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The cohomology of filiform Lie algebras of maximal rank ☆

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ABSTRACT

We describe the structure of the cohomology of the filiform Lie algebras L_n and Q_n as a module over their (2-dimensional) torus of derivations. Our approach relies on the fact that both filiform algebras have an ideal $\mathfrak{h}$ of codimension 1 for which the structure of its cohomology under the action of the Levi factor of the algebra of derivations of $\mathfrak{h}$ is known.

1. Introduction

A filiform Lie algebra is a nilpotent Lie algebra of maximal class. The rank of a filiform Lie algebra[1] is at most 2 and there are, up to isomorphism, only two families of filiform Lie algebras of rank 2 [7]:

$$\{L_n : n \geq 2\} \quad \text{and} \quad \{Q_n : n \geq 3,\ n \text{ odd}\}.$$

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[1] The rank of a nilpotent Lie algebra is the dimension of a maximal torus of derivations.

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Both L_n and Q_n are of dimension $n+1$ and their nonzero brackets of basis elements $X_0, X_1, \dots, X_n$ are as follows:

$$L_n: \quad [X_0, X_i] = X_{i+1}, \quad \text{for } 1 \le i \le n-1. \tag{1.1}$$

$$Q_n: \quad [X_0, X_i] = X_{i+1}, \quad \text{for } 1 \le i \le n-1, \tag{1.2}$$
$$[X_i, X_{n-i}] = (-1)^i X_n, \quad \text{for } 1 \le i \le n-1.$$

Filiforms Lie algebras play an important role in the study of nilpotent Lie algebras. M. Vergne [13] proved that for each n, the subset of filiform laws in dimension n constitute an open set in the variety of nilpotent Lie algebras of dimension n. In addition, she proved that a filiform Lie algebra $\mathfrak{n}$ is naturally graded if and only if $\mathfrak{n}$ is isomorphic to L_n or Q_n.

The cohomology of filiform Lie algebras has been studied in different contexts (see for instance [1,3,9,11]). In particular G. Armstrong and S. Sigg computed the Betti numbers of L_n and, more recently, H. Pouseele computed the Betti numbers of Q_n.

This paper is devoted to describe the structure of the cohomology of L_n and Q_n as a module over their (2-dimensional) torus of derivations. Our approach is the same in both cases, and relies on the fact that both filiform algebras have an ideal $\mathfrak{h}$ of codimension 1 for which the structure of its cohomology under the action of the Levi factor of the algebra of derivations of $\mathfrak{h}$ is known. This approach should also work in many other instances such as (nilpotent) extensions of nilradicals of parabolic subalgebras of semisimple Lie algebras.

We also give explicit cohomology classes for low degrees and we describe all their nonzero affine cohomology classes (given a 2-cocycle $\omega \in \Lambda^2\mathfrak{n}^*$, $[\omega]$ is affine if $[\omega] \neq 0$ and ω is nonzero in Center$(\mathfrak{n}) \wedge \mathfrak{n}$, see [3]).

In both cases, L_n and Q_n, the linear subspace $\langle X_1, \dots, X_n\rangle$ is an ideal of codimension 1 that we denote by $\mathfrak{h}_L$ and $\mathfrak{h}_Q$ respectively. Moreover, $\mathfrak{h}_L$ is an abelian Lie algebra of dimension n and $\mathfrak{h}_Q$ is isomorphic to the Heisenberg Lie algebra of dimension $n = 2m+1$. The goal of this paper is to describe the cohomology of these Lie algebras. We point out that L_1, L_2, $L_3 \simeq Q_3$ and Q_5 are respectively the nilradicals of the Borel subalgebras of type $A_1 \times A_1$, A_2, B_2 and G_2, and their cohomology is well understood from the classical results of Kostant. The Lie algebras L_n, $n \ge 4$ and Q_n, $n \ge 7$, are not Kostant nilradicals since they have rank 2.

The cohomology of L_n and Q_n can be described in terms of the cohomology of the ideals $\mathfrak{h}_L$ and $\mathfrak{h}_Q$ respectively, be means of the well known formulas

$$H^p(L_n) \simeq H^p(\mathfrak{h}_L)^{X_0} \oplus H^{p-1}(\mathfrak{h}_L)_{X_0}, \tag{1.3}$$

$$H^p(Q_n) \simeq H^p(\mathfrak{h}_Q)^{X_0} \oplus H^{p-1}(\mathfrak{h}_Q)_{X_0}, \tag{1.4}$$

which follow from the well known Dixmier's exact sequence [5]. In fact, given a Lie algebra $\mathfrak{g}$ with an ideal $\mathfrak{h}$ of codimension 1, each element D outside $\mathfrak{h}$ induces an exact sequence

$$\dots \to H^{p-1}(\mathfrak{h}) \xrightarrow{\theta(D)} H^{p-1}(\mathfrak{h}) \to H^p(\mathfrak{g}) \to H^p(\mathfrak{h}) \xrightarrow{\theta(D)} H^p(\mathfrak{h}) \to \dots$$

where the (connecting) morphism $\theta(D) = -\overline{D}^t$ is the *Lie derivative* of D ($\overline{D}$ is the canonical extension of $\operatorname{ad} D$ to the exterior algebra of $\mathfrak{h}$). Therefore it follows that, for every $p = 0, \dots, \dim \mathfrak{g}$,

$$H^p(\mathfrak{g}) = H^p(\mathfrak{h})^{\theta(D)} \oplus H^{p-1}(\mathfrak{h})_{\theta(D)} \otimes D^*, \tag{1.5}$$

where $D^* \in \mathfrak{g}^*$ is the linear functional that vanishes on $\mathfrak{h}$ dual to D. The linear spaces of coinvariants $H^{p-1}(\mathfrak{h})_{\theta(D)}$ and invariants $H^{p-1}(\mathfrak{h})^{\theta(D)}$ are of the same dimension and moreover, if $\theta(D)$ is diagonalizable (which is the case if $\operatorname{ad} D$ is), one can identify them canonically and thus

$$H^p(\mathfrak{g}) \simeq H^p(\mathfrak{h})^{\theta(D)} \oplus H^{p-1}(\mathfrak{h})^{\theta(D)} \otimes D^*, \tag{1.6}$$

$$H^p(\mathfrak{g}) \simeq H^p(\mathfrak{h})^{\theta(D)} \oplus H^{p-1}(\mathfrak{h})^{\theta(D)}. \tag{1.7}$$

The main tool in our description of the cohomology of L_n and Q_n is the representation theory of classical Lie algebras. In both cases we identify the algebra $\mathfrak{a}$ of semisimple derivations with a Cartan subalgebra of $\mathfrak{gl}_2(\mathbb{C})$ and we describe the $\mathfrak{a}$-module structure of the cohomology. The '$\mathfrak{sl}_2$-trick' has been already used to compute Lie algebra cohomology in many instances, see for example [1,2,6].

In the first case $H^p(\mathfrak{h}_L) = \Lambda^p V^*$ and the action of X_0 can be seen as the action of a raising operator of $\mathfrak{sl}_2$ on $\Lambda^p V^*$ where V is the irreducible representation of highest weight $n-1$. Thus $H^p(\mathfrak{h}_L)^{X_0}$ is the space of highest weight vectors of the $\mathfrak{sl}_2$-module $\Lambda^p V^*$. Using the Stanley character formula for this representation [12, Theorem 7.21.2] we complete our computation.

In the second case, considering $\mathfrak{h}_Q$ as the nilradical of a parabolic subalgebra of the symplectic Lie algebra $\mathfrak{sp}_{2m+2}$, with Levi factor $\mathfrak{sp}_{2m}$, one knows the $\mathfrak{sp}_{2m}$-module structure of $H^p(\mathfrak{h}_Q)$ [10,4]. As in the previous case, the action of X_0 can be seen as the action of a raising operator of an $\mathfrak{sl}_2 \hookrightarrow \mathfrak{sp}_{2m}$. The corresponding branching law allows us to complete our computation in this case.

2. Preliminaries and notation

All Lie algebras and representations are over the complex numbers.

2.1. The $\mathfrak{gl}_2$-module structure of ΛV_n^*

Let V_n be the irreducible representation of the Lie algebra $\mathfrak{sl}_2$ of highest weight n. Consider the identity matrix of $\mathfrak{gl}_2$ acting on V_n by the scalar r and denote this $\mathfrak{gl}_2$-module by $V_{n,r}$. The dimension of $V_{n,r}$ is $\dim V_{n,r} = \dim V_n = n + 1$.

Let $F, H, E \in \mathfrak{sl}_2$ be the usual basis of $\mathfrak{sl}_2$ with nonzero brackets

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H,$$

and let $I \in \mathfrak{gl}_2$ be the identity matrix.

Let $\{v_n, v_{n-2}, \dots, v_{-n+2}, v_{-n}\}$ be a basis of weight vectors of V_n and let us denote by $\mathcal{E}_n$, $\mathcal{H}_n$, $\mathcal{F}_n$ the $(n+1) \times (n+1)$-matrices representing E, H, F as operators on V_n. $\mathcal{H}_n$ is diagonal with entries $n - 2j$, $j = 0, \dots, n$, and we further assume that the basis was chosen so that

$$\mathcal{E}_n = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}, \qquad \mathcal{F}_n = \begin{pmatrix} 0 & & & & & \\ n & 0 & & & & \\ & 2(n-1) & 0 & & & \\ & & 3(n-2) & 0 & & \\ & & & \ddots & \ddots & \\ & & & & n & 0 \end{pmatrix}.$$

Let us now consider the $\mathfrak{gl}_2$-module structure of $\Lambda^p V_{n,1}^*$. The identity I acts by the scalar $-p$ and the $\mathfrak{sl}_2$-module structure of $\Lambda^p V_{n,1}^*$ can be read off from its character. Given an $\mathfrak{sl}_2$-module V, its character is

$$\mathrm{ch}_V(q) = \sum_{j \in \mathbb{Z}} c_{V,j} q^j,$$

where $c_{V,j}$ is the dimension of the H-eigenspace of eigenvalue j in V. Recall that $c_{V,j} = c_{V,-j}$ and that the multiplicity of the irreducible highest weight module of highest weight n in the decomposition of V, is equal to $c_{V,n} - c_{V,n+2}$.

Since $V_n^* \simeq V_n$ as $\mathfrak{sl}_2$-modules, $\Lambda^p V_n^* \simeq \Lambda^p V_n$ and thus they have the same character. Let ch_n^p be the character of $\Lambda^p V_n$ and let $c_{n,p,j}$, $n \in \mathbb{Z}_{\geq 0}$ and $p, j \in \mathbb{Z}$, be defined by

$$\mathrm{ch}_n^p(q) = \sum_{j \in \mathbb{Z}} c_{n,p,j} q^j.$$

It is not difficult to see that $c_{n,p,j} = 0$ if $|j| > p(n+1-p)$ or $j \not\equiv p(n+1-p) \bmod (2)$.

For $n \in \mathbb{Z}_{\geq 0}$ and $p = 0, \dots, n+1$, let $a_{n,p,j}$ be defined by

$$\Lambda^p V_n \simeq \bigoplus_{j=0}^{p(n+1-p)} a_{n,p,j} V_j.$$

We know that, for $0 \leq j \leq p(n+1-p)$,

$$a_{n,p,j} = c_{n,p,j} - c_{n,p,j+2} \tag{2.1}$$

and, in particular, $a_{n,p,j} = 0$ if $j \not\equiv p(n+1-p) \bmod (2)$. We find it convenient to extend the definition of $a_{n,p,j}$ to $j \in \mathbb{Z}_{\geq 0}$ and $p \in \mathbb{Z}$ by assuming that $a_{n,p,j} = 0$ if either $p < 0$, $p > n+1$ or $j > p(n+1-p)$. Moreover, since

$$\Lambda^p V_n \simeq \Lambda^{n+1-p} V_n, \tag{2.2}$$

as $\mathfrak{sl}_2$-modules, it follows that

$$c_{n,p,j} = c_{n,n+1-p,j} \quad \text{and} \quad a_{n,p,j} = a_{n,n+1-p,j}. \tag{2.3}$$

For each $\lambda = (\lambda_1, \ldots, \lambda_d)$ with $\lambda_1 \geq \ldots \geq \lambda_d$, the Schur functor $\mathcal{S}_\lambda$ associates to each representation V of $\mathfrak{sl}_2$ a representation $\mathcal{S}_\lambda(V)$. For $\lambda = (1^p) = (1, \ldots, 1)$, $\mathcal{S}_\lambda(V) = \Lambda^p(V)$. In the case $V = V_n$, a character formula for $\mathcal{S}_\lambda(V_n)$ can be derived from [12, Theorem 7.21.2]. For $\lambda = (1^p)$ it yields

$$\mathrm{ch}^p_{n-1}(q) = q^{-p(n-p)} \binom{[n]_{q^2}}{[p]_{q^2}},$$

where $[a]_q = 1 + q + \ldots + q^{a-1} = \frac{1-q^a}{1-q}$ is the q-analog of the number a, $[a]_q! = [a]_q![a-1]_q! \ldots [2]_q![1]_q!$ and $\binom{[a]_q}{[b]_q} = \frac{[a]_q!}{[b]_q![a-b]_q!}$. For instance,

$$\begin{aligned} \mathrm{ch}^2_3(q) &= q^{-4} \frac{(1-q^8)(1-q^6)}{(1-q^4)(1-q^2)} \\ &= q^4 + q^2 + 2 + q^{-2} + q^{-4}, \end{aligned}$$

and this implies that

$$\Lambda^2 V_3^* \simeq V_4 \oplus V_0.$$

More generally, it is not difficult to obtain that

$$\Lambda^2 V_n^* \simeq V_{2(n-1)} \oplus V_{2(n-3)} \oplus \ldots \oplus V_{1+(-1)^n}. \tag{2.4}$$

More precisely, let $\{v_n^*, v_{n-2}^*, \ldots, v_{-n+2}^*, v_{-n}^*\}$ be dual basis of the basis $\{v_n, v_{n-2}, \ldots, v_{-n+2}, v_{-n}\}$ of V_n introduced at the beginning of the section. Then, for $j = 0, \ldots, [\frac{n-1}{2}]$,

$$\sum_{i=0}^{j} (-1)^i v_{-n+2i}^* \wedge v_{-n+2+4j-2i}^*$$

is a highest weight vector in $\Lambda^2 V_{n,1}^*$ of weight $2n-2-4j$.

As far as we know, there are no very explicit formulas for the multiplicities $a_{n,p,j}$ for $p \geq 3$. The following table contains all $a_{n,p,j}$, for $n = 5, 6, 7, 8$ and $p = 3, 4$. Combined with (2.2) and (2.4) this yields the complete decomposition of $\Lambda^p V_n$, for $n \leq 8$ and all p.

j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\Lambda^3 V_5$				1		1				1											
$\Lambda^3 V_6$	1				1		1		1				1								
$\Lambda^3 V_7$				1		1		1		1		1				1					
$\Lambda^3 V_8$			1				2		1		1		1		1				1		
$\Lambda^4 V_7$	1				2				2		1		1				1				
$\Lambda^4 V_8$	1				2		1		2		1		2		1		1				1

2.2. The symplectic Lie algebra

Let V be a vector space of dimension $2m$ and let φ be a symplectic form on V (skew-symmetric bilinear and non-degenerate). Fix an ordered basis

$$\mathcal{B} = \mathcal{B}_m = \{e_m, e_{m-1}, \dots, e_1, e_{-1}, \dots, e_{-m}\}, \tag{2.5}$$

such that

$$\varphi(e_i, e_{-j}) = -\varphi(e_{-j}, e_i) = \delta_{ij}, \quad 1 \le i, j \le m;$$

where δ is the Kronecker delta.

The subgroup Sp_m of GL_{2m} that preserves the symplectic form φ is, by definition, the symplectic group. The Lie algebra of the Lie group Sp_m is the Lie subalgebra $\mathfrak{sp}_m$ of $\mathfrak{gl}_{2m}$ consisting of all the endomorphisms of V whose matrices, in the basis $\mathcal{B}$, are of the form

$$\begin{array}{|c|c|}\hline A & B \\ \hline C & -\hat{A} \\ \hline\end{array}, \qquad B = \hat{B},\ C = \hat{C},$$

where A, B and C are $m \times m$ matrices and $\hat{X}$ denotes the 'transpose' of X with respect to the secondary diagonal (from the upper-right corner to the lower-left corner), i.e.,

$$\hat{X}_{ij} = X_{(m+1-j)(m+1-i)}.$$

The Lie algebra $\mathfrak{sp}_m$ is simple of type C_m and its root structure is described as follows. For u and v in V, let $E_{u,v} \in \mathfrak{gl}(V)$ be the map defined by

$$E_{u,v}(x) = \varphi(x, v)u.$$

It is clear that $E_{u,v}E_{v',w} = \varphi(v', v)E_{u,w}$ and

$$\mathfrak{sp}_m = \langle E_{x,y} + E_{y,x} : x, y \in \mathcal{B} \rangle.$$

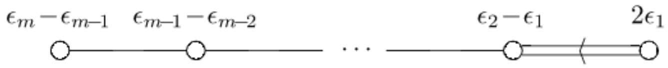
If

$$H_i = E_{e_i, e_{-i}} + E_{e_{-i}, e_i}, \quad i = 1 \dots m,$$

then the ordered set $\{H_m, H_{m-1}, \dots, H_1\}$ is a basis of a Cartan subalgebra of $\mathfrak{sp}_m$. If $\{\epsilon_m, \dots, \epsilon_1\}$ is the corresponding dual basis, then

$$\{\varepsilon_j - \varepsilon_i : 1 \le i < j \le n\} \cup \{\varepsilon_j + \varepsilon_i : 1 \le i < j \le n\} \cup \{2\varepsilon_j : 1 \le i \le n\}$$

is the set of positive roots corresponding to the triangular decomposition associated to the usual triangular decomposition of $\mathfrak{gl}_{2m}$ (upper triangular matrices are positive). The subset of simple roots is



2.3. The fundamental $\mathfrak{sp}_m$-modules and principal branchings

The fundamental weights of $\mathfrak{sp}_m$ are

$$\lambda_{m,p} = \epsilon_m + \ldots + \epsilon_{m+1-p}, \quad p = 1, \ldots, m.$$

For $p = 1, \ldots, m$, let $U_{2m,p}$ denote the irreducible $\mathfrak{sp}_m$-module corresponding to the dominant weight $\lambda_{m,p}$. We find it convenient to extend this notation by defining $U_{2m,0}$ as the trivial 1-dimensional $\mathfrak{sp}_m$-module and $U_{2m,p} = 0$ for negative p or $p > m$.

It is well known (see for instance [8, Corollary 5.5.16]) that $U_{2m,p}$ is a submodule of $\Lambda^p V$ ($\dim V = 2m$) for all $p = 0, 1, \ldots, m$. In fact

$$\Lambda^p V \simeq U_{2m,p} \oplus \Lambda^{p-2} V, \quad p = 0, 1, \ldots, m; \tag{2.6}$$

as $\mathfrak{sp}_m$-modules, or equivalently

$$\Lambda^p V \simeq \bigoplus_{j=0}^{[\frac{p}{2}]} U_{2m,p-2j}, \quad p = 0, 1, \ldots, m.$$

In particular, $\dim U_{2m,p} = \binom{2m}{p} - \binom{2m}{p-2}$. We also know that $\Lambda^p V \simeq \Lambda^{2m-p} V$ as $\mathfrak{sp}_m$-modules.

Let $\mathfrak{s}_m$ be the Lie subalgebra of $\mathfrak{sp}_m$ spanned by the (principal) $\mathfrak{s}$-triple $\{\mathcal{E}_{2m-1}, \mathcal{H}_{2m-1}, \mathcal{F}_{2m-1}\}$:

$$\mathcal{E}_{2m-1} = E_{e_1,e_1} + \sum_{i=1}^{m-1} E_{e_{i+1},e_{-i}} + E_{e_{-i},e_{i+1}},$$

$$\mathcal{H}_{2m-1} = \sum_{j=1}^{m} (2j-1) H_i,$$

$$\mathcal{F}_{2m-1} = -m^2 E_{e_{-1},e_{-1}} + \sum_{i=1}^{m-1} i(2m-i)(E_{e_i,e_{-i-1}} + E_{e_{-i-1},e_i}).$$

Notice that we keep the same notation for the $\mathfrak{s}$-triple introduced in Section 2.1, even though they are not exactly the same; however these two $\mathfrak{s}$-triples are conjugate by a diagonal matrix with ± 1's. In particular, $V \simeq V_{2m-1}$ as $\mathfrak{s}_m$-modules and the decomposition of $\Lambda^p V$ as $\mathfrak{s}_m$-module is given by

$$\Lambda^p V \simeq \bigoplus_{j=0}^{p(2m-p)} a_{2m-1,p,j} V_j, \quad p = 0, \dots, 2m,$$

where

$$a_{2m-1,p,j} = c_{2m-1,p,j} - c_{2m-1,p,j+2}.$$

It follows from (2.6) that

$$U_{2m,p} \simeq \bigoplus_{j=0}^{p(2m-p)} (a_{2m-1,p,j} - a_{2m-1,p-2,j}) V_j, \quad p = 0, 1, \dots, m, \tag{2.7}$$

is the decomposition of $U_{2m,p}$ as $\mathfrak{s}_m$-module. For instance,

$$U_{2m,1} \simeq V_{2m-1}, \tag{2.8}$$

$$U_{2m,2} \simeq V_{2(2m-2)} \oplus V_{2(2m-4)} \oplus \dots \oplus V_8 \oplus V_4 \tag{2.9}$$

and

	1	3	5	7	9	11	13	15	17	19	21	23	25
$U_{6,3}$		1			1								
$U_{8,3}$		1	1		1	1		1					
$U_{10,3}$		1	1	1	1	1	1	1	1		1		
$U_{10,5}$	1		1	1	1	1	1	1	1	1			1

	0	2	4	6	8	10	12	14	16	18	20	22	24
$U_{8,4}$			1		1	1			1				
$U_{10,4}$	1		1	1	2	1	2	1	1	1	1		1

3. The cohomology of L_n

Let $\mathfrak{l}_n = \mathfrak{gl}_2 \ltimes V_{n,1}$ and consider L_n as the following Lie subalgebra of $\mathfrak{l}_{n-1}$:

$$L_n \simeq \langle \mathcal{E}_{n-1} \rangle \ltimes V_{n-1,1}.$$

The Lie algebra L_n has rank 2, i.e., it has a 2-dimensional maximal torus of (semisimple) derivations. This torus can be chosen to be $\operatorname{ad}(\mathfrak{a})$, where

$$\mathfrak{a} = \langle H, I \rangle \subset \mathfrak{l}_{n-1}$$

is a Cartan subalgebra of $\mathfrak{gl}_2$. Their actions on L_n are given by

$$\begin{array}{c|cc} & \mathcal{E}_{n-1} & v_{n+1-2j} \\ \hline \mathrm{ad}(I) & 0 & v_{n+1-2j} \\ \mathrm{ad}(H)=\mathcal{H}_{n-1} & 2\mathcal{E}_{n-1} & (n+1-2j)v_{n+1-2j} \end{array} \tag{3.1}$$

where $j = 1, \dots, n$.

The 2-dimensional algebra $\mathfrak{a}$ of derivations of L_n acts on ΛL_n^* and on the cohomology of L_n. We now describe the $\mathfrak{a}$-weights appearing on ΛL_n^* and in the next subsection we will do so for $H^*(L_n)$.

3.1. The $\mathfrak{a}$-module structure of ΛL_n^*

The $\mathfrak{a}$-weights of L_n where given in (3.1). If we identify an element $\mu \in \mathfrak{a}^*$ with the pair $(\mu(H), \mu(I))$, the weights of L_n, with the corresponding weight vectors, are:

$$\begin{array}{cccccc} \mathcal{E}_{n-1} & v_{n-1} & v_{n-3} & \dots & v_{3-n} & v_{1-n} \\ (2,0) & (n-1,1) & (n-3,1) & \dots & (3-n,1) & (1-n,1) \end{array}$$

We denote by $\mu_e, \mu_{v_{n+1-2j}} \in \mathfrak{a}^*$ the $\mathfrak{a}$-weights of $\mathcal{E}_{n-1}$ and v_{n+1-2j}, $j = 1, \dots, n$, respectively. The multiplicity of a weight $\mu \in \mathfrak{a}^*$ in $\Lambda^p L_n^*$ is the number of ways in which $-\mu$ can be written as a sum of exactly p different weights of L_n. Let

$$P_n = \{\mu \in \mathfrak{a}^* : \mu \text{ is a weight of } \Lambda L_n^*\}.$$

It is not difficult to see that P_n is a convex polygon with $2(n+1)$ edges and it always has the following two symmetries:

$$\sigma_0(\mu) = -\mu_\Sigma - \mu \tag{3.2}$$

$$\sigma_1(\mu) = \bigl(-\mu(H), \mu(I)\bigr) - (2,0), \tag{3.3}$$

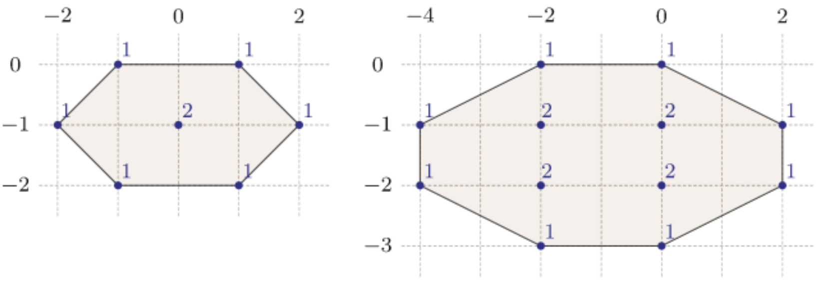
where $\mu_\Sigma = \mu_e + \sum_{j=1}^n \mu_{v_{n+1-2j}}$ is the sum of all weights in L_n.

The vertices of P_n, are $\nu_0, \dots, \nu_n, \sigma_0(\nu_0), \dots, \sigma_0(\nu_n)$, where the first ones are the negative of the sum of consecutive (with respect to the first coordinate) weights in L_n, that is

$$\nu_0 = 0,$$

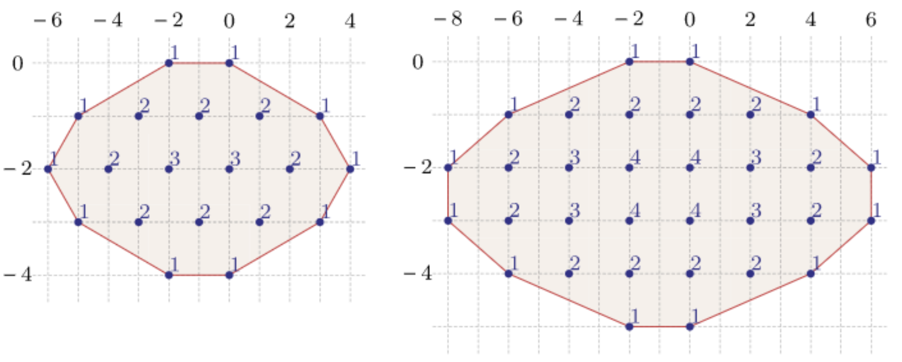
$$\nu_{k+1} = -\mu_e - \sum_{j=1}^{k} \mu_{v_{n+1-2j}}, \quad k = 0, \dots, n-1.$$

The polygons P_n, for $n = 2, 3$ (corresponding to L_2 and L_3), are shown in the following picture. We recall that L_2 and L_3 are isomorphic to the nilradicals of the Borel subalgebras of type A_2 and B_2 respectively.



Weights in ΛL_n^* for $n = 2, 3$.

The following picture shows the multiplicities of the weights appearing on ΛL_n^* for $n = 4$ and 5.



Weights in ΛL_n^* for $n = 4, 5$.

Given $p = 0, \ldots, n+1$, it is not difficult to read off, from these pictures, the $\mathfrak{a}$-weights appearing in $\Lambda^p L_n^*$, yet it is no automatic. Since μ_e is the only weight of L_n whose second coordinate, $\mu(I)$, is different from 1, the weights of $\Lambda^p L_n^*$ appear in the following two rows: $\mu(I) = p-1, p$.

In the next subsection we will describe which of these weights actually appear in the cohomology.

3.2. The $\mathfrak{a}$-module structure of $H^p(L_n)$

Given $\mu \in P_n \subset \mathfrak{a}^*$, let $H^p(L_n)_\mu$ be the $\mathfrak{a}$-isotopic component of $H^p(L_n)$ of weight μ.

The cohomology of L_n can be obtained from (1.6) as follows. If we denote

$$H^p_{\text{high}}(L_n) = \left(\Lambda^p V^*_{n-1,1}\right)^{\mathcal{E}_{n-1}},$$
$$H^p_{\text{low}}(L_n) = \left(\Lambda^{p-1} V^*_{n-1,1}\right)_{\mathcal{E}_{n-1}} \otimes \mathcal{E}^*_{n-1},$$

then

$$H^p(L_n)_\mu = H^p_{\text{high}}(L_n)_\mu \oplus H^p_{\text{low}}(L_n)_\mu.$$

Notice that $H^p_{\text{high}}(L_n)_\mu$ is the space of $\mathfrak{gl}_2$-highest weight vectors in $\Lambda^p V^*_{n-1,1}$ of weight μ and $H^p_{\text{low}}(L_n)_\mu$ is the space of $\mathfrak{gl}_2$-lowest weight vectors in $\Lambda^{p-1} V^*_{n-1,1}$ of weight $\mu + \mu_e$ (tensor $\mathcal{E}^*_{n-1}$).

Since μ is a highest weight of a $\mathfrak{gl}_2$-module if and only if $\sigma_1(\mu) + \mu_e$ is a lowest weight, we obtain that

$$H^p_{\text{high}}(L_n)_\mu \simeq H^{p+1}_{\text{low}}(L_n)_{\sigma_1(\mu)}. \tag{3.4}$$

Since $V^*_{n,1} \simeq V_{n,-1}$ and (2.2) one derives that

$$\begin{aligned} H^p_{\text{high}}(L_n)_\mu &\simeq H^{n+1-p}_{\text{low}}(L_n)_{\sigma_0(\mu)}, \\ H^p_{\text{low}}(L_n)_\mu &\simeq H^{n+1-p}_{\text{high}}(L_n)_{\sigma_0(\mu)}. \end{aligned} \tag{3.5}$$

We have proved the following theorem that describes the multiplicity of each $\mu \in P_n$ in $H^*(L_n)$.

Theorem 3.1. *The subset of $\mu \in P_n \subset \mathfrak{a}^*$ appearing in $H^*(L_n)$ is stable by σ_0 and σ_1 and Eqs. (3.5) and (3.4) hold. Given $\mu \in P_n$, then*

(1) $H^p_{\text{high}}(L_n)_\mu = 0$ unless $p = 0, \dots, n$, $\mu(I) = -p$ and $\mu(H) = j$ with $j \in \mathbb{Z}_{\geq 0}$. In this case

$$\dim H^p_{\text{high}}(L_n)_\mu = a_{n-1,p,j}.$$

(2) $H^p_{\text{low}}(L_n)_\mu = 0$ unless $p = 1, \dots, n+1$, $\mu(I) = -p+1$ and $\mu(H) = -j-2$ with $j \in \mathbb{Z}_{\geq 0}$. In this case

$$\dim H^p_{\text{low}}(L_n)_\mu = a_{n-1,p-1,j}.$$

The above theorem allows us to express the dimension of the cohomology groups in combinatorial terms as follows. We know from (2.1) that

$$\begin{aligned} \dim H^p_{\text{high}}(L_n) &= \sum_{j \geq 0} a_{n-1,p,j} \\ &= c_{n-1,p,0} + c_{n-1,p,1}, \end{aligned}$$

and similarly

$$\dim H^p_{\text{low}}(L_n) = c_{n-1,p-1,0} + c_{n-1,p-1,1}.$$

Recall that $c_{n-1,p,j}$ is the dimension of the H-eigenspace of eigenvalue j in $\Lambda^p V_{n-1}$ and hence it is the number of different ways to express j as a sum of exactly p different numbers from the set $\{n-1, n-3, \dots, 3-n, 1-n\}$. In particular, $c_{n-1,p,0} = 0$ (resp. $c_{n-1,p,1} = 0$) if $p(n-p)$ is odd (resp. even).

Given $a, b, c \in \mathbb{Z}_{\geq 0}$, with $b \geq a$ and $c \geq \frac{b(b+1)}{2}$, let $Q(a,b,c)$ be the number of partitions of c in exactly b different parts, not exceeding a. It is straightforward to see that, if $j \equiv p(n-p) \bmod 2$, then

$$c_{n-1,p,j} = Q\left(n, p, \tfrac{p(n+1)+j}{2}\right). \tag{3.6}$$

Corollary 3.2. *The dimension of the cohomology groups of L_n are given by*

$$\dim H^p_{\text{high}}(L_n) = Q\left(n, p, \left\lceil \tfrac{p(n+1)}{2} \right\rceil\right),$$
$$\dim H^p_{\text{low}}(L_n) = Q\left(n, p-1, \left\lceil \tfrac{(p-1)(n+1)}{2} \right\rceil\right).$$

We recall that $Q(a,b,c) = \mathcal{P}(a-b+1, b, c-\binom{b}{2})$, where $\mathcal{P}(x,y,z)$ is the number of partitions of z in exactly y parts, not exceeding x. For low cohomology degrees, the following corollary describes explicitly a basis of the cohomology groups (see also (2.4)). The dimension of $H^p(L_n)$, for $p = 0, 1, 2$, already appeared in [13].

Corollary 3.3. *If $p = 0, 1, 2$, then $H^p(L_n)$ is multiplicity free for all m and the weights appearing in it are given by the following table:*

	$p=0$	$p=1$	$p=2$
$H^p_{\text{high}}(L_n)$	$(0,0)$	$(n-1,-1)$	$(2n-4,-2), (2n-8,-2), \dots (1-(-1)^n, -2)$
$H^p_{\text{low}}(L_n)$		$(-2,0)$	$(-n-1,-1)$.

In particular,

$$\dim H^p(L_n) = \begin{cases} 1, & \text{if } p = 0; \\ 2, & \text{if } p = 1; \\ [\frac{n}{2}] + 1, & \text{if } p = 2. \end{cases}$$

Moreover, in terms of the basis

$$\{\mathcal{E}^*_{n-1}, v^*_{n-1}, v^*_{n-3}, \dots, v^*_{-n+3}, v^*_{-n+1}\}$$

of L^*_n, bases for the first cohomology groups $H^p(L_n)$, $p = 0, 1, 2$, are

$H^0(L_n)$: $\{1\}$,

$H^1(L_n)$: $\{v^*_{-n+1}\} \cup \{\mathcal{E}^*_{n-1}\}$,

$$H^2(L_n): \left\{ \sum_{i=0}^{j} (-1)^i v^*_{-n+1+2i} \wedge v^*_{-n+3+4j-2i} \right\}_{j=0}^{[\frac{n-2}{2}]} \cup \left\{ v^*_{n-1} \wedge \mathcal{E}^*_{n-1} \right\}.$$

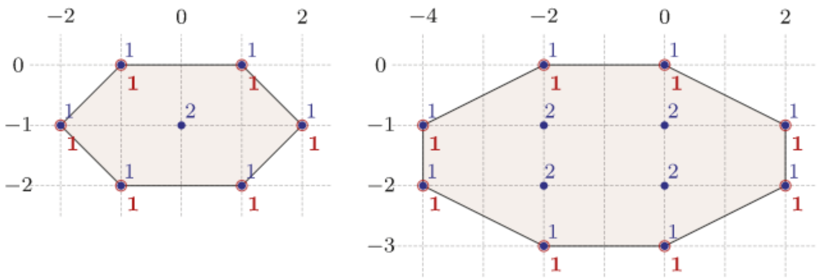
Remark 3.4. Any 2-cohomology class that, expressed in the basis given in the previous corollary has nonzero component in $v^*_{n-1} \wedge \mathcal{E}^*_{n-1}$, is affine (see [3]).

3.3. Diagrams of cohomology weights

We now present some pictures that describe the $\mathfrak{a}$-module structure of the cohomology of L_n for small n, illustrating our previous results.

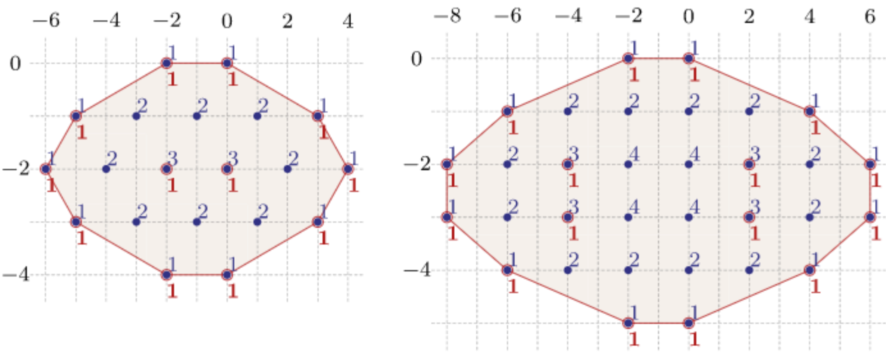
We indicate in bold the multiplicities in the cohomology of L_n (we keep the multiplicities in ΛL^*_n). The p-cohomology corresponds to the bold dots in row $-p$ on the right half side of the picture (these are highest) and the bold dots in row $-p+1$ on the left half side of the picture (these are lowest).

The first picture shows the multiplicities in the cohomology of L_2 and L_3, the two cases in which this filiform Lie algebra is a Borel subalgebra of a semisimple Lie algebra. As it is well known for these cases, the cohomology only appears in the vertices of P_n.

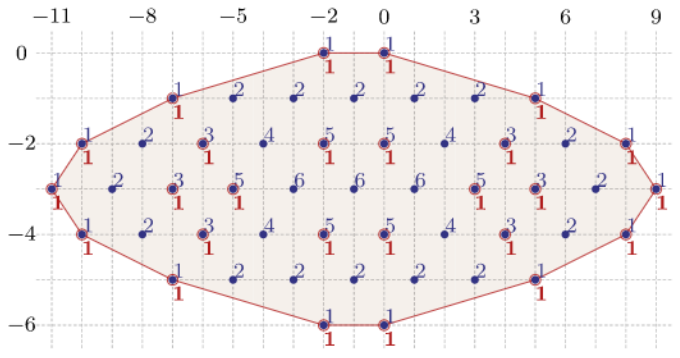


Weights in $H^*(L_n)$ for $n = 2, 3$.

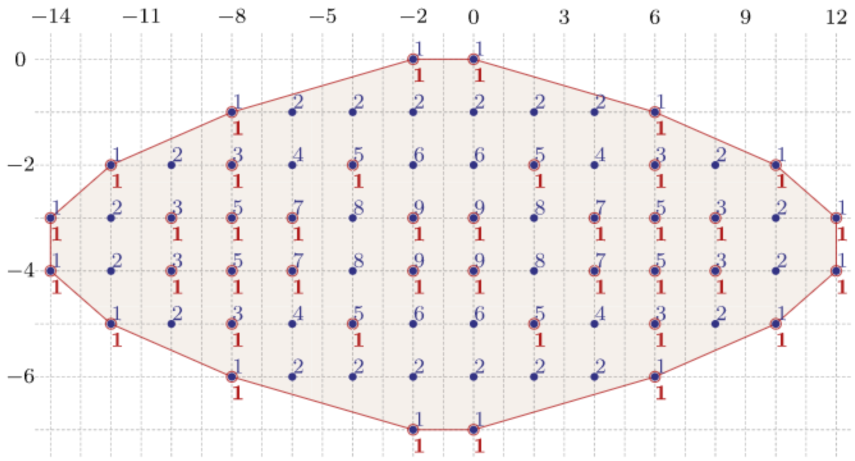
Next we show the multiplicities in the cohomology of L_n for $n = 4$ and 5. Now the cohomology also appears inside P_n.



Weights in $H^*(L_n)$ for $n = 4, 5$.

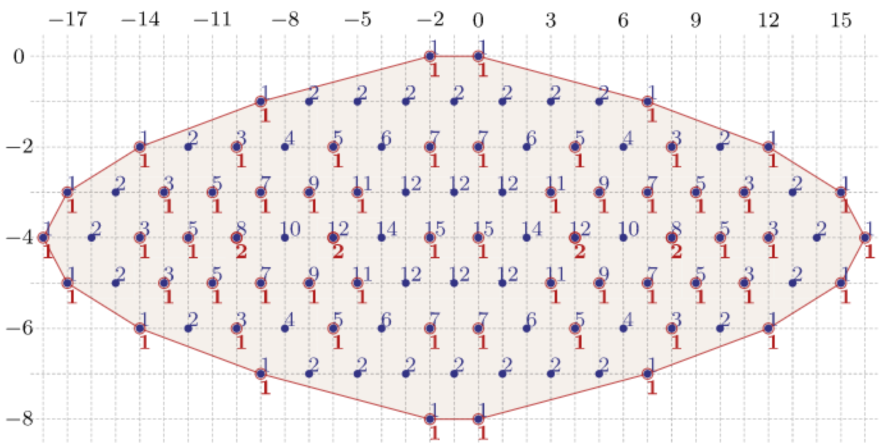


Weights in $H^*(L_6)$.



Weights in $H^*(L_7)$

In all the previous examples the cohomology is multiplicity free, but this is no longer true for $n \geq 8$.



Weights in $H^*(L_7)$

4. The cohomology of Q_n

4.1. *The Heisenberg Lie algebra $\mathfrak{h}_m$ as a nilradical in $\mathfrak{sp}_{m+1}$*

Let

$$X_i = E_{e_{m+1},e_{-i}} + E_{e_{-i},e_{m+1}}, \quad i = 1,\dots,m;$$

$$Y_i = E_{e_{m+1},e_i} + E_{e_i,e_{m+1}}, \quad i = 1,\dots,m;$$

$$Z = E_{e_{m+1},e_{m+1}}.$$

It is clear that these are root vectors of $\mathfrak{sp}_{m+1}$, and the roots of X_i, Y_i, Z are, respectively, $\epsilon_{m+1} - \epsilon_i$, $\epsilon_{m+1} + \epsilon_i$ and $2\epsilon_{m+1}$. Since

$$[X_i, Y_j] = 2\delta_{ij} Z,$$

it follows that

$$\mathfrak{h}_m = \langle X_1,\dots,X_m,Y_1,\dots,Y_m,Z\rangle,$$

is a Lie subalgebra of $\mathfrak{sp}_{m+1}$ isomorphic to the Heisenberg Lie algebra of dimension $m+1$. This presents the Heisenberg Lie algebra as the nilradical of the parabolic subalgebra $\mathfrak{p}_m$ of $\mathfrak{sp}_{m+1}$ described below

$$\mathfrak{p}_m = \begin{array}{|c|c|c|c|} \hline a & v & w & z \\ \hline 0 & A & B & \hat{w} \\ \hline 0 & C & -\hat{A} & -\hat{v} \\ \hline 0 & 0 & 0 & -a \\ \hline \end{array}, \qquad B = \hat{B},\ C = \hat{C}.$$

The structure of this parabolic subalgebra is $\mathfrak{p}_m = \mathfrak{l}_m \ltimes \mathfrak{h}_m$ with $\mathfrak{l}_m \simeq \langle H_{m+1}\rangle \oplus \mathfrak{sp}_m$. In particular, we can consider the $\mathfrak{sp}_m$-module $U_{2m,p}$ as $\mathfrak{l}_m$-module by defining a scalar action of H_{m+1}, say a and we denote this $\mathfrak{l}_m$-module by $U^a_{2m,p}$.

It follows from Kostant's description of the cohomology of the nilradicals of parabolic subalgebras [10] that $H^p(\mathfrak{h}_m)$ is an irreducible $\mathfrak{l}_m$-module of highest weight μ_p where (see also [4])

$$\mu_p = \begin{cases} 0, & \text{if } p = 0; \\ -p\epsilon_{m+1} + \epsilon_m + \dots + \epsilon_{m+1-p}, & \text{if } p = 1,\dots,m; \\ -(p+1)\epsilon_{m+1} + \epsilon_m + \dots + \epsilon_{p-m}, & \text{if } p = m+1,\dots,2m; \\ -(2m+2)\epsilon_{m+1}, & \text{if } p = 2m+1. \end{cases}$$

Taking into account the decomposition $\mathfrak{l}_m \simeq \langle H_{m+1}\rangle \oplus \mathfrak{sp}_m$, we can rephrase Kostant's result in this case as follows. A dominant weight of $\mathfrak{l}_m$ is determined by a pair (λ, a) where λ is a dominant weight of $\mathfrak{sp}_m$ and a is the scalar by which H_{m+1} acts. In this terms,

$$\mu_p = \begin{cases} (0,0), & \text{if } p = 0; \\ (\lambda_{m,p}, -p), & \text{if } p = 1, \dots, m; \\ (\lambda_{m,2m+1-p}, -p-1), & \text{if } p = m+1, \dots, 2m; \\ (0, -(2m+2)), & \text{if } p = 2m+1. \end{cases}$$

Therefore,

$$H^p(\mathfrak{h}_m) \simeq \begin{cases} U_{2m,p}^{-p}, & \text{if } p = 0, \dots, m; \\ U_{2m,2m+1-p}^{-p-1}, & \text{if } p = m+1, \dots, 2m+1 \end{cases} \tag{4.1}$$

as $\mathfrak{l}_m$-modules.

4.2. The filiform Lie algebra Q_{2m+1} as a 1-extension by $\mathfrak{h}_m$

We keep regarding $\mathfrak{h}_m$ as the nilradical of $\mathfrak{sp}_{m+1}$ described in Section 4.1. We will look at $\mathfrak{sp}_m$ as the Lie subalgebra of $\mathfrak{sp}_{m+1}$ induced by the inclusion $\mathcal{B}_m \subset \mathcal{B}_{m+1}$ (see 2.5). The elements

$$\mathcal{E}_{2m+1} = E_{e_1,e_1} + \sum_{i=1}^{m} E_{e_{i+1},e_{-i}} + E_{e_{-i},e_{i+1}}$$

$$\mathcal{E}_{2m-1} = E_{e_1,e_1} + \sum_{i=1}^{m-1} E_{e_{i+1},e_{-i}} + E_{e_{-i},e_{i+1}}$$

where also introduced in Section 2.3. Notice that $\mathcal{E}_{2m+1} = \mathcal{E}_{2m-1} + X_m$ and $\mathcal{E}_{2m-1} \in \mathfrak{l}_m$. It is straightforward to see that

$$\begin{aligned} Q_{2m+1} &\simeq \operatorname{ad}(\mathcal{E}_{2m+1}) \ltimes \mathfrak{h}_m, \\ &\simeq \operatorname{ad}(\mathcal{E}_{2m-1}) \ltimes \mathfrak{h}_m, \\ &\simeq \langle \mathcal{E}_{2m-1}\rangle + \mathfrak{h}_m \subset \mathfrak{sp}_{m+1}. \end{aligned}$$

In terms of matrices, the element $e\mathcal{E}_{2m-1} + \sum_{i=1}^m x_i X_i + \sum_{i=1}^m y_i Y_i + zZ$ is

$$
\begin{array}{|c|ccccc|ccccc|c|}
\hline
0 & x_m & x_{m-1} & \dots & x_2 & x_1 & -y_1 & -y_2 & \dots & -y_{m-1} & -y_m & -z \\
\hline
 & 0 & e & & & & & & & & & -y_m \\
 & & 0 & e & & & & & & & & -y_{m-1} \\
 & & & \ddots & \ddots & & & & & & & \vdots \\
 & & & & 0 & e & & & & & & -y_2 \\
 & & & & & 0 & -e & & & & & -y_1 \\
\hline
 & & & & & & 0 & -e & & & & -x_1 \\
 & & & & & & & 0 & -e & & & -x_2 \\
 & & & & & & & & \ddots & \ddots & & \vdots \\
 & & & & & & & & & 0 & -e & -x_{m-1} \\
 & & & & & & & & & & 0 & -x_m \\
\hline
 & & & & & & & & & & & 0 \\
\hline
\end{array}
\tag{4.2}
$$

and this depicts a faithful representation of Q_{2m+1}.

The Lie algebra Q_{2m+1} has rank 2, i.e., it has a 2-dimensional maximal torus of (semisimple) derivations. This torus can be chosen to be $\mathrm{ad}(\mathfrak{a})$, where

$$\mathfrak{a} = \langle \mathcal{H}_{2m-1}, H_{m+1} \rangle \subset \mathfrak{l}_m$$

and the actions on Q_{2m+1} are given by

$$
\begin{array}{r|cccc}
 & \mathcal{E}_{2m-1} & X_i & Y_i & Z \\
\hline
\mathrm{ad}(H_{m+1}) & 0 & X_i & Y_i & 2Z \\
\mathrm{ad}(\mathcal{H}_{2m-1}) & 2\mathcal{E}_{2m-1} & (2i-1)X_i & (-2i+1)Y_i & 0
\end{array}
\tag{4.3}
$$

This 2-dimensional torus $\mathfrak{a}$ of derivations of Q_{2m+1} acts on ΛQ_{2m+1}^* and on the cohomology of Q_{2m+1}. We will now describe the $\mathfrak{a}$-weights appearing on ΛQ_{2m+1}^* and in Section 4.4 we will do so for $H^*(Q_{2m+1})$.

4.3. The $\mathfrak{a}$-module structure of ΛQ_{2m+1}^*

The $\mathfrak{a}$-weights of Q_{2m+1} were given in table (4.3). Let

$$\mu_e, \mu_{x_m}, \dots, \mu_{x_1}, \mu_{y_1}, \dots, \mu_{y_m}, \mu_z \in \mathfrak{a}^*$$

be, respectively, the $\mathfrak{a}$-weights of $\mathcal{E}_{2m-1}$, $X_m, \dots, X_1$, $Y_1, \dots, Y_m$ and Z. Notice that $\{\frac{1}{2}\mu_e, \frac{1}{2}\mu_z\}$ is the dual basis of $\{\mathcal{H}_{2m-1}, H_{m+1}\}$.

If we identify an element $\mu \in \mathfrak{a}^*$ with the pair $(\mu(\mathcal{H}_{2m-1}), \mu(H_{m+1}))$, the weights of Q_{2m+1}, with the corresponding weight vectors, are:

$\mu(H_{m+1})$	$\mathcal{E}_{2m-1}$	X_m	$\ldots$	X_2	X_1	Z	Y_1	Y_2	$\ldots$	Y_m
0	$(2,0)$									
1		$(2m-1,1)$	$\ldots$	$(3,1)$	$(1,1)$		$(-1,1)$	$(-3,1)$	$\ldots$	$(1-2m,1)$
2						$(0,2)$				

The multiplicity of a weight $\mu \in \mathfrak{a}^*$ in ΛQ^*_{2m+1} is the number of ways in which $-\mu$ can be written as a sum of different weights of Q_{2m+1} (exactly p summands for the multiplicity in $\Lambda^p Q^*_{2m+1}$). Let

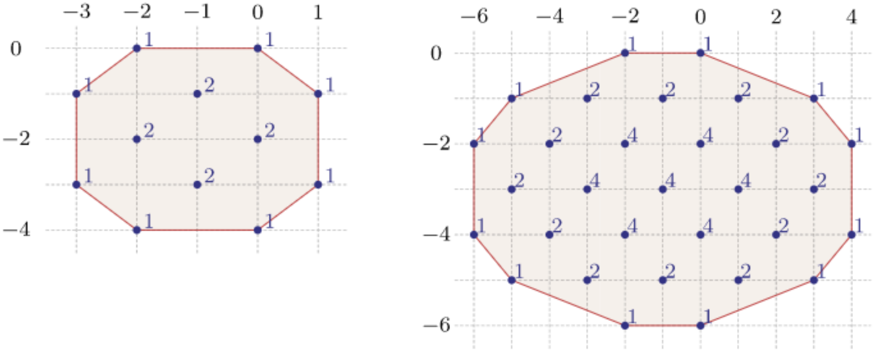
$$P_m = \{\mu \in \mathfrak{a}^* : \mu \text{ has positive multiplicity in } \Lambda Q^*_{2m+1}\}.$$

It is not difficult to see that P_m is a convex polygon with $4(m+1)$ edges. This polygon has two obvious symmetries:

$$\sigma_0(\mu) = -\mu_\Sigma - \mu \tag{4.4}$$

$$\sigma_1(\mu) = \big(-\mu(\mathcal{H}_{2m-1}), \mu(H_{m+1})\big) - (2,0) \tag{4.5}$$

where $\mu_\Sigma = \mu_e + \mu_z + \sum_{j=1}^m \mu_{x_j} + \sum_{j=1}^m \mu_{y_j}$ is the sum of all weights in Q_{2m+1}. The polygons P_m, for $m = 1, 2$ (corresponding to Q_3 and Q_5), are shown in the following pictures. We point out that Q_3 and Q_5 are isomorphic to the nilradicals of the Borel subalgebras of type B_2 and G_2 respectively.



Weights in ΛQ^*_{2m+1} for $m = 1, 2$.

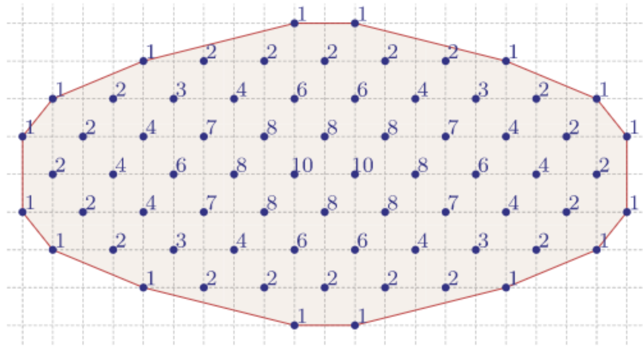
The vertices of P_m, are $\nu_0, \ldots, \nu_{2m+1}, \sigma_0(\nu_0), \ldots, \sigma_0(\nu_{2m+1})$, where the first ones are the negative of the sum of consecutive (with respect to the first coordinate) weights in Q_{2m+1}, that is

$$\nu_0 = 0,$$

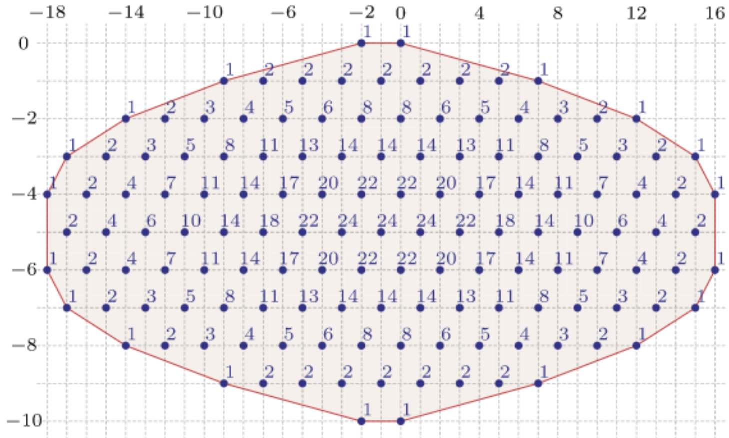
$$\nu_{k+1} = -\mu_e - \sum_{j=m+1-k}^{m} \mu_{x_j}, \quad k = 0, \ldots, m,$$

$$\nu_{m+k+1} = -\mu_e - \sum_{j=1}^{m} \mu_{x_j} - \sum_{j=1}^{k} \mu_{y_j}, \quad k = 1, \ldots, m.$$

The following pictures show the multiplicities of the weights appearing on ΛQ^*_{2m+1}, for $m = 3$ and 4.



Weights in $\Lambda^* Q_7$.



Weights in $\Lambda^* Q_9$.

Given $p = 0, \dots, 2m+2$, it is not difficult to read off, from these pictures, the $\mathfrak{a}$-weights appearing in $\Lambda^p Q^*_{2m+1}$, yet it is no automatic. Since μ_e and μ_z are the only weights of Q_{2m+1} whose second coordinate, $\mu(H_{m+1})$, is different from 1, the weights of $\Lambda^p Q^*_{2m+1}$ appear in the following three rows: $\mu(H_{m+1}) = j$, $j = p-1, p, p+1$.

In the next subsection we will describe which of these weights actually appear in the cohomology.

4.4. The $\mathfrak{a}$-module structure of $H^p(Q_{2m+1})$

We will regard $\mathfrak{a}$ as a Cartan subalgebra of $\tilde{\mathfrak{s}}_m$, where

$$\tilde{\mathfrak{s}}_m = \langle H_{m+1} \rangle \oplus \mathfrak{s}_m = \langle H_{m+1}, \mathcal{F}_{2m-1}, \mathcal{H}_{2m-1}, \mathcal{E}_{2m-1} \rangle.$$

It is clear that $\tilde{\mathfrak{s}}_m$ is a Lie subalgebra of $\mathfrak{l}_m \subset \mathfrak{sp}_{m+1}$ which is isomorphic to $\mathfrak{gl}_2$. Given $\mu \in P_m \subset \mathfrak{a}^*$, let $H^p(Q_{2m+1})_\mu$ be the $\mathfrak{a}$-isotopic component of $H^p(Q_{2m+1})$ of weight μ.

It follows from (1.6) that

$$H^p(Q_{2m+1}) = H^p(\mathfrak{h}_m)^{\mathcal{E}_{2m-1}} \oplus H^{p-1}(\mathfrak{h}_m)_{\mathcal{E}_{2m-1}} \otimes \mathcal{E}^*_{2m-1}.$$

This implies that $H^p(Q_{2m+1})_\mu$ is the space of $\tilde{\mathfrak{s}}_m$-highest weight vectors in $H^p(\mathfrak{h}_m)$ of weight μ and the space of $\tilde{\mathfrak{s}}_m$-lowest weight vectors in $H^{p-1}(\mathfrak{h}_m)$ of weight $\mu + \mu_e$. Let us denote

$$\begin{aligned} H^p_{\text{high}}(Q_{2m+1}) &= H^p(\mathfrak{h}_m)^{\mathcal{E}_{2m-1}}, \\ H^p_{\text{low}}(Q_{2m+1}) &= H^{p-1}(\mathfrak{h}_m)_{\mathcal{E}_{2m-1}} \otimes \mathcal{E}^*_{2m-1}. \end{aligned}$$

Noticing that μ is a highest weight of an $\tilde{\mathfrak{s}}_m$-module if and only if $\sigma_1(\mu) + \mu_e$ is a lowest weight, we obtain that

$$H^p_{\text{high}}(Q_{2m+1})_\mu \simeq H^{p+1}_{\text{low}}(Q_{2m+1})_{\sigma_1(\mu)}. \tag{4.6}$$

Now recall from (4.1) that

$$H^p(\mathfrak{h}_m) \simeq \begin{cases} U^{-p}_{2m,p}, & \text{if } p = 0, \dots, m; \\ U^{-p-1}_{2m,2m+1-p}, & \text{if } p = m+1, \dots, 2m+1; \end{cases}$$

as $\mathfrak{l}_m$-modules. This decomposition shows that

$$\begin{aligned} H^p_{\text{high}}(Q_{2m+1})_\mu &\simeq H^{2m+2-p}_{\text{low}}(Q_{2m+1})_{\sigma_0(\mu)}, \\ H^p_{\text{low}}(Q_{2m+1})_\mu &\simeq H^{2m+2-p}_{\text{high}}(Q_{2m+1})_{\sigma_0(\mu)}. \end{aligned} \tag{4.7}$$

Combining these results with (2.7) we obtain the following theorem.

Theorem 4.1. *The subset of $\mu \in P_m \subset \mathfrak{a}^*$ appearing in $H^*(Q_{2m+1})$ is stable by σ_0 and σ_1 and Eqs. (4.7) and (4.6) hold.*

(1) $H^p_{\text{high}}(Q_{2m+1})_\mu = 0$ unless one of the following two cases holds:
   (i) $p = 0, \dots, m$, $\mu(H_{m+1}) = -p$ and $\mu(\mathcal{H}_{2m+1}) = -j$ with $j \in \mathbb{Z}_{\geq 0}$; and in this case

$$\dim H^p_{\text{high}}(Q_{2m+1})_\mu = a_{2m-1,p,j} - a_{2m-1,p-2,j}.$$

   (ii) $p = m+1, \dots, 2m+1$, $\mu(H_{m+1}) = -p-1$ and $\mu(\mathcal{H}_{2m+1}) = -j$ with $j \in \mathbb{Z}_{\geq 0}$; and in this case

$$\dim H^p_{\text{high}}(Q_{2m+1})_\mu = a_{2m-1,p-1,j} - a_{2m-1,p+1,j}.$$

(2) $H^p_{\mathrm{low}}(Q_{2m+1})_\mu = 0$ *unless one of the following two cases holds:*

(i) $p = 1, \ldots, m+1$, $\mu(H_{m+1}) = -p+1$ *and* $\mu(\mathcal{H}_{2m+1}) = -j-2$ *with* $j \in \mathbb{Z}_{\geq 0}$*; and in this case*

$$\dim H^p_{\mathrm{low}}(Q_{2m+1})_\mu = a_{2m-1,p-1,j} - a_{2m-1,p-3,j}.$$

(ii) $p = m+2, \ldots, 2m+2$, $\mu(H_{m+1}) = -p$ *and* $\mu(\mathcal{H}_{2m+1}) = -j-2$ *with* $j \in \mathbb{Z}_{\geq 0}$*; and in this case*

$$\dim H^p_{\mathrm{low}}(Q_{2m+1})_\mu = a_{2m-1,p-2,j} - a_{2m-1,p,j}.$$

This theorem allows us to express the dimension of the cohomology groups in combinatorial terms as follows. As we did after Theorem 3.1, in this case we obtain:

$$\dim H^p_{\mathrm{high}}(Q_{2m+1}) = \begin{cases} c_{2m-1,p,0} - c_{2m-1,p-2,0}, & 0 \leq p \leq m,\ p \text{ even}; \\ c_{2m-1,p,1} - c_{2m-1,p-2,1}, & 0 \leq p \leq m,\ p \text{ odd}; \\ c_{2m-1,p-1,0} - c_{2m-1,p+1,0}, & m+1 \leq p \leq 2m+1,\ p \text{ even}; \\ c_{2m-1,p-1,1} - c_{2m-1,p+1,1}, & m+1 \leq p \leq 2m+1,\ p \text{ odd}; \end{cases}$$

and

$$\dim H^p_{\mathrm{low}}(Q_{2m+1}) = \begin{cases} c_{2m-1,p-1,1} - c_{2m-1,p-3,1}, & 1 \leq p \leq m+1,\ p \text{ even}; \\ c_{2m-1,p-1,0} - c_{2m-1,p-3,0}, & 1 \leq p \leq m+1,\ p \text{ odd}; \\ c_{2m-1,p-2,1} - c_{2m-1,p,1}, & m+2 \leq p \leq 2m+2,\ p \text{ even}; \\ c_{2m-1,p-2,0} - c_{2m-1,p,0}, & m+2 \leq p \leq 2m+2,\ p \text{ odd}. \end{cases}$$

Corollary 4.2. *If* $p = 0, 1, 2$*, then* $H^p(Q_{2m+1})$ *is multiplicity free for all* m *and the weights appearing in it are given by the following table:*

	$p=0$	$p=1$	$p=2$
$H^p_{\mathrm{high}}(Q_{2m+1})$	$(0,0)$	$(2m-1,-1)$	$(4m-4,-2),(4m-8,-2),\ldots(4,-2)$
$H^p_{\mathrm{low}}(Q_{2m+1})$		$(-2,0)$	$(-2m-1,-1)$.

In particular,

$$\dim H^p(Q_{2m+1}) = \begin{cases} 1, & \text{if } p = 0; \\ 2, & \text{if } p = 1; \\ m, & \text{if } p = 2. \end{cases}$$

Moreover, in terms of the basis

$$\{\mathcal{E}^*_{2m-1}, X^*_m, \dots, X^*_1, Y^*_1, \dots, Y^*_m, Z^*\}$$

of Q^*_{2m+1}, *dual to the basis* $\{\mathcal{E}_{2m-1}, X_m, \dots, X_1, Y_1, \dots, Y_m, Z\}$, *bases for the first cohomology groups* $H^p(Q_{2m+1})$, $p = 0,1,2$, *are*

$$\begin{aligned}
&H^0(Q_{2m+1}) : \{1\},\\
&H^1(Q_{2m+1}) : \{Y^*_m\} \cup \{\mathcal{E}^*_{2m-1}\},\\
&H^2(Q_{2m+1}) : \{X^*_m \wedge \mathcal{E}^*_{2m-1}\} \cup \left\{\sum_{i=0}^{j} (-1)^i Y^*_{m-i} \wedge Y^*_{m-1-2j+i}\right\}_{j=0}^{[\frac{m-2}{2}]}\\
&\qquad \cup \left\{\sum_{i=0}^{2j-m+1} (-1)^i Y^*_{m-i} \wedge X^*_{-m+2+2j-i}\right.\\
&\qquad \left. + \sum_{i=2j+2-m}^{j} (-1)^i Y^*_{m-i} \wedge Y^*_{m-1-2j+i}\right\}_{j=[\frac{m}{2}]}^{m-2}.
\end{aligned}$$

Remark 4.3. Any 2-cohomology class that, expressed in the basis given in the previous corollary has nonzero component in $X^*_m \wedge \mathcal{E}^*_{2m-1}$, is affine (see [3]).

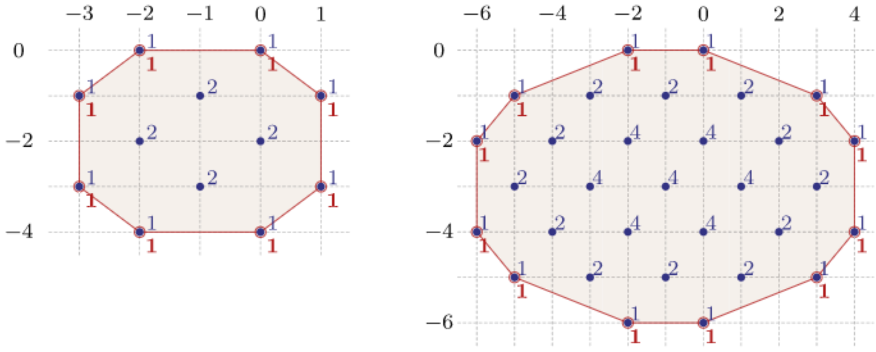
4.5. Diagrams of cohomology weights

We now present some pictures that describe the $\mathfrak{a}$-module structure of the cohomology of Q_n, for small n, illustrating our previous results.

We indicate in bold the multiplicities in the cohomology of Q_{2m+1} (we keep the multiplicities in ΛQ^*_{2m+1}). The p-cohomology can be described as follows:

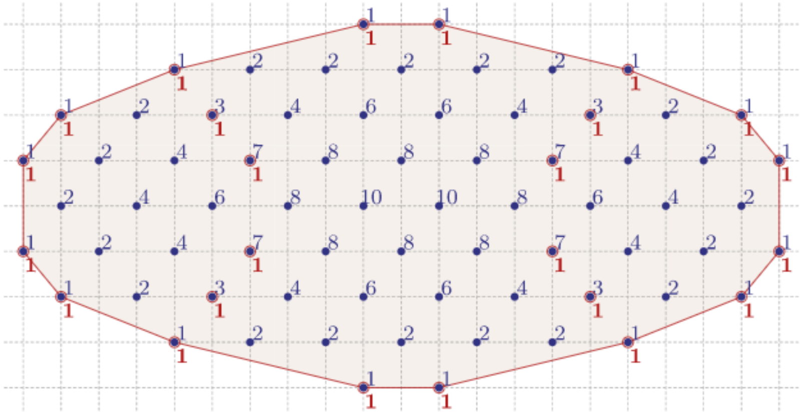
- If $p = 0, \dots, m$, then the p-cohomology corresponds to the bold dots in row $-p$ on the right half side of the picture (these are highest) and the bold dots in row $-p+1$ on the left half side of the picture (these are lowest).
- If $p = m+1$, then the p-cohomology corresponds to the bold dots in row $-m-1$ on the right half side of the picture (these are highest) and the bold dots in row $-m+1$ on the left half side of the picture (these are lowest).
- If $p = m+2, \dots, 2m+2$, then the p-cohomology corresponds to the bold dots in row $-p-1$ on the right half side of the picture (these are highest) and the bold dots in row $-p$ on the left half side of the picture (these are lowest).

This first picture shows the multiplicities in the cohomology of Q_3 and Q_5, the two cases in which this filiform Lie algebra is a Borel subalgebra of a semisimple Lie algebra. As we know for these cases, the cohomology only appears in the vertices of P_m.

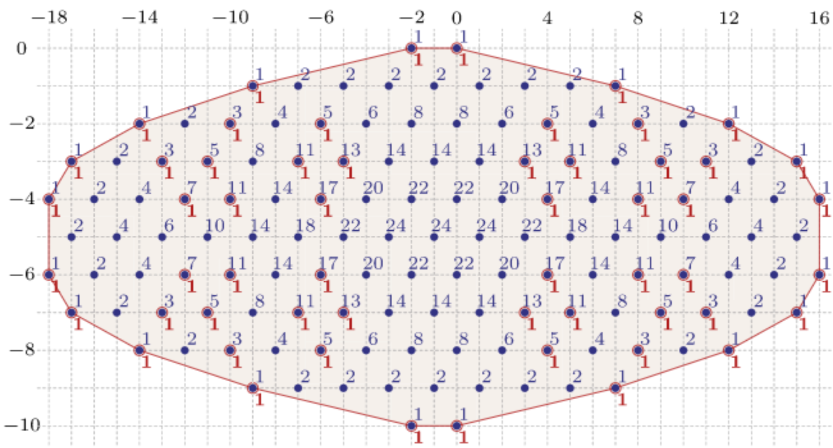


Weights in $H^*(Q_{2m+1})$ for $m = 1, 2$.

Next we show the multiplicities in the cohomology of Q_m for $m = 3$ and 4. Now the cohomology also appears inside P_m.

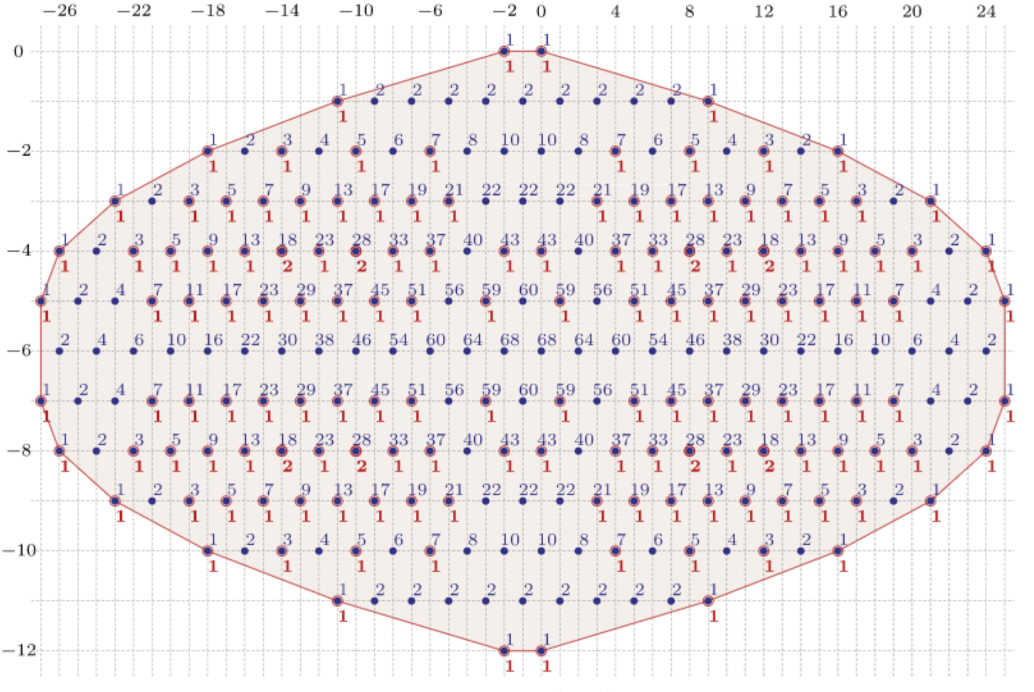


Weights in $H^*(Q_7)$.



Weights in $H^*(Q_9)$.

In all the previous examples the cohomology is multiplicity free, but this is no longer true for $m \geq 5$.



Weights in $H^*(Q_{11})$.

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