathbf{e}^1)}{\sqrt{2}},\frac{(\mathbf{e}^0+\mathbf{e}^1)}{\sqrt{2}},\frac{(\mathbf{e}^2+i\,\mathbf{e}^3)}{\sqrt{2}},\frac{(\mathbf{e}^2-i\,\mathbf{e}^3)}{\sqrt{2}}\right\}. \tag{17}$$

$\{\mathbf{n}^a\}$ is a null basis, $\mathbf{n}^a\cdot\mathbf{n}^a=0$, but it is not orthogonal ($\mathbf{l}\cdot\mathbf{n}=-\mathbf{m}\cdot\overline{\mathbf{m}}=1$). Then Eq. (1) can be rewritten in this new basis as

$$g_{\mu\nu}=\eta_{ab}\,n^a_\mu\,n^b_\nu, \tag{18}$$

where η_{ab} now reads

$$\eta_{ab}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{pmatrix}. \tag{19}$$

In terms of the tensorial product of the elements of the null tetrad the metric reads $\mathbf{g}=\mathbf{n}\otimes\mathbf{l}+\mathbf{l}\otimes\mathbf{n}-\mathbf{m}\otimes\overline{\mathbf{m}}-\overline{\mathbf{m}}\otimes\mathbf{m}$. A local Lorentz boost along the direction of $\mathbf{e}^1$, with parameter $\gamma(x^\mu)=\cosh[\lambda(x^\mu)]$, is the transformation $\{\mathbf{l},\mathbf{n}\}\mapsto\{\exp[-\lambda(x^\mu)]\,\mathbf{l},\exp[\lambda(x^\mu)]\,\mathbf{n}\}$, which clearly does not modify the form of the metric tensor. The null tetrad approach to get $T=0$ exploits the freedom in the choice of the function $\lambda(x^\mu)$. The strategy is the following: (i) determine the null tetrad $\{\mathbf{n}^a\}$, (ii) apply the transformation in the $\{\mathbf{l},\mathbf{n}\}$-sector, (iii) impose the condition that the torsion scalar T be zero. This approach was successfully implemented for the Kerr geometry [41]; here we will show that it is also useful in McVittie spacetime.[3]

Then we compute the null tetrad associated with the metric given in Eq. (16) with coordinates (t,R,θ,ϕ) and we perform a radial boost yielding

$$n^a{}_\mu=\frac{1}{\sqrt{2}}\begin{pmatrix}e^{-\lambda(t,R,\theta)}\left(\sqrt{1-2m/R}+RH(t)\right)&-e^{-\lambda(t,R,\theta)}\frac{1}{\sqrt{1-2m/R}}&0&0\\ e^{\lambda(t,R,\theta)}\left(\sqrt{1-2m/R}-RH(t)\right)&e^{\lambda(t,R,\theta)}\frac{1}{\sqrt{1-2m/R}}&0&0\\0&0&R&i\,R\sin\theta\\0&0&R&-i\,R\sin\theta\end{pmatrix}, \tag{20}$$

where the function λ has to be determined. For this null tetrad, we determine the torsion scalar by means of Eq. (4):

$$T=-6\,H(t)^2+2\,R^{-2}-4\,R^{-1}\,\partial_t\lambda. \tag{21}$$

By imposing the requirement that the torsion scalar vanishes, we solve the differential equation to obtain

$$\lambda(t,R)=\frac{t}{2R}-\frac{3R}{2}\int H^2(t)\,\mathrm{d}t. \tag{22}$$

Note that λ allows for an additive function of (R,θ) without affecting the result $T=0$.

In summary, according to Eq. (9), we conclude that $f(T)$ gravity is not able to deform McVittie metric since we have just found a solution leading to a vanishing torsion scalar which consistently solves the dynamical equations (notice the scaling of the Newton constant κ),

$$2\kappa\rho=f(T=0)+6\,H^2(t)\,f'(T=0) \\ =6\,H^2(t)\,f'(T=0), \tag{23}$$

$$2\kappa(p+\rho)=-4\,f'(T=0)\,\dot{H}(t)\,\frac{1}{\sqrt{1-2m/R}}, \tag{24}$$

where, due to the non-diagonal form of the metric tensor (16), the stress-energy tensor for a perfect fluid becomes

$$T^\mu{}_\nu=\begin{pmatrix}\rho&(\rho+p)\,R\,H(t)\sqrt{1-2m/R}&0&0\\0&-p&0&0\\0&0&-p&0\\0&0&0&-p\end{pmatrix}, \tag{25}$$

with ρ and p the energy density and the pressure, respectively [57].

5 Cosmology with $T=0$

As has been extensively studied in the literature, flat FLRW cosmology accepts the naive diagonal tetrad (in the Cartesian chart) as a suitable solution in the context of $f(T)$ gravity, which leads to $T=-6H^2(t)$ [17,32]. However, McVittie metric for $m=0$ reduces to FLRW metric. Therefore, we have also obtained the outstanding result that there exists a tetrad $\{\mathbf{e}^a\}$ having $T=0$ in flat FLRW spacetime, from which it immediately follows that such a tetrad is also a solution of $f(T)$ gravity (assuming $f(T=0)\neq 0$, and taking care

[3] In both cases, the success of the null tetrad approach seems to reside in the fact that the radial boost does not mix the sector $\{t,r\}$ with the angular part $\{\theta,\phi\}$.

of the scaling of Newton constant). By replacing $m = 0$ in Eq. (20), we find that the null tetrad associated with $\{\mathbf{e}^a\}$ is

$$n^a{}_\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-\lambda(t,R)}\,(1+RH(t)) & -e^{-\lambda(t,R)} & 0 & 0 \\ e^{\lambda(t,R)}\,(1-RH(t)) & e^{\lambda(t,R)} & 0 & 0 \\ 0 & 0 & R & iR\sin\theta \\ 0 & 0 & R & -iR\sin\theta \end{pmatrix}, \tag{26}$$

where $\lambda(t, R)$ is the function given in Eq. (22). According to Eq. (24), the dynamical equations for $m = 0$ become

$$\begin{aligned} 2\kappa\rho &= f(T=0) + 6\,H^2(t)\,f'(T=0) \\ &= 6\,H^2(t)\,f'(T=0), \end{aligned} \tag{27}$$

$$2\kappa(p+\rho) = -4\,f'(T=0)\,\dot{H}(t). \tag{28}$$

This remarkable result will be thoroughly studied in a forthcoming article [78]. At first sight, solutions providing the same metric tensor but different torsion scalars suggest the involvement of the extra degrees of freedom characteristic of $f(T)$ gravity. They constitute different admissible parallelizations of FLRW geometry.

To compare the two $f(T)$ solutions, firstly, we will get the orthonormal tetrad $e^a{}_\mu$ associated with the null tetrad given in Eq. (26), and then we will write it in the original chart $(t, |\mathbf{x}|)$. In fact, by performing the inverse coordinate transformation of Eq. (14), which reduces to $R = a(t)\,|\mathbf{x}| = a(t)\,r$ for $m = 0$, we will get $e^a{}_{\mu'}$, in terms of the spherical radial coordinate r. The first step is achieved through a transformation $L^a{}_b$ such that (see Eq. (17))

$$\begin{aligned} e^a{}_\mu = L^a{}_b\, n^b{}_\mu &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -i & i \end{pmatrix} \\ &\times \begin{pmatrix} e^{-\lambda}\,(1+RH(t)) & -e^{-\lambda} & 0 & 0 \\ e^{\lambda}\,(1-RH(t)) & e^{\lambda} & 0 & 0 \\ 0 & 0 & R & i\,R\,\sin\theta \\ 0 & 0 & R & -i\,R\,\sin\theta \end{pmatrix} \\ &= \begin{pmatrix} \cosh\lambda - R\,H(t)\sinh\lambda & \sinh\lambda & 0 & 0 \\ \sinh\lambda - R\,H(t)\cosh\lambda & \cosh\lambda & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & R\,\sin\theta \end{pmatrix}, \end{aligned} \tag{29}$$

where we set $\lambda \equiv \lambda(t, R)$ to abbreviate the notation. In the second step, we will perform the coordinate transformation $x^\mu = (t, R, \theta, \phi) \longrightarrow x^{\mu'} = (t, r, \theta, \phi)$. The tetrad $\{\mathbf{e}^a\}$ is a geometrical object independent of the coordinate choice,

$$\mathbf{e}^a = e^a{}_\mu\,\mathrm{d}x^\mu = e^a{}_{\mu'}\,\mathrm{d}x^{\mu'}, \tag{30}$$

but its components $e^a{}_\mu$ will change. According to Eq. (14), the 1-forms $\mathrm{d}x^\mu$ and $\mathrm{d}x^{\mu'}$ will not change except for $\mathrm{d}R = \dot{a}\,r\,\mathrm{d}t + a\,\mathrm{d}r$. Then one finds

$$e^a{}_{\mu'} = \begin{pmatrix} \cosh\lambda & a(t)\,\sinh\lambda & 0 & 0 \\ \sinh\lambda & a(t)\,\cosh\lambda & 0 & 0 \\ 0 & 0 & a(t)r & 0 \\ 0 & 0 & 0 & a(t)\,r\,\sin\theta \end{pmatrix}. \tag{31}$$

The tetrad (31) is the one having $T = 0$, as written in the usual spherical coordinates. On the other hand, there exists a tetrad which has $T = -6H^2$; this tetrad is diagonal in the Cartesian chart, but in spherical coordinates it reads

$$e^{a'}{}_{\mu'} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a(t)\,\sin\theta\,\cos\varphi & a(t)\,r\,\cos\theta\,\cos\varphi & -a\,(t)\,r\,\sin\theta\,\sin\varphi \\ 0 & a(t)\,\sin\theta\,\sin\varphi & a(t)\,r\,\cos\theta\,\sin\varphi & a(t)\,r\,\sin\theta\,\cos\varphi \\ 0 & a(t)\,\cos\theta & -a(t)\,r\,\sin\theta & 0 \end{pmatrix}. \tag{32}$$

Of course, there should exist a local Lorentz transformation connecting the tetrads (31) and (32), since they describe the same FLRW metric. In fact, the local Lorentz transformation is

$$\Lambda^{a'}{}_b = \begin{pmatrix} \cosh\lambda & \sinh\lambda & 0 & 0 \\ \sinh\lambda\,\sin\theta\,\cos\phi & \cosh\lambda\,\sin\theta\,\cos\phi & \cos\theta\,\cos\phi & -\sin\phi \\ \sinh\lambda\,\sin\theta\,\sin\phi & \cosh\lambda\,\sin\theta\,\sin\phi & \cos\theta\,\sin\phi & \cos\phi \\ \sinh\lambda\,\cos\theta & \cosh\lambda\,\cos\theta & -\sin\theta & 0 \end{pmatrix}, \tag{33}$$

i.e. $\mathbf{e}^{a'} = \Lambda^{a'}{}_b\,\mathbf{e}^b$, which can be separated into the product of two rotations and one boost

$$\begin{aligned} \Lambda^{a'}{}_b = &\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &\times \begin{pmatrix} \cosh\lambda & \sinh\lambda & 0 & 0 \\ \sinh\lambda & \cosh\lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{34}$$

In Ref. [42] it was shown that each solution to the dynamical equations for $f(T)$ allow for a set of local Lorentz remnant symmetries. Such symmetries are generated by Lorentz transformations accomplishing the condition

$$\mathrm{d}(\epsilon_{abcd}\,\mathbf{e}^a \wedge \mathbf{e}^b \wedge \eta^{de}\Lambda^c{}_{f'}\mathrm{d}\Lambda^{f'}{}_e) = 0. \tag{35}$$

Our pair $(\mathbf{e}^a, \Lambda^{a'}{}_b)$ does not satisfy this relationship. This is because, although both tetrads have consistent equations of motion, their associated torsion scalars are different; so the transformation (33) is not a symmetry of the action. Also, they have different classification concerning to the *n-closed-area-frame* (n-CAF), distinction firstly introduced in the same paper. Remember that a solution of $f(T)$ gravity is n-CAF if n of the six pairs $(\mathbf{e}^a, \mathbf{e}^b)$ satisfy the equation

$$\mathrm{d}(\mathbf{e}^a \wedge \mathbf{e}^b) = 0. \tag{36}$$

In this particular case, it can be proven that the tetrad (31) is a 1-CAF, since the only combination that is zero is the one with ($\mathbf{e}^0$, $\mathbf{e}^1$). On the other hand, the tetrad (32) is a 3-CAF [42]. This means that the second one allows for three independent *local* Lorentz transformations leaving T unchanged, while the first solution admits only the local boost associated with the remnant freedom of the function λ. On the contrary, the transformation (33) is not a remnant symmetry of the $f(T)$ dynamical equations because it changes the torsion scalar T.

6 Conclusions

The main purpose of $f(T)$ gravity would be finding low/high-energy deformations of Einstein gravity to address its shortcomings in a geometric framework. However, the null tetrad approach shows that it is rather easy to get TEGR solutions with constant torsion scalar. These solutions survive in $f(T)$ gravity, associated with modified Newton and cosmological constants. This is the way we followed to show that McVittie geometry is a solution to $f(T)$ gravity. Remarkably, by taking $m = 0$ we also obtained a new consistent solution for FLRW universe in $f(T)$ gravity, which has $T = 0$. To the best of our knowledge, this is the first time that a consistent cosmological solution with a vanishing torsion scalar is introduced in the literature. The fact that the torsion scalar differs from $-6H^2$ could be a manifestation of the extra degrees of freedom of the theory, which is a topic under consideration presently [78].

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