

Pure gauge QCD and holography

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Holographic models for the pure gauge QCD vacuum are explored. The holographic renormalization of these models is considered as required by a phenomenological approach that takes the β -functions of the models as the only input. This approach is done taking the dilaton as the coordinate orthogonal to the border. This choice greatly simplifies the analysis and gives a geometrical interpretation for the fixed points of the renormalization group flow. Examples are constructed that present asymptotic freedom, confinement of static quarks, either with vanishing or non-vanishing gluon condensate G_2 . The latter models require an extension of the dilaton-gravity models already considered in the literature. This extension is also determined by the only input, i.e. the β -function. In addition the restrictions imposed by the trace anomaly equation are studied. In doing so a holographic derivation of this equation is presented.

I. INTRODUCTION

The relation between large N gauge theories and string theory [28] together with the AdS/CFT correspondence [1, 13, 21, 29] have opened new insights into strongly interacting gauge theories. The application of these ideas to QCD has received significant attention since those breakthroughs. From the phenomenological point of view, the so called AdS/QCD approach has produced very interesting results in spite of the strong assumptions involved in its formulation [5, 6, 9, 12, 25, 26]. It seems important to further proceed investigating these ideas and refining the current understanding of a possible QCD gravity dual.

In this work the phenomenological properties of a 5-dimensional holographic model for the vacuum of pure gauge QCD are investigated. The interest in studying that sector of the strong interaction gauge theory stems from the fact that in spite of its simplicity, it is expected to have the key features of QCD, namely asymptotic freedom and confinement. From the holographic point of view, it is interesting to inquire if it is possible, by an adequate choice of the 5-dimensional action for background fields, to obtain a reasonable description of both the IR and UV properties of pure gauge QCD. The model studied in this work is a simple and phenomenologically motivated extension of 5-dimensional dilaton gravity already considered in [14–16][11]. The approach employed in this work take as input the β -function of the model and study the resulting phenomenology. In doing so the holographic renormalization of the theory is considered and adapted to the phenomenological motivations of this work. The study mentioned above is greatly simplified by taking the dilaton, or equivalently the t'Hooft coupling, as the coordinate orthogonal to the border. This choice is not only useful but also leads to an interesting geometrical interpretation of the border theory β -function

zeros.

The features and results of this work are summarized as follows,

- A simple extension of 5-dimensional dilaton-gravity models is considered as holographic duals of 4-dimensional pure gauge theories.
- The holographic renormalization of these models is considered and adapted to the phenomenological approach. In this approach the β -functions of the models is the only input.
- Taking the dilaton as the radial coordinate greatly simplifies the phenomenological analysis. In addition it allows to identify borders of the space with zeros of the β -function.
- Models are constructed with vanishing and non-vanishing gluon condensate G_2 in the UV. Models with asymptotic freedom and $G_2 \neq 0$ necessarily depart from former dilaton-gravity models and motivate the extension of these models mentioned above.
- The restrictions imposed by the trace anomaly equation (TAE) on these models is studied. In order to understand the meaning of this equation in holographic terms, it is derived by requiring the independence of the 5-dimensional action on the renormalization point. In the framework of the present model such restrictions can be fulfilled and provide strong constraints on the possible models.

The paper is organized as follows. Section II presents the class of models to be considered and introduces an adequately defined superpotential. Section III describes the holographic renormalization of these models for the case of "vacuum" metrics. Section IV deals with the phenomenological approach, the choice of coordinates mentioned above and the expression in this framework of confinement and the gluon condensate. In addition, section V presents models with have vanishing gluon condensate in the UV. This motivates the extension mentioned

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above, which is presented in section VI, together with concrete models constructed out of the perturbative β -function, which present confinement and a non-vanishing gluon condensate. Section VII derives the TAE in holographic language and studies its phenomenological consequences. Section VIII presents some concluding remarks and further interesting issues to be studied.

II. ACTION AND EQUATIONS OF MOTION

The action for the models to be considered is given by,

$$S_{d+1} = \frac{N^2}{8\pi^2} \left(\int_{M_{d+1}} d^{d+1}x \sqrt{g} [-R + V(\phi) + \frac{1}{2}L(\phi) g^{ab} \partial_a \phi \partial_b \phi] - 2 \int_{M_d} d^d x \sqrt{h} K \right), \quad a, b = 0, \dots, d \quad (\text{II.1})$$

it is worth noting that this action for $L(\phi) = 1$ reduces to the one considered in [14, 16][11]. Below, models will be considered with either $L(\phi) = \pm 1$ or given in terms of the β -function of the border theory. The factor in front of the parenthesis in (II.1) is the one employed in [3], where the N refers to the border theory gauge group, namely $SU(N)$. In this scheme lengths are measured in units of the metric radius, that in these units is one[30]. The equations of motion are,

$$E_{ab} - \frac{1}{2} L(\phi) \partial_a \phi \partial_b \phi + \frac{1}{4} g_{ab} L(\phi) (\partial \phi)^2 + \frac{1}{2} g_{ab} V(\phi) = 0 \quad (\text{II.2})$$

$$\partial_a (\sqrt{g} g^{ab} \partial_b \phi) - \sqrt{g} \frac{\partial V(\phi)}{\partial \phi} - \frac{1}{2} \frac{\partial L(\phi)}{\partial \phi} g^{ab} \partial_a \phi \partial_b \phi = 0 \quad (\text{II.3})$$

since the focus is on the vacuum of the boundary field theory, only metrics and scalar fields having flat boundary space isometry invariance are considered, thus only solutions for the metric and scalar field of the following general form are considered,

$$ds^2 = du^2 + e^{2A(u)} \eta_{\mu\nu} dx^\mu dx^\nu, \quad \mu, \nu = 0, \dots, d-1 \\ \phi = \phi(u) \quad (\text{II.4})$$

the equations of motion for this choice reduce to,

$$d(d-1)A'^2 - \frac{L}{2}\phi'^2 + V(\phi) = 0 \quad (\text{II.5})$$

$$A'' - \frac{1}{\xi} L \phi'^2 = 0 \quad (\text{II.6})$$

$$dL A' \phi' + (L \phi')' - \frac{\partial V}{\partial \phi} - \frac{1}{2} \frac{\partial L}{\partial \phi} \phi'^2 = 0 \quad (\text{II.7})$$

where $\xi = 2(1-d)$ and the $'$ denotes derivative respect to u .

A. Superpotential

If a superpotential $W(\phi)$ is defined by,

$$A' = W(\phi) \quad (\text{II.8})$$

then,

$$A'' = \phi' \frac{\partial W}{\partial \phi}$$

comparing with (II.6) leads to,

$$\phi' = \frac{\xi}{L} \frac{\partial W}{\partial \phi} \quad (\text{II.9})$$

replacing (II.8) and (II.9) in (II.5) leads to,

$$d(d-1)W^2 - \frac{\xi^2}{2L} \left(\frac{\partial W}{\partial \phi} \right)^2 + V = 0 \quad (\text{II.10})$$

equation (II.7) in terms of W is,

$$d\xi W \frac{\partial W}{\partial \phi} + \frac{\xi}{L} \frac{\partial W}{\partial \phi} \frac{\partial}{\partial \phi} \left(\xi \frac{\partial W}{\partial \phi} \right) - \frac{1}{2} \frac{\partial L}{\partial \phi} \left(\frac{\xi}{L} \frac{\partial W}{\partial \phi} \right)^2 - \frac{\partial V}{\partial \phi} = 0 \quad (\text{II.11})$$

noting that,

$$\frac{1}{2} \frac{\partial}{\partial \phi} \left(\frac{\xi^2}{L} \left(\frac{\partial W}{\partial \phi} \right)^2 \right) = \frac{\xi}{L} \frac{\partial W}{\partial \phi} \frac{\partial}{\partial \phi} \left(\xi \frac{\partial W}{\partial \phi} \right) - \frac{1}{2} \frac{\partial L}{\partial \phi} \left(\frac{\xi}{L} \frac{\partial W}{\partial \phi} \right)^2$$

shows that (II.11) is the same as,

$$\frac{\partial}{\partial \phi} \left[\frac{d\xi}{2} W^2 - V + \frac{\xi^2}{2L} \left(\frac{\partial W}{\partial \phi} \right)^2 \right] = 0$$

thus (II.10) implies (II.11), if everything is described in terms of the superpotential W defined by equations (II.8) and (II.9).

B. The on-shell action

In an analogous way as in [11] the on-shell $d+1$ -dimensional action can be computed and is given by,

$$S_{d+1} = \frac{N^2}{8\pi^2} \xi W(\phi) \sqrt{g}, \quad \sqrt{g} = e^{dA} \quad (\text{II.12})$$

The energy-momentum tensor is obtained recalling that, according to the correspondence,

$$S = \int_{M_d} d^d x \sqrt{g} g_{\mu\nu} T^{\mu\nu}, \quad (\text{II.13})$$

leading to,

$$T_\mu^\mu = \frac{2}{\sqrt{g}} g_{\mu\nu} \frac{\delta S_{d+1}}{\delta g_{\mu\nu}^R} = -d \frac{N^2}{4\pi^2} \xi W(u) \quad (\text{II.14})$$

this result coincides with one obtained from the expression of this trace in terms of the extrinsic curvature [4][2][23] on the border of this space.

III. HOLOGRAPHIC RENORMALIZATION

A basic idea in the holographic renormalization group is the relation between the radial coordinate (orthogonal to the border theory coordinates) and the energy scale at which the border theory is observed. This relation is given by,

$$\mu = e^{2A(u)} \quad (\text{III.1})$$

and this last factor has the information about how the scale of energies depends on the radial coordinate. This is a direct consequence of the form (II.4) of the metric in these coordinates.

Replacing the solutions of the equations of motion for $A(u)$ and $\phi(u)$ in S_{d+1} leads to a divergent expression when $u \rightarrow \infty$, the UV border. This expression can be regularized [29] by introducing a cut-off u_0 in the coordinate u . Next a subtraction procedure is employed. It is important that this procedure does not spoil the symmetries of the theories under consideration. For example it should be such that in the AdS case it leads to an expression for the renormalized on-shell five dimensional action such that it implies a vanishing trace of the border theory energy-momentum tensor. That is, in the AdS case, conformal symmetry should be maintained by the renormalization procedure. This can be achieved by considering the leading asymptotic behavior of the solutions and defining the renormalized action S_{d+1}^R by subtracting to the five dimensional on-shell bare action, the counterterm action S_{d+1}^C defined by evaluating the bare action with these asymptotic expressions and with the same cut-off u_0 as above. This procedure was proposed in [17] and applied to holographic models in [2, 23], it leads to a finite expression for the subtracted action and it preserves conformal symmetry in the AdS case. Thus,

$$S_{d+1}^R = S_{d+1} - S_{d+1}^C$$

As mentioned at the end of the last section the trace of the energy-momentum tensor derived from the on-shell bare gravitational action is given by,

$$T_\mu^\mu = \frac{2}{\sqrt{g}} g_{\mu\nu} \frac{\delta S_{d+1}}{\delta g_{\mu\nu}^R} = -d \frac{N^2}{4\pi^2} \xi W(u)$$

Based on this bare relation, the renormalized trace of the energy momentum tensor ${}^R T_\mu^\mu$ and the renormalized superpotential W^R are defined by,

$${}^R T_\mu^\mu = \frac{2}{\sqrt{g}} g_{\mu\nu} \frac{\delta S_{d+1}^R}{\delta g_{\mu\nu}^R} = -d \frac{N^2}{4\pi^2} \xi W^R(u)$$

given the renormalized superpotential, other quantities are defined by maintaining the bare relations. For example the renormalized warp factor can be obtained by integrating the equation,

$$\frac{dA^R}{du} = W^R(u) \Rightarrow A^R(u) = C_A + \int du W^R(u)$$

the integration constant C_A is fixed by giving its value for a given point u_R of the radial coordinate [31]. This value u_R plays the role of the renormalization scale or subtraction point in field theory renormalization. As usual physical results should be independent of u_R . As will be shown below, for the case of confining solutions, the constant C_A is related to the string tension of the linear potential between static quarks.

The phenomenological approach employed in this work is to take the β -function as input of the model. The beta function of the gauge theory can be obtained from the AdS/CFT identification of the $SU(N)$ Yang-Mills renormalized coupling g_{YM} with the dilaton profile according to [1]:

$$\frac{\lambda}{N} = \frac{g_{YM}^2}{4\pi} = e^\phi \quad (\text{III.2})$$

where λ is the renormalized t'Hooft coupling. Using also (III.1) the beta function is given by,

$$\beta(\lambda) = \frac{d\lambda}{d \log \mu} = \frac{d\lambda}{dA^R} \quad (\text{III.3})$$

The models considered in this work have $L(\phi)$ equal to ± 1 or given in terms of the β -function, thus $L(\phi)$, as the β -function itself, is subject to no renormalization. Indeed, in the phenomenological approach given below they are considered as external input. From now on all indices R indicating renormalized quantities will not be included, every quantity appearing in the following equations is a renormalized one, unless otherwise is stated.

IV. PHENOMENOLOGICAL APPROACH AND CHOICE OF COORDINATES

As mentioned above the models considered below take as input the β -function depending on the t'Hooft coupling λ defined in (III.2). From this input the exponent A appearing in the warp factor that defines the metric (II.4) can be obtained as a function of λ . Indeed, the second equality in (III.3) implies that,

$$A(\lambda) = c_A + \int \frac{d\lambda}{\beta(\lambda)} \quad (\text{IV.1})$$

furthermore the superpotential W can be obtained as a function of λ as follows,

$$\begin{aligned} W &= A' = \frac{dA}{d\phi} \phi' = \frac{\lambda \xi}{\beta(\lambda) L} \frac{\partial W}{\partial \phi} \\ &\Downarrow \\ \ln W &= c_W + \int d\lambda \frac{L(\lambda) \beta(\lambda)}{\xi \lambda^2} \end{aligned} \quad (\text{IV.2})$$

where in the third equality (II.9) was employed. Thus knowing the beta function as a function of λ determines the superpotential and the solution of the equations of motion for the warp factor as a function of λ .

The physical properties of the models considered in this work, in the domain wall coordinates employed in (II.4), require the computation of both $A(u)$ and $\phi(u)$. The dependence of the dilaton on u can in principle be obtained from (III.3) and (IV.2),

$$\phi' = \frac{\beta(\lambda)W(\lambda)}{\lambda} \Rightarrow u = C + \int \frac{d\lambda}{\beta(\lambda)W(\lambda)} \quad (\text{IV.3})$$

and obtaining the inverse function of the r.h.s in the last equation. Therefore the obtention of a closed analytical expression for $\phi(u)$ is not always possible in general. However there is an alternative approach which is quite interesting in itself and permits to obtain the physical results without having to deal with inversions of the sort mentioned above. This approach consists in taking the dilaton as the radial coordinate, indeed the metric (II.4) can be written in terms of this coordinate as follows,

$$\begin{aligned} ds^2 &= du^2 + e^{2A(u)} \eta_{ij} dx^i dx^j \\ &= \frac{d\phi^2}{\phi'(\phi)^2} + e^{2A(\phi)} \eta_{\mu\nu} dx^\mu dx^\nu \\ &= \frac{d\lambda^2}{\beta(\lambda)^2 W(\lambda)^2} + e^{2A(\lambda)} \eta_{\mu\nu} dx^\mu dx^\nu \quad (\text{IV.4}) \end{aligned}$$

where the last equality follows from (IV.3). Recalling that the borders of the space are given by the singularities and zeros of the coefficients in the metric, it becomes clear that *the zeros of the beta function correspond to borders of the space*. This last statement is very interesting because it gives a clear geometrical interpretation of an important field theoretical quantity. It should be remarked that its derivation is very simple, quite general and not restricted to the models under consideration. Indeed only the identification (III.2) and the consideration of global metrics (II.4), that correspond to studying the vacuum of the border field theory, are employed to derive this interpretation.

Next, the condition for confinement and the computation of the gluon condensate are considered in this scheme.

A. Confinement

This is studied by computing the static quark potential through the consideration of a rectangular Wilson loop. This is done using the Nambu-Goto action as in [22]. For the case of a non-AdS metric this study is considered in [20] where a condition for confinement is given in terms of the metric tensor components. For the metric in (IV.4) the following two functions are considered in [20],

$$f(\lambda) = e^{2A_S(\lambda)}, \quad g(\lambda) = \frac{e^{2A_S(\lambda)}}{\beta(\lambda)W(\lambda)}$$

where $A_S(\lambda)$ denotes the string frame warp exponent given by,

$$A_S(\lambda) = A(\lambda) + \frac{2}{d-1} \phi(\lambda)$$

The condition states that there is a linear potential between static quarks if $f(\lambda)$ has a minimum λ_0 and the string tension is given by $f(\lambda_0)$. According to (IV.1) the function $f(\lambda)$ is given in terms of the β function by,

$$f(\lambda) = e^{2(c_A + f \frac{d\lambda}{\beta(\lambda)} + \frac{2}{d-1} \ln(\frac{\lambda}{\lambda_0}))}$$

the condition for having an extremum being,

$$\frac{df(\lambda)}{d\lambda} = 0 = e^{2(c_A + f \frac{d\lambda}{\beta(\lambda)} + \frac{2}{d-1} \ln(\frac{\lambda}{\lambda_0}))} \left(\frac{1}{\beta(\lambda)} + \frac{2}{(d-1)\lambda} \right)$$

thus for a non-vanishing string tension the condition for having a extremum at λ_0 is,

$$\beta(\lambda_0) = -\frac{(d-1)}{2} \lambda_0 \quad (\text{IV.5})$$

with the string tension given by,

$$\sigma = e^{2(c_A + f \frac{d\lambda}{\beta(\lambda)} + \frac{2}{d-1} \ln(\frac{\lambda}{\lambda_0}))} |_{\lambda=\lambda_0}$$

this relation is remarkable in the sense that it gives a condition for confinement and a expression for the string tension, which make no reference to its higher dimensional holographic origin[32].

B. The gluon condensate

Among important fundamental non-perturbative effects in QCD, the existence of a non-vanishing gluon condensate G_2 was early on identified [27]. It has important manifestations in hadron phenomenology, and there are indications of its non-vanishing from lattice QCD [8, 19]. However it should be mentioned that it is not completely obvious from the data that it should be non-vanishing[18].

The gluon condensate G_2 is proportional to the v.e.v. of $Tr[F^2]$, i.e.[33],

$$G_2 = \frac{-1}{2\pi^2} \langle Tr[F^2] \rangle \quad (\text{IV.6})$$

this quantity is an approximate renormalization group invariant in the UV. Indeed the trace anomaly equation[34],

$$T^\mu_\mu = -\frac{N}{8\pi} \frac{\beta(\lambda)}{\lambda^2} Tr[F^2] = \frac{\pi N}{4} \frac{\beta(\lambda)}{\lambda^2} G_2$$

shows that in the UV, where the β -function is well approximated by $\beta(\lambda) = -b_0 \lambda^2$, the gluon condensate is up to a constant the same as the trace of the energy momentum tensor. From the holographic point of view

G_2 can be computed using the following source-operator correspondence,

$$\frac{1}{g_{YM}^2} = \frac{e^{-\phi}}{4\pi} \rightsquigarrow \frac{1}{2} Tr[F^2]$$

leading to

$$\frac{2}{\sqrt{g}} \frac{\delta S_{d+1}^R}{\delta \left(\frac{1}{g_{YM}^2} \right)} = \langle Tr[F^2] \rangle = -4\pi \frac{\lambda^2}{N} \left(\frac{2}{\sqrt{g}} \frac{\delta S_{d+1}^R}{\delta \lambda} \right) \quad (IV.7)$$

thus giving the following holographic expression for G_2 ,

$$G_2 = \frac{2}{\pi} \frac{\lambda^2}{N} \left(\frac{2}{\sqrt{g}} \frac{\delta S_{d+1}^R}{\delta \lambda} \right) = \frac{\lambda^2 N}{2\pi^3} \xi \frac{dW(\lambda)}{d\lambda} \quad (IV.8)$$

where in the last equality the renormalized version of (II.12) was employed.

V. EXAMPLES WITH $L = \pm 1$

In this section two sample models are considered. One of them, referred to as non-perturbative, has been considered in [11] where it was chosen so as to be able to analytically perform the inversion mentioned in the last section. This inversion should be implemented for example to obtain $\phi(u)$ from (IV.3). Thus in this example it is possible to check explicitly that the physical results regarding confinement and the gluon condensate are the same using the domain wall coordinate u or the t'Hooft coupling as a coordinate. In addition this example is asymptotically AdS. On the perturbative side, the two loop perturbative beta function is considered. For this example the inversions mentioned above can not be performed analytically, however all the physical information that is obtained for the first example can be also obtained in this case thanks to the choice of coordinates appearing in the previous section

A. Non-perturbative

The beta function of this example is given by[11],

$$\beta(\lambda) = -\frac{\alpha \lambda \ln \frac{\lambda}{N}}{1 - \frac{\alpha}{2\xi} \ln^2 \frac{\lambda}{N}}$$

this implies, using (IV.1) and (IV.2), that $A(\lambda)$ and $W(\lambda)$ are given in this case by,

$$\begin{aligned} A(\lambda) &= C_A - \frac{1}{24} \ln^2 \frac{\lambda}{N} - \frac{1}{\alpha} \ln \left(\log \frac{\lambda}{N} \right) \\ W(\lambda) &= e^{C_W} \left(12 + \alpha \ln^2 \frac{\lambda}{N} \right) \end{aligned} \quad (V.1)$$

this β -function has a zero at $\lambda = N$ where $W(\lambda)$ is finite, thus indicating that there is a border of the space at

that value. This border is an ultraviolet one since the energy scale $\mu = e^{2A}$ diverges at $\lambda = N$. On the other hand for $\lambda \rightarrow \infty$, the β -function diverges and the energy scale goes to zero signaling an infrared border. Having vanishing Yang-Mills coupling in the UV indicates the identification,

$$\frac{\lambda}{N} - 1 = \frac{g_{YM}^2}{4\pi} \quad (V.2)$$

this is different form (III.2) and produces a change in the expression (IV.8) which for the identification (V.2) is,

$$G_2 = \frac{2N}{\pi} \left(\frac{\lambda}{N} - 1 \right)^2 \left(\frac{2}{\sqrt{g}} \frac{\delta S_{d+1}^R}{\delta \lambda} \right)$$

1. Confinement

Condition (IV.5) in this case leads to,

$$-\frac{\alpha \lambda_0 \log \frac{\lambda_0}{N}}{1 - \frac{\alpha}{2\xi} \log^2 \frac{\lambda_0}{N}} = -\frac{(d-1)}{2} \lambda_0$$

whose solutions for $d = 4$ and $N = 3$ are ,

$$\lambda_0^\pm = e^{\frac{4\alpha \pm 2\sqrt{-3\alpha + 4\alpha^2 + \alpha \text{Log}[3]}}{\alpha}}$$

the - sign corresponding to a minimum of $A_s(\lambda)$ the + to a maximum . Thus the string tension is given by ,

$$\begin{aligned} \sigma &= e^{2A_s(\lambda_0^-)} \\ &= e^{2C_A - \frac{1}{12} \text{Log} \left[e^{4 - \frac{2\sqrt{\alpha(-3+4\alpha)}}{\alpha}} \right]^2} \times \\ &\quad \left(e^{4 - \frac{2\sqrt{\alpha(-3+4\alpha)}}{\alpha}} \right)^{4/3} \left(4 - \frac{2\sqrt{\alpha(-3+4\alpha)}}{\alpha} \right)^{-\frac{2}{\alpha}} \end{aligned}$$

for $\alpha = 1$ these expressions reduce to,

$$\lambda_0^-|_{\alpha=1} = 3e^2, \quad \sigma|_{\alpha=1} = \frac{1}{4} e^{\frac{7}{3} + 2C_A},$$

The same calculation can be done in terms of the coordinate u obtaining the same results. The connection between both being given explicitly by the change of coordinates[11],

$$\lambda = N e^{\phi(u)} = N e^{C e^{-\alpha u}}$$

with the constant C_A related to C by $C_A = \frac{\ln C}{\alpha}$. This relation shows that the ultraviolet $\lambda \rightarrow N$ corresponds for $\alpha > 0$ to $u \rightarrow \infty$ and the infrared $\lambda \rightarrow \infty$ to $u \rightarrow -\infty$ and interchanged if $\alpha < 0$.

2. Gluon condensate

Using (II.12) and (IV.8), the following expression for the gluon condensate is obtained,

$$\begin{aligned} G_2(\lambda) &= \frac{(\lambda - N)^2 N}{2\pi^3} \xi \frac{\partial W(\lambda)}{\partial \lambda} \\ &= \frac{e^{C_W} \alpha N \xi}{\pi^3} \frac{(\lambda - N)^2}{\lambda} \ln\left(\frac{\lambda}{N}\right) \end{aligned}$$

whose power series expansion in the UV, i.e. for $\lambda \rightarrow N$ is,

$$\begin{aligned} G_2(\lambda) &= \frac{e^{C_W} \alpha N \xi}{\pi^3} \left(\frac{(\lambda - N)^3}{N^2} - \frac{(\lambda - N)^4}{2N^3} + \dots \right) \\ &= \frac{e^{C_W} \alpha N^2 \xi}{\pi^3} \left(\left(\frac{g_{YM}^2}{4\pi} \right)^3 - \frac{1}{2} \left(\frac{g_{YM}^2}{4\pi} \right)^4 + \dots \right) \end{aligned}$$

showing that in this case G_2 goes to zero in the UV.

B. Perturbative

The 2-loop perturbative β -function is considered,

$$\beta_{2l}(\lambda) = -\frac{11}{6\pi} \lambda^2 - \frac{17}{12\pi^2} \lambda^3$$

using (IV.1) and (IV.2), $A(\lambda)$ and $W(\lambda)$ are obtained,

$$\begin{aligned} A(\lambda) &= C_A - \frac{12}{22} \pi^2 \left[-\frac{1}{\pi \lambda} \right. \\ &\quad \left. - \frac{17}{22\pi^2} (\text{Log}[\lambda] - \text{Log}[22\pi + 17\lambda]) \right] \end{aligned}$$

$$W(\lambda) = \exp\left(C_W - \frac{\lambda(44\pi + 17\lambda)}{24\pi^2 \xi} \right) \quad (\text{V.3})$$

the energy scale is given by,

$$\mu = e^{A(\lambda)} = e^{C_A + \frac{6\pi}{11\lambda}} \left(\frac{\lambda}{e^{C_A + \frac{6\pi}{11\lambda}}} \right)^{51/121}$$

which shows that $\lambda \rightarrow 0$ corresponds to the UV and that there is a minimum energy scale to be explored by this model given by,

$$\lim_{\lambda \rightarrow \infty} \mu = \frac{e^{C_A}}{17^{51/121}}$$

1. Confinement

Condition (IV.5) in this case leads to a cubic equation,

$$\frac{(d-1)}{2} \lambda_0 - \frac{11}{6\pi} \lambda_0^2 - \frac{17}{12\pi^2} \lambda_0^3 = 0$$

the solution that corresponds to a minimum of $A_S(\lambda)$ is,

$$\lambda_0 = \frac{1}{17} \left(-11 + \sqrt{19 + 102d} \right) \pi \Big|_{d=4} = 1.785$$

with the string tension given by,

$$\sigma = e^{2A_S(\lambda_0)} \Big|_{d=4} = 0.27 e^{0.96 + C_A} \left(\frac{1}{N} \right)^{2/3}$$

2. The gluon condensate

Using (IV.8) and (V.3) leads to,

$$\begin{aligned} G_2 &= \frac{\lambda^2 N}{2\pi^3} \xi \frac{\partial W(\lambda)}{\partial \lambda} \\ &= -\frac{\lambda^2 N}{2\pi^3} \frac{e^{C_W - \frac{\lambda(44\pi + 17\lambda)}{24\pi^2 \xi}} (22\pi + 17\lambda)}{12\pi^2} \end{aligned}$$

whose series expansion around $\lambda \rightarrow 0$ is,

$$G_2 = -\frac{11(e^{C_W} N) \lambda^2}{12\pi^4} - \frac{(e^{C_W} N(-121 + 51\xi)) \lambda^3}{72(\pi^5 \xi)}$$

which shows that also in this case the gluon condensate vanishes in the UV ($\lambda \rightarrow 0$).

VI. NON-VANISHING GLUON CONDENSATE AND $L(\lambda)$

The superpotential $W(\lambda)$ can be expressed in terms of the gluon condensate $G_2(\lambda)$ using (IV.8),

$$W(\lambda) = \tilde{C}_w + \frac{2\pi^3}{\xi N} \int d\lambda \frac{G_2(\lambda)}{\lambda^2} \quad (\text{VI.1})$$

In addition the relation (IV.2) allows to express the β -function in terms of the superpotential[35],

$$\beta = \frac{\xi \lambda^2}{LW} \frac{dW}{d\lambda} \quad (\text{VI.2})$$

using (VI.1) the β -function is given in terms of G_2 by

$$\beta = \frac{G_2(\lambda)}{L(\lambda) \left(\tilde{C}_w + \frac{2\pi^3}{\xi N} \left(\int d\lambda \frac{G_2(\lambda)}{\lambda^2} \right) \right)} \frac{2\pi^3}{N}$$

This equation shows that if a theory has $L = \pm 1$ and a non-vanishing gluon condensate in the UV, then the beta function near $\lambda = 0$ grows linearly with λ with positive(negative) coefficient. Indeed taking $G_2(\lambda)$ to be a constant g_2 leads to,

$$\beta_{g_2}^{L=\pm 1} = \pm \frac{g_2}{\tilde{C}_w - \frac{2\pi^3 g_2}{\xi N \lambda}} \Big|_{\lambda \rightarrow 0} \simeq \mp \frac{\xi N \lambda}{2\pi^3} \Big|_{d=4} = \pm \frac{6N\lambda}{2\pi^3}$$

Next it is shown that it is possible to choose expressions for $L(\lambda)$ solely determined by the β -function that lead to a non-vanishing gluon condensate in the UV, without restricting the choice of the β -function. Two choices will be considered.

1. $L(\lambda) = \frac{-\xi\lambda}{\beta(\lambda)}$. This choice leads to,

$$W(\lambda) = \exp\left(C_W - \int \frac{d\lambda}{\lambda}\right) = \frac{e^{C_W}}{\lambda}$$

which gives,

$$G_2 = -\frac{e^{C_W} N \xi}{2\pi^3} \Big|_{d=4} = \frac{6 e^{C_W} N}{2\pi^3}$$

it is noteworthy that the constant gluon condensate determines the integration constant C_W .

2. $L(\lambda) = \frac{-\xi\lambda}{\beta_{as}(\lambda)}$. Where $\beta_{as}(\lambda)$ denotes the asymptotic expression of $\beta(\lambda)$ for $\lambda \rightarrow 0$. This gives,

$$W(\lambda) = \exp\left(C_W - \int \frac{d\lambda}{\lambda} \frac{\beta(\lambda)}{\beta_{as}(\lambda)}\right)$$

by definition of $\beta_{as}(\lambda)$ it holds that,

$$\frac{\beta(\lambda)}{\beta_{as}(\lambda)} = 1 + f(\lambda), \quad \lim_{\lambda \rightarrow 0} f(\lambda) = 0$$

leading to the following expression for G_2 ,

$$G_2(\lambda) = \frac{-e^{C_W} N \xi}{2\pi^3} [1 + f(\lambda)] \exp\left(-\int d\lambda f(\lambda)\right)$$

which in the UV has a constant value given, as in the previous case, by,

$$\lim_{\lambda \rightarrow 0} G_2(\lambda) = \frac{-e^{C_W} N \xi}{2\pi^3}$$

VII. HOLOGRAPHY AND THE TRACE ANOMALY EQUATION

A. Derivation of the TAE

In this section it is shown that the TAE can be obtained, in the holographic context, by requiring the independence on the renormalization scale u_R of the renormalized on-shell five dimensional action as a function of its boundary values. This requirement is analogous to the one employed [24] for deriving the TAE solely in terms of the border field theory.

This requirement is,

$$\frac{dS_{d+1}(A, \phi)}{du_R} = 0 \Rightarrow \frac{\delta S_{d+1}}{\delta A} A' + \frac{\delta S_{d+1}^R}{\delta \phi} \phi' = 0$$

which, recalling that,

$$-\frac{2}{\sqrt{g}} g_{\mu\nu} \frac{\delta \sqrt{g}}{\delta g_{\mu\nu}} = \frac{\delta \sqrt{g}}{\delta A}, \quad \sqrt{g} = e^{dA}$$

and that (III.2) implies,

$$\frac{\delta S_{d+1}}{\delta \phi} = -\frac{1}{g_{YM}^2} \frac{\delta S_{d+1}}{\delta(1/g_{YM}^2)}$$

leads to,

$$-\frac{2}{\sqrt{g}} g_{\mu\nu} \frac{\delta S_{d+1}}{\delta g_{\mu\nu}} - \frac{\phi'}{A'} \frac{1}{g_{YM}^2} \left(\frac{2}{\sqrt{g}} \frac{\delta S_{d+1}}{\delta(1/g_{YM}^2)} \right) = 0$$

recalling the holographic version of the border theory energy-momentum tensor $T_{\mu\nu}$ vacuum expectation value (II.14), the definition of the β -function (III.3), which implies $\beta(g_{YM}^2) = g_{YM}^2 \frac{d\phi}{dA} = g_{YM}^2 \frac{\phi'}{A'}$ and the correspondence (IV.7) leads to,

$$\langle T_{\mu}^{\mu} \rangle = -\frac{\beta(g_{YM}^2)}{g_{YM}^4} \langle Tr[F^2] \rangle \quad (\text{VII.1})$$

which is the TAE.

B. Phenomenology and TAE

Replacing (II.14), and using (II.12) to compute the derivative in the r.h.s. of (VII.1) leads to,

$$W(\lambda) = -\frac{1}{d} \beta(\lambda) \frac{dW(\lambda)}{d\lambda} \quad (\text{VII.2})$$

comparing this equation with (IV.2) leads an expression of $L(\lambda)$ in terms of the β -function,

$$L(\lambda) = -\frac{d\xi\lambda^2}{\beta(\lambda)^2} \quad (\text{VII.3})$$

the solution of (VII.2) for $W(\lambda)$ is,

$$W = e^{C_W - d \int \frac{d\lambda}{\beta(\lambda)}} \quad (\text{VII.4})$$

replacing (VII.2) and (VII.4) in (IV.8) implies,

$$G_2 = -d\xi \frac{\lambda^2 N}{2\pi^3} \frac{e^{C_W - d \int \frac{d\lambda}{\beta(\lambda)}}}{\beta(\lambda)} \quad (\text{VII.5})$$

Taking $\beta(\lambda) = d\lambda$ leads to a non-vanishing UV condensate given by,

$$G_2 = -\xi \frac{N}{2\pi^3} e^{C_W} \Big|_{d=4} = \frac{18}{2\pi^3} e^{C_W}$$

However, taking for $\beta(\lambda)$ the 1-loop result $-\frac{11}{6\pi}\lambda^2$ leads to,

$$G_2 = d \frac{N}{2\pi^3} \xi \left(\frac{6\pi}{11} \right) e^{C_W - \frac{6\pi d}{11\lambda}}$$

which leads to $G_2 = 0$ in the UV.

This results indicate that in the framework of the models considered in this work and described by the $d+1$ -dimensional action given by (II.1), the requirements imposed by the TAE can be met. They restrict $L(\phi)$ to fulfill (VII.3). The case of vanishing gluon condensate $G_2 = 0$, corresponds to β -functions such that,

$$\lim_{\lambda \rightarrow 0} \frac{e^{-d \int \frac{d\lambda}{\beta(\lambda)}}}{\beta(\lambda)} = 0$$

and is fulfilled by the perturbative β -functions. For $G_2 \neq 0$ the β -function should include a term linear in λ with positive coefficient.

VIII. CONCLUSIONS AND OUTLOOK

In this work the description of the pure gauge QCD vacuum by means of a 5-dimensional two derivative dilaton gravity like holographic theory has been considered. This has been done taking as only input the β -function of the border gauge field theory. It has been shown that models can be constructed with the following properties:

1. Asymptotic freedom and UV behavior matching the perturbative β -function.
2. Confinement of static quarks.

Regarding the gluon condensate G_2 it has been shown that in order to have $G_2 \neq 0$ in models with property 1. and 2., an extension of the two derivative dilaton gravity model considered in the literature should be included. In this work this is done by including a dilaton dependent factor in front of the dilaton kinetic term, which is either a constant or given in terms of the β -function. Furthermore the validity or not of the trace anomaly equation(TAE) has been considered in this work. It is shown that in this framework, fulfillment of the TAE and properties 1. and 2. can only be matched with $G_2 = 0$. Regarding these results it should be emphasized that, particularly for the case of G_2 and the TAE, the models considered are tested in the UV regime which is the

regime where it is not at all clear that they should work. Indeed, regarding these properties, the present work can be considered as an exploration of the limitations of these models in the UV.

Interesting issues that deserve further study include:

- The models considered in this work include the five dimensional metric as a dynamical variable. A natural next step is to use the geometries determined by these models as background metrics for 5-dimensional holographic theories with fields corresponding to the basic observables of QCD. The expectation is that the description of both IR and UV properties should be improved in comparison with more crude assumptions[6, 9].
- Consideration of higher order in $1/\sqrt{N}$ corrections. This should be considered as a means to study the stability of the results under corrections. These corrections, from the point of view of a more fundamental string theory motivating these background field actions, correspond to corrections in α' , the inverse of the string tension. They correspond also to higher order curvature corrections and the more fundamental string theory should provide precise expressions for the higher curvature-higher dilaton derivatives terms.

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- [30] Including adequate powers of the metric radius gives any quantity in natural units. In this respect it is worth noting that the potential $V(\phi)$ has dimensions of length to the -2 .
- [31] This corresponds to choosing one of the classical solutions of the equations of motion that differ from one another by the value of this constant[7, 10].
- [32] In ref. [16], a condition for confinement is given in terms of the IR asymptotics of the beta function, this is done for geometries that are asymptotically AdS. The condi-

tion given in (IV.5) reduces to the one in [16] for these cases. The point in considering this extension is that as argued in [11] the matching with the perturbative QCD β -function in the UV leads, in the framework of these models, to a metric that is not asymptotically AdS. More precisely, the leading term is AdS(proportional to u in domain wall coordinates) but there are additional terms(logarithmic in u) which are not AdS and diverge when approaching the UV border ($u \rightarrow \infty$).

- [33] The field strength tensor and pure gauge theory action employed in (IV.6) are given by,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] , \mathcal{L} = \frac{1}{2g^2} Tr[F^2]$$

in terms of the rescaled field $A_\mu^a = g\tilde{A}_\mu^a$, the field strength, Lagrangian density and gluon condensate are given by,

$$\tilde{F}_{\mu\nu}^a = \partial_\mu \tilde{A}_\nu^a - \partial_\nu \tilde{A}_\mu^a + gf^{abc}[\tilde{A}_\mu^b, \tilde{A}_\nu^c] , \mathcal{L} = \frac{1}{2} Tr[\tilde{F}^2]$$

$$G_2 = \frac{-g^2}{2\pi^2} \langle Tr[\tilde{F}^2] \rangle = -\frac{1}{2\pi^2} \langle Tr[F^2] \rangle$$

thus this definition of G_2 is the same as the one in ([8]).

- [34] This relation is at the basis of the gluon condensate lattice computations[8].
- [35] Eq. (VI.2) shows that given a β -function W is determined up to a multiplicative constant, e^{C_w} .