Differential and Integral Equations Volume xx, Number xxx, , Pages xx–xx

TUG-OF-WAR GAMES AND PARABOLIC PROBLEMS WITH SPATIAL AND TIME DEPENDENCE

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Abstract. In this paper, we use probabilistic arguments (Tug-of-War games) to obtain the existence of viscosity solutions to a parabolic problem of the form

$$\begin{cases} K_{(x,t)}(Du)u_t(x,t) = \frac{1}{2}\langle D^2uJ_{(x,t)}(Du), J_{(x,t)}(Du)(x,t)\rangle & \text{in } \Omega_T, \\ u(x,t) = F(x) & \text{on } \Gamma, \end{cases}$$

where $\Omega_T = \Omega \times (0,T]$ and Γ is its parabolic boundary. This problem can be viewed as a version with spatial and time dependence of the evolution problem given by the infinity Laplacian,

$$u_t(x,t) = \langle D^2u(x,t)\frac{Du}{|Du|}(x,t), \frac{Du}{|Du|}(x,t)\rangle .$$

1. Introduction

Our goal in this article is to look for parabolic PDEs that may arise as continuous values of Tug-of-War games when one takes into account the number of plays that the players play and considering sets of possible movements that may depend on space and time. In this way, we obtain what we can call a natural way of defining a *parabolic problem involving the infinity Laplacian with spatial and time dependence.*

Accepted for publication: September 2013.
AMS Subject Classifications: 35B05, 35J25, 35J69.
Leandro M. Del Pezzo was supported by CONICET (Argentina) PIP 5478/1438 tand Julio D. Rossi by MTM2008-05824, (Spain).

Solutions to the infinity Laplacian $\langle D^2u(x,t)\frac{Du}{|Du|}(x,t),\, \frac{Du}{|Du|}(x,t)\rangle = 0$ appear naturally when one considers absolutely minimizing Lipschitz extensions (AMLE) of a Lipschitz function F, defined on the boundary; see the survey [3] and [9] (see also [2, 6, 11, 12, 13]). This equation (and also the $p-$Laplacian) was related to continuous values of Tug-of-War games, see [21]. See also [2, 4, 5, 14, 16, 17, 18, 20, 22] and, for numerical approximations, [19].

The evolution problem given by the infinity Laplacian, is given by

$$v_t(x,t) = \Big\langle D^2v(x,t)\frac{Dv}{|Dv|}(x,t),\, \frac{Dv}{|Dv|}(x,t)\Big\rangle. \tag{1.1}$$

For existence, asymptotic behaviour and further properties of the solutions, we refer to [1, 10].

Recently, see [18], probabilistic methods (based on Tug-of-War games) were used to obtain mean value characterizations of solutions to parabolic PDEs, including the equation (1.1). The Tug-of-War game that is related to this equation, see [18], can be briefly described as follows: a Tug-of-War game is a two-person, zero-sum game. That is, two players are in contest and the total earnings of one are the losses of the other. Let T be a positive constant and Ω be a bounded smooth open subset of $\mathbb{R}^N$. We consider the parabolic cylinder $\Omega_T = \Omega\times(0,T]$ with the parabolic boundary $\Gamma = \partial\Omega\times[0,T]\cup\Omega\times\{0\}$ and, for a fixed $\eta > 0$, we define a strip around the parabolic boundary $\Gamma_\eta = \Omega_\eta \times (-\eta^2, 0] \cup \Theta_\eta \times (0,T]$, where $\Omega_\eta = \big\{x \in \mathbb{R}^N \colon \operatorname{dist}(x,\Omega) \le \eta\big\}$ and $\Theta_\eta = \{x \in \mathbb{R}^N \setminus \Omega\colon \operatorname{dist}(x,\partial\Omega) \le \eta\}$. Let $F : \Gamma_\eta \to \mathbb{R}$ be a Lipschitz continuous function (the final payoff function). The rules of the game are the following: At the initial time, t_0, a token is placed at a point $x_0 \in \Omega$. Then, a (fair) coin is tossed and the winner of the toss is allowed to move the game position to any $x_1 \in \overline{B_\epsilon(x_0)}$ and the time is decreased by $c\epsilon^2$ (c is just a normalizing constant, see [18]). At each turn, the coin is tossed again, and the winner chooses a new game state $x_k \in \overline{B_\epsilon(x_{k-1})}$ while time decreases at each time $c\epsilon^2$. Once the token has reached some $(x_\tau,\tau) \in \Gamma_\eta$, the game ends and the first player earns $F(x_\tau,\tau)$ (while the second player earns $-F(x_\tau,\tau)$). This game has an expected value $u_\epsilon(x_0)$ (called the value of the game) that verifies the Dynamic Programming Principle (DPP),

$$u_\epsilon(x,t) = \tfrac{1}{2}\sup_{y\in\overline{B_\epsilon(x)}} u_\epsilon(y,t-c\epsilon^2) + \tfrac{1}{2}\inf_{y\in\overline{B_\epsilon(x)}} u_\epsilon(y,t-c\epsilon^2) \tag{1.2}$$

for every $(x,t) \in \Omega \times (0,T)$. In the above equation, it is understood that $u_\epsilon(x,t) = F(x,t)$ for $(x,t) \in \Gamma_\eta$. This formula can be intuitively explained from the fact that the first player tries to maximize the expected outcome (and has probability 1/2 of selecting the next state of the game) while the second tries to minimize the expected outcome (and also has probability 1/2 of choosing the next position). As $\epsilon \to 0$, we have that $u_\epsilon \rightrightarrows v$ uniformly and this limit v (that is called the continuous value of the game) turns out to be a viscosity solution to (1.1) with the Dirichlet boundary condition $v(x,t) = F(x,t)$, for $(x,t) \in \Gamma$. The fact that the limit is a solution to the equation can be intuitively explained as follows: for a smooth function ϕ with non-zero gradient, the maximum in $\overline{B_\epsilon(x)}$ is attained at a point on the boundary of the ball $\partial B_\epsilon(x)$ that lies close to the direction of the gradient, that is, the location of the maximum is close to $x+\epsilon D\phi(x)/|D\phi(x)|$. Analogously, the minimum is close to $x - \epsilon D\phi(x)/|D\phi(x)|$ and; hence, the DPP, equation (1.2), for the smooth function ϕ reads as

$$\begin{aligned}\phi(x,t) - \phi(x,t-c\epsilon^2) \sim \tfrac{1}{2}\phi\Big(x + \epsilon\frac{D\phi(x,t-c\epsilon^2)}{|D\phi(x,t-c\epsilon^2)|}, t - c\epsilon^2\Big) \\ +\tfrac{1}{2}\phi\Big(x - \epsilon\frac{D\phi(x,t-c\epsilon^2)}{|D\phi(x,t-c\epsilon^2)|}, t - c\epsilon^2\Big) - \phi(x,t-c\epsilon^2),\end{aligned}$$

that is a discretization of the equation. Note that the right hand side is a discretization of the second derivative in the direction of the gradient. This formal calculation can be fully justified when one works in the viscosity sense, see [18].

As we have mentioned, our goal in this paper is to show that one can obtain existence of viscosity solutions to more general parabolic equations when one allows the possible movements of the players. To be more precise, our main concern in this paper is to answer the following question:

What are the PDEs that can be obtained as continuous values of Tug-of-War games when we replace the ball $\overline{B_\epsilon(x)}$ with a more general family of sets $\mathcal{A}_\epsilon(x,t)$?

To answer this question, we have to assume certain conditions on the family of sets $\mathcal{A}_\epsilon(x,t)$ and the way that they behave as $\epsilon \to 0$ (see Section 2 for details). If we play the same game described before with the possible positions given by the sets $\mathcal{A}_\epsilon(x,t)$, the DPP reads as

$$u_\epsilon(x,t) = \frac{1}{2}\sup_{(y,s)\in\mathcal{A}_\epsilon(x,t)} u_\epsilon(y,s) + \frac{1}{2}\inf_{(y,s)\in\mathcal{A}_\epsilon(x,t)} u_\epsilon(y,s).$$

Following our previous discussion for the case of balls, we can guess that the limit PDE as $\epsilon \to 0$ will depend on the point at which a smooth function ϕ with non-zero gradient attains its maximum (and its minimum) in $\mathcal{A}_\epsilon(x,t)$.

Our conditions on the sets $\mathcal{A}_\epsilon(x,t)$ are such that there is a preferred direction where the maxima and the minima of a smooth function ϕ with non-zero gradient are closely located when $\epsilon \to 0$. This preferred direction depends on the spatial location and on the gradient of ϕ at that point. We call such direction $J_{(x,t)}(D\phi(x,t))$. Also, due to scaling properties of the sets, there is a preferred time that depends on x, t, and $D\phi$. We refer to this as $K_{(x,t)}(D\phi(x,t))$. With this in mind our main result reads as follows:

Under adequate assumptions on the family of sets $\mathcal{A}_\epsilon(x,t)$, there is a uniform limit (along a subsequence) as $\epsilon \to 0$ of the values of the game, v, that is a viscosity solution to

$$K_{(x,t)}(Du(x,t))u_t(x,t) = \frac{1}{2}\langle D^2u(x,t)J_{(x,t)}(Du(x,t)), J_{(x,t)}(Du(x,t))\rangle$$

in $\Omega_T = \Omega \times (0,T]$, with boundary condition $u(x,t) = F(x)$ on the parabolic boundary Γ.

Uniqueness for this general problem and regularity issues seem delicate and are left open.

Organization of the paper. In Section 2, we describe the Tug-of-War game, introduce the precise set conditions that we assume on the family of the set $\mathcal{A}_\varepsilon(x,t)$, state the DPP for our game and prove that the game has a value and the comparison principle for values of the game. In Section 3, we prove that the $\varepsilon-$value of the game converge uniformly to a continuous function; finally, in Section 4, we show that the limit is a viscosity solution to our parabolic equation.

Throughout this paper, the points in $\mathbb{R}^N$ are denoted by $x = (x^1, \dots, x^N)$, $|\cdot|$ denote the $2-$norm in $\mathbb{R}^N$, $\langle \cdot, \cdot \rangle$ denote the usual inner product of $\mathbb{R}^N$, the ball of center $x_0 \in \mathbb{R}^N$ and radius $\rho > 0$ is denoted by $B(x_0, \rho)$ and $\pi_1, \pi_2 : \mathbb{R}^{N+1} \to \mathbb{R}$ denote the projections with respect to the $x-$axis and $t-$axis respectively. Finally, let $\mathbb{S}^N$ denote the space of symmetric $N \times N$ matrices.

2. Description of the game

Now, we describe the Tug-of-War game, following [16, 18].

Let $F : \Gamma_\eta \to \mathbb{R}$ be a bounded Borel function, F is called the final payoff function.

Tug-of-War game with spatial and time dependence. A Tug-of-War game is a zero-sum game between two players (Player I and Player II). At the beginning a token is placed at a point $(x_0, t_0) \in \Omega_T$ and we fix $\varepsilon > 0$. Then, the players toss a fair coin and the winner decides a new game state (x_1, t_1) in a set $\mathcal{A}_\varepsilon(x_0, t_0)$, that depends on the position (x_0, t_0) and will be defined later. Then, the coin is tossed again and the winner chooses a new game state $(x_2, t_2) \in \mathcal{A}_\varepsilon(x_1, t_1)$. They continue playing the game until the token hits the parabolic boundary strip Γ_ε. At the end of the game, Player II pays Player I the amount given by the payoff function F, that is, Player I earns $F(x_\tau, t_\tau)$ and the Player II earns $-F(x_\tau, t_\tau)$, where τ is the number of rounds (a stopping time) that takes the game to end. Later, we will show that $0 < \tau < +\infty$ (see Remark 2.2). This procedure yields a sequence of game states $(x_0, t_0), (x_1, t_1), \dots, (x_\tau, t_\tau)$, where every (x_k, t_k), except (x_0, t_0), are random variables, depending on the coin tosses and the strategies adopted by the players. A strategy S_I for Player I is a collection of measurable mappings $S_I = \{S_I^k\}_{k=1}^{\tau}$, such that the next game position is

$$S_I^{k+1}((x_0, t_0), (x_1, t_1), \dots, (x_k, t_k)) = (x_{k+1}, t_{k+1}) \in \mathcal{A}_\varepsilon(x_k, t_k),$$

if Player I wins the coin toss, given the partial history $((x_0, t_0), (x_1, t_1), \dots, (x_\tau, t_\tau))$. Similarly, Player II plays according to the strategy S_{II}. The next game position $(x_{k+1}, t_{k+1}) \in \mathcal{A}_\varepsilon(x_k, t_k)$, given the history $((x_0, t_0), (x_1, t_1), \dots, (x_k, t_k))$, is selected according to a probability distribution

$$p(\cdot|(x_0, t_0), (x_1, t_1), \dots, (x_k, t_k))$$

which, in our case, is given by the fair coin toss.

The fixed starting point (x_0, t_0), the domain Ω_T and the strategies S_I and S_{II} determine a unique probability measure $\mathbb{P}^{x_0}_{S_I, S_{II}}$ on the space of plays $(\Omega_T \cup \Gamma_\varepsilon)^\infty$. We denote by $\mathbb{E}^{(x_0,t_0)}_{S_I, S_{II}}$ the corresponding expectation. If S_I and S_{II} denote the strategies adopted by the Player I and II respectively, given $(x_0, t_0) \in \Omega_T$, the expected payoff is given by $\mathbb{E}^{(x_0,t_0)}_{S_I, S_{II}}[F(x_\tau, t_\tau)]$. The $\varepsilon-$value for the Player I, when starting from (x_0, t_0), is then defined as

$$u_I^\varepsilon(x_0, t_0) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}^{(x_0,t_0)}_{S_I, S_{II}}[F(x_\tau, t_\tau)],$$

while the $\varepsilon-$value of the game for the Player II is given by

$$u_{II}^\varepsilon(x_0, t_0) = \inf_{S_{II}} \sup_{S_I} \mathbb{E}^{(x_0,t_0)}_{S_I, S_{II}}[F(x_\tau, t_\tau)].$$

Now, we will describe the family of subsets of $\Omega_T \cup \Gamma_\varepsilon$ that encode the possible movements of the game.

We consider a family of sets $\{\mathcal{A}(x, t)\}_{(x,t) \in \Omega_T}$ with the following properties: For every $(x, t) \in \Omega_T$,

A1. $\mathcal{A}(x, t)$ is a compact subset of $B(0, 1) \times [-c/2, c/2]$ $(0 < c < 1)$ such that $(0, 0) \in \mathcal{A}(x, t)$;

A2. For all $s \in \pi_2(\mathcal{A}(x,t))$, the set $\mathcal{A}^s(x,t) := \big\{y \in \mathbb{R}^N : (y,s) \in \mathcal{A}(x,t)\big\}$ is symmetric with respect to the origin;

A3. Continuity respect to (x,t) : Given $(x,t) \in \Omega_T$, if $\{(x_n,t_n)\}_{n\in\mathbb{N}} \subset \Omega_T$ and $(x_n,t_n) \to (x,t)$ as $n \to \infty$, then for every $(y,s) \in \mathcal{A}(x,t)$, there exist $(y_n,s_n) \in \mathcal{A}(x_n,t_n)$, such that $(y_n,s_n) \to (y,s)$ as $n \to \infty$. Moreover, if $(y_n,s_n) \in \mathcal{A}(x_n,t_n)$ and $(y_n,s_n) \to (y,s)$ as $n \to \infty$, then $(y,s) \in \mathcal{A}(x,t)$;

A4. For every $v \in \mathbb{R}^N \setminus \{0\}$, there exists a unique $(z,r) \in \mathcal{A}(x,t)$ such that

$$\min\{\langle v,y\rangle : y \in \pi_1(\mathcal{A}(x,t))\} = \langle v,z\rangle.$$

From now on, $J_{(x,t)}(v)$ and $I_{(x,t)}(v)$ denote the point z and the time r respectively. Observe that

$$\langle v, J_{(x,t)}(v)\rangle \neq 0,$$

and

$$(J_{(x,t)}(\lambda v), I_{(x,t)}(\lambda v)) = (J_{(x,t)}(v), I_{(x,t)}(v))$$

for any $\lambda > 0$. Therefore, $(J_{(x,t)}(v), I_{(x,t)}(v))$ depends only in the direction of v. Moreover, $(-J_{(x,t)}(v), I_{(x,t)}(v)) \in \mathcal{A}(x,t)$

$$\max\{\langle v,y\rangle : y \in \pi_1(\mathcal{A}(x,t))\} = \langle v, -J_{(x,t)}(v)\rangle.$$

In addition, we require that,

$$J_{(x,t)} : \partial B(0,1) \to \partial\pi_1(\mathcal{A}(x,t))$$

is surjective.

Example 2.1. We now give some examples of possible choices of sets $\mathcal{A}(x,t)$.

(1) For any $(x,t) \in \Omega_T$, we define

$$\mathcal{A}_1(x,t) := \left\{(y,s) \in B(0,1) \times \left[-\tfrac{c}{2}, \tfrac{c}{2}\right] : |y|^2 + |s|^2 \leq \rho^2\right\},$$

where $0 < \rho < \min\{1, c/2\}$. For this family of sets,

$$(J_{(x,t)}(v), I_{(x,t)}(v)) = \left(-\tfrac{\rho v}{|v|}, 0\right)$$

for all $(x,t) \in \Omega_T$ and $v \in \mathbb{R}^N \setminus \{0\}$.

(2) For any $(x,t) \in \Omega_T$, we define

$$\mathcal{A}_2(x,t) := \left\{(y,s) \in B(0,1) \times \left[0, \tfrac{c}{2}\right] : |y|^2 \leq \tfrac{2\rho s}{c}\right\}$$

where $0 < \rho < 1$. Then

$$(J_{(x,t)}(v), I_{(x,t)}(v)) = \left(-\tfrac{\rho v}{|v|}, \tfrac{c}{2}\right) \text{ for all } (x,t) \in \Omega_T \text{ and } v \in \mathbb{R}^N \setminus \{0\}.$$

Then, we define the set of possible movements for any $(x,t)\in\Omega_T$. Given $(x,t)\in\Omega_T$ and $\varepsilon>0$ small, the set of possible movements in (x,t) is given by a scaled version of the original family of sets. We let

$$\mathcal{A}_\varepsilon(x,t):=\left\{(x,t)+\left(\varepsilon y,\varepsilon^2\tfrac{1-c}{c}s-\varepsilon^2\tfrac{c+1}{2}\right):(y,s)\in\mathcal{A}(x,t)\right\}. \qquad (2.1)$$

In the rest of this section, we only assume that the family $\{\mathcal{A}(x,t)\}_{(x,t)\in\Omega_T}$ has property **A1**. The rest of properties will be used in the following sections.

Remark 2.2. Since, by assumption **A1** the time t_k decreases at least $c\varepsilon^2$ at each round of the game, given $(x,t)\in\Omega_T$ we have that $0\le\tau(x,t)<\frac{t}{c\varepsilon^2}+1$. Then, the number of rounds that the player need to end the game when starting from (x_0,t_0) is finite.

We have a Dynamic Programming Principle for our game. For the proof, see [15, Chapter 3].

Lemma 2.3 (DPP). *The $\varepsilon-$value of the game for the Player I satisfies*

$$\begin{cases} u_I^\varepsilon(x,t)=\frac12\displaystyle\sup_{(y,s)\in\mathcal{A}_\varepsilon(x,t)}u_I^\varepsilon(y,s)+\frac12\inf_{(y,s)\in\mathcal{A}_\varepsilon(x,t)}u_I^\varepsilon(y,s) & \text{if } (x,t)\in\Omega_T,\\ u_I^\varepsilon(x,t)=F(x,t) & \text{if } (x,t)\in\Gamma_\varepsilon.\end{cases}$$

The $\varepsilon-$value function for the Player II, $u_{II}^\varepsilon(x,t)$, satisfies the same equations.

Our next goal is to state a comparison principle for the $\varepsilon-$values functions and then, we will show that the game has a value.

Definition 2.4. *A function v is a subsolution of DPP if*

$$v(x,t)\ge\frac12\sup_{(y,s)\in\mathcal{A}_\varepsilon(x,t)}v(y,s)+\frac12\inf_{(y,s)\in\mathcal{A}_\varepsilon(x,t)}v(y,s)$$

for $(x,t)\in\Omega_T$. Respectively, the supersolutions are defined by reversing the inequality for v, that is

$$v(x,t)\le\frac12\sup_{(y,s)\in\mathcal{A}_\varepsilon(x,t)}v(y,s)+\frac12\inf_{(y,s)\in\mathcal{A}_\varepsilon(x,t)}v(y,s)$$

for $(x,t)\in\Omega_T$.

Theorem 2.5 (Comparison Principle). *Let $\Omega\subset\mathbb{R}^N$ be a bounded open set and v be a subsolution (supersolution) of DPP and $v\le F$ ($v\ge F$) in Γ_ε, we have that $u_{II}^\varepsilon\ge v$ ($u_I^\varepsilon\le v$) in Ω_T.*

Hence, we have that u_I^ε (u_{II}^ε) is the lowest (largest) function that satisfies the DPP with boundary values F.

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and F a given payoff function in Γ_η. Then, the game has a $\varepsilon-$value, i.e., $u_I^\varepsilon = u_{II}^\varepsilon$.*

The proofs of above theorems are analogous to the proofs of Theorem 4.4 and Theorem 4.5 in [18], respectively.

Observe that, using Theorems 2.5 and 2.6, we have that there exists a unique function that verify the DPP with a fixed boundary datum.

Theorem 2.7. *Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth open set and F a given payoff function in Γ_η. There exists a unique function u_ε in Ω_T that verifies the DPP with boundary values F. Moreover, the function u_ε coincides with the $\varepsilon-$value of the game.*

Theorems 2.5 and 2.7 imply the comparison principle for functions that verify the DPP.

Theorem 2.8. *Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth open set and v, u be functions verifying the DPP with boundary values H and F in Γ_ε respectively. Then, if $H \leq F$, we have that $v \leq u$ in Ω_T.*

As a consequence, we get that solutions to the DPP are uniformly bounded for bounded ε.

Corollary 2.9. *Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth open set and u be a function verifying the DPP with boundary values F in Γ_ε. Then,*

$$\inf_{\Gamma_\varepsilon} F \leq u(x,t) \leq \sup_{\Gamma_\varepsilon} F \quad \textit{for any } (x,t) \in \Omega_T.$$

3. Uniform convergence

In this section, we prove that, extracting a subsequence if necessary, we have uniform convergence of u_ε as $\varepsilon \to 0$. To this end, we adapt some ideas from [7] and we use the following modification of Arzela–Ascoli lemma, see [17] for the proof.

Lemma 3.1. *Let $\{f_\varepsilon : \overline{\Omega_T} \to \mathbb{R}, \varepsilon > 0\}$ be a set of functions such that:*

(1) *There exist a positive constant C so that $|f_\varepsilon(x,t)| < C$ for every $\varepsilon > 0$ and every $(x,t) \in \overline{\Omega_T}$.*
(2) *Given $\nu > 0$, there exist positive constants r_0 and ε_0 such that for any $\varepsilon < \varepsilon_0$ and any $(x,t), (y,s) \in \overline{\Omega_T}$ with $|x-y| + |t-s| < r_0$, it holds that $|f_\varepsilon(x,t) - f_\varepsilon(y,s)| < \nu$.*

Then, there exists a uniformly continuous function $f : \overline{\Omega_T} \to \mathbb{R}$ and a subsequence still denoted by $\{f_\varepsilon\}_{\varepsilon>0}$, such that $f_\varepsilon \to f$ uniformly in $\overline{\Omega_T}$ as $\varepsilon \to 0$.

Now, let $\eta > 0$ and $F : \Gamma_\eta \to \mathbb{R}$ be a bounded Borel function. We consider the family of functions $\{u_\varepsilon\}_{\varepsilon>0}$, where u_ε are the $\varepsilon-$values of the game with payoff function F for each $\varepsilon > 0$. Observe that, by Corollary 2.9, we have that

$$|u_\varepsilon(x,t)| \leq \sup_{(w,s)\in\Gamma_\eta} |F(w,s)| \quad \forall (z,s) \in \overline{\Omega_T}. \tag{3.1}$$

Therefore, the family $\{u_\varepsilon\}_{\varepsilon>0}$ satisfies the first condition in Lemma 3.1. Then, to prove the uniform convergence of the family, we only need to show that $\{u_\varepsilon\}_{\varepsilon>0}$ satisfies also the second condition of the lemma. To prove this, we need to use properties **A1-A2** for the family of sets $\{\mathcal{A}(x,t)\}_{(x,t)\in\Omega_T}$.

Remark 3.2. Observe that, by **A2**, any linear function $l(x) = \langle v, x\rangle + b$ is a solution of the DDP (with $F(x,t) = l(x)$ in Γ_ε), where $v \in \mathbb{R}^N$ and $b \in \mathbb{R}$.

Lemma 3.3. *Let Ω be a bounded convex domain with $\partial\Omega \in C^2$ and positive curvature, $f : \Omega_\eta \to \mathbb{R}$ be a Lipschitz continuous function and assume that the family of sets $\{\mathcal{A}(x,t)\}_{(x,t)\in\Omega_T}$ satisfies the properties* **A1**–**A2**. *Then, if we take $F(x,t) = f(x)$ as our payoff function, given $\nu > 0$, there exists positive constants r_0 and ε_0, such that for any $\varepsilon < \varepsilon_0$ and any $(x,t),(y,s) \in \overline{\Omega_T}$ with $|x-y|+|t-s| < r_0$, it holds that $|u_\varepsilon(x,t)-u_\varepsilon(y,s)| < \nu$, where u_ε is the $\varepsilon-$value of the game with boundary value $F(x,t)$, i.e., $\{u_\varepsilon\}_{\varepsilon>0}$ satisfies the second condition of the Lemma* 3.1.

Proof. We divide the proof in four cases.

Case 1. The case $(x,t),(y,s) \in \Gamma$ is a consequence of the fact that f is assumed to be Lipschitz.

Case 2. Now, we study the case $(x,t) \in \Omega_T$ and $(y,t) \in \Gamma_\varepsilon$ with $y \in \Theta_\varepsilon$. As in the proof of [7, Lemma 14], using that $\partial\Omega \in C^2$, we can choose an hyperplane Π_0, such that Π_0 is tangent to Ω at some point $y_0 \in \partial\Omega$ and y lies in the outward normal direction to $\partial\Omega$ at y_0. Via a translation and rotation of the coordinate axes, we can assume that $y_0 = 0$ and $\Pi_0 = \{x^N = 0\}$. Moreover, using that $\partial\Omega$ has positive curvature, there exist a positive constants k and K such that for $U = B(0,k) \times \{x \in \mathbb{R}^N\colon -k < x^N < k\}$, we have that

$$\Omega \cap U \subset \Big\{x \in \mathbb{R}^N\colon x^N \leq -K \sum_{i=1}^{N-1} (x^i)^2\Big\}.$$

On the other hand, by the definition of Θ_ε, if $x \in \Theta_\varepsilon \cap U$, there exists $z \in \partial\Omega \cap U$ such that $|x - z| \leq \varepsilon$. Then, for any $\delta > 0$ if $-8\delta < z^N < 8\delta$ and

$0 < \varepsilon < \left(\frac{8\delta}{K}\right)^{1/2}$, we have that

$$K \sum_{i=1}^{N-1} (x^i)^2 \leq 24\delta + K\varepsilon^2 < 32\delta.$$

Then, for any $0 < \delta < \frac{1}{2K}$ and $0 < \varepsilon < \left(\frac{8\delta}{K}\right)^{1/2}$

$$\mathcal{C}_{\delta,\varepsilon} \subset \left\{ w \in \mathbb{R}^{N-1} \colon K \sum_{i=1}^{N-1} (w^i)^2 < 32\delta \right\} \times (-8\delta, 8\delta) \subset B(0, \rho_\delta)$$

where $\mathcal{C}_{\delta,\varepsilon} := \Theta_\varepsilon \cap \{x \in \mathbb{R}^N \colon -8\delta < x^N < 8\delta\}$ and $\rho_\delta = \left(\frac{64\delta}{K}\right)^{1/2}$. Therefore,

$$f(x) \leq \alpha := \sup_{z \in \Theta_\varepsilon \cap B(0,\rho_\delta)} f(z) \quad \forall x \in \mathcal{C}_{\delta,\varepsilon}.$$

Now, we consider the function $v : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $v(x,t) = ax^N + b$, where a, b are given by $a = -\frac{\beta-\alpha}{4\delta+\varepsilon}$, $b = \frac{4\delta\alpha+\varepsilon\beta}{4\delta+\varepsilon}$ with $\beta = \sup_{z\in\Omega_\eta} f(z)$. Observe that v is decreasing with respect to the space variable x^N, $v(x,t) \equiv \beta$ on $\{x \in \mathbb{R}^N \colon x^N = -4\delta\}$, $v(x,t) \equiv \alpha$ on $\{x \in \mathbb{R}^N \colon x^N = \varepsilon\}$ and, by Remark 3.2, v is a solution of the DPP.

If $a = 0$, then we have $\alpha = \beta$. Then, by Corollary 2.9,

$$u_\varepsilon(x,t) \leq \alpha \quad \forall (x,t) \in \Omega_T \cup \Gamma_\varepsilon.$$

Now, we consider the case $a \neq 0$. We observe that, if we take

$$\Omega' = \Omega \cap \{x \in \mathbb{R}^N \colon -4\delta < x^N < 0\}, \quad \Omega'_\varepsilon = \{x \in \mathbb{R}^N \colon \mathrm{dis}(x, \Omega') \leq \varepsilon\},$$

$$\Gamma'_\varepsilon = \left(\Omega'_\varepsilon \times (-\varepsilon^2, 0]\right) \cup \left((\Omega'_\varepsilon \setminus \Omega') \times (0,T]\right),$$

we have that v and u_ε are solutions of DPP in $\Omega'_T = \Omega \times (0,T]$ with payoff functions v and u_ε in Γ'_ε respectively. Since, by definition of v, $u_\varepsilon(x,t) \leq v(x,t)$ in Γ'_ε, using Theorem 2.8, we have that

$$u_\varepsilon(x,t) \leq v(x,t) \text{ in } \Omega'_\varepsilon \times (-\varepsilon^2, T].$$

On the other hand, there exists $\varepsilon_1 > 0$ (depending of δ, α and β), such that $v(x,t) \leq \alpha + \frac{1}{2}(\beta - \alpha)$ in $(\Omega \cap \{x \in \mathbb{R}^N \colon -\delta - \varepsilon < x^N < \varepsilon\}) \times (-\varepsilon^2, T]$ for all $\varepsilon < \varepsilon_1$. Then, by an iterative process, we have that for any $m \in \mathbb{N}$ there exists $\varepsilon_m > 0$ (depending of δ, α and β), such that

$$u_\varepsilon(x,t) \leq \alpha + (\tfrac{1}{2})^m(\beta - \alpha)$$

in $(\Omega \cap \{x \in \mathbb{R}^N \colon -\delta/4^m - \varepsilon < x^N < \varepsilon\}) \times (-\varepsilon^2, T]$, for all $\varepsilon < \varepsilon_m$.

The argument needed to obtain an analogous lower bound is similar.

On the other hand, since f is Lipschitz, we have

$$|\alpha - f(y)| \leq C\delta^{1/2} \quad \forall y \in B(0,\delta).$$

Therefore, given $\nu > 0$, we can choose small $\delta, \varepsilon > 0$ and large enough $m \in \mathbb{N}$ such that $x \in \Omega$ and $y \in \Theta_\varepsilon$ with $|x-y| < \delta/4^m$ it holds

$$|u_\varepsilon(x,t) - F(y,s)| = |u_\varepsilon(x,t) - f(y)| < \nu \quad \forall t \in (0,T]\, \forall s \in (-\varepsilon^2, T]. \quad (3.2)$$

Case 3. The case $(x,t) \in \Omega_T$ and $(y,s) \in \Omega \times (-\varepsilon^2, 0]$. First, we assume that $x = y$ and Player I follows a strategy S_I^x where he points to y and Player II follows any strategy. Then, $M_k = |x_k - x|^2 - k\varepsilon^2$ is a supermartingale. Indeed,

$$\mathbb{E}^{(x,t)}_{S_I^x, S_{II}}[|x_k - x|^2 | x, x_1, \dots, x_{k-1}] \leq |x_{k-1} - x|^2 + \varepsilon^2.$$

Then, by the optimal stopping theorem and Remark 2.2, we have

$$\mathbb{E}^{(x,t)}_{S_I^x, S_{II}}[|x_\tau - x|^2] \leq C(t + \varepsilon^2)$$

where C is a constant independent of x and ε. Thus, by Jensen's inequality, we get

$$\mathbb{E}^{(x,t)}_{S_I^x, S_{II}}[|x_\tau - x|] \leq C(t + \varepsilon^2)^{\frac{1}{2}} \leq C(t^{\frac{1}{2}} + \varepsilon).$$

Hence,

$$F(x,s) - LC(t^{\frac{1}{2}} + \varepsilon) \leq \mathbb{E}^{(x,t)}_{S_I^x, S_{II}}[F(x_\tau, t_\tau)] \leq F(x,s) + LC(t^{\frac{1}{2}} + \varepsilon)$$

where L is the Lipschitz constant of f. Then,

$$\begin{aligned} u_\varepsilon(x,t) &= \sup_{S_I} \inf_{S_{II}} \mathbb{E}^{(x,t)}_{S_I, S_{II}}[F(x_\tau, t_\tau)] \\ &\geq \inf_{S_{II}} \mathbb{E}^{(x,t)}_{S_I^x, S_{II}}[F(x_\tau, t_\tau)] \geq F(x,s) - LC(t^{1/2} + \varepsilon). \end{aligned}$$

Therefore,

$$u_\varepsilon(x,t) - F(x,s) = u_\varepsilon(x,t) - f(x) \geq -C(t^{1/2} + \varepsilon).$$

Similarly, choosing for Player II the strategy where he points to x, we have that

$$u_\varepsilon(x,t) - f(x) \leq C(t^{1/2} + \varepsilon).$$

Thus, we conclude that

$$|u_\varepsilon(x,t) - f(x)| \leq C(t^{1/2} + \varepsilon).$$

Finally, if $x \neq y$, we utilize the above inequality and we have that

$$|u_\varepsilon(x,t) - u_\varepsilon(y,s)| = |u_\varepsilon(x,t) - f(y)|$$

$$\leq |u_\varepsilon(x,t) - f(x)| + |f(x) - f(y)| \leq C(|x-y| + t^{\frac{1}{2}} + \varepsilon).$$

Therefore, by (3.2) and the above inequality, given $\nu > 0$, there exist $\varepsilon_0, r_0 > 0$ so that

$$|u_\varepsilon(x,t) - u_\varepsilon(y,s)| \leq \nu \tag{3.3}$$

for all $\varepsilon < \varepsilon_0$ and for any $(x,t) \in \Omega_T$ and $(y,s) \in \Gamma_\varepsilon$, such that $|x-y| + |t-s| < r_0$.

Case 4. Finally, we study the case $(x,t), (y,s) \in \Omega_T$. We consider, as in the proof of [18, Lemma 17],

$$\hat{\Omega}_T = \left\{ (z,t) \in \Omega_T \colon d((z,t),\Gamma) > \frac{r_0}{3} \right\}.$$

where $d((z,t),\Gamma) = \inf\{|z-y| + |t-s| \colon (y,s) \in \Gamma\}$, and the boundary strip

$$\hat{\Gamma} = \left\{ (x,t) \in \overline{\Omega_T} \colon d((z,t),\Gamma) \leq \frac{r_0}{3} \right\}.$$

Let $(x,t), (y,s) \in \Omega_T$ such that $|x-y| + |t-s| < \frac{r_0}{3}$. First, if $(x,t), (y,s) \in \hat{\Gamma}$, by comparison the values (x,t) and (y,s) to the nearby boundary values and using (3.3), we have $|u_\varepsilon(x,t) - u_\varepsilon(y,s)| \leq \nu$ for all $\varepsilon < \varepsilon_0$.

Finally, the case $(x,t), (y,s) \in \hat{\Omega}_T$. Without loss of generality, we can assume that $t > s$. Define

$$\hat{F}(z,h) = u_\varepsilon(z - x + y, h - t + s) + 3\nu, \quad \text{for } (z,h) \in \hat{\Gamma}.$$

Then, by the reasoning above,

$$\hat{F}(z,h) \geq u_\varepsilon(z,h) \quad \forall (z,h) \in \hat{\Gamma}.$$

Let $\hat{u}_\varepsilon$ be a solution of DPP in $\hat{\Omega}_T$ with the boundary values $\hat{F}$ in $\hat{\Gamma}$. By comparison principle and uniqueness, we have

$$u_\varepsilon(x,t) \leq \hat{u}_\varepsilon(x,t) = u_\varepsilon(y,s) + 3\nu.$$

The reverse bound follows by a similar argument. □

Now, by (3.1) and using Lemma 3.1 and Lemma 3.3, we get the main result of this section.

Theorem 3.4. *Under the same hypothesis in Lemma* 3.3, *let* $\{u_\varepsilon\}_{\varepsilon>0}$ *be the family of solution of DPP in* Ω_T *with a fixed Lipschitz continuous datum* $F(x,t) = f(x)$ *in* Γ. *Then, there exists a subsequence still denoted by* $\{u_\varepsilon\}_{\varepsilon>0}$ *and a uniformly continuous function* u *such that,* $u_\varepsilon \to u$ *uniformly in* $\overline{\Omega}_T$ *as* $\varepsilon \to 0^+$.

4. The limit equation

Throughout this section, Ω is a bounded convex domain with $\partial\Omega \in C^2$ and positive curvature, $f : \Omega_\eta \to \mathbb{R}$ is a Lipschitz continuous function, and we take $F(x,t) = f(x)$ as our payoff function. We assume that the family of sets $\{\mathcal{A}(x,t)\}_{(x,t)\in\Omega_T}$ satisfies the full set of properties **A1**–**A4**.

The aim of this section is to prove that the function u, given by Theorem 3.4, is a viscosity solution of the following PDE

$$\begin{cases} G(D^2u(x,t), \nabla u(x,t), u_t(x,t), x, t) = 0 & \text{in } \Omega_T, \\ u(x,t) = F(x,t) & \text{in } \Gamma, \end{cases} \tag{4.1}$$

where D^2u is the Hessian matrix of u and $G : \mathbb{S}^N \times \mathbb{R}^N \times \mathbb{R} \times \Omega_T \to \mathbb{R}^N$ is defined by

$$G(M,v,s,x,t) = \begin{cases} (-\frac{1-c}{c} I_{(x,t)}(v) + \frac{c+1}{2})s - \frac{1}{2}\langle M J_{(x,t)}(v), J_{(x,t)}(v)\rangle & \text{if } v \neq 0, \\ (-\frac{1-c}{c} \hat{I}_{(x,t)}(s) + \frac{c+1}{2})s & \text{if } v = 0, \end{cases}$$

where $J_{(x,t)}$, $I_{(x,t)}$ are defined in Section 2 and $\hat{I}_{(x,t)}(s)$ is defined as the unique time, such that

$$(0, \hat{I}_{(x,t)}(s)) \in \mathcal{A}(x,t), \qquad \hat{I}_{(x,t)}(s)s = \min\{rs \colon (0,r) \in \mathcal{A}(x,t)\}$$

if $s \in \mathbb{R}\setminus\{0\}$ and $\hat{I}_{(x,t)}(0) := 0$.

First, we will give the precise definition of viscosity solution to (4.1) following [8]. We denote by G^* and G_* the upper and lower semicontinuous envelopes of G respectively, i.e.,

$$G^*(M,v,s,x,t) := \limsup_{\varepsilon\to 0} \Big\{ G(\hat{M},\hat{v},\hat{s},\hat{x},\hat{t}) : (\hat{M},\hat{v},\hat{s},\hat{x},\hat{t}) \in \mathbb{C}_\varepsilon(M,v,s,x,t) \Big\},$$

where

$$\mathbb{C}_\varepsilon(M,v,s,x,t) := \Big\{ \|M-\hat{M}\| + |s-\hat{s}| + |v-\hat{v}| + |x-\hat{x}| + |t-\hat{t}| < \varepsilon \Big\}$$

and $G_*(M,v,s,x,t) := -(-G)^*(M,v,s,x,t)$ for every $(M,v,s,x,t) \in \mathbb{S}^N \times \mathbb{R}^N \times \mathbb{R} \times \Omega_T$.

Definition 4.1. *A function $u \in C(\overline{\Omega_T})$ is a viscosity solution to* (4.1) *if $u(x,t) = F(x,t)$ on Γ and the following two conditions hold:*

(i) *For every $\phi \in C^{2,1}(\overline{\Omega_T})$, such that $u - \phi$ has a strict minimum at $(x_0,t_0) \in \Omega_T$ we have*

$$G^*(D^2\phi(x_0,t_0), \nabla\phi(x_0,t_0), \phi_t(x_0,t_0), x_0, t_0) \geq 0;$$

(ii) *For every* $\phi \in C^{2,1}(\overline{\Omega_T})$, *such that* $u - \phi$ *has a strict maximum at* $(x_0, t_0) \in \Omega_T$, *we have*

$$G_*(D^2\phi(x_0,t_0), \nabla\phi(x_0,t_0), \phi_t(x_0,t_0), x_0, t_0) \le 0.$$

Now, we characterize the upper and lower envelopes for the function G.

Lemma 4.2. *For any* $(M, v, s, x, t) \in \mathbb{S}^N \times \mathbb{R}^N \times \mathbb{R} \times \Omega_T$, *we have*

$$G^*(M,v,s,x,t) = \begin{cases} G(M,v,s,x,t) & \text{if } v \neq 0, \\ \max\limits_{(z,r)\in\mathcal{A}(x,t)} \left\{ \left(-\frac{1-c}{c} r + \frac{c+1}{2}\right) s - \frac{1}{2}\langle Mz, z\rangle \right\} & \text{if } v = 0, \end{cases}$$

and

$$G_*(M,v,s,x,t) = \begin{cases} G(M,v,s,x,t) & \text{if } v \neq 0, \\ \min\limits_{(z,r)\in\mathcal{A}(x,t)} \left\{ \left(-\frac{1-c}{c} r + \frac{c+1}{2}\right) s - \frac{1}{2}\langle Mz, z\rangle \right\} & \text{if } v = 0. \end{cases}$$

Proof. We only prove the characterization for G^*, the proof for G_* is similar.

Step 1. First, we prove that $G^*(M,v,s,x,t) = G(M,v,s,x,t)$ if $v \neq 0$. Let $(M_n, v_n, s_n, x_n, t_n) \to (M, v, s, x, t)$. As $v \neq 0$, we can assume that $v_n \neq 0$. Then, by definition, for any $n \in \mathbb{N}$,

$$\begin{aligned} &G(M_n, v_n, s_n, x_n, t_n) \\ &= \left(-\tfrac{1-c}{c} I_{(x_n,t_n)}(v_n) + \tfrac{c+1}{2}\right) s_n - \tfrac{1}{2}\langle M_n J_{(x_n,t_n)}(v_n), J_{(x_n,t_n)}(v_n)\rangle. \end{aligned}$$

Since $(J_{(x_n,t_n)}(v_n), I_{(x_n,t_n)}(v_n)) \in \mathcal{A}(x_n, t_n) \subset B(0,1) \times [-\frac{c}{2}, \frac{c}{2}]$ for every $n \in \mathbb{N}$, there exists a subsequence still denote by $\{(J_{(x_n,t_n)}(v_n), I_{(x_n,t_n)}(v_n))\}_{n\in\mathbb{N}}$ and $(y, r) \in B(0,1) \times [-\frac{c}{2}, -\frac{c}{2}]$, such that

$$(J_{(x_n,t_n)}(v_n), I_{(x_n,t_n)}(v_n)) \to (y, r) \text{ as } n \to +\infty.$$

Moreover, by **A3**, $(y, r) \in \mathcal{A}(x, t)$. Then, by definition of $(J_{(x,t)}(v), I_{(x,t)}(v))$, we have that

$$\langle v, y\rangle \ge \langle v, J_{(x,t)}(v)\rangle. \tag{4.2}$$

On the other hand, by **A3**, there exist $(y_n, r_n) \in \mathcal{A}(x_n, t_n)$ such that

$$(y_n, r_n) \to (J_{(x,t)}(v), I_{(x,t)}(v)) \text{ as } n \to +\infty.$$

Thus, by definition of $(J_{(x_n,t_n)}(v_n), I_{(x_n,t_n)}(v_n))$, we have

$$\langle v_n, y_n\rangle \ge \langle v_n, J_{(x_n,t_n)}(v_n)\rangle \quad \forall n \in \mathbb{N}.$$

Then, taking limit as $n \to +\infty$ and using (4.2), we get $\langle v, y\rangle = \langle v, J_{(x,t)}(v)\rangle$. Thus, by **A4**, $y = J_{(x,t)}(v)$ and $r = I_{(x,t)}(s)$, we have

$$G(M_n, v_n, s_n, x_n, t_n) \to G(M, v, s, x, t)$$

as $n \to +\infty$ and; therefore, $G^*(M, v, s, x, t) = G(M, v, s, x, t)$ if $v \neq 0$.

Step 2. Now, we consider the case $v = 0$ and we show that

$$G^*(M, 0, s, x, t) \le \max_{(z,r)\in\mathcal{A}(x,t)} \left\{ \left(-\tfrac{1-c}{c} r + \tfrac{c+1}{2}\right) s - \tfrac{1}{2}\langle Mz, z\rangle \right\}.$$

Let $(M_n, v_n, s_n, x_n, t_n) \to (M, 0, s, x, t)$. If $v_n = 0$ for n large,

$$G(M_n, v_n, s_n, x_n, t_n) = \left(-\tfrac{1-c}{c} \hat{I}_{(x_n,t_n)}(s_n) + \tfrac{c+1}{2}\right) s_n$$

Then, as in step 1, extracting a subsequence still denoted

$$\{(M_n, v_n, s_n, x_n, t_n)\}_{n\in\mathbb{N}},$$

we have that $(0, \hat{I}_{(x_n,t_n)}(s_n)) \to (0, r_0) \in \mathcal{A}(x, t)$, and; therefore,

$$G(M_n, v_n, s_n, x_n, t_n) \to \left(-\tfrac{1-c}{c} r_0 + \tfrac{c+1}{2}\right) s \tag{4.3}$$

If $v_n \neq 0$ for n large,

$$\begin{aligned} G(M_n, v_n, s_n, x_n, t_n) = &\left(-\tfrac{1-c}{c} I_{(x_n,t_n)}(v_n) + \tfrac{c+1}{2}\right) s_n \\ &-\tfrac{1}{2}\langle M_n J_{(x_n,t_n)}(v_n), J_{(x_n,t_n)}(v_n)\rangle. \end{aligned}$$

Then, arguing again as in step 1, extracting a subsequence that we still denote by $\{(M_n, v_n, s_n, x_n, t_n)\}_{n\in\mathbb{N}}$, we have that there exist $(w, h) \in \mathcal{A}(x, t)$ such that

$$G(M_n, v_n, s_n, x_n, t_n) \to \left(-\tfrac{1-c}{c} h + \tfrac{c+1}{2}\right) s - \tfrac{1}{2}\langle Mz, z\rangle. \tag{4.4}$$

as $n \to +\infty$. Thus, by (4.3) and (4.4),

$$G^*(M, 0, s, x, t) \le \max_{\mathcal{A}(x,t)} \left\{ \left(-\tfrac{1-c}{c} r + \tfrac{c+1}{2}\right) s - \tfrac{1}{2}\langle Mz, z\rangle \right\}.$$

Step 3. Finally, we prove that

$$\max_{(z,r)\in\mathcal{A}(x,t)} \left\{ \left(-\tfrac{1-c}{c} r + \tfrac{c+1}{2}\right) s - \tfrac{1}{2}\langle Mz, z\rangle \right\} \le G^*(M, 0, s, x, t).$$

Since $\mathcal{A}(x, t)$ is a compact, there exists (Z, R), such that

$$\max_{(z,r)\in\mathcal{A}(x,t)} \left\{ \left(-\tfrac{1-c}{c} s + \tfrac{c+1}{2}\right) r - \tfrac{1}{2}\langle Mz, z\rangle \right\} = \left(-\tfrac{1-c}{c} s + \tfrac{c+1}{2}\right) R - \tfrac{1}{2}\langle MZ, Z\rangle.$$

First, we suppose that $Z = 0$. If $s \neq 0$, we have that $\hat{I}_{(x,t)}(s) = R$ and then

$$G(M, 0, s, x, t) = \left(-\tfrac{1-c}{c} R + \tfrac{c+1}{2}\right) s = \max_{\mathcal{A}(x,t)} \left\{ \left(-\tfrac{1-c}{c} s + \tfrac{c+1}{2}\right) r - \tfrac{1}{2}\langle Mz, z\rangle \right\}.$$

If $s = 0$, then

$$\max_{(z,r)\in\mathcal{A}(x,t)} \left\{ \left(-\tfrac{1-c}{c} r + \tfrac{c+1}{2}\right) s - \tfrac{1}{2}\langle Mz, z\rangle \right\} = 0 = G(M, 0, 0, x, t).$$

Therefore, if $Z=0$

$$\max_{(z,r)\in\mathcal{A}(x,t)}\left\{\left(-\tfrac{1-c}{c}r+\tfrac{c+1}{2}\right)s-\tfrac{1}{2}\langle Mz,z\rangle\right\}\leq G^*(M,0,s,x,t).$$

If $Z\neq 0$, without loss generality, we can assume that $(Z,R)\in\partial\mathcal{A}(x,t)$. By **A4**, $J_{(x,t)}:\partial B(0,1)\to\partial\pi_1(\mathcal{A}(x,t))$ is subjective, then there exists $v\in B(0,1)$, such that $J_{(x,t)}(v)=Z$. Thus, using again **A4**,

$$\left(J_{(x,t)}\left(\tfrac{v}{n}\right),I_{(x,t)}\left(\tfrac{v}{n}\right)\right)=(J_{(x,t)}(v),I_{(x,t)}(v))=(Z,R)$$

for all $n\in\mathbb{N}$

$$\begin{aligned}G\left(M,\tfrac{v}{n},s,x,t\right)&=\left(-\tfrac{1-c}{c}I_{(x,t)}\left(\tfrac{v}{n}\right)+\tfrac{c+1}{2}\right)s-\tfrac{1}{2}\langle MJ_{(x,t)}\left(\tfrac{v}{n}\right),J_{(x,t)}\left(\tfrac{v}{n}\right)\rangle\\&=\left(-\tfrac{1-c}{c}I_{(x,t)}(v)+\tfrac{c+1}{2}\right)s-\tfrac{1}{2}\langle MJ_{(x,t)}(v),J_{(x,t)}(v)\rangle\\&=\left(-\tfrac{1-c}{c}R+\tfrac{c+1}{2}\right)s-\tfrac{1}{2}\langle MZ,Z\rangle.\end{aligned}$$

Hence,

$$\max_{(z,r)\in\mathcal{A}(x,t)}\left\{\left(-\tfrac{1-c}{c}r+\tfrac{c+1}{2}\right)s-\tfrac{1}{2}\langle Mz,z\rangle\right\}\leq G\left(M,0,s,x,t\right).$$

The proof is now completed. □

Theorem 4.3. *If the values of the game* $\{u_\varepsilon\}_{\varepsilon>0}$ *uniform converge to* $u\in C(\overline{\Omega_T})$, *then* u *is a viscosity solution to* (4.1) *in the sense of Definition* 4.1.

Proof. We begin by observing that, as $u_\varepsilon\rightrightarrows u$ and $u_\varepsilon=F$ on $\partial\Omega_T$, we have that $u=F$ on Γ.

Now, we prove that if $\phi\in C^{2,1}(\overline{\Omega_T})$ and $u-\phi$ has a strict local minimum at (x_0,t_0), then

$$G^*(D^2\phi(x_0,t_0),\nabla\phi(x_0,t_0),\phi_t(x_0,t_0),x_0,t_0)\geq 0.$$

As $u-\phi$ has a strict local minimum at (x_0,t_0), we have that

$$u(x,t)-\phi(x,t)>u(x_0,t_0)-\phi(x_0,t_0)\quad(x,t)\neq(x_0,t_0).$$

Then, by the uniform convergence of u_ε to u, there exists a sequence $(x_\varepsilon,t_\varepsilon)\to(x_0,t_0)$, such that

$$u_\varepsilon(x,t)-\phi(x,t)\geq u_\varepsilon(x_\varepsilon,t_\varepsilon)-\phi(x_\varepsilon,t_\varepsilon)-o(\varepsilon^2)$$

for every (x,t) in a fixed neighborhood of (x_0,t_0). Hence,

$$\max_{(y,s)\in\mathcal{A}_\varepsilon(x_\varepsilon,t_\varepsilon)}u_\varepsilon(y,s)\geq\max_{(y,s)\in\mathcal{A}_\varepsilon(x_\varepsilon,t_\varepsilon)}\phi(y,s)+u_\varepsilon(x_\varepsilon,t_\varepsilon)-\phi(x_\varepsilon,t_\varepsilon)-o(\varepsilon^2),$$
$$\min_{(y,s)\in\mathcal{A}_\varepsilon(x_\varepsilon,t_\varepsilon)}u_\varepsilon(y,s)\geq\min_{(y,s)\in\mathcal{A}_\varepsilon(x_\varepsilon,t_\varepsilon)}\phi(y,s)+u_\varepsilon(x_\varepsilon,t_\varepsilon)-\phi(x_\varepsilon,t_\varepsilon)-o(\varepsilon^2).$$

By Theorem 2.7, we have that

$$\begin{aligned} u_\varepsilon(x_\varepsilon,t_\varepsilon) =& \tfrac{1}{2}\left\{ \max_{(y,s)\in\mathcal{A}_\varepsilon(x_\varepsilon,t_\varepsilon)} u_\varepsilon(y,s) + \min_{(y,s)\in\mathcal{A}_\varepsilon(x_\varepsilon,t_\varepsilon)} u_\varepsilon(y,s)\right\} \\ \geq& \tfrac{1}{2}\left\{ \max_{(y,s)\in\mathcal{A}_\varepsilon(x_\varepsilon,t_\varepsilon)} \phi(y,s) + \min_{(y,s)\in\mathcal{A}_\varepsilon(x_\varepsilon,t_\varepsilon)} \phi(y,s)\right\} \\ &+ u_\varepsilon(x_\varepsilon,t_\varepsilon) - \phi(x_\varepsilon,t_\varepsilon) - o(\varepsilon^2). \end{aligned}$$

Therefore,

$$\phi_\varepsilon(x_\varepsilon,t_\varepsilon) \geq \tfrac{1}{2}\left\{ \max_{(y,s)\in\mathcal{A}_\varepsilon(x_\varepsilon,t_\varepsilon)} \phi(y,s) + \min_{(y,s)\in\mathcal{A}_\varepsilon(x_\varepsilon,t_\varepsilon)} \phi(y,s)\right\} - o(\varepsilon^2). \tag{4.5}$$

Now, let $(x_\varepsilon^m, t_\varepsilon^m) \in \mathcal{A}_\varepsilon(x_\varepsilon,t_\varepsilon)$, such that

$$\min_{(y,s)\in\mathcal{A}_\varepsilon(x_\varepsilon,t_\varepsilon)} \phi(y,s) = \phi(x_\varepsilon^m, t_\varepsilon^m) \tag{4.6}$$

and let $\widetilde{x_\varepsilon^m}$ by the symmetrical point of x_ε^m respect to x_ε, that is $\widetilde{x_\varepsilon^m} = 2x_\varepsilon - x_\varepsilon^m$. Observe that,

$$\widetilde{x_\varepsilon^m} - x_\varepsilon = x_\varepsilon - x_\varepsilon^m, \tag{4.7}$$

and by **A2**, we have that

$$(\widetilde{x_\varepsilon^m}, t_\varepsilon^m) \in \mathcal{A}_\varepsilon(x_\varepsilon,t_\varepsilon). \tag{4.8}$$

As $(x_\varepsilon^m, t_\varepsilon^m), (\widetilde{x_\varepsilon^m}, t_\varepsilon^m) \in \mathcal{A}_\varepsilon(x_\varepsilon,t_\varepsilon)$, by (2.1), there exists $(y_\varepsilon^m, s_\varepsilon^m) \in \mathcal{A}(x_\varepsilon,t_\varepsilon)$, such that

$$\begin{aligned} (x_\varepsilon^m, t_\varepsilon^m) &= (x_\varepsilon,t_\varepsilon) + \left(\varepsilon y_\varepsilon^m, \varepsilon^2\tfrac{1-c}{c}s_\varepsilon^m - \varepsilon^2\tfrac{c+1}{2}\right), \\ (\widetilde{x_\varepsilon^m}, t_\varepsilon^m) &= (x_\varepsilon,t_\varepsilon) + \left(-\varepsilon y_\varepsilon^m, \varepsilon^2\tfrac{1-c}{c}s_\varepsilon^m - \varepsilon^2\tfrac{c+1}{2}\right). \end{aligned} \tag{4.9}$$

Then, using (4.5),(4.6) and (4.8), we have

$$\phi(x_\varepsilon,t_\varepsilon) \geq \tfrac{1}{2}\left\{\phi(\widetilde{x_\varepsilon^m}, t_\varepsilon^m) + \phi(x_\varepsilon^m, t_\varepsilon^m)\right\} - o(\varepsilon^2).$$

Now, consider the Taylor expansion of second order of $\phi(\cdot, t_\varepsilon^m)$ and using (4.7) and (4.9), we have that

$$\phi(x_\varepsilon,t_\varepsilon) \geq \phi(x_\varepsilon, t_\varepsilon^m) + \frac{\varepsilon^2}{2}\langle D^2\phi(x_\varepsilon,t_\varepsilon^m)y_\varepsilon^m, y_\varepsilon^m\rangle + o(\varepsilon^2).$$

Then,

$$\frac{\phi(x_\varepsilon,t_\varepsilon) - \phi(x_\varepsilon,t_\varepsilon^m)}{\varepsilon^2} \geq \frac{1}{2}\langle D^2\phi(x_\varepsilon,t_\varepsilon^m)y_\varepsilon^m, y_\varepsilon^m\rangle + \frac{o(\varepsilon^2)}{\varepsilon^2},$$

and using the Taylor expansion of first order of $\phi(x_\varepsilon,\cdot)$ and using (4.9), we get

$$-\left(\tfrac{1-c}{c}s_\varepsilon^m-\tfrac{c+1}{2}\right)\phi_t(x_\varepsilon,t_\varepsilon)\geq\tfrac{1}{2}\langle D^2\phi(x_\varepsilon,t_\varepsilon^m)y_\varepsilon^m,y_\varepsilon^m\rangle+o(1). \qquad (4.10)$$

On the other hand, since $(y_\varepsilon^m,s_\varepsilon^m)\in\mathcal{A}(x_\varepsilon,t_\varepsilon)\subset B(0,1)\times[-\frac{c}{2},\frac{c}{2}]$ for all $\varepsilon>0$, there exists a subsequence, still denoted by $\{(y_\varepsilon^m,s_\varepsilon^m)\}_{\varepsilon>0}$, such that

$$(y_\varepsilon^m,s_\varepsilon^m)\to(y_0,s_0)\in B(0,1)\times[-\frac{c}{2},\frac{c}{2}] \qquad (4.11)$$

as $\varepsilon\to0^+$. Moreover, $(y_0,s_0)\in\mathcal{A}(x_0,y_0)$ due to **A3**. Thus, taking limit in (4.10) as $\varepsilon\to0^+$, we have that

$$0\leq\left(-\tfrac{1-c}{c}s_0+\tfrac{c+1}{2}\right)\phi_t(x_0,t_0)-\tfrac{1}{2}\langle D^2\phi(x_0,t_0)y_0,y_0\rangle. \qquad (4.12)$$

In the case that $\nabla\phi(x_0,t_0)=0$, we have

$$\begin{aligned}0&\leq\left(-\tfrac{1-c}{c}s_0+\tfrac{c+1}{2}\right)\phi_t(x_0,t_0)-\tfrac{1}{2}\langle D^2\phi(x_0,t_0)y_0,y_0\rangle\\&\leq\max_{(y,s)\in\mathcal{A}(x_0,t_0)}\left\{\left(-\tfrac{1-c}{c}s+\tfrac{c+1}{2}\right)\phi_t(x_0,t_0)-\tfrac{1}{2}\langle D^2\phi(x_0,t_0)y,y\rangle\right\}\\&=G^*(D^2\phi(x_0,t_0),\nabla\phi(x_0,t_0),\phi_t(x_0,t_0),x_0,t_0).\end{aligned}$$

Now, we study the case $\nabla\phi(x_0,t_0)\neq0$. We claim that

$$(y_0,s_0)=(J_{(x,t)}(\nabla\phi(x_0,t_0)),I_{(x,t)}(\nabla\phi(x_0,t_0))).$$

From this claim and (4.12), we have that

$$\begin{aligned}0&\leq\left(-\tfrac{1-c}{c}s_0+\tfrac{c+1}{2}\right)\phi_t(x_0,t_0)-\tfrac{1}{2}\langle D^2\phi(x_0,t_0)y_0,y_0\rangle\\&=\left(-\tfrac{1-c}{c}s_0+\tfrac{c+1}{2}\right)\phi_t(x_0,t_0)\\&\quad-\tfrac{1}{2}\langle D^2\phi(x_0,t_0)J_{(x,t)}(\nabla\phi(x_0,t_0)),J_{(x,t)}(\nabla\phi(x_0,t_0))\rangle\\&=G^*(D^2\phi(x_0,t_0),\nabla\phi(x_0,t_0),\phi_t(x_0,t_0),x_0,t_0).\end{aligned}$$

Now, we prove the claim. First we observe that

$$\langle\nabla\phi(x_0,t_0),y_0\rangle\geq\langle\nabla\phi(x_0,t_0),J_{(x_0,t_0)}(\nabla\phi(x_0,t_0))\rangle$$

due to $(y_0,t_0)\in\mathcal{A}(x_0,t_0)$.

On the other hand, by **A3**, there exists $(y_\varepsilon,s_\varepsilon)\in\mathcal{A}(x_\varepsilon,t_\varepsilon)$, such that

$$(y_\varepsilon,s_\varepsilon)\to(J_{(x_0,t_0)}(\nabla\phi(x_0,t_0),I_{(x_0,t_0)}(\nabla\phi(x_0,t_0))\text{ as }\varepsilon\to0^+. \qquad (4.13)$$

Then,

$$\phi(z_\varepsilon,r_\varepsilon)\geq\phi(x_\varepsilon^m,t_\varepsilon^m)$$

where $(z_\varepsilon, r_\varepsilon) = (x_\varepsilon, t_\varepsilon) + \left(\varepsilon y_\varepsilon, \varepsilon^2 \frac{1-c}{c} s_\varepsilon - \varepsilon^2 \frac{c+1}{2}\right)$. Using (4.11) and (4.13), we get

$$0 \le \frac{\phi(z_\varepsilon, r_\varepsilon) - \phi(x_\varepsilon^m, t_\varepsilon^m)}{\varepsilon} \to \langle \nabla\phi(x_0,t_0), J_{(x_0,t_0)}(\nabla\phi(x_0,t_0)) - y_0\rangle$$

as $\varepsilon \to 0^+$. Thus,

$$\langle \nabla\phi(x_0,t_0), y_0\rangle = \langle \nabla\phi(x_0,t_0), J_{(x_0,t_0)}(\nabla\phi(x_0,t_0))\rangle,$$

and by **A4**, we have $(y_0, s_0) = (J_{(x_0,t_0)}(\nabla\phi(x_0,t_0)), I_{(x_0,t_0)}(\nabla\phi(x_0,t_0)))$. □

Example 4.4. We now give some examples.

(1) If we take the family of sets $\{\mathcal{A}_1(x,t)\}_{(x,t)\in\Omega_T}$, where $\mathcal{A}_1(x,t)$ is defined in Example 2.1, we have that

$$G(M,v,s,x,t) = \begin{cases} \frac{c+1}{2} s - \frac{\rho^2}{2|v|^2}\langle Mv, v\rangle & \text{if } v \neq 0, \\ \frac{1-c}{c}\rho|s| + \frac{c+1}{2} s & \text{if } v = 0. \end{cases}$$

(2) Let $\{\mathcal{A}_2(x,t)\}_{(x,t)\in\Omega_T}$ be the family of sets defined in Example 2.1, then

$$G(M,v,s,x,t) = \begin{cases} cs - \frac{\rho^2}{2|v|^2}\langle Mv, v\rangle & \text{if } v \neq 0, \\ \frac{c+1}{2} s & \text{if } v = 0. \end{cases}$$

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