Spherical Functions of Fundamental K-Types Associated with the n-Dimensional Sphere

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Abstract. In this paper, we describe the irreducible spherical functions of fundamental K-types associated with the pair $(G, K) = (\mathrm{SO}(n+1), \mathrm{SO}(n))$ in terms of matrix hypergeometric functions. The output of this description is that the irreducible spherical functions of the same K-fundamental type are encoded in new examples of classical sequences of matrix-valued orthogonal polynomials, of size 2 and 3, with respect to a matrix-weight W supported on $[0, 1]$. Moreover, we show that W has a second order symmetric hypergeometric operator D.

Key words: matrix-valued spherical functions; matrix orthogonal polynomials; the matrix hypergeometric operator; n-dimensional sphere

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1 Introduction

The theory of spherical functions dates back to the classical papers of É. Cartan and H. Weyl; they showed that spherical harmonics arise in a natural way from the study of functions on the n-dimensional sphere $S^n - \mathrm{SO}(n+1)/\mathrm{SO}(n)$. The first general results in this direction were obtained in 1950 by Gel'fand, who considered zonal spherical functions of a Riemannian symmetric space G/K. In this case we have a decomposition $G - KAK$. When the Abelian subgroup A is one dimensional, the restrictions of zonal spherical functions to A can be identified with hypergeometric functions, providing a deep and fruitful connection between group representation theory and special functions. In particular when G is compact this gives a one to one correspondence between all zonal spherical functions of the symmetric pair (G, K) and a sequence of orthogonal polynomials.

In light of this remarkable background it is reasonable to look for an extension of the above results, by considering matrix-valued irreducible spherical functions on G of a general K-type. This was accomplished for the first time in the case of the complex projective plane $P_2(\mathbb{C}) - \mathrm{SU}(3)/\mathrm{U}(2)$ in [5]. This seminal work gave rise to a series of papers including [6, 7, 8, 10, 14, 15, 16, 17, 18, 19], where one considers matrix valued spherical functions associated to a compact symmetric pair (G, K) of rank one, arriving at sequences of matrix valued orthogonal polynomials of one real variable satisfying an explicit three-term recursion relation, which are also eigenfunctions of a second order matrix differential operator (bispectral property).

The very explicit results contained in this paper are obtained for certain K-types, namely the fundamental K-types. Also, the detailed construction of sequences of matrix orthogonal polynomials out of these irreducible spherical functions, following the general pattern established in [5], gives new examples of *classical sequences* of matrix-valued orthogonal polynomials of size 2 and 3. For the general notions concerning matrix-valued orthogonal polynomials see [9].

Interesting generalizations of these sequences are given in [20], where the coefficients of the three term recursion relation satisfied by them is exhibited.

The present paper is an outgrowth of the results of [25, Chapter 5] and we are currently working on the extension of these results for the spherical functions of any K-type associated with the n-dimensional sphere. Using [23], one can obtain the corresponding results for the spherical functions of any K-type associated with n-dimensional real projective space. The starting point is to describe the irreducible spherical functions associated with the pair $(G, K) -$ $(\mathrm{SO}(n+1), \mathrm{SO}(n))$ in terms of eigenfunctions of a matrix linear differential operator of order two. The output of this description is that the irreducible spherical functions of the same fundamental K-type are encoded in a sequence of matrix valued orthogonal polynomials.

Briefly the main results of this paper are the following. After some preliminaries, in Section 3 we study the eigenfunctions of an operator Δ on G, which is closely related to the Casimir operator. Every spherical function Φ has to be eigenfunction of this operator Δ; considering the KAK-decomposition

$$\mathrm{SO}(n+1) - \mathrm{SO}(n)\mathrm{SO}(2)\mathrm{SO}(n)$$

and choosing an appropriate coordinate y on an open subset of A, we translate the condition $\Delta\Phi - \lambda\Phi$, $\lambda \in \mathbb{C}$, into a matrix valued differential equation $\widetilde{D}H - \lambda H$ on the open interval $(0, 1)$, where H is the restriction of Φ to $\mathrm{SO}(2)$. The property of the spherical functions

$$\Phi(xgy) - \pi(x)\Phi(g)\pi(y), \qquad g \in G, \qquad x, y \in K,$$

tell us that Φ is determined by its K-type and the function H.

In Section 4 we first explicitly describe all the irreducible spherical functions of the symmetric pair $(G, K) - (\mathrm{SO}(n+1), \mathrm{SO}(n))$ with M-irreducible K-types, with $M - \mathrm{SO}(n-1)$, the centralizer of the subgroup A in K; we give these expressions in terms of the hypergeometric function $_2F_1$.

In Section 5 the operator $\widetilde{D}$ is studied in detail when the K-types correspond to fundamental representations. Certain K-fundamental types are M irreducible, and therefore they were already considered en Section 4; besides, when n is odd there is a particular fundamental K-type which has three M-submodules, this case is studied in the last section of this work. For the rest of the cases we considered separately when n is even and when n is odd. Although, in both cases we worked with the concrete realizations of the fundamental representations considering the exterior powers of the standard representation of $\mathrm{SO}(n)$:

$$\Lambda^1(\mathbb{C}^n),\ \Lambda^2(\mathbb{C}^n),\ \ldots,\ \Lambda^{\ell-1}(\mathbb{C}^n),$$

with $n - 2\ell$ or $n - 2\ell + 1$.

In Section 6 we conjugate the operator $\widetilde{D}$, by using the polynomial function

$$\Psi(y) - \begin{pmatrix} 2y - 1 & 1 \\ 1 & 2y - 1 \end{pmatrix},$$

whose columns correspond to irreducible spherical functions, in order to obtain a matrix-valued hypergeometric operator $D - \Psi^{-1}\widetilde{D}\Psi$:

$$DP - y(1 - y)P'' + (C - yU)P' - VP,$$

with

$$C - \begin{pmatrix} (n/2 + 1) & 1 \\ 1 & (n/2 + 1) \end{pmatrix}, \qquad U - (n+2)I, \qquad V - \begin{pmatrix} p & 0 \\ 0 & n - p \end{pmatrix}.$$

Then, we study all the possible eigenvalues corresponding to irreducible spherical functions and all the polynomial eigenfunctions of D.

In Section 7, for any fundamental K-type $(\Lambda^k(\mathbb{C}^n))$ with $1 \leq p \leq \ell - 1$, we find a matrix-weight W, which is a scalar multiple of

$$W - (y(1-y))^{n/2-1} \begin{pmatrix} p(2y-1)^2 + n - p & n(2y-1) \\ n(2y-1) & (n-p)(2y-1)^2 + p \end{pmatrix},$$

such that D is a symmetric operator with respect to the inner product defined among continuous vector-valued functions on $[0,1]$ by

$$\langle P_1, P_2 \rangle_W - \int_0^1 P_2^*(y) W(y) P_1(y) dy.$$

Also we prove that every spherical function gives a vector polynomial eigenfunction P of D. Therefore we obtain the following explicit expression of P in terms of the matrix hypergeometric function for any irreducible spherical function

$$P(y) - \sum_{j-0}^{w} \frac{y^j}{j!} [C; U; V + \lambda]_j P(0),$$

see Theorem 7.6.

In Section 8 for each pair (n, p) we construct a sequence of matrix orthogonal polynomials $\{P_w\}_{w \geq 0}$ of size 2 with respect to the weight function W, which are eigenfunctions of the symmetric differential operator D. Namely,

$$DP_w - P_w \begin{pmatrix} \lambda(w,0) & 0 \\ 0 & \lambda(w,1) \end{pmatrix},$$

where

$$\lambda(w, \delta) - \begin{cases} -w(w+n+1) - p & \text{if} \quad \delta - 0, \\ -w(w+n+1) - n + p & \text{if} \quad \delta - 1. \end{cases}$$

Finally, in Section 9 we develop the same techniques in order to obtain analogous results for irreducible spherical functions of the particular K-fundamental type $\Lambda^\ell(\mathbb{C}^n)$ for which we have three M-submodules instead of only two. This only occurs when n is of the form $2\ell + 1$.

It is worth to notice that, unlike the other cases, the 3×3 matrix-weight built here does reduce to a smaller size.

2   Preliminaries

2.1   Spherical functions

Let G be a locally compact unimodular group and let K be a compact subgroup of G. Let $\hat{K}$ denote the set of all equivalence classes of complex finite dimensional irreducible representations of K; for each $\delta \in \hat{K}$, let ξ_δ denote the character of δ, $d(\delta)$ the degree of δ, i.e. the dimension of any representation in the class δ, and $\chi_\delta - d(\delta)\xi_\delta$. We shall choose once and for all the Haar measure dk on K normalized by $\int_K dk - 1$.

We shall denote by V a finite dimensional vector space over the field $\mathbb{C}$ of complex numbers and by of all linear transformations of V into V. Whenever we refer to a topology on such a vector space we shall be talking about the unique Hausdorff linear topology on it.

Definition 2.1. A spherical function Φ on G of type $\delta \in \hat{K}$ is a continuous function on G with values in $\mathrm{End}(V)$ such that

   i) $\Phi(e) - I$ (I is the identity transformation);

   ii) $\Phi(x)\Phi(y) - \int_K \chi_\delta(k^{-1})\Phi(xky)dk$ for all $x, y \in G$.

The reader can find a number of general results in [21] and [4]. For our purpose it is appropriate to recall the following facts.

Proposition 2.2 ([21, Proposition 1.2]). *If $\Phi : G \longrightarrow \mathrm{End}(V)$ is a spherical function of type δ then:*

   i) $\Phi(k_1 g k_2) - \Phi(k_1)\Phi(g)\Phi(k_2)$, for all $k_1, k_2 \in K$, $g \in G$;

   ii) $k \mapsto \Phi(k)$ is a representation of K such that any irreducible subrepresentation belongs to δ.

Concerning the definition, let us point out that the spherical function Φ determines its type univocally (Proposition 2.2) and let us say that the number of times that δ occurs in the representation $k \mapsto \Phi(k)$ is called the height of Φ.

A spherical function $\Phi : G \longrightarrow \mathrm{End}(V)$ is called irreducible if V has no proper subspace invariant by $\Phi(g)$ for all $g \in G$.

If G is a connected Lie group, it is not difficult to prove that any spherical function $\Phi : G \longrightarrow \mathrm{End}(V)$ is differentiable (C^∞), and moreover that it is analytic. Let $D(G)$ denote the algebra of all left invariant differential operators on G and let $D(G)^K$ denote the subalgebra of all operators in $D(G)$ which are invariant under all right translations by elements in K.

In the following proposition (V, π) will be a finite dimensional representation of K such that any irreducible subrepresentation belongs to the same class $\delta \in \hat{K}$.

Proposition 2.3. *A function $\Phi : G \longrightarrow \mathrm{End}(V)$ is a spherical function of type δ if and only if*

   i) Φ is analytic;

   ii) $\Phi(k_1 g k_2) - \pi(k_1)\Phi(g)\pi(k_2)$, for all $k_1, k_2 \in K$, $g \in G$, and $\Phi(e) - I$;

  iii) $[D\Phi](g) - \Phi(g)[D\Phi](e)$, for all $D \in D(G)^K$, $g \in G$.

Moreover, we have that the eigenvalues $[D\Phi](e)$, $D \in D(G)^K$, characterize the spherical functions Φ as stated in the following proposition.

Proposition 2.4 ([21, Remark 4.7]). *Let $\Phi, \Psi : G \longrightarrow \mathrm{End}(V)$ be two spherical functions on a connected Lie group G of the same type $\delta \in K$. Then $\Phi - \Psi$ if and only if $(D\Phi)(e) - (D\Psi)(e)$ for all $D \in D(G)^K$.*

Let us observe that if $\Phi : G \longrightarrow \mathrm{End}(V)$ is a spherical function, then $\Phi : D \mapsto [D\Phi](e)$ maps $D(G)^K$ into $\mathrm{End}_K(V)$ ($\mathrm{End}_K(V)$ denotes the space of all linear maps of V into V which commutes with $\pi(k)$ for all $k \in K$) defining a finite dimensional representation of the associative algebra $D(G)^K$. Moreover, the spherical function is irreducible if and only if the representation $\Phi : D(G)^K \longrightarrow \mathrm{End}_K(V)$ is irreducible. We quote the following result from [19].

Proposition 2.5 ([19, Proposition 2.5]). *Let G be a connected reductive linear Lie group. Then the following properties are equivalent:*

   i) $D(G)^K$ is commutative;

   ii) every irreducible spherical function of (G, K) is of height one.

In this paper the pair (G, K) is $(\mathrm{SO}(n+1), \mathrm{SO}(n))$. Then, it is known that $D(G)^K$ is an Abelian algebra; moreover, $D(G)^K$ is isomorphic to $D(G)^G \otimes D(K)^K$ (see in [13, Theorem 10.1] or [1]), where $D(G)^G$ (resp. $D(K)^K$) denotes the subalgebra of all operators in $D(G)$ (resp. $D(K)$) which are invariant under all right translations by elements in G (resp. K).

An immediate consequence of this is that all irreducible spherical functions of our pair (G, K) are of height one.

Spherical functions of type δ (see in [21, Section 3]) arise in a natural way upon considering representations of G. If $g \mapsto U(g)$ is a continuous representation of G, say on a finite dimensional vector space E, then

$$P_\delta = \int_K \chi_\delta\left(k^{-1}\right) U(k)\, dk$$

is a projection of E onto $P_\delta E = E(\delta)$. If $P_\delta \neq 0$ the function $\Phi : G \longrightarrow \mathrm{End}(E(\delta))$ defined by

$$\Phi(g)a = P_\delta U(g)a, \qquad g \in G, \qquad a \in E(\delta), \tag{2.1}$$

is a spherical function of type δ. In fact, if $a \in E(\delta)$ we have

$$\Phi(x)\Phi(y)a = P_\delta U(x) P_\delta U(y)a = \int_K \chi_\delta\left(k^{-1}\right) P_\delta U(x) U(k) U(y)a\, dk$$

$$= \left(\int_K \chi_\delta\left(k^{-1}\right) \Phi(xky)\, dk\right) a.$$

If the representation $g \mapsto U(g)$ is irreducible then the associated spherical function Φ is also irreducible. Conversely, any irreducible spherical function on a compact group G arises in this way from a finite dimensional irreducible representation of G.

2.2   Root space structure of $\mathfrak{so}(n, \mathbb{C})$

Let E_{ik} denote the square matrix with a 1 in the ik-entry and zeros elsewhere; and let us consider the matrices

$$I_{ki} = E_{ik} - E_{ki}, \qquad 1 \leq i, k \leq n.$$

Then, the set $\{I_{ki}\}_{i<k}$ is a basis of the Lie algebra $\mathfrak{so}(n)$. These matrices satisfy the following commutation relations

$$[I_{ki}, I_{rs}] = \delta_{ks} I_{ri} + \delta_{ri} I_{sk} + \delta_{is} I_{kr} + \delta_{rk} I_{is}.$$

If we assume that $k > i$, $r > s$ then we have

$$[I_{ki}, I_{is}] = I_{sk}, \qquad [I_{ki}, I_{rk}] = I_{ri}, \qquad [I_{ki}, I_{ri}] = I_{kr}, \qquad [I_{ki}, I_{ks}] = I_{is},$$

and all the other brackets are zero. From this it easily follows that the set

$$\{I_{p\,p-1} : 2 \leq p \leq n\}$$

generates the Lie algebra $\mathfrak{so}(n)$.

Proposition 2.6. *Given $n \in \mathbb{N}$, we have that the operator*

$$Q_n = \sum_{1 \leq i, k \leq n} I_{ki}^2 \in D(\mathrm{SO}(n))$$

is right invariant under $\mathrm{SO}(n)$, i.e.

$$Q_n \in D(\mathrm{SO}(n))^{\mathrm{SO}(n)}, \qquad \forall\, n \in \mathbb{N}_0.$$

Proof. To prove that Q_n is right invariant under $\mathrm{SO}(n)$ it is enough to prove that $\dot{I}_{p,p-1}(Q_n) = 0$ for all $2 \le p \le n$. We have

$$\dot{I}_{p,p-1}(Q_n) = \sum_{1 \le i,k \le n} \left([I_{p,p-1}, I_{ki}]I_{ki} + I_{ki}[I_{p,p-1}, I_{ki}] \right).$$

Then

$$\dot{I}_{p,p-1}(Q_n) = \sum_{1 \le i \le n} \left(I_{ip}I_{p-1,i} + I_{p-1,i}I_{ip} \right) + \sum_{1 \le k \le n} \left(I_{k,p-1}I_{kp} + I_{kp}I_{k,p-1} \right)$$
$$+ \sum_{1 \le k \le n} \left(I_{pk}I_{k,p-1} + I_{k,p-1}I_{pk} \right) + \sum_{1 \le i \le n} \left(I_{p-1,i}I_{p,i} + I_{p,i}I_{p-1,i} \right) = 0.$$

This proves the proposition.   ■

2.3   The operator $Q_{2\ell}$

Let us assume that $n = 2\ell$. We look at a root space decomposition of $\mathfrak{so}(n)$ in terms of the basis elements I_{ki}, $1 \le i < k \le n$.

The linear span

$$\mathfrak{h} = \langle I_{21}, I_{43}, \dots, I_{2\ell,2\ell-1} \rangle_{\mathbb{C}}$$

is a Cartan subalgebra of $\mathfrak{so}(n, \mathbb{C})$. To find the root vectors it is convenient to visualize the elements of $\mathfrak{so}(n, \mathbb{C})$ as $\ell \times \ell$ matrices of 2×2 blocks. Thus $\mathfrak{h}$ is the subspace of all diagonal matrices of 2×2 skew-symmetric blocks. The subspaces of all matrices A with a block A_{jk} of size two, $1 \le j < k \le \ell$, in the place (j,k) and $-A_{jk}^{\mathrm{t}}$ in the place (k,j) with zeros in all other places, are $\mathrm{ad}(\mathfrak{h})$-stable. Let

$$H = i(x_1 I_{21} + \cdots + x_\ell I_{2\ell,2\ell-1}) \in \mathfrak{h},$$

for $x_1, \dots, x_\ell \in \mathbb{R}$. Then $[H, A] = \lambda(H)A$, $\forall H \in \mathfrak{h}$, if and only if for every A_{jk} we have

$$x_j(H)iI_{2j,2j-1}A_{jk} - x_k(H)iA_{jk}I_{2k,2k-1} = \lambda(H)A_{jk}, \qquad \forall H \in \mathfrak{h}.$$

Up to a scalar, the nontrivial solutions of these linear equations are the following:

$$A_{jk} = \begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix} \qquad \text{with corresponding} \quad \lambda = \mp(x_j + x_k),$$

$$A_{jk} = \begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix} \qquad \text{with corresponding} \quad \lambda = \mp(x_j - x_k).$$
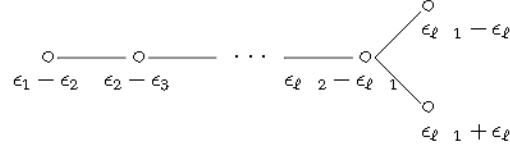
Let $\epsilon_j \in \mathfrak{h}^*$ be defined by $\epsilon_j(H) = x_j$ for $1 \le j \le \ell$. Then for $1 \le j < k \le \ell$, the following matrices are root vectors of $\mathfrak{so}(2\ell, \mathbb{C})$:

$$X_{\epsilon_j + \epsilon_k} = I_{2k-1,2j-1} - I_{2k,2j} - i(I_{2k-1,2j} + I_{2k,2j-1}),$$
$$X_{-\epsilon_j - \epsilon_k} = I_{2k-1,2j-1} - I_{2k,2j} + i(I_{2k-1,2j} + I_{2k,2j-1}),$$
$$X_{\epsilon_j - \epsilon_k} = I_{2k-1,2j-1} + I_{2k,2j} - i(I_{2k-1,2j} - I_{2k,2j-1}),$$
$$X_{-\epsilon_j + \epsilon_k} = I_{2k-1,2j-1} + I_{2k,2j} + i(I_{2k-1,2j} - I_{2k,2j-1}). \tag{2.2}$$

Thus, if we choose the following set of positive roots

$$\Delta^+ = \{\epsilon_j + \epsilon_k, \epsilon_j - \epsilon_k : 1 \le j < k \le \ell\},$$

then the Dynkin diagram of $\mathfrak{so}(2\ell,\mathbb{C})$ is D_ℓ:

By looking at the 2×2 blocks A_{jk} of the different roots, namely

$$X_{\epsilon_j+\epsilon_k} - \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \qquad X_{-\epsilon_j-\epsilon_k} - \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix},$$

$$X_{\epsilon_j-\epsilon_k} - \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \qquad X_{-\epsilon_j+\epsilon_k} - \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix},$$

it is easy to obtain the following inverse relations

$$I_{2k-1,2j-1} - \tfrac{1}{4}\big(X_{\epsilon_j+\epsilon_k} + X_{-\epsilon_j-\epsilon_k} + X_{\epsilon_j-\epsilon_k} + X_{-\epsilon_j+\epsilon_k}\big),$$
$$I_{2k,2j} - \tfrac{1}{4}\big(-X_{\epsilon_j+\epsilon_k} - X_{-\epsilon_j-\epsilon_k} + X_{\epsilon_j-\epsilon_k} + X_{-\epsilon_j+\epsilon_k}\big),$$
$$I_{2k,2j-1} - \tfrac{i}{4}\big(X_{\epsilon_j+\epsilon_k} - X_{-\epsilon_j-\epsilon_k} - X_{\epsilon_j-\epsilon_k} + X_{-\epsilon_j+\epsilon_k}\big),$$
$$I_{2k-1,2j} - \tfrac{i}{4}\big(X_{\epsilon_j+\epsilon_k} - X_{-\epsilon_j-\epsilon_k} + X_{\epsilon_j-\epsilon_k} - X_{-\epsilon_j+\epsilon_k}\big).$$

From this it follows that

$$I_{2k-1,2j-1}^2 + I_{2k,2j}^2 + I_{2k,2j-1}^2 + I_{2k-1,2j}^2$$
$$- \tfrac{1}{4}\big(X_{\epsilon_j+\epsilon_k}X_{-\epsilon_j-\epsilon_k} + X_{-\epsilon_j-\epsilon_k}X_{\epsilon_j+\epsilon_k} + X_{\epsilon_j-\epsilon_k}X_{-\epsilon_j+\epsilon_k} + X_{-\epsilon_j+\epsilon_k}X_{\epsilon_j-\epsilon_k}\big).$$

Therefore

$$Q_{2\ell} - \sum_{1\le j\le\ell} I_{2j,2j-1}^2 + \frac{1}{4}\sum_{1\le j<k\le\ell}\big(X_{\epsilon_j+\epsilon_k}X_{-\epsilon_j-\epsilon_k} + X_{-\epsilon_j-\epsilon_k}X_{\epsilon_j+\epsilon_k}$$
$$+ X_{\epsilon_j-\epsilon_k}X_{-\epsilon_j+\epsilon_k} + X_{-\epsilon_j+\epsilon_k}X_{\epsilon_j-\epsilon_k}\big).$$

Now using the expressions in (2.2) we get

$$[X_{\epsilon_j+\epsilon_k}, X_{-\epsilon_j-\epsilon_k}] - -4i(I_{2j,2j-1} + I_{2k,2k-1}),$$
$$[X_{\epsilon_j-\epsilon_k}, X_{-\epsilon_j+\epsilon_k}] - -4i(I_{2j,2j-1} - I_{2k,2k-1}).$$

Thus $Q_{2\ell}$ becomes

$$Q_{2\ell} - \sum_{1\le j\le\ell} I_{2j,2j-1}^2 - 2\sum_{1\le j\le\ell}(\ell-j)iI_{2j,2j-1}$$
$$+ \sum_{1\le j<k\le\ell}\tfrac{1}{2}\big(X_{-\epsilon_j-\epsilon_k}X_{\epsilon_j+\epsilon_k} + X_{-\epsilon_j+\epsilon_k}X_{\epsilon_j-\epsilon_k}\big). \tag{2.3}$$

2.4   The operator $Q_{2\ell+1}$

Now we look at a root space decomposition of $\mathfrak{so}(n)$ in terms of the basis elements I_{ki}, $1 \le i < k \le n$ when $n - 2\ell + 1$.

The linear span

$$\mathfrak{h} - \langle I_{21}, I_{43}, \ldots, I_{2\ell,2\ell-1}\rangle_{\mathbb{C}}$$

is a Cartan subalgebra of $\mathfrak{so}(n,\mathbb{C})$. To find the root vectors it is convenient to visualize the elements of $\mathfrak{so}(n,\mathbb{C})$ as $\ell \times \ell$ matrices of 2×2 blocks occupying the left upper corner of the

square matrices of size $2\ell + 1$, with the last column (respectively row) made up of ℓ columns (respectively rows) of size two and a zero in the place $(2\ell + 1, 2\ell + 1)$. The subspaces of all matrices A with a block A_{jk}, $1 \leq j < k \leq \ell$, in the place (j, k), with the block $-A_{jk}^t$ in the place (k, j) and with zeros in all other places, are ad($\mathfrak{h}$)-stable. Also the subspaces of all matrices B with a column B_j of size two, $1 \leq j \leq \ell$, in the place $(j, \ell + 1)$, with the row $-B_j^t$ in the place $(\ell + 1, j)$ and with zeros in all other places, are ad($\mathfrak{h}$)-stable.

On the other hand $[H, B] - \lambda B$ if and only if

$$x_j i I_{2j,2j-1} B_j - \lambda B_j.$$

Up to a scalar this linear equation has two linearly independent solutions:

$$B_j - \begin{pmatrix} 1 \\ \pm i \end{pmatrix} \qquad \text{with corresponding} \quad \lambda - \mp x_j,$$

Let $\epsilon \in \mathfrak{h}^*$ be defined by $\epsilon(H) - x_j$ for $1 \leq j \leq \ell$. Then for $1 \leq j < k \leq \ell$ and $1 \leq r \leq \ell$, the following matrices are root vectors of $\mathfrak{so}(2\ell + 1, \mathbb{C})$:

$$X_{\epsilon_j + \epsilon_k} - I_{2k-1,2j-1} - I_{2k,2j} - i(I_{2k-1,2j} + I_{2k,2j-1}),$$
$$X_{-\epsilon_j - \epsilon_k} - I_{2k-1,2j-1} - I_{2k,2j} + i(I_{2k-1,2j} + I_{2k,2j-1}),$$
$$X_{\epsilon_j - \epsilon_k} - I_{2k-1,2j-1} + I_{2k,2j} - i(I_{2k-1,2j} - I_{2k,2j-1}),$$
$$X_{-\epsilon_j + \epsilon_k} - I_{2k-1,2j-1} + I_{2k,2j} + i(I_{2k-1,2j} - I_{2k,2j-1}),$$
$$X_{\epsilon_r} - I_{n,2r-1} - iI_{n,2r},$$
$$X_{-\epsilon_r} - I_{n,2r-1} + iI_{n,2r}.$$

Thus, if we choose the following set of positive roots

$$\Delta^+ - \{\epsilon_r, \epsilon_j + \epsilon_k, \epsilon_j - \epsilon_k : 1 \leq r \leq \ell, 1 \leq j < k \leq \ell\},$$

then the Dynkin diagram of $\mathfrak{so}(2\ell + 1, \mathbb{C})$ is B_ℓ:

By looking at the 2×1 columns of the different roots, namely

$$X_{\epsilon_j} - \begin{pmatrix} 1 \\ -i \end{pmatrix}, \qquad X_{-\epsilon_j} - \begin{pmatrix} 1 \\ i \end{pmatrix},$$

it is easy to obtain the following inverse relations

$$I_{n,2r-1} - \tfrac{1}{2}(X_{\epsilon_r} + X_{-\epsilon_r}), \qquad I_{n,2r} - \tfrac{i}{2}(X_{\epsilon_r} - X_{-\epsilon_r}).$$

From this it follows that

$$I_{n,2r-1}^2 + I_{n,2r}^2 - \tfrac{1}{2}(X_{\epsilon_r} X_{-\epsilon_r} + X_{-\epsilon_r} X_{\epsilon_r}) - -i I_{2r,2r-1} + X_{-\epsilon_r} X_{\epsilon_r},$$

since $[X_{\epsilon_r}, X_{-\epsilon_r}] - -2i I_{2r,2r-1}$. Therefore we have that

$$Q_{2\ell+1} - \sum_{1 \leq j \leq 2\ell} I_{n,j}^2 + Q_{2\ell} - \sum_{1 \leq r \leq 2\ell} (-i I_{2r,2r-1} + X_{-\epsilon_r} X_{\epsilon_r}) + Q_{2\ell}.$$

Then

$$Q_{2\ell+1} - \sum_{1 \leq j \leq \ell} I_{2j,2j-1}^2 - \sum_{1 \leq j \leq \ell} (2(\ell - j) + 1) i I_{2j,2j-1}$$
$$+ \sum_{1 \leq j < k \leq \ell} \tfrac{1}{2}\left(X_{-\epsilon_j - \epsilon_k} X_{\epsilon_j + \epsilon_k} + X_{-\epsilon_j + \epsilon_k} X_{\epsilon_j - \epsilon_k}\right) + \sum_{1 \leq r \leq 2\ell} X_{-\epsilon_r} X_{\epsilon_r}. \tag{2.4}$$

2.5 Gel'fand–Tsetlin basis

For any n we identify the group $SO(n)$ with a subgroup of $SO(n+1)$ in the following way: given $k \in SO(n)$ we have

$$k \simeq \begin{pmatrix} k & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} \in SO(n+1).$$

Let $T_{\mathbf{m}}$ be an irreducible unitary representation of $SO(n)$ with highest weight $\mathbf{m}$ and let $V_{\mathbf{m}}$ be the space of this representation. Highest weights $\mathbf{m}$ of these representations are given by the ℓ-tuples of integers $\mathbf{m} - \mathbf{m}_n - (m_{1n}, \dots, m_{\ell n})$ for which

$$m_{1n} \geq m_{2n} \geq \cdots \geq m_{\ell-1,n} \geq |m_{\ell n}| \qquad \text{if} \quad n - 2\ell,$$
$$m_{1n} \geq m_{2n} \geq \cdots \geq m_{\ell n} \geq 0 \qquad \text{if} \quad n - 2\ell + 1,$$

and m_{jn} are all integers.

The restriction of the representation $T_{\mathbf{m}}$ of the group $SO(2\ell + 1)$ to the subgroup $SO(2\ell)$ decomposes into the direct sum of all representations $T_{\mathbf{m}'}$, $\mathbf{m}' - \mathbf{m}_{n-1} - (m_{1,n-1}, \dots, m_{\ell,n-1})$ for which the betweenness conditions

$$m_{1,2\ell+1} \geq m_{1,2\ell} \geq m_{2,2\ell+1} \geq m_{2,2\ell} \geq \cdots \geq m_{\ell,2\ell+1} \geq m_{\ell,2\ell} \geq -m_{\ell,2\ell+1}$$

are satisfied. For the restriction of the representations $T_{\mathbf{m}}$ of $SO(2\ell)$ to the subgroup $SO(2\ell - 1)$ the corresponding betweenness conditions are

$$m_{1,2\ell} \geq m_{1,2\ell-1} \geq m_{2,2\ell} \geq m_{2,2\ell-1} \geq \cdots \geq m_{\ell-1,2\ell} \geq m_{\ell-1,2\ell-1} \geq |m_{\ell,2\ell}|.$$

All multiplicities in the decompositions are equal to one (see [24, p. 362]).

If we continue this procedure of restriction of irreducible representations successively to the subgroups

$$SO(n - 2) > SO(n - 3) > \cdots > SO(2),$$

then we finally get one dimensional representations of the group $SO(2)$. If we take a unit vector in each one of these one dimensional representations we get an orthonormal basis of the representation space $V_{\mathbf{m}}$. Such a basis is called a Gel'fand Tsetlin basis. The elements of a Gel'fand Tsetlin basis $\{v(\mu)\}$ of the representation $T_{\mathbf{m}}$ of $SO(n)$ are labelled by the Gel'fand Tsetlin patterns $\mu - (\mathbf{m}_n, \mathbf{m}_{n-1}, \dots, \mathbf{m}_3, \mathbf{m}_2)$, where the betweenness conditions are depicted in the following diagrams.

If $n - 2\ell + 1$

$$\mu - \begin{bmatrix} m_{1n} & & m_{2n} & \cdots\cdots & m_{\ell n} & & -m_{\ell n} \\ & m_{1,n-1} & & \cdots\cdots\cdots & & m_{\ell,n-1} & \\ & & \cdots\cdots\cdots\cdots\cdots\cdots & & & \\ & & m_{15} & m_{25} & & -m_{25} & \\ & & & m_{14} & m_{24} & & \\ & & & m_{13} & & -m_{13} & \\ & & & & m_{12} & & \end{bmatrix}.$$

If $n - 2\ell$

$$\mu - \begin{bmatrix} m_{1n} & & m_{2n} & \cdots\cdots & & m_{\ell n} & \\ & m_{1,n-1} & & \cdots\cdots & m_{\ell-1,n-1} & & -m_{\ell-1,n-1} \\ & & \cdots\cdots\cdots\cdots\cdots\cdots & & & \\ & & m_{15} & m_{25} & & -m_{25} & \\ & & & m_{14} & m_{24} & & \\ & & & m_{13} & & -m_{13} & \\ & & & & m_{12} & & \end{bmatrix}.$$

The chain of subgroups $SO(n-1) > SO(n-2) > \cdots > SO(2)$ defines the orthonormal basis $\{v(\mu)\}$ uniquely up to multiplication of the basis elements by complex numbers of absolute value one.

2.6 An explicit expression for $\dot{\pi}(Q_n)$

Since $Q_n \in D(SO(n))^{SO(n)}$, given $\dot{\pi} \in \hat{SO}(n)$ it follows that $\dot{\pi}(Q_n)$ commutes with $\pi(k)$ for all $k \in SO(n)$. Hence, by Schur's Lemma $\dot{\pi}(Q_n) - \lambda I$. From expressions (2.3) and (2.4) we can give the explicit value of λ in terms of the highest weight of π, by computing $\dot{\pi}(Q_n)$ on a highest weight vector.

Proposition 2.7. *Let (π, V_π) be an irreducible representation of $SO(2\ell)$ of highest weight $\mathbf{m} - (m_1, m_2, \ldots, m_\ell)$. Then, $\dot{\pi}(Q_{2\ell}) - \lambda I$, with*

$$\lambda - \sum_{1 \leq j \leq \ell} \left(-m_j^2 - 2(\ell - j)m_j \right). \tag{2.5}$$

Proposition 2.8. *Let (π, V_π) be an irreducible representation of $SO(2\ell + 1)$ of highest weight $\mathbf{m} - (m_1, m_2, \ldots, m_\ell)$. Then, $\dot{\pi}(Q_{2\ell+1}) - \lambda I$, with*

$$\lambda - \sum_{1 \leq j \leq \ell} \left(-m_j^2 - (2(\ell - j) + 1)m_j \right). \tag{2.6}$$

3 The differential operator Δ

We shall look closely at the left invariant differential operator Δ of $SO(n+1)$ defined by

$$\Delta - \sum_{j-1}^{n} I_{n+1,j}^2,$$

in order to study its eigenfunctions and eigenvalues. Later we will use all this to understand the irreducible spherical functions of fundamental K-types associated with the pair $(G, K) - (SO(n+1), SO(n))$.

Proposition 3.1. *Let $G - SO(n+1)$ and $K - SO(n)$. Let us consider the following left invariant differential operator of G*

$$\Delta - \sum_{j-1}^{n} I_{n+1,j}^2.$$

Then Δ is also right invariant under K.

Proof. This is a direct consequence of the identity

$$Q_{n+1} - Q_n + \Delta$$

and Proposition 2.6. ∎

Let us define the one-parameter subgroup A of G as the set of all elements of the form

$$a(s) - \begin{pmatrix} I_{n-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cos s & \sin s \\ \mathbf{0} & -\sin s & \cos s \end{pmatrix}, \qquad -\pi \leq s \leq \pi, \tag{3.1}$$

where I_{n-1} denotes the identity matrix of size $n-1$, and let $M - \mathrm{SO}(n-1)$ be the centralizer of A in K.

Now we want to get the expression of $[\Delta\Phi](a(s))$ for any smooth function Φ on G with values in $\mathrm{End}(V_\pi)$ such that $\Phi(kgk') - \pi(k)\Phi(g)\pi(k')$ for all $g \in G$ and all $k, k' \in K$.

We have

$$\left[I^2_{n+1,j}\Phi\right](a(s)) - \frac{\partial^2}{\partial t^2}\Phi(a(s)\exp t I_{n+1,j})\Big|_{t-0}.$$

Hence, we will use the decomposition $G - KAK$ to write $a(s)\exp t I_{n+1,j} - k(s,t)a(s,t)h(s,t)$, with $k(s,t), h(s,t) \in K$ and $a(s,t) \in A$.

Let us take on $A \setminus \{a(\pi)\}$ the coordinate function $x(a(s)) - s$, with $-\pi < s < \pi$, and let

$$F(s) - F(x(a(s))) - \Phi(a(s)).$$

From now on we will assume that $-\pi < s, t, s+t < \pi$.

If $j - n$ we have $a(s)\exp t I_{n+1,n} - a(s)a(t) - a(s+t)$. Thus we may take

$$a(s,t) - a(s+t), \qquad k(s,t) - h(s,t) - e.$$

Since $x(a(s+t)) - s+t$, we obtain

$$\left[I^2_{n+1,n}\Phi\right](a(s)) - \frac{\partial^2}{\partial t^2}\Phi(a(s)\exp t I_{n+1,n})\Big|_{t-0} - \frac{\partial^2}{\partial t^2}\Phi(a(s+t))\Big|_{t-0}$$

$$- \frac{\partial^2}{\partial t^2}F(s+t)\Big|_{t-0} - F''(s).$$

For $1 \leq j \leq n-1$, when $s \notin \mathbb{Z}\pi$, we may take

$$k(s,t) - \begin{pmatrix} I_{j-1} & 0 & 0 & 0 & 0 \\ 0 & \frac{\sin s \cos t}{\sqrt{1-\cos^2 s \cos^2 t}} & 0 & \frac{\sin t}{\sqrt{1-\cos^2 s \cos^2 t}} & 0 \\ 0 & 0 & I_{n-j-1} & 0 & 0 \\ 0 & \frac{-\sin t}{\sqrt{1-\cos^2 s \cos^2 t}} & 0 & \frac{\sin s \cos t}{\sqrt{1-\cos^2 s \cos^2 t}} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$h(s,t) - \begin{pmatrix} I_{j-1} & 0 & 0 & 0 & 0 \\ 0 & \frac{\sin s}{\sqrt{1-\cos^2 s \cos^2 t}} & 0 & \frac{-\cos s \sin t}{\sqrt{1-\cos^2 s \cos^2 t}} & 0 \\ 0 & 0 & I_{n-j-1} & 0 & 0 \\ 0 & \frac{\cos s \sin t}{\sqrt{1-\cos^2 s \cos^2 t}} & 0 & \frac{\sin s}{\sqrt{1-\cos^2 s \cos^2 t}} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$a(s,t) - \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & \cos s \cos t & \sqrt{1-\cos^2 s \cos^2 t} \\ 0 & -\sqrt{1-\cos^2 s \cos^2 t} & \cos s \cos t \end{pmatrix}.$$

Then, for $0 < s < \pi$, we have $x(a(s,t)) - \arccos(\cos s \cos t)$ and

$$\frac{\partial}{\partial t}x(a(s,t)) - \frac{\cos s \sin t}{\sqrt{1-\cos^2 s \cos^2 t}}.$$

From here we get

$$\frac{\partial}{\partial t}x(a(s,t))\Big|_{t-0} - 0 \qquad \text{and} \qquad \frac{\partial^2}{\partial t^2}x(a(s,t))\Big|_{t-0} - \frac{\cos s}{\sin s}.$$

Thus

$$\frac{\partial}{\partial t}\Phi(a(s,t))\Big|_{t=0} - F'(s)\frac{\partial}{\partial t}x(a(s,t))\Big|_{t=0} = 0 \qquad \text{and} \qquad \frac{\partial^2}{\partial t^2}\Phi(a(s,t))\Big|_{t=0} = \frac{\cos s}{\sin s}F'(s).$$

We observe that $k(s,0) = h(s,0) = e$ and that $a(s,0) = a(s)$. Then

$$[I_{nj}^2\Phi](a(s)) = \frac{\partial^2}{\partial t^2}\pi(k(s,t))\Big|_{t=0}\Phi(a(s)) + 2\frac{\partial}{\partial t}\pi(k(s,t))\Big|_{t=0}\frac{\partial}{\partial t}\Phi(a(s,t))\Big|_{t=0}$$

$$+ 2\frac{\partial}{\partial t}\pi(k(s,t))\Big|_{t=o}\Phi(a(s))\frac{\partial}{\partial t}\pi(h(s,t))\Big|_{t=0} + \frac{\partial^2}{\partial t^2}\Phi(a(s,t))\Big|_{t=0}$$

$$+ 2\frac{\partial}{\partial t}\Phi(a(s,t))\Big|_{t=0}\frac{\partial}{\partial t}\pi(h(s,t))\Big|_{t=0} + \Phi(a(s))\frac{\partial^2}{\partial t^2}\pi(h(s,t))\Big|_{t=0}.$$

We also have

$$\frac{\partial}{\partial t}\pi(k(s,t))\Big|_{t=0} = \dot\pi\left(\frac{\partial}{\partial t}k(s,t)\Big|_{t=0}\right) = \frac{1}{\sin s}\dot\pi(I_{n,j}),$$

and

$$\frac{\partial}{\partial t}\pi(h(s,t))\Big|_{t=0} = \dot\pi\left(\frac{\partial}{\partial t}h(s,t)\Big|_{t=0}\right) = -\frac{\cos s}{\sin s}\dot\pi(I_{n,j}).$$

We will need the following proposition, whose proof appears in the Appendix and its idea is taken from [5].

Proposition 3.2. *If* $A(s,t) = k(s,t)$ *or* $A(s,t) = h(s,t)$, *then in either case for* $0 < s < \pi$, *we have*

$$\frac{\partial^2(\pi\circ A)}{\partial t^2}\Big|_{t=0} = \dot\pi\left(\frac{\partial A}{\partial t}\Big|_{t=0}\right)^2.$$

Moreover in each case, for $1 \le j \le n-1$ *and* $0 < s < \pi$, *we have*

$$\frac{\partial^2}{\partial t^2}\pi(k(s,t))\Big|_{t=0} = \frac{1}{\sin^2 s}\dot\pi(I_{n,j})^2, \qquad \frac{\partial^2}{\partial t^2}\pi(h(s,t))\Big|_{t=0} = \frac{\cos^2 s}{\sin^2 s}\dot\pi(I_{n,j})^2.$$

Now we obtain the following corollaries.

Corollary 3.3. *Let* Φ *be any smooth function on* G *with values in* $\mathrm{End}(V_\pi)$ *such that* $\Phi(kgk') = \pi(k)\Phi(g)\pi(k')$ *for all* $g \in G$ *and all* $k, k' \in K$. *Then, if* $F(s) = \Phi(a(s))$, *for* $0 < s < \pi$ *we have*

$$[\Delta\Phi](a(s)) = F''(s) + (n-1)\frac{\cos s}{\sin s}F'(s) + \frac{1}{\sin^2 s}\sum_{j=1}^{n-1}\dot\pi(I_{n,j})^2 F(s)$$

$$- 2\frac{\cos s}{\sin^2 s}\sum_{j=1}^{n-1}\dot\pi(I_{n,j})F(s)\dot\pi(I_{n,j}) + \frac{\cos^2 s}{\sin^2 s}F(s)\sum_{j=1}^{n-1}\dot\pi(I_{n,j})^2.$$

Corollary 3.4. *Let* Φ *be an irreducible spherical function on* G *of type* $\pi \in \hat{K}$. *Then, if* $F(s) = \Phi(a(s))$, *we have*

$$F''(s) + (n-1)\frac{\cos s}{\sin s}F'(s) + \frac{1}{\sin^2 s}\sum_{j=1}^{n-1}\dot\pi(I_{n,j})^2 F(s)$$

$$- 2\frac{\cos s}{\sin^2 s}\sum_{j=1}^{n-1}\dot\pi(I_{n,j})F(s)\dot\pi(I_{n,j}) + \frac{\cos^2 s}{\sin^2 s}F(s)\sum_{j=1}^{n-1}\dot\pi(I_{n,j})^2 = \lambda F(s),$$

for some $\lambda \in \mathbb{C}$ *and* $0 < s < \pi$.

Notice that the expression in Corollary 3.4 generalizes the very well known situation when the K-type is the trivial one, as we state in the following corollary (cf. [11, p. 403, equation (10)]).

Corollary 3.5. *Let* Φ *be an irreducible spherical function on* G *of the trivial K-type. Then, for* $F(s) - \Phi(a(s))$ *we have*

$$F''(s) + (n-1)\frac{\cos s}{\sin s}F'(s) - \lambda F(s),$$

for some $\lambda \in \mathbb{C}$ *and* $0 < s < \pi$.

Let us make the change of variables $y - (1+\cos s)/2$, with $0 < s < \pi$; then $0 < y < 1$. We also have $\cos s - 2y - 1$, $\sin^2 s - 4y(1-y)$ and $\frac{d}{dy} - -\frac{\sin s}{2}$. If we let $H(y) - F(s)$, i.e.

$$H(y) - \Phi(a(s)), \qquad \text{with} \quad \cos s - 2y - 1,$$

we obtain

$$F'(s) - -\frac{\sin s}{2}H'(s), \qquad F''(s) - \frac{\sin^2 s}{4}H''(y) - \frac{\cos s}{2}H'(y).$$

In terms of this new variable Corollary 3.4 becomes

Corollary 3.6. *Let* Φ *be an irreducible spherical function on* G *of type* $\pi \in \hat{K}$. *Then, if* $H(y) - \Phi(a(s))$ *with* $y - (1+\cos s)/2$, *we have*

$$y(1-y)H''(y) + \frac{1}{2}n(1-2y)H'(y) + \frac{1}{4y(1-y)}\sum_{j-1}^{n-1}\dot{\pi}(I_{n,j})^2 H(y)$$

$$+ \frac{(1-2y)}{2y(1-y)}\sum_{j-1}^{n-1}\dot{\pi}(I_{n,j})H(y)\dot{\pi}(I_{n,j}) + \frac{(1-2y)^2}{4y(1-y)}H(y)\sum_{j-1}^{n-1}\dot{\pi}(I_{n,j})^2 - \lambda H(y),$$

for some $\lambda \in \mathbb{C}$ *and* $0 < y < 1$.

Remark 3.7. Let us notice that, for any $y \in (0,1)$, $H(y)$ is a scalar linear transformation when restricted to any M-submodule, see Proposition 2.2. Therefore, if m is the number of M-submodules contained in (V,π), we consider the vector valued function $H : (0,1) \to \mathbb{C}^m$ whose entries are given by those scalar values that $H(y)$ takes on every M-submodule.

If the End(V)-valued function H satisfies the differential equation given in Corollary 3.6, then the vector valued function H satisfies

$$y(1-y)H''(y) + \frac{1}{2}n(1-2y)H'(y) + \frac{1}{4y(1-y)}N_1 H(y)$$

$$+ \frac{(1-2y)}{2y(1-y)}EH(y) + \frac{(1-2y)^2}{4y(1-y)}N_2 H(y) - \lambda H(y),$$

where E, N_1 and N_2 are matrices of size $m \times m$.

Even more, since $\sum_{j-1}^{n-1} I_{n,j}^2 - Q_n - Q_{n-1}$, Proposition 2.6 implies $\sum_{j-1}^{n-1} I_{n,j}^2 \in D(\mathrm{SO}(n))^{\mathrm{SO}(n-1)}$, therefore $\sum_{j-1}^{n-1}\dot{\pi}(I_{n,j})^2$ is scalar valued when restricted to any M-submodule. Hence, $N_1 - N_2$ and the equation above is equivalent to

$$y(1-y)H''(y) + \frac{n}{2}(1-2y)H'(y) + \frac{(1-2y)}{2y(1-y)}EH(y) + \frac{1+(1-2y)^2}{4y(1-y)}NH(y) - \lambda H(y), \quad (3.2)$$

where N is a diagonal matrix of size $m \times m$. To obtain an explicit expression of E for any K-type is a very serious matter; in the following sections we shall find explicitly the expressions of E and N, for certain K-types.

Remark 3.8. It is worth to observe that from (2.5) and (2.6) we can immediately obtain every entry of the diagonal matrix N.

4   The K-types which are M-irreducible

Let $K - \mathrm{SO}(n)$, $M - \mathrm{SO}(n-1)$, with $n - 2\ell+1$, and let $\mathbf{m}_n - (m_{1n}, \ldots, m_{\ell n})$ be a K-type such that $V_{\mathbf{m}}$ is irreducible as M-module. The highest weights $\mathbf{m}_{n-1}$ of the M-submodules of $V_{\mathbf{m}}$ are those that satisfies the following intertwining relations

$$
\begin{array}{cccccc}
m_{1n} & & m_{2n} & \cdots & m_{\ell,n} & -m_{\ell n} \\
& m_{1,n-1} & & \cdots & \cdots & m_{\ell,n-1}
\end{array}.
$$

Since $V_{\mathbf{m}}$ is irreducible as M-module it follows that $m_{1n} - \cdots - m_{\ell,n} - 0$. The converse is also true, therefore $V_{\mathbf{m}}$ is M-irreducible if and only if it is the trivial representation.

Let now consider the case $K - \mathrm{SO}(n)$, $M - \mathrm{SO}(n-1)$, with $n - 2\ell$ and let $\mathbf{m}_n - (m_{1n}, \ldots, m_{\ell n})$ be a K-type such that $V_{\mathbf{m}}$ is irreducible as M-module. The highest weights $\mathbf{m}_{n-1}$ of the M-submodules of $V_{\mathbf{m}}$ are those that satisfies the following intertwining relations

$$
\begin{array}{cccccc}
m_{1n} & & m_{2n} & \cdots & m_{\ell-1,n} & m_{\ell n} \\
& m_{1,n-1} & & \cdots & m_{\ell-1,n-1} & -m_{\ell-1,n-1}
\end{array}.
$$

Since $V_{\mathbf{m}}$ is irreducible as M-module it follows that $m_{1n} - \cdots - m_{\ell-1,n} - d$ and $m_{\ell n} - d - j$ with $0 \leq j \leq 2d$, since $m_{\ell-1,n} \geq |m_{\ell n}|$. This implies that $m_{1,n-1} - \cdots - m_{\ell-2,n-1} - d$ and $m_{\ell-1,n-1} - q$ with $d \geq q \geq \max\{d - j, j - d\}$. Thus, if $0 \leq j \leq d$ we have $d \geq q \geq d - j$ and by irreducibility we must have $j - 0$. Similarly if $d \leq j \leq 2d$ we have $d \geq q \geq j - d$ and by irreducibility we must have $j - 2d$. Therefore $\mathbf{m}_n - d\alpha$ or $\mathbf{m}_n - d\beta$, where

$$
\alpha - (1, \ldots, 1), \qquad \beta - (1, \ldots, 1, -1).
$$

The converse is also true, therefore $V_{\mathbf{m}}$ is M-irreducible if and only if $\mathbf{m}_n - d\alpha$ or $\mathbf{m}_n - d\beta$ for any $d \in \mathbb{N}_0$.

If Φ is an irreducible spherical function on $\mathrm{SO}(n + 1)$ of type π, whose highest weight is $\mathbf{m}_n - d\alpha$ or $\mathbf{m}_n - d\beta$, then from Corollary 3.6 we get that the associated function H satisfies

$$
y(1 - y)H''(y) + \ell(1 - 2y)H'(y) + \frac{1-y}{y}\sum_{j-1}^{n-1}\dot{\pi}(I_{nj})^2 H(y) - \lambda H(y).
$$

To compute $\sum_{j-1}^{n-1}\dot{\pi}(I_{nj})^2$ we write $\sum_{j-1}^{n-1}\dot{\pi}(I_{nj})^2 - \dot{\pi}(Q_n - Q_{n-1})$.

Let us first consider $\mathbf{m}_n - d\alpha$. If $v \in V_{\mathbf{m}_n}$ is a highest weight vector, then

$$
\dot{\pi}(Q_n)v - -d\ell(d + \ell - 1)v \qquad \text{and} \qquad \dot{\pi}(Q_{n-1})v - -d(\ell - 1)(d + \ell - 1)v,
$$

see (2.5) and (2.6). Therefore

$$
\sum_{j-1}^{n-1}\dot{\pi}(I_{nj})^2 v - -d(d + \ell - 1)v.
$$

Let us now consider $\mathbf{m}_n - d\beta$. If $v \in V_{\mathbf{m}_n}$ is a highest weight vector, then $\dot{\pi}(Q_n)v - -2d\ell(d + \ell - 1)v$ as before, and $\dot{\pi}(Q_{n-1})v - -2d(\ell - 1)(d + \ell - 1)v$ as before because in both cases $\mathbf{m}_{n-1}$ is the same.

Therefore if $\mathbf{m}_n - (d, \ldots, d, \pm d)$ we have

$$\sum_{j-1}^{n-1} \dot{\pi}(I_{nj})^2 v - -d(d + \ell - 1)v.$$

Hence, if Φ is an irreducible spherical function on $SO(n+1)$, $n - 2\ell$, of type $\mathbf{m}_n - (d, \ldots, d, \pm d) \in \mathbb{C}^\ell$, then the associated scalar value function $H - h$ satisfies

$$y(1-y)h''(y) + \ell(1-2y)h'(y) - \frac{d(d+\ell-1)(1-y)}{y}h(y) - \lambda h(y). \tag{4.1}$$

Let us now compute the eigenvalue λ corresponding to the spherical function of type $\pi \in \hat{SO}(2\ell)$, of highest weight $\mathbf{m}_n - d\alpha$, associated with the irreducible representation $\tau \in SO(2\ell+1)$, of highest weight $\mathbf{m}_{n+1} - (w, d, \ldots, d) \in \mathbb{C}^\ell$. If $v \in V_{\mathbf{m}_{n+1}}$ is a highest weight vector, then from (2.6) we have

$$\dot{\tau}(Q_{n+1})v - -(w(w+2\ell-1) + d(\ell-1)(d+\ell-1))\,v.$$

If $v \in V_{\mathbf{m}_n}$ is a highest weight vector, then from (2.5) we have

$$\dot{\tau}(Q_n)v - \dot{\pi}(Q_n)v - -d\ell(d+\ell-1)v.$$

Since $\Delta - Q_{n+1} - Q_n$ it follows that

$$\lambda - -w(w+2\ell-1) + d(d+\ell-1).$$

To solve (4.1) we write $h - y^\alpha f$. Then we get

$$y(1-y)y^\alpha f'' + (2\alpha(1-y) + \ell(1-2y))y^\alpha f'$$
$$+ (\alpha(\alpha-1)(1-y) + \ell\alpha(1-2y) - d(d+\ell-1)(1-y))y^{\alpha-1}f - \lambda y^\alpha f.$$

Thus the indicial equation is $\alpha(\alpha-1) + \ell\alpha - d(d+\ell-1) - 0$ and $\alpha - d$ is one of its solutions. If we take $h - y^d f$, then we obtain

$$y(1-y)f'' + (2d+\ell - 2(d+\ell)y)f' - d\ell f - \lambda f.$$

If we replace $\lambda - -w(w+2\ell-1) + d(d+\ell-1)$ we get

$$y(1-y)f'' + (2d+\ell - 2(d+\ell)y)f' - (d-w)(2\ell+d+w-1)f - 0.$$

Let $a - d - w$, $b - 2\ell + d + w - 1$, $c - 2d + \ell$ then the above equation becomes

$$y(1-y)f'' + (c - (1+a+b)y)f' - abf - 0.$$

A fundamental system of solutions of this equation near $y - 0$ is given by the following functions

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; y\right), \qquad y^{1-c}\,{}_2F_1\left(\begin{matrix} a-c+1, b-c+1 \\ 2-c \end{matrix}; y\right).$$

Since $h - y^d f$ is bounded near $y - 0$ it follows that

$$h(y) - u y^d\,{}_2F_1\left(\begin{matrix} d-w, 2\ell+d+w-1 \\ 2d+\ell \end{matrix}; y\right),$$

where the constant u is determined by the condition $h(1) - 1$.

Remark 4.1. Let $h_w - h_w(y)$, $w \geq d$, be the function h above. Then h_w is a polynomial of degree w. Moreover observe that the function y^d used to hypergeometrize (4.1) is precisely h_d.

Let us now compute the eigenvalue λ corresponding to the spherical function of type $\mathbf{m}_n - d\beta$ associated with an irreducible representation τ of $\mathrm{SO}(n+1)$ of highest weight $\mathbf{m}_{n+1} - (w, d, \ldots, d) \in \mathbb{C}^\ell$. If $v \in V_{\mathbf{m}_{n+1}}$ is a highest weight vector, we obtain $\dot{\tau}(Q_{n+1})v - -(w(w+2\ell-1) + d(\ell-1)(d+\ell-1))v$.

If $v \in V_{\mathbf{m}_n}$ is a highest weight vector, then $\dot{\pi}(Q_n)v - -d\ell(d+\ell-1)v$ as above, because $Q_n v$ does not depend on the sign of the last coordinate of $\mathbf{m}_n$. Since $\Delta - Q_{n+1} - Q_n$ we also have

$$\lambda - -w(w+2\ell-1) + d(d+\ell-1).$$

Therefore we have proved the following result.

Theorem 4.2. *The scalar valued functions $H - h$ associated with the irreducible spherical functions on $\mathrm{SO}(n+1)$, $n - 2\ell$, of $\mathrm{SO}(n)$-type $\mathbf{m}_n - (d, \ldots, d, \pm d) \in \mathbb{C}^\ell$, are parameterized by the integers $w \geq d$ and are given by*

$$h_w(y) - u y^d \,_2F_1\left(\begin{matrix} d-w, 2\ell+d+w-1 \\ 2d+\ell \end{matrix} ; y\right)$$

where the constant u is determined by the condition $h_w(1) - 1$.

5 The operator Δ for fundamental K-types

We are interested in finding a more explicit expression of the differential equation given in Corollary 3.6:

$$y(1-y)H''(y) + \frac{1}{2}n(1-2y)H'(y) + \frac{1}{4y(1-y)}\sum_{j-1}^{n-1}\dot{\pi}(I_{n,j})^2 H(y)$$

$$+ \frac{(1-2y)}{2y(1-y)}\sum_{j-1}^{n-1}\dot{\pi}(I_{n,j})H(y)\dot{\pi}(I_{n,j}) + \frac{(1-2y)^2}{4y(1-y)}H(y)\sum_{j-1}^{n-1}\dot{\pi}(I_{n,j})^2 - \lambda H(y),$$

for certain representations $\pi \in \hat{\mathrm{SO}}(n)$, including those that are fundamental.

The obvious place to start to look for irreducible representations of $\mathrm{SO}(n)$ is among the exterior powers of the standard representation of $\mathrm{SO}(n)$. It is known that $\Lambda^p(\mathbb{C}^{2\ell})$ are irreducible $\mathrm{SO}(2\ell)$-modules for $p - 1, \ldots, \ell-1$, and that $\Lambda^\ell(\mathbb{C}^{2\ell})$ splits into the direct sum of two irreducible submodules. While in the odd case $\Lambda^p(\mathbb{C}^{2\ell+1})$ are irreducible $\mathrm{SO}(2\ell+1)$-modules for $p - 1, \ldots, \ell$. See Theorems 19.2 and 19.14 in [3].

Moreover, $\Lambda^p(\mathbb{C}^n)$ and $\Lambda^{n-p}(\mathbb{C}^n)$ are isomorphic $\mathrm{SO}(n)$-modules. In fact, if $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is the canonical basis of $\mathbb{C}^n$, then the linear map $\xi : \Lambda^p(\mathbb{C}^n) \to \Lambda^{n-p}(\mathbb{C}^n)$ defined by

$$\xi(\mathbf{e}_{u_1} \wedge \cdots \wedge \mathbf{e}_{u_p}) - (-1)^{u_1 + \cdots + u_p}\mathbf{e}_{v_1} \wedge \cdots \wedge \mathbf{e}_{v_{n-p}},$$

where $u_1 < \cdots < u_p$ and $v_1 < \cdots < v_{n-p}$ are complementary ordered set of indices, is an $\mathrm{SO}(n)$-isomorphism.

All these statements can be established directly upon observing that the elements $I_{ki} - E_{ki} - E_{ik}$ with $1 \leq i < k \leq n$ form a basis of the Lie algebra $\mathfrak{so}(n)$, and that

$$I_{ki}\mathbf{e}_k - \mathbf{e}_i, \qquad I_{ki}\mathbf{e}_i - -\mathbf{e}_k \qquad \text{and} \qquad I_{ki}\mathbf{e}_j - 0 \qquad \text{if} \quad j \neq k, i.$$

We will refer to the irreducible $\mathrm{SO}(2\ell)$-modules $\Lambda^p(\mathbb{C}^{2\ell})$ for $p - 1, \ldots, \ell-1$, respectively, the irreducible $\mathrm{SO}(2\ell+1)$-modules $\Lambda^p(\mathbb{C}^{2\ell+1})$ for $p - 1, \ldots, \ell$, as the fundamental $\mathrm{SO}(2\ell)$-modules, respectively, as the fundamental $\mathrm{SO}(2\ell+1)$-modules, for reasons that will be clarified in the following Sections 5.1 and 5.2.

5.1   The even case: $K = \mathrm{SO}(2\ell)$

First we will study the case $n - 2\ell$, with $\ell > 2$. The fundamental weights of $\mathfrak{so}(2\ell, \mathbb{C})$ are

$$\lambda_p - \epsilon_1 + \cdots + \epsilon_p, \qquad 1 \le p \le \ell - 2,$$
$$\lambda_{\ell-1} - \tfrac{1}{2}(\epsilon_1 + \cdots + \epsilon_{\ell-1} - \epsilon_\ell), \qquad \lambda_\ell - \tfrac{1}{2}(\epsilon_1 + \cdots + \epsilon_{\ell-1} + \epsilon_\ell).$$

Here we will consider the fundamental K-modules

$$\Lambda^1(\mathbb{C}^n), \ \Lambda^2(\mathbb{C}^n), \ \dots, \ \Lambda^{\ell-1}(\mathbb{C}^n).$$

We will show that the highest weight of $\Lambda^p(\mathbb{C}^n)$ is $\epsilon_1 + \cdots + \epsilon_p$ for $1 \le p \le \ell - 1$. Observe that $\lambda_{\ell-1}$ and λ_ℓ are not analytically integral and therefore they will not be considered, although we will also consider the K-module with highest weight $\lambda_{\ell-1} + \lambda_\ell - \epsilon_1 + \cdots + \epsilon_{\ell-1}$. Notice that we have already considered the cases $2\lambda_{\ell-1}$ and $2\lambda_\ell$ in Section 4, which are M-irreducible. We will also show that the fundamental K-modules are direct sum of two irreducible M-submodules.

In order to obtain the explicit expression of E in (3.2) for a given irreducible representation π of $K - \mathrm{SO}(n)$, of highest weight $\varepsilon_1 + \cdots + \varepsilon_p$, we are interested to compute

$$\sum_{j-1}^{n-1} \dot\pi(I_{nj}) P_s \dot\pi(I_{nj})\big|_{V_r} - \lambda(r,s) I_{V_r},$$

with $r, s - 0, 1$ corresponding to the two M-submodules V_0 and V_1 of the representation π, associated with $\mathbf{m}_{n-1} - (1, \dots, 1, 0, \dots, 0) \in \mathbb{C}^{\ell-1}$ with $p-1$ and p ones, respectively (see the betweenness conditions in Section 2.5); being P_0 and P_1 the respective projections.

Let us consider the standard action of $K - \mathrm{SO}(n)$ on $V - \mathbb{C}^n$, and take the canonical basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. Then we have the irreducible K-module $\Lambda^p(V)$ for $1 \le p \le \ell - 1$. The vector $(\mathbf{e}_1 - i\mathbf{e}_2) \wedge (\mathbf{e}_3 - i\mathbf{e}_4) \wedge \cdots \wedge (\mathbf{e}_{2p-1} - i\mathbf{e}_{2p})$ is the unique, up to a scalar, dominant vector and its weight is $(1, \dots, 1, 0, \dots, 0) \in \mathbb{C}^\ell$ with p ones. Then, if V' is the subspace generated by $\{\mathbf{e}_1, \dots, \mathbf{e}_{n-1}\}$, $\Lambda^p(V)$ is the direct sum of two M-submodules, namely

$$\Lambda^p(V) - V_0 \oplus V_1 - \Lambda^{p-1}(V') \wedge \mathbf{e}_n \oplus \Lambda^p(V') \tag{5.1}$$

whose highest weights are $(1, \dots, 1, 0, \dots, 0) \in \mathbb{C}^{\ell-1}$ with $p-1$ ones and $(1, \dots, 1, 0, \dots, 0) \in \mathbb{C}^{\ell-1}$ with p ones, respectively. It is easy to see that $(\mathbf{e}_1 - i\mathbf{e}_2) \wedge (\mathbf{e}_3 - i\mathbf{e}_4) \wedge \cdots \wedge (\mathbf{e}_{2p-3} - i\mathbf{e}_{2p-2}) \wedge \mathbf{e}_n$ is an M-highest weight vector in $\Lambda^{p-1}(V') \wedge \mathbf{e}_n$ and that $(\mathbf{e}_1 - i\mathbf{e}_2) \wedge (\mathbf{e}_3 - i\mathbf{e}_4) \wedge \cdots \wedge (\mathbf{e}_{2p-1} - i\mathbf{e}_{2p})$ is an M highest weight vector in $\Lambda^p(V')$.

To get $\lambda(0,0)$ it is enough to compute

$$\sum_{j-1}^{n-1} \dot\pi(I_{nj}) P_0 \dot\pi(I_{nj})(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n).$$

Since we have that $\dot\pi(I_{nj})(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n) - \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_j$ we obtain $P_0 \dot\pi(I_{nj})(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n) - 0$ and $\lambda(0,0) - 0$.

To get $\lambda(0,1)$ it is enough to compute

$$\sum_{j-1}^{n-1} \dot\pi(I_{nj}) P_1 \dot\pi(I_{nj})(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n).$$

We have

$$P_1 \dot\pi(I_{nj})(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n) - \begin{cases} 0 & \text{if} \quad 1 \le j \le p-1, \\ \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_j & \text{if} \quad p \le j \le n-1. \end{cases}$$

Therefore we have

$$\dot\pi(I_{nj})P_1\dot\pi(I_{nj})(\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_{p-1}\wedge\mathbf{e}_n) - \begin{cases} 0 & \text{if}\quad 1\le j\le p-1, \\ -\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_{p-1}\wedge\mathbf{e}_n & \text{if}\quad p\le j\le n-1. \end{cases}$$

Hence $\lambda(0,1) - -(n-p)$.

Similarly, to get $\lambda(1,0)$ it is enough to compute

$$\sum_{j-1}^{n-1}\dot\pi(I_{nj})P_0\dot\pi(I_{nj})(\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_p).$$

We have

$$\dot\pi(I_{nj})(\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_p) - \begin{cases} -\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_n\wedge\cdots\wedge\mathbf{e}_p & \text{if}\quad 1\le j\le p, \\ 0 & \text{if}\quad p+1\le j\le n-1, \end{cases}$$

where $\mathbf{e}_n$ appears in the j-place. Therefore

$$\dot\pi(I_{nj})P_0\dot\pi(I_{nj})(\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_p) - \begin{cases} -\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_p & \text{if}\quad 1\le j\le p, \\ 0 & \text{if}\quad p+1\le j\le n-1. \end{cases}$$

Hence $\lambda(1,0) - -p$.

Also it is clear now that $\sum_{j-1}^{n-1}\dot\pi(I_{nj})P_1\dot\pi(I_{nj})(\mathbf{e}_1\wedge\cdots\wedge\mathbf{e}_p) - 0$, hence $\lambda(1,1) - 0$.

Therefore, when π is the standard representation of K in $\Lambda^p(V)$, $1\le p\le\ell-1$, we have

$$(\lambda(r,s))_{0\le r,s\le 1} - \begin{pmatrix} 0 & p-n \\ -p & 0 \end{pmatrix}.$$

Therefore, we obtain a more explicit version of Corollary 3.6 using (3.2) and Remark 3.8.

Corollary 5.1. *Let Φ be an irreducible spherical function on G of type $\pi\in\hat{SO}(n)$, $n-2\ell$. If the highest weight of π is of the form $(1,\ldots,1,0,\ldots,0)\in\mathbb{C}^\ell$, with p ones, $1\le p\le\ell-1$, then the function $H:(0,1)\to\mathrm{End}(\mathbb{C}^2)$ associated with Φ satisfies*

$$y(1-y)H''(y)+\frac{1}{2}n(1-2y)H'(y)+\frac{1+(1-2y)^2}{4y(1-y)}\begin{pmatrix} p-n & 0 \\ 0 & -p \end{pmatrix}H(y)$$
$$+\frac{(1-2y)}{2y(1-y)}\begin{pmatrix} 0 & p-n \\ -p & 0 \end{pmatrix}H(y)-\lambda H(y),$$

for some $\lambda\in\mathbb{C}$.

5.2   The odd case: $K = SO(2\ell+1)$

We now study the case $n - 2\ell+1$, with $\ell\ge 1$. The fundamental weights of $\mathfrak{so}(2\ell+1,\mathbb{C})$ are

$$\lambda_p - \epsilon_1+\cdots+\epsilon_p,\qquad 1\le p\le\ell-1,$$
$$\lambda_\ell - \tfrac{1}{2}(\epsilon_1+\cdots+\epsilon_\ell).$$

Here we will consider the fundamental K-modules

$$\Lambda^1(\mathbb{C}^n),\ \Lambda^2(\mathbb{C}^n),\ \ldots,\ \Lambda^\ell(\mathbb{C}^n).$$

We will show that the highest weight of $\Lambda^p(\mathbb{C}^n)$ is $\epsilon_1 + \cdots + \epsilon_p$ for $1 \leq p \leq \ell$. Also we will establish that $\Lambda^p(\mathbb{C}^n)$ splits into the direct sum of two M-submodules for $1 \leq p \leq \ell - 1$, while $\Lambda^\ell(\mathbb{C}^n)$ splits into the sum of three M-submodules; for this reason it will be treated separately in Section 8.

Observe that λ_ℓ is not analytically integral and therefore it will not be considered, although we will consider the K-module with highest weight $2\lambda_\ell$.

As in the even case we are interested in computing

$$\sum_{j-1}^{n-1} \dot\pi(I_{nj})P_s\dot\pi(I_{nj})\Big|_{V_r} - \lambda(r,s)I_{V_r},$$

with $r,s - 0,1$ corresponding to the two M-submodules V_0 and V_1 of the representation π, corresponding to $\mathbf{m}_{n-1} - (1,\ldots,1,0,\ldots,0) \in \mathbb{C}^\ell$ with $p-1$ and p ones respectively (see the betweenness conditions in Section 2.5). Being P_0 and P_1 the respective projections.

Let us consider the standard action of $K - \mathrm{SO}(n)$ on $V - \mathbb{C}^n$, and take the canonical basis $\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$. Then we have the irreducible K-module $\Lambda^p(V)$ for $1 \leq p \leq \ell - 1$. The vector $(\mathbf{e}_1 - i\mathbf{e}_2) \wedge (\mathbf{e}_3 - i\mathbf{e}_4) \wedge \cdots \wedge (\mathbf{e}_{2p-1} - i\mathbf{e}_{2p})$ is the unique, up to a scalar, dominant vector and its weight is $(1,\ldots,1,0,\ldots,0) \in \mathbb{C}^\ell$ with p ones. Then, if V' is the subspace generated by $\{\mathbf{e}_1,\ldots,\mathbf{e}_{n-1}\}$, $\Lambda^p(V)$ is the direct sum of two irreducible M-submodules, namely

$$\Lambda^p(V) - V_0 \oplus V_1 - \Lambda^{p-1}(V') \wedge \mathbf{e}_n \oplus \Lambda^p(V') \tag{5.2}$$

of highest weights $(1,\ldots,1,0,\ldots,0) \in \mathbb{C}^\ell$ with $p-1$ ones, and $(1,\ldots,1,0,\ldots,0) \in \mathbb{C}^\ell$ with p ones, respectively. It is easy to see that $(\mathbf{e}_1 - i\mathbf{e}_2) \wedge (\mathbf{e}_3 - i\mathbf{e}_4) \wedge \cdots \wedge (\mathbf{e}_{2p-3} - i\mathbf{e}_{2p-2}) \wedge \mathbf{e}_n$ is an M-highest weight vector in $\Lambda^{p-1}(V') \wedge \mathbf{e}_n$ and that $(\mathbf{e}_1 - i\mathbf{e}_2) \wedge (\mathbf{e}_3 - i\mathbf{e}_4) \wedge \cdots \wedge (\mathbf{e}_{2p-1} - i\mathbf{e}_{2p})$ is an M highest weight vector in $\Lambda^p(V')$.

To get $\lambda(0,0)$ it is enough to compute

$$\sum_{j-1}^{n-1} \dot\pi(I_{nj})P_0\dot\pi(I_{nj})(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n).$$

Since we have that $\dot\pi(I_{nj})(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n) - \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_j$, we obtain $P_0\dot\pi(I_{nj})(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n) - 0$ and $\lambda(0,0) - 0$.

To get $\lambda(0,1)$ it is enough to compute

$$\sum_{j-1}^{n-1} \dot\pi(I_{nj})P_1\dot\pi(I_{nj})(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n).$$

We have

$$P_1\dot\pi(I_{nj})(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n) - \begin{cases} 0 & \text{if } 1 \leq j \leq p-1, \\ \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_j & \text{if } p \leq j \leq n-1. \end{cases}$$

Therefore

$$\dot\pi(I_{nj})P_1\dot\pi(I_{nj})(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n) - \begin{cases} 0 & \text{if } 1 \leq j \leq p-1, \\ -\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n & \text{if } p \leq j \leq n-1. \end{cases}$$

Hence $\lambda(0,1) - -(n-p)$.

Similarly, to get $\lambda(1,0)$ it is enough to compute

$$\sum_{j-1}^{n-1} \dot\pi(I_{nj})P_0\dot\pi(I_{nj})(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_p).$$

We have that

$$\dot\pi(I_{nj})(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_p) = \begin{cases} -\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n \wedge \cdots \wedge \mathbf{e}_p & \text{if} \quad 1 \le j \le p, \\ 0 & \text{if} \quad p+1 \le j \le n-1, \end{cases}$$

where $\mathbf{e}_n$ appears in the j-place. Therefore

$$\dot\pi(I_{nj})P_0\dot\pi(I_{nj})(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_p) = \begin{cases} -\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_p & \text{if} \quad 1 \le j \le p, \\ 0 & \text{if} \quad p+1 \le j \le n-1. \end{cases}$$

Hence $\lambda(1,0) = -p$.

Also it is clear now that $\sum_{j=1}^{n-1} \dot\pi(I_{nj})P_1\dot\pi(I_{nj})(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_p) = 0$, hence $\lambda(1,1) = 0$.

Therefore, when π is the standard representation of K in $\Lambda^p(V)$, $1 \le p \le \ell - 1$, we have

$$(\lambda(r,s))_{0 \le r,s \le 1} = \begin{pmatrix} 0 & p-n \\ -p & 0 \end{pmatrix}.$$

Therefore, we obtain a more explicit version of Corollary 3.6 using (3.2) and Remark 3.8.

Corollary 5.2. *Let Φ be an irreducible spherical function on G of type $\pi \in \hat{SO}(n)$, $n = 2\ell + 1$. If the highest weight of π is of the form $(1,\ldots,1,0,\ldots,0) \in \mathbb{C}^\ell$, with p ones, $1 \le p \le \ell - 1$, then the function $H : (0,1) \to \mathrm{End}(\mathbb{C}^2)$ associated with Φ satisfies*

$$y(1-y)H''(y) + \frac{1}{2}n(1-2y)H'(y) + \frac{1+(1-2y)^2}{4y(1-y)}\begin{pmatrix} p-n & 0 \\ 0 & -p \end{pmatrix}H(y)$$
$$+ \frac{(1-2y)}{2y(1-y)}\begin{pmatrix} 0 & p-n \\ -p & 0 \end{pmatrix}H(y) = \lambda H(y),$$

for some $\lambda \in \mathbb{C}$.

6 The spherical functions of fundamental K-types

Let $n = 2\ell$, the irreducible spherical functions of K-type

$$\mathbf{m}_n = (1,\ldots,1,0,\ldots,0) \in \mathbb{C}^\ell,$$

with p ones, $1 \le p \le \ell - 1$, are those associated with the irreducible representations of G of highest weights of the form $\mathbf{m}_{n+1} = (w+1,1,\ldots,1,\delta,0,\ldots,0) \in \mathbb{C}^\ell$ that interlaces $\mathbf{m}_n$,

$$\begin{array}{ccccccccc} w+1 & & 1 & \ldots & 1 & \delta & 0 & \ldots & 0 \\ & 1 & \ldots & & \ldots & 1 & 0 & \ldots & \ldots & 0 \end{array}.$$

We now consider the K-module $\Lambda^p(\mathbb{C}^n)$ which has highest weight $\mathbf{m}_n$.

For $w = 0$ and $\delta = 0$ we consider the G-module $\Lambda^p(\mathbb{C}^{n+1})$ whose highest weight is $\mathbf{m}_{n+1}$, and we have the following K-module decomposition

$$\Lambda^p(\mathbb{C}^{n+1}) = \Lambda^p(\mathbb{C}^n) \oplus \Lambda^{p-1}(\mathbb{C}^n) \wedge \mathbf{e}_{n+1},$$

where $\Lambda^p(\mathbb{C}^n)$ is the sum of two $SO(n-1)$-modules:

$$\Lambda^p(\mathbb{C}^n) = \Lambda^p(\mathbb{C}^{n-1}) \oplus \Lambda^{p-1}(\mathbb{C}^{n-1}) \wedge \mathbf{e}_n.$$

We observe that

$$a(s)(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n) - \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge (\cos s\,\mathbf{e}_n - \sin s\,\mathbf{e}_{n+1})$$
$$- \cos s(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n) - \sin s(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_{n+1}).$$

Hence, if Φ_0 is the spherical function associated with the irreducible representation of G of highest weight $\mathbf{m}_{n+1} - (1, 1, \ldots, 1, \delta, 0, \ldots, 0) \in \mathbb{C}^\ell$ with $\delta - 0$, we have that

$$\Phi_0(a(s))(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n) - \cos s(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n).$$

Also we have that $a(s)(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_p) - \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_p$. Thus the vector valued function $F_0(s)$ given by the irreducible spherical function $\Phi_0(a(s))$ is

$$F_0(s) - \begin{pmatrix} \cos s \\ 1 \end{pmatrix}.$$

For $w - 0$ and $\delta - 1$ we consider the G-module $\Lambda^{p+1}(\mathbb{C}^{n+1})$ whose highest weight $\mathbf{m}_{n+1}$, and for $1 \le p \le \ell - 1$ we have the following K-module decomposition

$$\Lambda^{p+1}(\mathbb{C}^{n+1}) - \Lambda^{p+1}(\mathbb{C}^n) \oplus \Lambda^p(\mathbb{C}^n) \wedge \mathbf{e}_{n+1},$$

where $\Lambda^p(\mathbb{C}^n) \wedge \mathbf{e}_{n+1}$ is the sum of two $\mathrm{SO}(n-1)$-modules:

$$\Lambda^p(\mathbb{C}^n) \wedge \mathbf{e}_{n+1} - \Lambda^p(\mathbb{C}^{n-1}) \wedge \mathbf{e}_{n+1} \oplus \Lambda^{p-1}(\mathbb{C}^{n-1}) \wedge \mathbf{e}_n \wedge \mathbf{e}_{n+1}.$$

We observe that

$$a(s)(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n \wedge \mathbf{e}_{n+1}) - \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge (\sin s\,\mathbf{e}_n + \cos s\,\mathbf{e}_{n+1})$$
$$- \sin s(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n) + \cos s(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_{n+1}).$$

Hence, if Φ_1 is the spherical function associated with the irreducible representation of G of highest weight $\mathbf{m}_{n+1} - (1, 1, \ldots, 1, \delta, 0, \ldots, 0) \in \mathbb{C}^\ell$ with $\delta - 1$, we have that $\Phi_1(a(s))(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n \wedge \mathbf{e}_{n+1}) - \cos s(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n \wedge \mathbf{e}_{n+1})$. Also we have that

$$a(s)(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_p \wedge \mathbf{e}_{n+1}) - \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_{p-1} \wedge \mathbf{e}_n \wedge \mathbf{e}_{n+1}.$$

Thus the vector valued function $F_1(s)$ given by the irreducible spherical function $\Phi_1(a(s))$ is

$$F_1(s) - \begin{pmatrix} 1 \\ \cos s \end{pmatrix}.$$

Definition 6.1. We shall consider the 2×2 matrix-valued function $\Psi - \Psi(y)$, for $0 < y < 1$, whose columns are given by the functions $H_0(y) - F_0(s)$ and $H_1(y) - F_1(s)$, with $\cos s - 2y - 1$:

$$\Psi(y) - \begin{pmatrix} 2y - 1 & 1 \\ 1 & 2y - 1 \end{pmatrix}. \tag{6.1}$$

Since the functions $H_0(y)$ and $H_1(y)$ are associated with irreducible spherical functions, they satisfy the differential equation given in Corollary 5.1; moreover, the respective eigenvalues are $\lambda - -p$ and $\lambda - p - n$. Therefore, we have

$$y(1-y)\Psi'' + \frac{1}{2}n(1-2y)\Psi' + \frac{1+(1-2y)^2}{4y(1-y)} \begin{pmatrix} p-n & 0 \\ 0 & -p \end{pmatrix} \Psi$$
$$+ \frac{(1-2y)}{2y(1-y)} \begin{pmatrix} 0 & p-n \\ -p & 0 \end{pmatrix} \Psi - \Psi \begin{pmatrix} -p & 0 \\ 0 & p-n \end{pmatrix}.$$

Furthermore, it is easy to check that the function $\Psi(y)$ also satisfy the equation above even when n is odd.

Theorem 6.2. *The function Ψ can be used to obtain a hypergeometric differential equation from the one given in Corollaries 5.1 and 5.2. Precisely, if H is a vector-valued solution of the differential equation in Corollaries 5.1 or 5.2, with eigenvalue λ, then $P - \Psi^{-1}H$ is a solution of $DP - \lambda P$, where D is the hypergeometric differential operator given by*

$$DP - y(1-y)P'' - \left(\begin{matrix} (\frac{n}{2}+1)(2y-1) & -1 \\ -1 & (\frac{n}{2}+1)(2y-1) \end{matrix} \right) P' - \left(\begin{matrix} p & 0 \\ 0 & n-p \end{matrix} \right) P.$$

Proof. By hypothesis we have that

$$y(1-y)H''(y) + \frac{1}{2}n(1-2y)H'(y) + \frac{1+(1-2y)^2}{4y(1-y)} \left(\begin{matrix} p-n & 0 \\ 0 & -p \end{matrix} \right) H(y)$$

$$+ \frac{(1-2y)}{2y(1-y)} \left(\begin{matrix} 0 & p-n \\ -p & 0 \end{matrix} \right) H(y) - \lambda H(y),$$

Then, writing $H - \Psi P$, we have

$$y(1-y)P'' + \left(2y(1-y)\Psi^{-1}\Psi' + \frac{n}{2}(1-2y)I \right) P'$$

$$+ \Psi^{-1} \left(y(1-y)\Psi'' + \frac{n}{2}(1-2y)\Psi' + \frac{1+(1-2y)^2}{4y(1-y)} \left(\begin{matrix} p-n & 0 \\ 0 & -p \end{matrix} \right) \Psi \right.$$

$$\left. + \frac{(1-2y)}{2y(1-y)} \left(\begin{matrix} 0 & p-n \\ -p & 0 \end{matrix} \right) \Psi \right) P - \lambda P.$$

Now we compute

$$2y(1-y)\Psi^{-1}\Psi' - \frac{4y(1-y)}{4y(y-1)} \left(\begin{matrix} 2y-1 & -1 \\ -1 & 2y-1 \end{matrix} \right) - - \left(\begin{matrix} 2y-1 & -1 \\ -1 & 2y-1 \end{matrix} \right).$$

Therefore

$$y(1-y)P'' - \left(\begin{matrix} (\frac{n}{2}+1)(2y-1) & -1 \\ -1 & (\frac{n}{2}+1)(2y-1) \end{matrix} \right) P' - \left(\begin{matrix} \lambda+p & 0 \\ 0 & \lambda+n-p \end{matrix} \right) P - 0.$$

This completes the proof of the theorem. ■

6.1 Δ-eigenvalues of spherical functions

As we said, when $n - 2\ell$ the irreducible spherical functions of the pair $(\mathrm{SO}(n+1), \mathrm{SO}(n))$, of type $\mathbf{m}_n - (1, \ldots, 1, 0 \ldots, 0) \in \mathbb{C}^\ell$ with p ones, $1 \le p \le \ell-1$ are those associated with the irreducible representations τ of G of highest weights of the form $\mathbf{m}_{n+1} - (w+1, 1, \ldots, 1, \delta, 0, \ldots, 0) \in \mathbb{C}^\ell$ with $p-1$ ones, such that the following pattern holds

$$\begin{matrix} w+1 & & 1 & \ldots & 1 & \delta & 0 & \ldots & 0 \\ & 1 & \ldots & & \ldots & 1 & 0 & \ldots & & \ldots & 0 \end{matrix} .$$

Let $\Phi_{w,\delta}$ be the corresponding spherical function. Then $\Delta\Phi_{w,\delta} - \lambda\Phi_{w,\delta}$, where the eigenvalue $\lambda - \lambda_n(w,\delta)$ can be computed from the expression $\Delta - Q_{n+1} - Q_n$. If $v \in V_{\mathbf{m}_{n+1}}$ is a highest weight vector from (2.6) we have

$$\dot{\tau}(Q_{2\ell+1})v - -\left((w+1)^2 + (2\ell-1)(w+1) + (2\ell-p)(p-1) + 2\delta(\ell-p) \right)v.$$

If $v \in V_{\mathbf{m}_{2\ell}}$ is a highest weight vector, then from (2.5) we have

$$\dot{\pi}(Q_n)v - -p(2\ell-p)v.$$

Since $\Delta - Q_{n+1} - Q_n$ it follows that

$$\lambda_{2\ell}(w,\delta) - -(w+1)^2 - (2\ell - 1)(w+1) + (2\ell - p) - 2\delta(\ell - p)$$

Analogously, we obtain that the eigenvalues of the spherical functions $\Phi_{w,\delta}$ of the pair $(\mathrm{SO}(2\ell + 2), \mathrm{SO}(2\ell + 1))$ are of the form

$$\lambda_{2\ell+1}(w,\delta) - -(w+1)(w+2\ell+1) + 2\ell - p + 1 - \delta 2(\ell - p) - \delta^2,$$

here δ is 0 or 1 when we are in the cases $1 \le p < \ell$ but δ could also be -1 in the particular case $p - \ell$.

Therefore, we have that the eigenvalues of the spherical functions $\Phi_{w,\delta}$ of the pair $(\mathrm{SO}(n+1), \mathrm{SO}(n))$ are of the form

$$\lambda_n(w,\delta) - \begin{cases} -w(w+n+1) - p & \text{if } \delta - 0, \\ -w(w+n+1) - n + p & \text{if } \delta - \pm 1. \end{cases} \tag{6.2}$$

6.2 Polynomial eigenfunctions of the hypergeometric operator D

Let D be the differential operator on the real line introduced in Theorem 6.2:

$$DP - y(1-y)P'' + (C - yU)P' - VP, \tag{6.3}$$

with

$$C - \begin{pmatrix} (n/2+1) & 1 \\ 1 & (n/2+1) \end{pmatrix}, \qquad U - (n+2)I, \qquad V - \begin{pmatrix} p & 0 \\ 0 & n-p \end{pmatrix},$$

where n is of the form 2ℓ or $2\ell + 1$ for $\ell \in \mathbb{N}$ and $1 \le p < \ell$.

We will study the $\mathbb{C}^2$-vector valued polynomial eigenfunctions of D.

The equation $DP - \lambda P$ is an instance of a matrix hypergeometric differential equation studied in [22]. Since the eigenvalues of C, $n/2$ and $n/2 + 2$, are not in $-\mathbb{N}_0$ the function P is determined by $P_0 - P(0)$. For $|y| < 1$ it is given by

$$P(y) - {}_2H_1 \begin{pmatrix} U, V+\lambda \\ C \end{pmatrix}; y \end{pmatrix} P_0 - \sum_{j-0}^{\infty} \frac{y^j}{j!} [C; U; V+\lambda]_j P_0, \qquad P_0 \in \mathbb{C}^2,$$

where the symbol $[C; U; V+\lambda]_j$ is inductively defined by

$$[C; U; V+\lambda]_0 - 1,$$
$$[C; U; V+\lambda]_{j+1} - (C+j)^{-1}\left(j(U+j-1) + V + \lambda\right)[C; U; V+\lambda]_j,$$

for all $j \ge 0$.

Therefore, we have that there exists a polynomial solution if and only if the coefficient $[C; U; V+\lambda]_{j+1}$ is a singular matrix for some $j \in \mathbb{Z}$. Since the matrix $C+j$ is invertible for all $j \in \mathbb{N}_0$, we have that there is a polynomial solution of degree j for $DP - \lambda P$ if and only if there exists $P_0 \in \mathbb{C}^2$ such that $[C; U; V+\lambda]_j P_0 \ne 0$ and $(j(U+j-1) + V + \lambda)[C; U; V+\lambda]_j P_0 - 0$.

Now we easily observe that the only possible values for λ such that $j(U+j-1) + V + \lambda$ has non trivial kernel are those given in (6.2). Then, if $\lambda - -w(w+n+1) - p$, it is easy to check that the first and only j for which $j(U+j-1) + V + \lambda$ is singular is $j - w$, and its kernel (of dimension 1) is the subspace generated by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Analogously, if $\lambda - -w(w+n+1) - n + p$, it is easy to check that the first and only j for which $j(U+j-1) + V + \lambda$ is singular is $j - w$, and its kernel (of dimension 1) is the subspace generated by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively. Therefore we have the following result.

Theorem 6.3. *For a given $\ell \in \mathbb{N}$ take $n - 2\ell$ or $2\ell + 1$ and $1 \le p \le \ell - 1$, then the polynomial eigenfunctions of*

$$DP - y(1-y)P'' + (C - yU)P' - VP,$$

with

$$C - \begin{pmatrix} (n/2+1) & 1 \\ 1 & (n/2+1) \end{pmatrix}, \qquad U - (n+2)I, \qquad V - \begin{pmatrix} p & 0 \\ 0 & n-p \end{pmatrix}$$

have eigenvalues $-w(w+n+1) - p$ or $-w(w+n+1) - n + p$, with $w \in \mathbb{N}_0$; in both cases the degree of the polynomial is w with leading coefficient a multiple of $\binom{1}{0}$ or $\binom{0}{1}$, respectively.

7   The inner product

Given a finite dimensional irreducible representation π of K in the vector space V_π let $(C(G) \otimes \mathrm{End}(V_\pi))^{K \times K}$ be the space of all continuous functions $\Phi : G \longrightarrow \mathrm{End}(V_\pi)$ such that $\Phi(k_1 g k_2) - \pi(k_1)\Phi(g)\pi(k_2)$ for all $g \in G$, $k_1, k_2 \in K$. Let us equip V_π with an inner product such that $\pi(k)$ becomes unitary for all $k \in K$. Then we introduce an inner product in the vector space $(C(G) \otimes \mathrm{End}(V_\pi))^{K \times K}$ by defining

$$\langle \Phi_1, \Phi_2 \rangle - \int_G \mathrm{tr}(\Phi_1(g)\Phi_2(g)^*)dg,$$

where dg denote the Haar measure on G normalized by $\int_G dg - 1$, and where $\Phi_2(g)^*$ denotes the adjoint of $\Phi_2(g)$ with respect to the inner product in V_π.

By using Schur's orthogonality relations for the unitary irreducible representations of G, it follows that if Φ_1 and Φ_2 are non equivalent irreducible spherical functions, then they are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle$, i.e.

$$\langle \Phi_1, \Phi_2 \rangle - 0.$$

Recall that, given an irreducible spherical function Φ of type π of the pair (G, K), the function $\Phi(a(s))$ is scalar valued when restricted to any $\mathrm{SO}(n-1)$-module (see (3.1) for $a(s)$). We shall denote by m the number of $\mathrm{SO}(n-1)$-submodules of π, and by $d_1, d_2, \ldots, d_m$ the respective dimensions of each one of those submodules.

In particular, if Φ_1 and Φ_2 are two irreducible spherical functions of type $\pi \in \hat{K}$, we consider the vector valued functions $H_1(y)$ and $H_2(y)$ given by the diagonal matrix valued functions $\Phi_1(a(s))$ and $\Phi_2(a(s))$ (see Remark 3.7), with $y - (\cos s + 1)/2$, respectively, denoting

$$H_1(y) - (h_1(y), \ldots, h_m(y))^t, \qquad H_2(y) - (f_1(y), \ldots, f_m(y))^t.$$

Proposition 7.1. *If Φ_1, Φ_2 are two irreducible spherical functions of type $\pi \in \hat{K}$ then*

$$\langle \Phi_1, \Phi_2 \rangle - \frac{(n-1)!!}{(n-2)!!} \frac{2}{\omega_*} \sum_{i=1}^m d_i \int_0^1 (y(1-y))^{n/2-1} h_i(y)\overline{f_i(y)}dy,$$

with $\omega_* - \pi$ if n is even and $\omega_* - 2$ if n is odd.

Proof. Let $A - \exp \mathbb{R}I_{n+1,n}$ be the Lie subgroup of G of all elements of the form

$$a(s) - \exp sI_{n+1,n} - \begin{pmatrix} I_{n-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cos s & \sin s \\ \mathbf{0} & -\sin s & \cos s \end{pmatrix}, \qquad s \in \mathbb{R},$$

where I_{n-1} denotes the identity matrix of size $n - 1$.

Now [12, Theorem 5.10, p. 190] establishes that for every $f \in C(G/K)$ and a suitable constant c_*

$$\int_{G/K} f(gK) dg_K - c_* \int_{K/M} \left(\int_{-\pi}^{\pi} \delta_*(a(s)) f(ka(s)K) ds \right) dk_M,$$

where dg_K and dk_M are respectively the invariant measures on G/K and K/M normalized by $\int_{G/K} dg_K - \int_{K/M} dk_M - 1$ and the function $\delta_* : A \longrightarrow \mathbb{R}$ is defined by

$$\delta_*(a(s)) - \prod_{\nu \in \Sigma^+} |\sin is\nu(I_{n+1,n})|,$$

with Σ^+ the set of those positive roots whose restrictions to $\mathfrak{a}$, the Lie algebra of A, are not zero. In our case we have $\delta_*(a(s)) - |\sin^{n-1} s|$.

To find the value of c_* we consider the function $f \equiv 1$, having then

$$1 - 2c_* \int_0^\pi \sin^{n-1} s ds.$$

Since

$$\int \sin^{n-1} s ds - -\frac{1}{n-1} \sin^{n-2} s \cos s + \frac{n-2}{n-1} \int \sin^{n-3} ds,$$

we obtain that, for $n - 2\ell$ or $2\ell + 1$,

$$\int_0^\pi \sin^{n-1} s ds - \frac{n-2}{n-1} \frac{n-4}{n-3} \cdots \frac{n-2\ell+1}{n-2\ell+2} \int_0^\pi \sin^{n-2\ell} s ds.$$

Therefore

$$c_* - \frac{(n-1)!!}{(n-2)!!} \frac{1}{2\omega_*},$$

with $\omega_* - \pi$ for $n - 2\ell$ and $\omega_* - 2$ for $2\ell + 1$.

Since the function $g \mapsto \mathrm{tr}(\Phi_1(g)\Phi_2(g)^*)$ is invariant under left and right multiplication by elements in K, we have

$$\langle \Phi_1, \Phi_2 \rangle - \int_G \mathrm{tr}(\Phi_1(g)\Phi_2(g)^*) dg - 2c_* \int_0^\pi \sin^{n-1} s \, \mathrm{tr}\left(\Phi_1(a(s)\Phi_2(a(s))^*)\right) ds.$$

If we put $y - \frac{1}{2}(\cos s + 1)$ for $0 < s < \pi$ we have

$$\mathrm{tr}\left(\Phi_1(a(s)\Phi_2(a(s))^*)\right) - \sum_{i-1}^m d_i h_i(y) \overline{f_i(y)}.$$

Then

$$\langle \Phi_1, \Phi_2 \rangle - 4c_* \sum_{i-1}^m d_i \int_0^1 (4y(1-y))^{(n-2)/2} h_i(y) \overline{f_i(y)} dy,$$

and the proposition follows. ∎

Proposition 7.2. *If $\Phi_1, \Phi_2 \in (C^\infty(G) \otimes \mathrm{End}(V_\pi))^{K \times K}$ then*

$$\langle \Delta\Phi_1, \Phi_2 \rangle - \langle \Phi_1, \Delta\Phi_2 \rangle.$$

Proof. If we apply a left invariant vector field $X \in \mathfrak{g}$, to the function on G given by $g \mapsto \mathrm{tr}(\Phi_1(g)\Phi_2(g)^*)$, and then we integrate over G we obtain

$$0 - \int_G \mathrm{tr}\left((X\Phi_1)(g)\Phi_2(g)^*\right) dg + \int_G \mathrm{tr}\left(\Phi_1(g)(X\Phi_2)(g)^*\right) dg.$$

Therefore $\langle X\Phi_1, \Phi_2 \rangle - -\langle \Phi_1, X\Phi_2 \rangle$. Now let $\tau : \mathfrak{g}_\mathbb{C} \longrightarrow \mathfrak{g}_\mathbb{C}$ be the conjugation of $\mathfrak{g}_\mathbb{C}$ with respect to the real linear form $\mathfrak{g}$. Then $-\tau$ extends to a unique antilinear involutive * operator on $D(G)$ such that $(D_1 D_2)^* - D_2^* D_1^*$ for all $D_1, D_2 \in D(G)$. This follows easily from the fact that the universal enveloping algebra over $\mathbb{C}$ of $\mathfrak{g}$ is canonically isomorphic to $D(G)$. Then it follows that $\langle D\Phi_1, \Phi_2 \rangle - \langle \Phi_1, D^*\Phi_2 \rangle$.

Finally, it is easy to verify that $\Delta^* - \Delta$. ∎

7.1 Spherical functions as polynomial solutions of $DP = \lambda P$

Let us consider $\widetilde{D}$, the differential operator on $(0,1)$ introduced in Corollaries 5.1 and 5.2:

$$y(1-y)H''(y) + \frac{1}{2}n(1-2y)H'(y) + \frac{1+(1-2y)^2}{4y(1-y)}\begin{pmatrix} p-n & 0 \\ 0 & -p \end{pmatrix} H(y)$$

$$+ \frac{(1-2y)}{2y(1-y)}\begin{pmatrix} 0 & p-n \\ -p & 0 \end{pmatrix} H(y) - \lambda H(y), \tag{7.1}$$

Recall that the operator D that appears in (6.3) extends the differential operator $D - \Psi \widetilde{D} \Psi^{-1}$ to the whole real line, where

$$\Psi(y) - \begin{pmatrix} 2y-1 & 1 \\ 1 & 2y-1 \end{pmatrix}$$

is the matrix function given in (6.1) and used in Theorem 6.2.

We want to focus our attention on the following vector spaces of $\mathbb{C}^2$-valued analytic functions on $(0,1)$:

$$S_\lambda - \left\{ H - H(y) : \widetilde{D}H - \lambda H, \ H\left(\tfrac{\cos s + 1}{2}\right) \text{ analytic at } s - 0 \right\},$$
$$W_\lambda - \left\{ P - P(y) : DP - \lambda P, \text{ analytic on } [0,1] \right\}.$$

From Theorem 6.2 we know that the correspondence $P \mapsto \Psi P$ is an injective linear map from W_λ into S_λ. Now we want to prove that this map is bijective.

Theorem 7.3. *The linear map $P \mapsto \Psi P$ is an isomorphism from W_λ onto S_λ.*

Proof. A vector valued function $P \in W_\lambda$ is an eigenfunction of the hypergeometric operator D. Since it is analytic at $y - 1$ it is determined by $P(1)$, therefore $\dim(W_\lambda) - 2$.

On the other hand, if $H \in S_\lambda$ then there is a function $F(s)$ analytic at $s - 0$, such that it extends the function $H\left(\tfrac{\cos s + 1}{2}\right)$ defined on $(0, \pi)$. Then, F satisfies the following differential equation

$$F''(s) + (n-1)\frac{\cos s}{\sin s}F'(s) + \frac{1+\cos^2 s}{\sin^2 s}\begin{pmatrix} p-n & 0 \\ 0 & -p \end{pmatrix} F(s)$$

$$- 2\frac{\cos s}{\sin^2 s}\begin{pmatrix} 0 & p-n \\ -p & 0 \end{pmatrix} F(s) - \lambda F(s),$$

or equivalently

$$\sin^2 s F''(s) + \frac{n-1}{2}\sin(2s)F'(s) + (2-\sin^2 s)\begin{pmatrix} p-n & 0 \\ 0 & -p \end{pmatrix} F(s)$$

$$- 2\cos s \begin{pmatrix} 0 & p-n \\ -p & 0 \end{pmatrix} F(s) - \lambda \sin^2 s F(s), \tag{7.2}$$

Let $a_j \in \mathbb{C}^2$ and $\alpha_j, \beta_j, \gamma_j \in \mathbb{C}$, for $j \geq 0$, be the Taylor coefficients of F, sin, $\sin^2$ and cos at $s - 0$:

$$F(s) - \sum_{j \geq 0} a_j s^j, \qquad \sin s - \sum_{j \geq 1} \alpha_j s^j,$$

$$F'(s) - \sum_{j \geq 0} a_{j+1}(j+1)s^j, \qquad \sin^2 s - \sum_{j \geq 2} \beta_j s^j,$$

$$F''(s) - \sum_{j \geq 0} a_{j+2}(j+2)(j+1)s^j, \qquad \cos s - \sum_{j \geq 0} \gamma_j s^j.$$

Therefore, from (7.2) we have

$$\sum_{j \geq 0} \left[\sum_{k-0}^{j-2} \beta_{j-k} a_{k+2}(k+2)(k+1) + \frac{n-1}{2} \sum_{k-0}^{j-1} 2^{j-k} \alpha_{j-k} a_{k+1}(k+1) + \begin{pmatrix} p-n & 0 \\ 0 & -p \end{pmatrix} \right.$$

$$\left. \times \left(2a_j - \sum_{k-0}^{j-2} \beta_{j-k} a_k \right) - 2 \begin{pmatrix} 0 & p-n \\ -p & 0 \end{pmatrix} \sum_{k-0}^{j} \gamma_{j-k} a_k \right] s^j - \lambda \sum_{j \geq 0} \left[\sum_{k-0}^{j-2} \beta_{j-k} a_k \right] s^j.$$

Hence, since $\beta_2 - \alpha_1 - \gamma_0 - 1$, we have that

$$\left[j(j-1) + (n-1)j + 2 \begin{pmatrix} p-n & -p+n \\ p & -p \end{pmatrix} \right] a_j$$

is a linear combination with matrix coefficients of $\{a_0, a_1, \ldots, a_{j-1}\}$; it is clear that for $j - 1$ and $j > 2$ the matrix above is non singular, therefore $\{a_0, a_2\}$ determine completely the sequence $\{a_j\}_{j \geq 0}$. Also it is clear that when $j - 0$ or 2, that matrix has nullity 1. Therefore we can conclude that $\dim(S_\lambda) - 2$. The theorem follows. ∎

Theorem 7.4. *Let H be the $\mathbb{C}^2$-valued analytic function on $(0,1)$ given by an irreducible spherical function Φ on G of fundamental K-type $(1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^\ell$, with p ones, $0 < p < \ell$. If $P - \Psi^{-1}H$, then P is polynomial.*

Proof. We know that the function H is analytic in $(0,1)$, and from Corollary 5.1 we know that it is an eigenfunction of the operator $\widetilde{D}$ (see (7.1)). Also we know that the function $H(\frac{1+\cos s}{2})$ is analytic at $s - 0$, since $\Phi(a(s))$ it is. Therefore from Theorem 7.3 the function $P - \Psi^{-1}H$ is an analytic eigenfunction of D on the closed interval $[0,1]$.

If we introduce the following matrix-weight function $V - V(y)$ supported on the interval $[0,1]$

$$V(y) - \frac{(n-1)!!}{(n-2)!!} \frac{2}{\omega_*} (y(1-y))^{n/2-1} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},$$

with $\omega_* - \pi$ if n is even and $\omega_* - 2$ if n is odd, then from Proposition 7.1 we have

$$\langle \Phi_0, \Phi_1 \rangle - \int_0^1 H_2^*(y) V(y) H_1^*(y) dy.$$

It follows from Propositions 7.1 and 7.2 that $\widetilde{D}$ is a symmetric operator with respect to the inner product defined among continuous vector-valued functions on $[0,1]$ by

$$\langle H_1, H_2 \rangle_V - \int_0^1 H_2^*(y) V(y) H_1(y) dy.$$

Then, since $D - \Psi^{-1}\widetilde{D}\Psi$, we have that D is a symmetric operator with respect to the inner product defined among continuous vector-valued functions on $[0,1]$ by

$$\langle P_1, P_2 \rangle_W - \int_0^1 P_2^*(y) W(y) P_1(y) dy,$$

where

$$W - \Psi^* V \Psi.$$

Actually, we have that (W, D) is a classical pair in the sense of [7], see also [2]. As the weight W has finite moments there exists a sequence $\{Q_r\}_{r \geq 0}$ of 2×2 matrix-valued orthonormal polynomials, such that $DQ_r - Q_r\Lambda_r$ where Λ_r is a real diagonal matrix (for precise definitions and general facts on matrix-valued orthogonal polynomials see [5] and [2]).

Let $\{e_1, e_2\}$ be the canonical basis of $\mathbb{C}^2$. Then

$$\langle Q_r e_j, Q_s e_i \rangle_W - e_i^* \left(\int_0^1 Q_s^*(y) W(y) Q_r^*(y) dy \right) e_j - e_i^* \delta_{si} I e_j - \delta_{r,s} \delta_{i,j}.$$

Therefore, for $r \geq 0$, $j - 1, 2$, $\{Q_r e_j\}$ is a family of $\mathbb{C}^2$-valued orthonormal polynomials such that

$$D(Q_r e_j) - (DQ_r)e_j - (Q_r\Lambda_r)e_j - Q_r(\Lambda_r e_j) - \lambda_r^j(Q_r e_j),$$

where $\Lambda_r - \mathrm{diag}(\lambda_r^1, \lambda_r^2)$.

Now we write our function $P - \Psi^{-1} H$ as $P - \sum_{r,j} a_{r,j} Q_r e_j$, where $a_{r,j} - \langle P, Q_r e_j \rangle_W$. Since P is analytic on $[0,1]$ the sum converges not only in the L^2-norm but also in the topology based on uniform convergence of sequences of functions and their successive derivatives.

Therefore,

$$\lambda P - DP - \sum_{r,j} a_{r,j} \lambda_r^j Q_r e_j.$$

Then $a_{r,j} - 0$ if $\lambda_r^j \not\equiv \lambda$. Since $\dim W_\lambda - 2$ it follows that P is a polynomial. ∎

Remark 7.5. It is easy to see from (5.1) and (5.2) that the dimensions of the M-submodules of the fundamental representation of K with highest weight of the form $(1, \ldots, 1, 0 \ldots, 0)$, with p ones, are given by

$$d_1 - \frac{(n-1)!}{(p-1)!(n-p)!}, \qquad d_2 - \frac{(n-1)!}{p!(n-1-p)!},$$

therefore the weight W is given by

$$W - \frac{(n-1)!!}{(n-2)!!} \frac{2}{\omega_*} \frac{(n-1)!}{p!(n-p)!} (y(1-y))^{n/2-1} \Psi^* \begin{pmatrix} p & 0 \\ 0 & n-p \end{pmatrix} \Psi,$$

with $\omega_* - \pi$ if n is even and $\omega_* - 2$ if n is odd. Then, W is a scalar multiple of

$$\begin{pmatrix} p(2y-1)^2 + n - p & n(2y-1) \\ n(2y-1) & (n-p)(2y-1)^2 + p \end{pmatrix}.$$

Even more, since $0 < p < \ell$ and $n - 2\ell, 2\ell + 1$ it follows that $p \neq n - p$. Then it can be proved that the weight W does not reduce to a smaller size, i.e., there is not any invertible matrix M such that $M^* W(y) M$ is diagonal for all $y \in [0,1]$.

For a given fundamental K type $\pi \in \hat{\mathrm{SO}}(n)$, $n - 2\ell$ or $2\ell + 1$, with highest weight of the form $(1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^\ell$ with p ones $(0 < p < \ell)$, let $\Phi_{w,\delta}$ denote the irreducible spherical function of the pair $(\mathrm{SO}(n+1), \mathrm{SO}(n))$ given by $\tau \in \hat{\mathrm{SO}}(n+1)$ with highest weight of the form $(w+1, 1, \ldots, 1, \delta, 0 \ldots, 0)$ with $p - 1$ ones.

Therefore, combining (6.2), Theorems 6.3 and 7.4 we have the following statement.

Theorem 7.6. *Given $w \in \mathbb{N}_0$, every irreducible spherical function $\Phi_{w,\delta}$ of the pair $(\mathrm{SO}(n+1)$, $\mathrm{SO}(n))$, with $n - 2\ell$ or $2\ell + 1$, of type $\mathbf{m}_n - (1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^\ell$ with p ones $(0 < p < \ell)$, corresponds to a vector valued function $P_{w,\delta}$ $(\delta - 0, 1)$, which is a polynomial of degree w; and the leading coefficients of $P_{w,0}$ and $P_{w,1}$ are multiples of $\binom{1}{0}$ and $\binom{0}{1}$ respectively. Precisely*

$$P_{w,\delta}(y) - \sum_{j-0}^{w} \frac{y^j}{j!} [C; U; V + \lambda]_j P_{w,\delta}(0),$$

with

$$C - \begin{pmatrix} (n/2+1) & 1 \\ 1 & (n/2+1) \end{pmatrix}, \qquad U - (n+2)I, \qquad V - \begin{pmatrix} p & 0 \\ 0 & n-p \end{pmatrix},$$

$$\lambda - \lambda_n(w, \delta) - \begin{cases} -w(w+n+1) - p & \text{if } \delta - 0, \\ -w(w+n+1) - n + p & \text{if } \delta - 1. \end{cases}$$

Even more, the value of $P_{w,\delta}(0)$ can be computed.

Proof. It only remains to prove that $P_{w,\delta}(0)$ can be computed.

Let us consider the case $\delta - 0$. We know from (6.2) and Theorem 6.3 that there is some $c \in \mathbb{C}$ such that

$$[C; U; V + \lambda]_w P_{w,0}(0) - c \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Since $[C; U; V + \lambda]_w$ is invertible, this c is univocally determined by the condition $\Phi(e) - I$, which implies

$$\Psi(1) \sum_{j-0}^{w} \frac{1}{j!} [C; U; V + \lambda]_j P_{w,0}(0) - \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Similarly, we can prove the same for $P_{w,1}(0)$.   ∎

Remark 7.7. It is worth to observe that for $w, w' \geq 0$ and $\delta, \delta' - 0, 1$, since $\langle P_{w,\delta}, P_{w',\delta'} \rangle_W - \langle \Phi_{w,\delta}, \Phi_{w',\delta'} \rangle$, we have that if $(w, \delta) \not= (w', \delta')$ then

$$\langle P_{w,\delta}, P_{w',\delta'} \rangle_W - 0.$$

Therefore, our construction encodes all equivalent classes of irreducible spherical functions of a fundamental K-type of highest weight λ_p, $0 < p < \ell$, in the orthogonal set of $\mathbb{C}^2$-valued polynomials $\{P_{w,0}, P_{w,1}\}$. The degree of $P_{w,0}$ and $P_{w,1}$ is w, and the leading coefficient is a multiple of $\binom{1}{0}$ or $\binom{0}{1}$, respectively.

8 Matrix valued orthogonal polynomials

8.1 Matrix valued orthogonal polynomials

In this subsection, given n of the form 2ℓ or $2\ell + 1$ with $\ell \in \mathbb{N}$, for a fixed $0 < p < \ell$ we shall construct a sequence of matrix-valued polynomials $\{P_w\}_{w \geq 0}$ directly related to irreducible spherical functions of type $\pi \in \hat{\mathrm{SO}}(n)$ of highest weight $\mathbf{m}_\pi - (1, \ldots, 1, 0 \ldots, 0) \in \mathbb{C}^\ell$, with p ones.

Given a nonnegative integer w and $\delta - 0, 1$, we can consider $\Phi_{w,\delta}$, the irreducible spherical function of type π associated with the irreducible representation $\tau \in \hat{\mathrm{SO}}(n+1)$ of highest weight of the form $\mathbf{m}_\tau - (w+1, 1, \ldots, 1, \delta, 0, \ldots, 0)$ with $p-1$ ones.

We insist on recalling that, since π has only two $\mathrm{SO}(n-1)$-submodules, we can interpret the diagonal matrix-valued function $\Phi_{w,\delta}(a(s))$, $s \in (0, \pi)$, as a 2 column vector function.

Now we consider the vector-valued function

$$P_{w,\delta} : \ (0,1) \to \mathbb{C}^2$$

given by the vector function $P_{w,\delta}(y) - \Psi^{-1}(y)\Phi_{w,\delta}(a(s))$, with $\cos(s) - 2y - 1$. Then, we define the matrix-valued function

$$P_w - P_w(y),$$

whose δ-th column ($\delta - 0, 1$) is given by the $\mathbb{C}^2$-valued polynomial $P_{w,\delta}(y)$.

Let consider the matrix-valued skew symmetric bilinear form defined among C^∞ 2×2 matrix-valued functions on $[0,1]$ by

$$\langle P, Q \rangle_W - \int_0^1 Q^*(y) W(y) P(y) dy,$$

where

$$W - \frac{(n-1)!!}{(n-2)!!} \frac{2}{\omega_*} \frac{(n-1)!}{p!(n-p)!} (y(1-y))^{n/2-1} \begin{pmatrix} p(2y-1)^2 + n - p & n(2y-1) \\ n(2y-1) & (n-p)(2y-1)^2 + p \end{pmatrix} .$$

See Remark 7.5. Then we state the following theorem.

Theorem 8.1. *The matrix-valued polynomial functions P_w, $w \geq 0$, form a sequence of orthogonal polynomials with respect to W, which are eigenfunctions of the symmetric differential operator D in (6.3). Moreover,*

$$DP_w - P_w \begin{pmatrix} \lambda(w,0) & 0 \\ 0 & \lambda(w,1) \end{pmatrix},$$

where

$$\lambda(w,\delta) - \begin{cases} -w(w+n+1) - p & \text{if } \ \delta - 0, \\ -w(w+n+1) - n + p & \text{if } \ \delta - 1. \end{cases}$$

Proof. From Theorem 6.2 we have that the δ-th column of P_w is an eigenfunction of the operator D with eigenvalue $\lambda(w, \delta)$, see (6.2) and (6.3). Therefore we have

$$DP_w - P_w \Lambda_w,$$

with

$$\Lambda_w - \begin{pmatrix} \lambda(w,0) & 0 \\ 0 & \lambda(w,1) \end{pmatrix} .$$

From Theorem 7.6 we know that each column of P_w is a polynomial function of degree w and, even more, that P_w is a polynomial whose leading coefficient is a nonsingular diagonal matrix.

Given w and w', non negative integers, by using Remark 7.7 we have

$$\langle P_{w'}, P_w \rangle_W = \int_0^1 P_w(y)^* W(y) P_{w'}(y) du = \sum_{\delta,\delta'=0}^1 \int_0^1 \left(P_{w,\delta}(y)^* W(y) P_{w',\delta'}(y) du \right) E_{\delta,\delta'}$$

$$= \sum_{\delta,\delta'=0}^1 \delta_{w,w'} \delta_{\delta,\delta'} \left(\int_0^1 P_{w,\delta}(y)^* W(y) P_{w',\delta'}(y) du \right) E_{\delta,\delta'}$$

$$= \delta_{w,w'} \sum_{\delta=0}^1 \int_0^1 \left(P_{w,\delta}(y)^* W(y) P_{w'\delta}(y) du, \right) E_{\delta,\delta},$$

which proves the orthogonality. Even more, it also shows us that $\langle P_w, P_w \rangle_W$ is a diagonal matrix. Also, making a few simple computations we have that

$$\langle DP_w, P_{w'} \rangle = \delta_{w,w'} \langle P_w, P_{w'} \rangle \Lambda_w = \delta_{w,w'} \Lambda_w^* \langle P_w, P_{w'} \rangle = \langle P_w, DP_{w'} \rangle,$$

for every $w, w' \in \mathbb{N}_0$, since Λ_w is real and diagonal. This concludes the proof of the theorem. ∎

9   The $\mathrm{SO}(2\ell+1)$-type with highest weight $2\lambda_\ell$

In this section $K = \mathrm{SO}(2\ell+1)$. We will focus on the particular case when the K-type is given by an irreducible representation π with highest weight $2\lambda_\ell = (1,1,\ldots,1)$. We will first see that such K-module is the direct sum of three M-submodules, and we will find similar results to those obtained for the fundamental K-types $\lambda_1, \ldots, \lambda_{\ell-1}$ that are direct sum of two M-submodules.

Let us consider the irreducible K-module $\Lambda^\ell(V)$, with $V = \mathbb{C}^n$, $n = 2\ell+1$. The vector $v = (\mathbf{e}_1 - i\mathbf{e}_2) \wedge (\mathbf{e}_3 - i\mathbf{e}_4) \wedge \cdots \wedge (\mathbf{e}_{2\ell-1} - i\mathbf{e}_{2\ell})$ is the unique, up to a scalar, dominant vector and its weight is $2\lambda_\ell = (1,1,\ldots,1)$.

It is not difficult to see that $\Lambda^\ell(V)$ is the sum of three M-irreducible submodules, namely

$$\Lambda^\ell(V) = V_1 \oplus V_0 \oplus V_{-1} \tag{9.1}$$

with respective highest weights $(1,\ldots,1), (1,\ldots,1,0), (1,\ldots,1,-1) \in \mathbb{C}^\ell$ and having $V_0 = \Lambda^{\ell-1}(V) \wedge \mathbf{e}_n$ and $V_1 \oplus V_{-1} \simeq \Lambda^\ell(\mathbb{C}^{n-1})$.

The vectors

$$v_1 = (\mathbf{e}_1 - i\mathbf{e}_2) \wedge (\mathbf{e}_3 - i\mathbf{e}_4) \wedge \cdots \wedge (\mathbf{e}_{2\ell-1} - i\mathbf{e}_{2\ell}),$$
$$v_0 = -(\mathbf{e}_1 - i\mathbf{e}_2) \wedge (\mathbf{e}_3 - i\mathbf{e}_4) \wedge \cdots \wedge (\mathbf{e}_{2\ell-3} - i\mathbf{e}_{2\ell-2}) \wedge \mathbf{e}_n,$$
$$v_{-1} = (\mathbf{e}_1 - i\mathbf{e}_2) \wedge (\mathbf{e}_3 - i\mathbf{e}_4) \wedge \cdots \wedge (\mathbf{e}_{2\ell-1} + i\mathbf{e}_{2\ell})$$

are M-highest weight vectors in V_1, V_0 and V_{-1}, respectively. Also let us call P_1, P_0 and P_{-1} the respective projections on V_1, V_0 and V_{-1}, according to the decomposition (9.1).

In order to obtain the explicit expression of E in (3.2) we are interested to compute

$$\sum_{j=1}^{n-1} \dot{\pi}(I_{nj}) P_s \dot{\pi}(I_{nj})\big|_{V_r} = \lambda(r,s) I_{V_r},$$

with $r, s = 1, 0, -1$ corresponding to the three M-submodules V_1, V_0 and V_{-1} of the representation π.

If $1 \leq j \leq \ell$, then

$$\dot{\pi}(I_{n,2j-1})(\mathbf{e}_{2k-1} - i\mathbf{e}_{2k}) - \begin{cases} 0 & \text{if } k \neq j, \\ -\mathbf{e}_n & \text{if } k - j, \end{cases}$$

$$\dot{\pi}(I_{n,2j})(\mathbf{e}_{2k-1} - i\mathbf{e}_{2k}) - \begin{cases} 0 & \text{if } k \neq j, \\ i\mathbf{e}_n & \text{if } k - j, \end{cases}$$

therefore, it is easy to see that $P_0\dot{\pi}(I_{n,2j-1})v_0 - P_0\dot{\pi}(I_{n,2j})v_0 - 0$ and that $P_r\dot{\pi}(I_{n,2j-1})v_s - P_r\dot{\pi}(I_{n,2j})v_s - 0$ when $s \pm 1$ and $r \pm 1$; i.e.

$$\lambda(0,0) - \lambda(-1,-1) - \lambda(1,-1) - \lambda(-1,1) - \lambda(1,1) - 0.$$

Furthermore, it is easy to see that, for $1 \leq j \leq \ell$ and r equal to 1 or -1, we have

$$\dot{\pi}(I_{n,2j-1})P_0\dot{\pi}(I_{n,2j-1})v_r + \dot{\pi}(I_{n,2j})P_0\dot{\pi}(I_{n,2j})v_r - -v_r,$$

then $\lambda(-1,0) - \lambda(1,0) - -\ell$. Therefore, it only remains to compute

$$\sum_{j-1}^{\ell} \left(\dot{\pi}(I_{n,2j-1})P_s\dot{\pi}(I_{n,2j-1})v_0 + \dot{\pi}(I_{n,2j})P_s\dot{\pi}(I_{n,2j})v_0\right),$$

for $s - \pm 1$.

To obtain $P_s\dot{\pi}(I_{n,k})v_0$ it is necessary to decompose $\dot{\pi}(I_{n,k})v_0$ according to the direct sum (9.1). We know that $\dot{\pi}(X_{-\varepsilon_j-\varepsilon_\ell})v_1 \in V_1$ and $\dot{\pi}(X_{-\varepsilon_j+\varepsilon_\ell})v_{-1} \in V_{-1}$; recall that

$$X_{-\epsilon_j-\epsilon_\ell} - I_{2\ell-1,2j-1} - I_{2\ell,2j} + i(I_{2\ell-1,2j} + I_{2\ell,2j-1}),$$
$$X_{-\epsilon_j+\epsilon_\ell} - I_{2\ell-1,2j-1} + I_{2\ell,2j} + i(I_{2\ell-1,2j} - I_{2\ell,2j-1}),$$

see (2.2). We have

$$\dot{\pi}\left(X_{-\varepsilon_j-\varepsilon_\ell}\right)(\mathbf{e}_{2j-1} - i\mathbf{e}_{2j}) - -2(\mathbf{e}_{2\ell-1} + i\mathbf{e}_{2\ell}),$$
$$\dot{\pi}\left(X_{-\varepsilon_j-\varepsilon_\ell}\right)(\mathbf{e}_{2\ell-1} - i\mathbf{e}_{2\ell}) - 2\left(\mathbf{e}_{2j-1} + i\mathbf{e}_{2j}\right),$$
$$\dot{\pi}\left(X_{-\varepsilon_j-\varepsilon_\ell}\right)(\mathbf{e}_{2k-1} - i\mathbf{e}_{2k}) - 0, \qquad \text{for } k \neq s, \ell.$$

Therefore, for $1 \leq j < \ell$,

$$\dot{\pi}\left(X_{-\varepsilon_j-\varepsilon_\ell}\right)v_1 - 2(\mathbf{e}_1 - i\mathbf{e}_2) \wedge \cdots \wedge \left(\mathbf{e}_{2(\ell-1)-1} - i\mathbf{e}_{2(\ell-1)}\right) \wedge \left(\mathbf{e}_{2j-1} + i\mathbf{e}_{2j}\right)$$
$$- 2(\mathbf{e}_1 - i\mathbf{e}_2) \wedge \cdots \wedge \left(\mathbf{e}_{2j-3} - i\mathbf{e}_{2j-2}\right) \wedge \left(\mathbf{e}_{2\ell-1} + i\mathbf{e}_{2\ell}\right) \wedge \left(\mathbf{e}_{2j+1} - i\mathbf{e}_{2j+2}\right) \wedge$$
$$\cdots \wedge \left(\mathbf{e}_{2\ell-1} - i\mathbf{e}_{2\ell}\right).$$

Similarly, for $1 \leq j < \ell$,

$$\dot{\pi}\left(X_{-\varepsilon_j+\varepsilon_\ell}\right)v_1 - 2(\mathbf{e}_1 - i\mathbf{e}_2) \wedge \cdots \wedge \left(\mathbf{e}_{2(\ell-1)-1} - i\mathbf{e}_{2(\ell-1)}\right) \wedge \left(\mathbf{e}_{2j-1} + i\mathbf{e}_{2j}\right)$$
$$+ 2(\mathbf{e}_1 - i\mathbf{e}_2) \wedge \cdots \wedge \left(\mathbf{e}_{2j-3} - i\mathbf{e}_{2j-2}\right) \wedge \left(\mathbf{e}_{2\ell-1} + i\mathbf{e}_{2\ell}\right) \wedge \left(\mathbf{e}_{2j+1} - i\mathbf{e}_{2j+2}\right) \wedge$$
$$\cdots \wedge \left(\mathbf{e}_{2\ell-1} - i\mathbf{e}_{2\ell}\right).$$

Hence, for $1 \leq j < \ell$, we have

$$\frac{-i}{8}\left(\dot{\pi}\left(X_{-\varepsilon_j-\varepsilon_\ell}\right)v_1 + \dot{\pi}\left(X_{-\varepsilon_j+\varepsilon_\ell}\right)v_{-1}\right)$$
$$- (\mathbf{e}_1 - i\mathbf{e}_2) \wedge \cdots \wedge \left(\mathbf{e}_{2(\ell-1)-1} - i\mathbf{e}_{2(\ell-1)}\right) \wedge \mathbf{e}_{2j} - \dot{\pi}(I_{n,2j})v_0,$$
$$\frac{1}{8}\left(\dot{\pi}\left(X_{-\varepsilon_j-\varepsilon_\ell}\right)v_1 + \dot{\pi}\left(X_{-\varepsilon_j+\varepsilon_\ell}\right)v_{-1}\right)$$

$$- (\mathbf{e}_1 - i\mathbf{e}_2) \wedge \cdots \wedge (\mathbf{e}_{2(\ell-1)-1} - i\mathbf{e}_{2(\ell-1)}) \wedge \mathbf{e}_{2j-1} - \dot\pi(I_{n,2j-1})v_0.$$

Then, for $1 \le j < \ell$,

$$\dot\pi\left(I_{n,2j-1}\right) P_1 \dot\pi\left(I_{n,2j-1}\right) v_0 - \tfrac{1}{8}\dot\pi\left(I_{n,2j-1}\right)\dot\pi\left(X_{-\varepsilon_j-\varepsilon_\ell}\right)v_1$$
$$- \tfrac{i}{2}\left(\mathbf{e}_1 - i\mathbf{e}_2\right) \wedge \cdots \wedge \mathbf{e}_{2j} \wedge \cdots \wedge \left(\mathbf{e}_{2(\ell-1)-1} - i\mathbf{e}_{2(\ell-1)}\right) \wedge \mathbf{e}_n,$$
$$\dot\pi\left(I_{n,2j}\right) P_1 \dot\pi\left(I_{n,2j}\right) v_0 - \tfrac{-i}{8}\dot\pi\left(I_{n,2j}\right)\dot\pi\left(X_{-\varepsilon_j-\varepsilon_\ell}\right)v_1$$
$$- -\tfrac{1}{2}\left(\mathbf{e}_1 - i\mathbf{e}_2\right) \wedge \cdots \wedge \mathbf{e}_{2j-1} \wedge \cdots \wedge \left(\mathbf{e}_{2(\ell-1)-1} - i\mathbf{e}_{2(\ell-1)}\right) \wedge \mathbf{e}_n.$$

Therefore, for $1 \le j < \ell$,

$$\dot\pi\left(I_{n,2j-1}\right) P_1 \dot\pi\left(I_{n,2j-1}\right) v_0 + \dot\pi\left(I_{n,2j}\right) v_0 P_1 \dot\pi\left(I_{n,2j}\right) v_0 - -\tfrac{1}{2}v_0.$$

Besides, for $j - \ell$ we have

$$\dot\pi(I_{n,2\ell})v_0 - \tfrac{1}{2i}(-v_1 + v_{-1}) \qquad \text{and} \qquad \dot\pi(I_{n,2\ell-1})v_0 - \tfrac{1}{2}(v_1 + v_{-1}).$$

Therefore, since

$$\dot\pi\left(I_{n,2\ell}\right) P_1 \dot\pi(I_{n,2\ell})v_0 - -\tfrac{1}{2i}\dot\pi\left(I_{n,2\ell}\right)v_1 - -\tfrac{1}{2}v_0,$$
$$\dot\pi\left(I_{n,2\ell-1}\right) P_1 \dot\pi(I_{n,2\ell-1})v_0 - \tfrac{1}{2}\dot\pi\left(I_{n,2\ell-1}\right)v_1 - -\tfrac{1}{2}v_0,$$

we have that

$$\sum_{j-0}^{n-1} \dot\pi\left(I_{n,j}\right) P_1 \dot\pi(I_{n,j})v_0 - -\frac{\ell+1}{2}v_0,$$

i.e.

$$\lambda(0,1) - -\frac{\ell+1}{2}.$$

Analogously we obtain

$$\lambda(0,-1) - -\frac{\ell+1}{2}.$$

Hence

$$(\lambda(r,s))_{-1 \le r,s \le 1} - \begin{pmatrix} 0 & -\ell & 0 \\ -\frac{\ell+1}{2} & 0 & -\frac{\ell+1}{2} \\ 0 & -\ell & 0 \end{pmatrix}.$$

Therefore, we obtain a more explicit version of Corollary 3.6 using (3.2) and Remark 3.8. Confront Corollary 5.2.

Corollary 9.1. *Let Φ be an irreducible spherical function on G of type $\pi \in \hat{\mathrm{SO}}(n)$, $n - 2\ell+1$. If the highest weight of π is of the form $(1,\ldots,1) \in \mathbb{C}^\ell$, then the function $H : (0,1) \to \mathbb{C}^3$ associated with Φ satisfies $\widetilde{D}H - \lambda H$, for some $\lambda \in \mathbb{C}$ with*

$$\widetilde{D}H - y(1-y)H''(y) + \frac{1}{2}n(1-2y)H'(y) + \frac{(1-2y)^2+1}{4y(1-y)}\begin{pmatrix} -\ell & 0 & 0 \\ 0 & -\ell-1 & 0 \\ 0 & 0 & -\ell \end{pmatrix} H(y)$$

$$+ \frac{(1-2y)}{2y(1-y)}\begin{pmatrix} 0 & -\ell & 0 \\ -\frac{\ell+1}{2} & 0 & -\frac{\ell+1}{2} \\ 0 & -\ell & 0 \end{pmatrix} H(y).$$

9.1   Spherical functions of $SO(2\ell+1)$-type $2\lambda_\ell$

Let $n-2\ell+1$, we now focus on the spherical functions $\Phi_{w,\delta}$ of type $\mathbf{m}_n - (1,\dots,1) \in \mathbb{C}^\ell$, which are associated with the irreducible representations of $SO(n+1)$ of highest weights of the form $\mathbf{m}_{n+1} - (w+1,1,\dots,1,\delta) \in \mathbb{C}^{\ell+1}$ such that the following pattern holds

$$
\begin{matrix}
w+1 & & 1 & \dots & 1 & & \delta \\
 & 1 & \dots & & \dots & 1 & & -1
\end{matrix}.
$$

As before we make the function Ψ whose columns are given by the spherical functions $\Phi_{0,\delta}$, $\delta - -1,0,1$. When $w-0$, this is calculable using [24, p. 364, equation (8)] or alternatively by considering the G-modules $\Lambda^{\ell+1}(\mathbb{C}^{n+1}) - V_1 \oplus V_{-1}$ and $\Lambda^\ell(\mathbb{C}^{n+1}) - V_0$ and working in the same way that we already did in the beginning of Section 6 for the 2×2 cases (here V_t, for $t-1,0,-1$, are the irreducible G-modules with highest weights $(1,\dots,1,t) \in \mathbb{C}^{\ell+1}$).

Therefore, if $\cos s - 2y - 1$ we have

$$
\Psi(y) - \begin{pmatrix} e^{is} & 1 & e^{-is} \\ 1 & \frac{1}{2}(e^{is}+e^{-is}) & 1 \\ e^{-is} & 1 & e^{is} \end{pmatrix}
$$

$$
- \begin{pmatrix} 2y-1+2i\sqrt{y-y^2} & 1 & 2y-1-2i\sqrt{y-y^2} \\ 1 & 2y-1 & 1 \\ 2y-1-2i\sqrt{y-y^2} & 1 & 2y-1+2i\sqrt{y-y^2} \end{pmatrix}.
$$

Each column of Ψ satisfies the differential equation given in Corollary 9.1. And it is easy to check that we have

$$
y(1-y)H''(y) + \frac{1}{2}n(1-2y)H'(y) + \frac{(1-2y)^2+1}{4y(1-y)} \begin{pmatrix} -\ell & 0 & 0 \\ 0 & -\ell-1 & 0 \\ 0 & 0 & -\ell \end{pmatrix} \Psi(y)
$$

$$
+ \frac{(1-2y)}{2y(1-y)} \begin{pmatrix} 0 & -\ell & 0 \\ -\frac{\ell+1}{2} & 0 & -\frac{\ell+1}{2} \\ 0 & -\ell & 0 \end{pmatrix} \Psi(y) - \Psi(y) \begin{pmatrix} -\ell-1 & 0 & 0 \\ 0 & -\ell & 0 \\ 0 & 0 & -\ell-1 \end{pmatrix}.
$$

Theorem 9.2. *The function Ψ can be used to obtain a hypergeometric differential equation from the one given in Corollary 9.1. Precisely, if H is a vector-valued solution of the differential equation in Corollary 9.1, with eigenvalue λ, then $P - \Psi^{-1}H$ is a solution of $DP - \lambda P$, where D is the hypergeometric differential operator given by*

$$
DP - y(1-y)P'' + (C-yU)P' - VP,
$$

with

$$
C - \begin{pmatrix} (n+2)/2 & 1/2 & 0 \\ 1 & (n+2)/2 & 1 \\ 0 & 1/2 & (n+2)/2 \end{pmatrix}, \qquad U - (n+2)I,
$$

$$
V - \begin{pmatrix} -\ell-1 & 0 & 0 \\ 0 & -\ell & 0 \\ 0 & 0 & -\ell-1 \end{pmatrix}.
$$

Proof. Let us write $H - \Psi P$. Then

$$
y(1-y)P'' + \left(2y(1-y)\Psi^{-1}\Psi' + \frac{n}{2}(1-2y)I\right)P'
$$

$$+ \Psi^{-1} \left(y(1-y)\Psi'' + \frac{n}{2}(1-2y)\Psi' + \frac{1+(1-2y)^2}{4y(1-y)} \begin{pmatrix} -\ell & 0 & 0 \\ 0 & -\ell-1 & 0 \\ 0 & 0 & -\ell \end{pmatrix} \Psi \right.$$

$$\left. + \frac{(1-2y)}{2y(1-y)} \begin{pmatrix} 0 & -\ell & 0 \\ -\frac{\ell+1}{2} & 0 & -\frac{\ell+1}{2} \\ 0 & -\ell & 0 \end{pmatrix} \Psi \right) P - \lambda P.$$

Now we compute

$$2y(1-y)\Psi^{-1}\Psi' + \frac{n}{2}(1-2y)I - -(n+2)yI + \begin{pmatrix} (n+2)/2 & 1/2 & 0 \\ 1 & (n+2)/2 & 1 \\ 0 & 1/2 & (n+2)/2 \end{pmatrix}.$$

Therefore

$$y(1-y)P'' + \left(-(n+2)yI + \begin{pmatrix} (n+2)/2 & 1/2 & 0 \\ 1 & (n+2)/2 & 1 \\ 0 & 1/2 & (n+2)/2 \end{pmatrix} \right) P'$$

$$+ \left(\begin{pmatrix} -\ell-1 & 0 & 0 \\ 0 & -\ell & 0 \\ 0 & 0 & -\ell-1 \end{pmatrix} - \lambda I \right) P - 0.$$

This completes the proof of the theorem. ∎

We obtain a similar result to Theorem 6.3, with an analogous proof:

Theorem 9.3. *For a given $\ell \in \mathbb{N}$ let $n - 2\ell + 1$, then the nonzero polynomial eigenfunctions of*

$$DP - y(1-y)P'' + (C - yU)P' - VP,$$

with

$$C - \begin{pmatrix} (n+2)/2 & 1/2 & 0 \\ 1 & (n+2)/2 & 1 \\ 0 & 1/2 & (n+2)/2 \end{pmatrix}, \qquad U - (n+2)I,$$

$$V - \begin{pmatrix} -\ell-1 & 0 & 0 \\ 0 & -\ell & 0 \\ 0 & 0 & -\ell-1 \end{pmatrix},$$

have eigenvalues $-w(w+n+1) - \ell$ or $-w(w+n+1) - \ell - 1$, with $w \in \mathbb{N}_0$. In both cases the degree of the polynomial is w and the leading coefficient can be any multiple of $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ or any linear combination of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, respectively.

Let us consider $\widetilde{D}$, the differential operator on $(0,1)$ introduced in Corollary 9.1:

$$\widetilde{D}H - y(1-y)H''(y) + \frac{1}{2}n(1-2y)H'(y)$$

$$+ \frac{(1-2y)^2+1}{4y(1-y)} \begin{pmatrix} -\ell & 0 & 0 \\ 0 & -\ell-1 & 0 \\ 0 & 0 & -\ell \end{pmatrix} H(y) + \frac{(1-2y)}{2y(1-y)} \begin{pmatrix} 0 & -\ell & 0 \\ -\frac{\ell+1}{2} & 0 & -\frac{\ell+1}{2} \\ 0 & -\ell & 0 \end{pmatrix} H(y).$$

Recall that the operator D that appears in Theorem 9.3 extends the differential operator $D - \Psi\widetilde{D}\Psi^{-1}$ to the whole real line.

We want to focus our attention on the following vector spaces of $\mathbb{C}^3$-valued analytic functions on $(0,1)$:

$$S_\lambda - \left\{ H - H(y) : \widetilde{D}H - \lambda H, \ H(\tfrac{\cos s+1}{2}) \text{ analytic at } s - 0 \right\},$$
$$W_\lambda - \left\{ P - P(y) : DP - \lambda P, \text{ analytic on } [0,1] \right\}.$$

From Theorem 9.2 we know that the correspondence $P \mapsto \Psi P$ is an injective linear map from W_λ into S_λ. In fact, $\Psi((\cos s + 1)/2)$ is analytic as a function of s and P is analytic at $y - 1$, hence $H((\cos s + 1)/2) - (\Psi P)((\cos s + 1)/2)$ is analytic at $s - 0$.

Then, we have an analogous result to Theorem 7.3, whose proof is quite similar and therefore we will omit it.

Theorem 9.4. *The linear map $P \mapsto \Psi P$ is an isomorphism from W_λ onto S_λ.*

Now, we can easily make a proof similar to that one of Theorem 7.4 in order to obtain next theorem.

Theorem 9.5. *Let H be the $\mathbb{C}^3$-valued analytic function on $(0,1)$ given by an irreducible spherical function Φ on $\mathrm{SO}(2\ell + 2)$ of fundamental $\mathrm{SO}(2\ell + 1)$-type $(1,\ldots,1) \in \mathbb{C}^\ell$. If $P - \Psi^{-1} H$, then P is polynomial.*

For a given fundamental K-type $\pi \in \hat{\mathrm{SO}}(n)$, $n - 2\ell + 1$, with highest weight $(1,\ldots,1) \in \mathbb{C}^\ell$, let $\Phi_{w,\delta}$ denote the irreducible spherical function of the pair $(\mathrm{SO}(n+1), \mathrm{SO}(n))$ given by $\tau \in \hat{\mathrm{SO}}(n+1)$ with highest weight of the form $(w+1,1,\ldots,1,\delta) \in \mathbb{C}^{\ell+1}$, $\delta - -1, 0, 1$.

Now, combining Theorems 9.3, 9.5 and the expression of the eigenvalue $\lambda_n(w,\delta)$ given in (6.2) we have the following statement.

Theorem 9.6. *Given $w \in \mathbb{N}$, every irreducible spherical function $\Phi_{w,\delta}$ of the pair $(\mathrm{SO}(n+1), \mathrm{SO}(n))$ with $n - 2\ell + 1$, of type $\mathbf{m}_n - (1,\ldots,1) \in \mathbb{C}^\ell$, corresponds to a vector-valued function $P_{w,\delta}$ $(w \geq 0, \delta - -1, 0, 1)$, which is a polynomial of degree w. The leading coefficients of $P_{w,0}$ is a multiple of $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and the leading coefficients of $P_{w,-1}$ and $P_{w,1}$ are both linear combinations of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Precisely*

$$P_{w,\delta}(y) - \sum_{j-0}^{w} \frac{y^j}{j!} [C; U; V + \lambda]_j P_{w,\delta}(0),$$

with

$$C - \begin{pmatrix} (n+2)/2 & 1/2 & 0 \\ 1 & (n+2)/2 & 1 \\ 0 & 1/2 & (n+2)/2 \end{pmatrix},$$

$$U - (n+2)I, \qquad V - \begin{pmatrix} -\ell-1 & 0 & 0 \\ 0 & -\ell & 0 \\ 0 & 0 & -\ell-1 \end{pmatrix},$$

$$\lambda - \lambda_n(w,\delta) - \begin{cases} -w(w+n+1) - \ell & \text{if} \quad \delta - 0, \\ -w(w+n+1) - \ell - 1 & \text{if} \quad \delta - \pm 1. \end{cases}$$

Even more, the value of $P_{w,\delta}(0)$ can be computed.

Proof. It only remains to prove that $P_{w,\delta}(0)$ can be computed.

Let us consider the case $\delta = 0$. We know from (6.2) and Theorem 9.3 that there is some $c \in \mathbb{C}$ such that

$$[C; U; V + \lambda]_w P_{w,0}(0) = c \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Since $[C; U; V + \lambda]_w$ is invertible this c is univocally determined by the condition $\Phi(e) = I$ which implies

$$\Psi(1) \sum_{j=0}^{w} \frac{1}{j!} [C; U; V + \lambda]_j P_{w,0}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Now let us consider the cases $\delta = \pm 1$. We know from (6.2) and Theorem 9.3 that

$$[C; U; V + \lambda]_w P_{w,\delta}(0) \in \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle;$$

since $[C; U; V + \lambda]_w$ is invertible, this condition tells us that $P_{w,\delta}(0)$ belongs to a plane which contains the origin and does not depend on δ.

Besides, the condition $\Phi_{w,\delta}(e) = I$, for $\delta = \pm 1$, tells us

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \sum_{j=0}^{w} \frac{1}{j!} [C; U; V + \lambda]_j P_{w,\delta}(0).$$

Then, $P_{w,\delta}(0)$ belongs to a plane, parallel to the kernel of

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \sum_{j=0}^{w} \frac{1}{j!} [C; U; V + \lambda]_j,$$

which does not contain the origin and does not depend on δ. Therefore we know that both $P_{w,1}(0)$ and $P_{w,-1}(0)$ are in the same straight line.

On the other hand, recall that we have

$$\Phi_{w,\delta}(a(s)) = \Psi\left(\frac{\cos s + 1}{2}\right) P_{w,\delta}\left(\frac{\cos s + 1}{2}\right),$$

where

$$a(s) = \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & \cos s & \sin s \\ 0 & -\sin s & \cos s \end{pmatrix},$$

then

$$\frac{d}{ds}\bigg|_{s=0} \Phi(a(s)) = \begin{pmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ -i & 0 & i \end{pmatrix} P_{w,\delta}(1).$$

From [24, p. 364, equation (8)] we can easily compute $\frac{d}{ds}\Phi_{w,\delta}(a(s))$ at $s = 0$, which is obtained by looking at the action of $\dot{\tau}(I_{n+1,n})$ and considering the corresponding projection, see (2.1); having then

$$\delta\frac{i(w+\ell+1)}{1+\ell}\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}i & 0 & -i\\0 & 0 & 0\\-i & 0 & i\end{pmatrix}\sum_{j=0}^{w}\frac{1}{j!}[C;U;V+\lambda]_j P_{w,\delta}(0).$$

This last condition establishes that $P_{w,1}(0)$ and $P_{w,-1}(0)$ are in two different and parallel planes, and the line mentioned above does not belong to any of them since each plane has to intersect it. Therefore the values of $P_{w,1}(0)$ and $P_{w,-1}(0)$ are univocally determined. ∎

9.2   Matrix-valued orthogonal polynomials of size 3

In this subsection, given n of the form $2\ell + 1$ with $\ell \in \mathbb{N}$, we shall construct a sequence of matrix-valued polynomials $\{P_w\}_{w\geq 0}$ directly related to irreducible spherical functions of type $\pi \in \hat{\mathrm{SO}}(n)$ of highest weight $\mathbf{m}_\pi = (1,\dots,1) \in \mathbb{C}^\ell$.

Given a nonnegative integer w and $\delta = -1, 0, 1$, we can consider $\Phi_{w,\delta}$, the irreducible spherical function of type π associated with the irreducible representation $\tau \in \hat{\mathrm{SO}}(n+1)$ of highest weight of the form $\mathbf{m}_\tau = (w+1, 1, \dots, 1, \delta)$.

We insist on recalling that, since π has only three $\mathrm{SO}(2\ell)$-submodules, we can interpret the diagonal matrix-valued function $\Phi_{w,\delta}(a(s))$, $s \in (0, \pi)$, as a 3 column vector function.

Now we consider the vector-valued function

$$P_{w,\delta}: \ (0,1) \to \mathbb{C}^3$$

given by the vector function $P_{w,\delta}(y) = \Psi^{-1}(y)\Phi_{w,\delta}(a(s))$, with $\cos(s) = 2y - 1$. Then, we define the matrix-valued function

$$P_w = P_w(y),$$

whose δ-th column ($\delta = -1, 0, 1$) is given by the $\mathbb{C}^3$-valued polynomial $P_{w,\delta}(y)$.

Let consider the matrix-valued skew symmetric bilinear form defined among continuous 3×3 matrix-valued functions on $[0, 1]$ by

$$\langle P, Q\rangle_W = \int_0^1 Q^*(y)W(y)P(y)dy,$$

where the 3×3 weight-matrix W is given by

$$W(y) = \frac{(n-1)!!}{(n-2)!!}(y(1-y))^{n/2-1}\Psi^*(y)\begin{pmatrix}d_1 & 0 & 0\\0 & d_2 & 0\\0 & 0 & d_3\end{pmatrix}\Psi(y)$$

with

$$d_1 = d_3 = \frac{(2\ell+1)!}{\ell!(\ell+2)!}, \qquad d_2 = \frac{(2\ell+1)!}{\ell!(\ell+1)!},$$

and

$$\Psi(y) = \begin{pmatrix}2y-1+2i\sqrt{y-y^2} & 1 & 2y-1-2i\sqrt{y-y^2}\\1 & 2y-1 & 1\\2y-1-2i\sqrt{y-y^2} & 1 & 2y-1+2i\sqrt{y-y^2}\end{pmatrix}$$

Let us recall that, from Proposition 7.1, we have

$$\langle\Phi_{w,\delta},\Phi_{w',\delta'}\rangle = \int_0^1 P_{w,\delta}^* W(y)P_{w',\delta'}dy.$$

Remark 9.7. Notice that W reduces to a smaller size: if $M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix}$ we have

$$MW(y)M^* = \frac{(n-1)!!}{(n-2)!!}(y(1-y))^{n/2-1}4$$

$$\times \begin{pmatrix} 2d_1(2y-1)^2 + d_2 & d_1(2y-1) + d_2(2y-1)/\sqrt{2} & 0 \\ d_1(2y-1)\sqrt{2} + d_2(2y-1)/\sqrt{2} & d_1 + d_2(2y-1)^2/2 & 0 \\ 0 & 0 & d_1 8(y-y^2) \end{pmatrix}.$$

Then we state the following theorem.

Theorem 9.8. *The matrix-valued polynomial functions P_w, $w \geq 0$, form a sequence of orthogonal polynomials with respect to W, which are eigenfunctions of the symmetric differential operator D from Theorem 9.2. Moreover,*

$$DP_w = P_w \begin{pmatrix} \lambda(w,-1) & 0 & 0 \\ 0 & \lambda(w,0) & 0 \\ 0 & 0 & \lambda(w,1) \end{pmatrix},$$

where

$$\lambda(w,\delta) = \begin{cases} -w(w+n+1) - p & \text{if} \quad \delta = 0, \\ -w(w+n+1) - n + p & \text{if} \quad \delta = \pm 1. \end{cases}$$

Proof. The proof is completely analogous to the proof of Theorem 8.1   ∎

Appendix

Proof of Proposition 3.2. For $|t|$ sufficiently small $A(s,t)$ is close to the identity of K, i.e. to the identity matrix I_n. So we can consider the function

$$X(s,t) = \log(A(s,t)) = B(s,t) - \frac{B(s,t)^2}{2} + \frac{B(s,t)^3}{3} - \cdots , \tag{9.2}$$

where $B(s,t) = A(s,t) - I_n$. Then

$$\pi(A(s,t)) = \pi(\exp X(s,t)) = \exp \dot{\pi}(X(s,t)) = \sum_{j \geq 0} \frac{\dot{\pi}(X(s,t))^j}{j!}.$$

Now we differentiate with respect to t to obtain

$$\frac{\partial(\pi \circ A)}{\partial t} = \dot{\pi}\left(\frac{\partial X}{\partial t}\right) + \frac{1}{2!}\dot{\pi}\left(\frac{\partial X}{\partial t}\right)\dot{\pi}(X) + \frac{1}{2!}\dot{\pi}(X)\dot{\pi}\left(\frac{\partial X}{\partial t}\right)$$

$$+ \frac{1}{3!}\dot{\pi}\left(\frac{\partial X}{\partial t}\right)\dot{\pi}(X)^2 + \frac{1}{3!}\dot{\pi}(X)\dot{\pi}\left(\frac{\partial X}{\partial t}\right)\dot{\pi}(X) + \frac{1}{3!}\dot{\pi}(X)^2\dot{\pi}\left(\frac{\partial X}{\partial t}\right) + \cdots . \tag{9.3}$$

Since $X(s,0) = 0$, if we differentiate (9.2) with respect to t and evaluate at $(s,0)$ we obtain

$$\frac{\partial^2(\pi \circ A)}{\partial t^2}\Big|_{t=0} = \dot{\pi}\left(\frac{\partial^2 X}{\partial t^2}\Big|_{t=0}\right) + \dot{\pi}\left(\frac{\partial X}{\partial t}\Big|_{t=0}\right)^2 .$$

To compute $\frac{\partial X}{\partial t}\Big|_{t=0}$ and $\frac{\partial^2 X}{\partial t^2}\Big|_{t=0}$ we differentiate (9.2) and we get

$$\frac{\partial X}{\partial t} = \frac{\partial B}{\partial t} - \frac{1}{2}\left(\frac{\partial B}{\partial t}\right)B - \frac{1}{2}B\left(\frac{\partial B}{\partial t}\right) + \frac{1}{3}\left(\frac{\partial B}{\partial t}\right)B^2 + \frac{1}{3}B\left(\frac{\partial B}{\partial t}\right)B + \frac{1}{3}B^2\left(\frac{\partial B}{\partial t}\right) + \cdots .$$

Since $B(s,0) - 0$ we have

$$\frac{\partial X}{\partial t}\Big|_{t-0} - \frac{\partial B}{\partial t}\Big|_{t-0} - \frac{\partial A}{\partial t}\Big|_{t-0}.$$

We also get

$$\frac{\partial^2 X}{\partial t^2}\Big|_{t-0} - \frac{\partial^2 A}{\partial t^2}\Big|_{t-0} - \left(\frac{\partial A}{\partial t}\Big|_{t-0}\right)^2.$$

Now we will first consider the case $A(s,t) - k(s,t)$. A direct computation yields to

$$\frac{\partial k}{\partial t} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-\sin s \sin t}{(1-\cos^2 s \cos^2 t)^{3/2}} & 0 & \frac{\sin^2 s \cos t}{(1-\cos^2 s \cos^2 t)^{3/2}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-\sin^2 s \cos t}{(1-\cos^2 s \cos^2 t)^{3/2}} & 0 & \frac{-\sin s \sin t}{(1-\cos^2 s \cos^2 t)^{3/2}} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

in particular $\frac{\partial k}{\partial t}\Big|_{t-0} - \frac{1}{\sin s}I_{n,j}$. Differentiating once more with respect to t and evaluating at $t - 0$ we obtain $\frac{\partial^2 k}{\partial t^2}\Big|_{t-0} - -\frac{1}{\sin^2 s}(E_{jj} + E_{n,n})$. Then we get

$$\frac{\partial^2 A}{\partial t^2}\Big|_{t-0} - \left(\frac{\partial A}{\partial t}\Big|_{t-0}\right)^2 - -\frac{1}{\sin^2 s}(E_{jj} + E_{n,n}) - \frac{1}{\sin^2 s}I_{n,j}^2 - 0.$$

Similarly when $A(s,t) - h(s,t)$ we obtain

$$\frac{\partial h}{\partial t} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-\sin s \cos^2 s \cos t \sin t}{(1-\cos^2 s \cos^2 t)^{3/2}} & 0 & \frac{-\cos s \cos t \sin^2 s}{(1-\cos^2 s \cos^2 t)^{3/2}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\cos s \cos t \sin^2 s}{(1-\cos^2 s \cos^2 t)^{3/2}} & 0 & \frac{-\sin s \cos^2 s \cos t \sin t}{(1-\cos^2 s \cos^2 t)^{3/2}} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

in particular $\frac{\partial h}{\partial t}\Big|_{t-0} - -\frac{\cos s}{\sin s}I_{n,j}$. Differentiating once more with respect to t and evaluating at $t - 0$ we obtain $\frac{\partial^2 h}{\partial t^2}\Big|_{t-0} - -\frac{\cos^2 s}{\sin^2 s}(E_{jj} + E_{n,n})$. Then we get

$$\frac{\partial^2 A}{\partial t^2}\Big|_{t-0} - \left(\frac{\partial A}{\partial t}\Big|_{t-0}\right)^2 - -\frac{\cos^2 s}{\sin^2 s}(E_{jj} + E_{n,n}) - \frac{\cos^2 s}{\sin^2 s}I_{n,j}^2 - 0.$$

Proposition follows. ∎

Acknowledgements

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