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More on the Terwilliger algebra of Johnson schemes

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ABSTRACT

In Levstein and Maldonado (2007), the Terwilliger algebra of the Johnson scheme $J(n, d)$ was determined when $n \geq 3d$. In this paper, we determine the Terwilliger algebra $\mathcal{T}$ for the remaining case $2d \leq n < 3d$.

1. Introduction

Let $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$ denote a commutative association scheme, where X is a finite set. Suppose $\mathrm{Mat}_X(\mathbb{C})$ denotes the algebra over $\mathbb{C}$ consisting of all matrices whose rows and columns are indexed by X. For each i, let A_i denote the binary matrix in $\mathrm{Mat}_X(\mathbb{C})$ whose (x, y)-entry is 1 if and only if $(x, y) \in R_i$. We call A_i the ith *adjacency matrix* of $\mathfrak{X}$. We abbreviate $A = A_1$, and call it the *adjacency matrix* of $\mathfrak{X}$. The subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ spanned by $A_0, A_1, \ldots, A_d$ is called the *Bose–Mesner algebra* of $\mathfrak{X}$, denoted by $\mathcal{B}$. Since $\mathcal{B}$ is commutative and generated by real symmetric matrices, it has a basis consisting of primitive idempotents, denoted by $E_0 = \frac{1}{|X|}J, E_1, E_2, \ldots, E_d$. For each $i \in \{0, 1, \ldots, d\}$, write

$$A_i = \sum_{j=0}^{d} p_i(j)E_j, \quad E_i = \frac{1}{|X|}\sum_{j=0}^{d} q_i(j)A_j.$$

The scalars $p_i(j)$ and $q_i(j)$ are called the *eigenvalues* and the *dual eigenvalues* of $\mathfrak{X}$, respectively.

Fix $x \in X$. For $0 \leq i \leq d$, let E_i^* denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ whose (y, y)-entry is defined by

$$(E_i^*)_{yy} = \begin{cases} 1, & \text{if } (x, y) \in R_i, \\ 0, & \text{otherwise.} \end{cases}$$

The subalgebra $\mathcal{T}(x)$ of $\mathrm{Mat}_X(\mathbb{C})$ generated by $A_0, A_1, \ldots, A_d; E_0^*, E_1^*, \ldots, E_d^*$ is called the *Terwilliger algebra* of $\mathfrak{X}$ with respect to x.

Terwilliger [12] first introduced the Terwilliger algebra of association schemes, which is an important tool in considering the structure of an association scheme. For more information, see [4,5,13,14]. The Terwilliger algebra is a finite-dimensional semisimple $\mathbb{C}$-algebra; it is difficult to determine its structure in general. The structures of the Terwilliger algebras of some

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association schemes have been determined; see [1,3] for group schemes, [15] for strongly regular graphs, [7,11] for Hamming schemes, [10] for Johnson schemes, [8] for odd graphs, and [9] for incidence graphs of Johnson geometry.

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$ and $\binom{[n]}{d}$ denote the collection of all d-element subsets of $[n]$. For $0 \le i \le d$, define $R_i = \{(x, y) \in \binom{[n]}{d} \times \binom{[n]}{d} \mid |x \cap y| = d - i\}$. Then $(\binom{[n]}{d}, \{R_i\}_{i=0}^{d})$ is a symmetric association scheme of class d, which is called the *Johnson scheme*, denoted by $J(n, d)$. Note that $J(n, d)$ is isomorphic to $J(n, n-d)$. So we always assume that $n \ge 2d$.

Since the automorphism group of $J(n, d)$ acts transitively on $\binom{[n]}{d}$, the isomorphism class of the Terwilliger algebra $\mathcal{T}(x)$ of $J(n, d)$ is independent of the choice of x in $\binom{[n]}{d}$. We will denote $\mathcal{T} := \mathcal{T}(x)$.

In [10], the Terwilliger algebra of the Johnson scheme $J(n, d)$ was determined when $n \ge 3d$. In this paper, we focus on the remaining case, and determine the Terwilliger algebra $\mathcal{T}$ of $J(n, d)$. In Section 2, we introduce intersection matrices and some useful identities. In Section 3, two families of subalgebras $\mathcal{M}^{(n,d)}$ and $\mathcal{N}$ of $\mathrm{Mat}_X(\mathbb{C})$ are constructed. In the last two sections, we show that $\mathcal{T} = \mathcal{M}^{(n,d)}$ when $2d < n < 3d$, and $\mathcal{T} = \mathcal{N}$ when $n = 2d$.

2. Intersection matrix

In this section we first introduce some useful identities for intersection matrices, then describe the adjacency matrix of the Johnson scheme $J(n, d)$ in terms of intersection matrices.

Let V be a set of cardinality v. Let $H_{i,j}^{r}(v)$ be a binary matrix whose rows and columns are indexed by the elements of $\binom{V}{i}$ and $\binom{V}{j}$ respectively, whose $\alpha_i\alpha_j$-entry is 1 if and only if $|\alpha_i \cap \alpha_j| = r$. We call $H_{i,j}^{r}(v)$ an intersection matrix. For simplicity, write $H_{i,j} := H_{i,j}^{\min(i,j)}$. Now we introduce some useful identities for intersection matrices.

Lemma 2.1 (*[6, Theorem 3]*)**.** *For $0 \le l \le \min(i, j)$ and $0 \le s \le \min(j, k)$,*

$$H_{i,j}^{l}(v)H_{j,k}^{s}(v) = \sum_{g=0}^{\min(i,k)} \left(\sum_{h=0}^{g} \binom{g}{h}\binom{i-g}{l-h}\binom{k-g}{s-h}\binom{v+g-i-k}{j+h-l-s} \right) H_{i,k}^{g}(v).$$

Lemma 2.2 (*[10, Lemma 4.5]*)**.** *Let v be a positive integer.*

(i) *For $0 \le i \le j \le l \le v$,*

$$H_{i,j}(v)H_{j,l}(v) = \binom{l-i}{l-j} H_{i,l}.$$

(ii) *For $0 \le \max(i, l) \le j \le v$,*

$$H_{i,j}(v)H_{j,l}(v) = \sum_{m=0}^{j-\max(i,l)} \binom{v-\max(i,l)-m}{j-\max(i,l)-m} H_{i,l}^{\min(i,l-m)}(v).$$

(iii) *For $0 \le j \le \min(i, l) \le v$,*

$$H_{i,j}(v)H_{j,l}(v) = \sum_{m=0}^{\min(i,l)-j} \binom{\min(i,l)-m}{j} H_{i,l}^{\min(i,l-m)}(v).$$

Pick $x \in \binom{[n]}{d}$. For $0 \le i \le d$, write $\Omega_i := \{y \in \binom{[n]}{d} \mid |x \cap y| = d - i\}$. Then we have the partition $\binom{[n]}{d} = \dot{\bigcup}_{i=0}^{d} \Omega_i$. Now we consider the mth adjacency matrix A_m of $J(n, d)$ as a block matrix with respect to this partition. Denote $(A_m)_{|\Omega_i \times \Omega_j}$ the submatrix of A_m with rows indexed by Ω_i and columns indexed by Ω_j.

In the remainder of this paper, we always assume that $I^{(v,k)}$ denotes the identity matrix of size $\binom{v}{k}$ and $A_m^{(v,k)}$ denotes the mth adjacency matrix of $J(v, k)$. In fact $A_m^{(v,k)} = H_{k,k}^{k-m}(v)$.

Lemma 2.3 (*[10, Lemmas 3.1, 3.5]*)**.** *Let A denote the adjacency matrix of $J(n, d)$. For $0 \le i < j \le d$, we have*

$$A_{|\Omega_i \times \Omega_i} = I^{(d,d-i)} \otimes A^{(n-d,i)} + A^{(d,d-i)} \otimes I^{(n-d,i)},$$
$$A_{|\Omega_i \times \Omega_{i+1}} = H_{d-i,d-i-1}(d) \otimes H_{i,i+1}(n-d),$$
$$A_{|\Omega_i \times \Omega_j} = 0, \quad \text{if } j \ge i+2,$$

where "⊗" denotes the Kronecker product of matrices.

3. Two algebras

Let $2d \leq n$ and $X = \binom{[n]}{d}$. Fix $x \in X$. In this section we shall construct two subalgebras $\mathcal{M}^{(n,d)}$ and $\mathcal{N}$ of $\mathrm{Mat}_X(\mathbb{C})$, which are in fact the Terwilliger algebras $\mathcal{T} = \mathcal{T}(x)$ of $J(n, d)$ in the cases where $2d < n < 3d$ and $J(2d, d)$, respectively. Hereafter the ground set of all matrices $H^l_{p,q}(d)$ in front of "⊗" is x and that of $H^l_{p,q}(n-d)$ behind "⊗" is $[n] \setminus x$.

Let $\mathcal{B}^{(v,k)}$ denote the Bose–Mesner algebra of $J(v, k)$, and $\{E_r^{(v,k)}\}_{r=0}^{\min(k,v-k)}$ denote its primitive idempotents. Let $p_i^{(v,k)}(j)$ be the eigenvalue of $A_i^{(v,k)}$ satisfying $A_i^{(v,k)} E_j^{(v,k)} = p_i^{(v,k)}(j) E_j^{(v,k)}$.

For $i, j = 0, 1, \ldots, d$, define the vector space

$$M_{i,j}^{(n,d)} = (\mathcal{B}^{(d,d-i)} H_{d-i,d-j}(d)) \otimes (\mathcal{B}^{(n-d,i)} H_{i,j}(n-d)).$$

Let $L : Y \longmapsto L(Y)$ be a map from $\bigcup_{i,j=0}^{d} M_{i,j}^{(n,d)}$ to $\mathrm{Mat}_X(\mathbb{C})$ such that for any $Y \in M_{i,j}^{(n,d)}$,

$$(L(Y))_{|\Omega_l \times \Omega_m} = \begin{cases} Y, & \text{if } l = i \text{ and } m = j, \\ 0, & \text{otherwise.} \end{cases}$$

Define

$$\mathcal{M}^{(n,d)} = \bigoplus_{i,j=0}^{d} L(M_{i,j}^{(n,d)}). \tag{1}$$

Lemma 3.1 *([10, Lemma 4.4]). Let $R(v, k, h) := \{r \mid H^r_{k,h}(v) \neq 0\}$. Then $\mathcal{B}^{(v,k)} H_{k,h}(v) = H_{k,h}(v) \mathcal{B}^{(v,h)} = \langle \{H^r_{k,h}(v)\}_{r \in R(v,k,h)} \rangle$.*

Lemma 3.1 implies that $H^r_{d-i,d-j}(d) \otimes H^s_{i,j}(n-d)$, $r \in R(d, d-i, d-j)$, $s \in R(n-d, i, j)$ is a basis of $M_{i,j}^{(n,d)}$. By Lemma 2.1, we observe that $\mathcal{M}^{(n,d)}$ is an algebra.

By Lemmas 2.3 and 3.1, the adjacency matrix A of $J(n, d)$ belongs to $\mathcal{M}^{(n,d)}$. Since each mth adjacency matrix of $J(n, d)$ may be written as a polynomial of A, one gets $\mathcal{B}^{(n,d)} \subseteq \mathcal{M}^{(n,d)}$. The fact that $E_m^* = L(H_{d-m,d-m}(d) \otimes H_{m,m}(n-d)) \in \mathcal{M}^{(n,d)}$ implies that $\mathcal{T}$ is a subalgebra of $\mathcal{M}^{(n,d)}$.

Next we construct another algebra $\mathcal{N}$. For $i, j = 0, 1, \ldots, d$, let $N_{i,j}$ be the vector space generated by

$$(E_r^{(d,d-i)} \otimes E_s^{(d,i)} + E_s^{(d,d-i)} \otimes E_r^{(d,i)})(H_{d-i,d-j}(d) \otimes H_{i,j}(d)), \quad 0 \leq r \leq s \leq \min(d, d-i). \tag{2}$$

Define

$$\mathcal{N} = \bigoplus_{i,j=0}^{d} L(N_{i,j}). \tag{3}$$

Observe that $\mathcal{N} \subseteq \mathcal{M}^{(2d,d)}$.

Lemma 3.2. *Each vector space $N_{i,j}$ has the basis*

$$H^{d-i-j+g}_{d-i,d-j}(d) \otimes H^h_{i,j}(d) + H^{d-i-j+h}_{d-i,d-j}(d) \otimes H^g_{i,j}(d), \quad g, h \in R(d, i, j),\ g \leq h. \tag{4}$$

Proof. By Lemma 3.1 we have that $\mathcal{B}^{(v,k)} H_{k,h}(v) = H_{k,h}(v) \mathcal{B}^{(v,h)} = \langle \{H^r_{k,h}(v)\}_{r \in R(v,k,h)} \rangle$.

Also note that it holds that

$$H^{d-i-j+g}_{d-i,d-j}(d) \otimes H^l_{i,j}(d) + H^{d-i-j+l}_{d-i,d-j}(d) \otimes H^g_{i,j}(d) = H^g_{i,j}(d) \otimes H^l_{i,j}(d) + H^l_{i,j}(d) \otimes H^g_{i,j}(d),$$

and that $\{H^g_{i,j}(d) \otimes H^h_{i,j}(d)\}_{g,h \in R(i,j)}$ are linearly independent, which implies that $\{H^g_{i,j}(d) \otimes H^h_{i,j}(d) + H^h_{i,j}(d) \otimes H^g_{i,j}(d)\}_{g,h \in R(d,i,j), g \leq h}$ are linearly independent. Hence, the desired result follows. □

Theorem 3.3. *Let $\mathcal{N}$ be as in (3). Then $\mathcal{N}$ is an algebra.*

Proof. It suffices to show that $N_{i,j} N_{j,k} \subseteq N_{i,k}$. By Lemma 2.1 there exist scalars $\alpha^g_{l,s}$ such that

$$H^l_{i,j}(d) H^s_{j,k}(d) = \sum_{g=0}^{\min(i,k)} \alpha^g_{l,s}\, H^g_{i,k}(d),$$

which implies that

$$\begin{aligned} &(H^m_{i,j}(d) \otimes H^n_{i,j}(d) + H^n_{i,j}(d) \otimes H^m_{i,j}(d))(H^s_{j,k}(d) \otimes H^t_{j,k}(d) + H^t_{j,k}(d) \otimes H^s_{j,k}(d)) \\ &\quad = \sum_{g=0}^{\min(i,k)} \sum_{l=0}^{\min(i,k)} (\alpha^g_{m,s}\, \alpha^l_{n,t} + \alpha^g_{m,t}\, \alpha^l_{n,s})(H^g_{i,k}(d) \otimes H^l_{i,k}(d) + H^l_{i,k}(d) \otimes H^g_{i,k}(d)), \end{aligned}$$

as desired. □

Lemma 3.4. *Let $\mathcal{T}$ be the Terwilliger algebra of $J(2d, d)$. Then $\mathcal{T}$ is a subalgebra of $\mathcal{N}$.*

Proof. Observe that $A_{|\Omega_i\times\Omega_{i+l}} = 0$ for $l \geq 2$. By Lemma 2.3, we obtain $A_{|\Omega_i\times\Omega_{i+1}} = H_{d-i,d-i-1}(d) \otimes H_{i,i+1}(d)$, and

$$\begin{aligned} A_{|\Omega_i\times\Omega_i} &= I^{(d,d-i)} \otimes A^{(d,i)} + A^{(d,d-i)} \otimes I^{(d,i)} \\ &= \sum_{r=0}^{\min(i,d-i)} \sum_{s=0}^{\min(i,d-i)} (p_1^{(d,i)}(r) + p_1^{(d,i)}(s))(E_r^{(d,d-i)} \otimes E_s^{(d,i)}) \\ &= \sum_{r=0}^{\min(i,d-i)} \sum_{s=0}^{\min(i,d-i)} \frac{p_1^{(d,i)}(r) + p_1^{(d,i)}(s)}{2} (E_r^{(d,d-i)} \otimes E_s^{(d,i)} + E_s^{(d,d-i)} \otimes E_r^{(d,i)}). \end{aligned}$$

Then $A \in \mathcal{N}$. Also we have that $E_m^* = L(H_{d-m,d-m}(d) \otimes H_{m,m}(d)) \in \mathcal{N}$, so $\mathcal{T} \subseteq \mathcal{N}$. □

We next introduce two mappings proposed in [10]: the lift map denoted by $\mathcal{L}_i$ and the pullback map denoted by $\mathcal{P}_i$.

For $0 \leq i < d$, define $\mathcal{L}_i$ to be the linear mapping from $M_{i,i}^{(n,d)}$ to $M_{i+1,i+1}^{(n,d)}$ satisfying

$$\mathcal{L}_i(E_r^{(d,d-i)} \otimes E_s^{(n-d,i)}) = (H_{d-i-1,d-i}(d)E_r^{(d,d-i)}H_{d-i,d-i-1}(d)) \otimes (H_{i+1,i}(n-d)E_s^{(n-d,i)}H_{i,i+1}(n-d));$$

for $0 < i \leq d$, define $\mathcal{P}_i$ to be the linear mapping from $M_{i,i}^{(n,d)}$ to $M_{i-1,i-1}^{(n,d)}$ satisfying

$$\mathcal{P}_i(E_r^{(d,d-i)} \otimes E_s^{(n-d,i)}) = (H_{d-i+1,d-i}(d)E_r^{(d,d-i)}H_{d-i,d-i+1}(d)) \otimes (H_{i-1,i}(n-d)E_s^{(n-d,i)}H_{i,i-1}(n-d)).$$

Note that the lift map is defined by premultiplying by $\big(H_{d-i-1,d-i}(d) \otimes H_{i+1,i}(n-d)\big)$ (that is equal to $A_{|\Omega_{i+1}\times\Omega_i}$ by Lemma 2.3) and post multiplying by $\big(H_{d-i,d-i-1}(d) \otimes H_{i,i+1}(n-d)\big)$ (that is equal to $A_{|\Omega_i\times\Omega_{i+1}}$ by Lemma 2.3). Since they belong to $\mathcal{T}_{|\Omega_{i+1}\times\Omega_i}$ and $\mathcal{T}_{|\Omega_i\times\Omega_{i+1}}$ respectively and since $\mathcal{T}$ is an algebra, then $\mathcal{L}_i(Y) \in \mathcal{T}_{|\Omega_{i+1}\times\Omega_{i+1}}$ for any $Y \in \mathcal{T}_{|\Omega_i\times\Omega_i}$. Similarly $\mathcal{P}_i(Y) \in \mathcal{T}_{|\Omega_{i-1}\times\Omega_{i-1}}$ for any $Y \in \mathcal{T}_{|\Omega_i\times\Omega_i}$.

By [2, p. 220], for $0 \leq j \leq k$,

$$p_1^{(v,k)}(j) = (k-j)(v-k-j) - j, \qquad p_k^{(v,k)}(j) = (-1)^j \binom{v-k-j}{k-j}. \tag{5}$$

Write $l_{v,k,j} = v - k + p_1^{(v,k)}(j)$ and $p_{v,k,j} = k + p_1^{(v,k)}(j)$.

Lemma 3.5 (*[10, Lemma 5.6]*).

$$\mathcal{L}_i(E_r^{(d,d-i)} \otimes E_s^{(n-d,i)}) = p_{d,d-i,r}\; l_{n-d,i,s}\; E_r^{(d,d-i-1)} \otimes E_s^{(n-d,i+1)},$$
$$\mathcal{P}_i(E_r^{(d,d-i)} \otimes E_s^{(n-d,i)}) = l_{d,d-i,r}\; p_{n-d,i,s}\; E_r^{(d,d-i+1)} \otimes E_s^{(n-d,i-1)}.$$

Here, $E_j^{(v,k)} = 0$ if $j > \min(k, v-k)$.

Corollary 3.6 (*[10, Corollary 5.7]*).

$$(E_r^{(d,d-i)}H_{d-i,d-j}(d)) \otimes (E_s^{(n-d,i)}H_{i,j}(n-d)) = (H_{d-i,d-j}(d)E_r^{(d,d-j)}) \otimes (H_{i,j}(n-d)E_s^{(n-d,j)}).$$

4. $\mathcal{T}$-algebra of $J(n, d)$ for $2d < n < 3d$

In this section we always assume that $2d < n < 3d$ and $\mathcal{M}^{(n,d)}$ is as in (1). We shall prove that $\mathcal{M}^{(n,d)}$ is the Terwilliger algebra $\mathcal{T}$ of $J(n, d)$.

For any real number a, we have

$$(A + aI)_{|\Omega_i\times\Omega_i} = \sum_{r=0}^{\min(i,d-i)} \sum_{s=0}^{\min(i,n-d-i)} (\mu_{i,r} + \lambda_{i,s} + a)E_r^{(d,d-i)} \otimes E_s^{(n-d,i)}, \tag{6}$$

where $\mu_{i,r} = p_1^{(d,d-i)}(r)$ and $\lambda_{i,s} = p_1^{(n-d,i)}(s)$. We always assume that a is a real number large enough such that the coefficients in (6) are positive.

Remark 1. If the coefficients in (6) are pairwise distinct, each $E_r^{(d,d-i)} \otimes E_s^{(n-d,i)}$ belongs to $\mathcal{T}_{|\Omega_i\times\Omega_i}$ since the left hand side of (6) and its powers belong to $\mathcal{T}_{|\Omega_i\times\Omega_i}$. By orthogonality of the idempotents

$$((A + aI)_{|\Omega_i\times\Omega_i})^j = \sum_{r=0}^{\min(i,d-i)} \sum_{s=0}^{\min(i,n-d-i)} (\mu_{i,r} + \lambda_{i,s} + a)^j E_r^{(d,d-i)} \otimes E_s^{(n-d,i)}, \tag{7}$$

obtaining a linear system of equations given by the powers of $(A + aI)_{|\Omega_i\times\Omega_i}$ as linear combinations of $E_r^{(d,d-i)} \otimes E_s^{(n-d,i)}$ with a Vandermonde matrix. See Section 5.1 in [10] for more explanation.

Theorem 4.1. *Suppose* $2d < n < 3d$*. Let* $\mathcal{T}$ *be the Terwilliger algebra of* $J(n, d)$ *and* $\mathcal{M}^{(n,d)}$ *be the algebra as in* (1)*. Then* $\mathcal{T} = \mathcal{M}^{(n,d)}$*.*

Proof. Since $L(M_{i,j}^{(n,d)}) = L(M_{i,i}^{(n,d)}(H_{d-i,d-j}(d) \otimes H_{i,j}(n-d)))$, by [10, Proposition 5.2] it is sufficient to prove that, for $0 \le r \le \min(d-i, i)$ and $0 \le s \le \min(n-d-i, i)$,

$$E_r^{(d,d-i)} \otimes E_s^{(n-d,i)} \in \mathcal{T}_{|\Omega_i \times \Omega_i}. \tag{8}$$

We shall prove (8) by induction on i (i decreases from d to 0).

Comment 1. *Since it holds that* $\mathcal{P}_{i+1}(Y) \in \mathcal{T}_{|\Omega_i \times \Omega_i}$ *for any* $Y \in \mathcal{T}_{|\Omega_{i+1} \times \Omega_{i+1}}$ *the strategy of the proof is to pull back those projectors* $E_r^{(d,d-i-1)} \otimes E_s^{(n-d,i+1)} \in \mathcal{T}_{|\Omega_{i+1} \times \Omega_{i+1}}$ *(whenever its pullback is different from zero) or separate the projectors as we explained in Remark 1.*

Induction: observe that $(A + aI)_{|\Omega_d \times \Omega_d} = \sum_{s=0}^{d}(\mu_{d,0} + \lambda_{d,s} + a)E_0^{(d,0)} \otimes E_s^{(n-d,d)}$. Since the parameters $\lambda_{d,s}$ are pairwise distinct, (8) holds for $i = d$.

Case 1. $\lceil \frac{n-d}{2} \rceil + 1 \le i \le d-1$.

Note that in this case $\min(d-i, i) = d-i$ and $\min(n-d-i, i) = n-d-i$.

For $0 \le r \le d-i-1$ and $0 \le s \le n-d-i-1$, by (5) one gets $l_{d,d-i-1,r} \ne 0$ and $p_{n-d,i+1,s} \ne 0$. By Lemma 3.5,

$$E_r^{(d,d-i)} \otimes E_s^{(n-d,i)} = l_{d,d-i-1,r}^{-1}\, p_{n-d,i+1,s}^{-1}\, \mathcal{P}_{i+1}(E_r^{(d,d-i-1)} \otimes E_s^{(n-d,i+1)}),$$

and so $E_r^{(d,d-i)} \otimes E_s^{(n-d,i)} \in \mathcal{T}_{|\Omega_i \times \Omega_i}$.

It is not possible to pull back $(E_{d-i}^{(d,d-i-1)} \otimes E_s^{(n-d,i+1)})$ nor also $(E_r^{(d,d-i-1)} \otimes E_{n-d-i}^{(n-d,i+1)})$.

By (6), we have

$$\begin{aligned}(A + aI)_{|\Omega_i \times \Omega_i} &- \sum_{r=0}^{d-i-1} \sum_{s=0}^{n-d-i-1} (\mu_{i,r} + \lambda_{i,s} + a)E_r^{(d,d-i)} \otimes E_s^{(n-d,i)} \\ &= \sum_{s=0}^{n-d-i} (\mu_{i,d-i} + \lambda_{i,s} + a)E_{d-i}^{(d,d-i)} \otimes E_s^{(n-d,i)} + \sum_{r=0}^{d-i-1} (\mu_{i,r} + \lambda_{i,n-d-i} + a)E_r^{(d,d-i)} \otimes E_{n-d-i}^{(n-d,i)}.\end{aligned}$$

It follows that the right hand side of the equality belongs to $\mathcal{T}_{|\Omega_i \times \Omega_i}$. In order to show that (8) holds, it suffices to show that each term belongs to $\mathcal{T}_{|\Omega_i \times \Omega_i}$. Observe that there do not exist three coefficients with the same value. If there exists a term whose coefficient is different from other coefficients, then this term belongs to $\mathcal{T}_{|\Omega_i \times \Omega_i}$. Next suppose that there exist two terms with the same coefficient. Suppose that $\mu_{i,d-i} + \lambda_{i,q} + a = \mu_{i,u} + \lambda_{i,n-d-i} + a$. Then $E_{d-i}^{(d,d-i)} \otimes E_q^{(n-d,i)} + E_u^{(d,d-i)} \otimes E_{n-d-i}^{(n-d,i)}$ belongs to $\mathcal{T}_{|\Omega_i \times \Omega_i}$, and by Lemma 3.5 its image under $\mathcal{P}_i$ is

$$(i + \mu_{i,d-i})(i + \lambda_{i,q})E_{d-i}^{(d,d-i+1)} \otimes E_q^{(n-d,i-1)} + (i + \mu_{i,u})(i + \lambda_{i,n-d-i})E_u^{(d,d-i+1)} \otimes E_{n-d-i}^{(n-d,i-1)}.$$

Suppose $(i+\mu_{i,d-i})(i+\lambda_{i,q}) = (i+\mu_{i,u})(i+\lambda_{i,n-d-i})$. Since $\mu_{i,d-i}+\lambda_{i,q} = \mu_{i,u}+\lambda_{i,n-d-i}$, one gets $\mu_{i,d-i}\lambda_{i,q} = \mu_{i,u}\lambda_{i,n-d-i}$. It follows that $(\mu_{i,d-i} - \lambda_{i,n-d-i})(\mu_{i,d-i} - \mu_{i,u}) = 0$, a contradiction to $\mu_{i,d-i} \ne \lambda_{i,n-d-i}$ and $\mu_{i,d-i} \ne \mu_{i,u}$. Therefore, we have

$$(i + \mu_{i,d-i})(i + \lambda_{i,q}) \ne (i + \mu_{i,u})(i + \lambda_{i,n-d-i}),$$

which implies that both $E_{d-i}^{(d,d-i+1)} \otimes E_q^{(n-d,i-1)}$ and $E_u^{(d,d-i+1)} \otimes E_{n-d-i}^{(n-d,i-1)}$ belong to $\mathcal{T}_{|\Omega_{i-1} \times \Omega_{i-1}}$. Computing their image under $\mathcal{L}_{i-1}$, by Lemma 3.5 again $E_{d-i}^{(d,d-i)} \otimes E_q^{(n-d,i)}$ and $E_u^{(d,d-i)} \otimes E_{n-d-i}^{(n-d,i)}$ belong to $\mathcal{T}_{|\Omega_i \times \Omega_i}$, as desired.

Case 2. $i = \lceil \frac{n-d}{2} \rceil$.

We divide our discussion into two subcases.

Case 2.1. $n-d$ is odd. By Lemma 3.5, for any $E_r^{(d,d-i)} \otimes E_s^{(n-d,i)} \in M_{i,i}^{(n,d)}$,

$$\mathcal{P}_i(E_r^{(d,d-i)} \otimes E_s^{(n-d,i)}) = l_{d,d-i,r}\, p_{n-d,i,s}\, E_r^{(d,d-i+1)} \otimes E_s^{(n-d,i-1)} \ne 0.$$

Similar to the proof in Case 1, (8) holds.

Case 2.2. $n-d$ is even. For $0 \le r \le d-i-1$ and $0 \le s \le n-d-i-1$, by Lemma 3.5,

$$E_r^{(d,d-i)} \otimes E_s^{(n-d,i)} = l_{d,d-i-1,r}^{-1}\, p_{n-d,i+1,s}^{-1}\, \mathcal{P}_{i+1}(E_r^{(d,d-i-1)} \otimes E_s^{(n-d,i+1)}),$$

which implies that $E_r^{(d,d-i)} \otimes E_s^{(n-d,i)} \in \mathcal{T}_{|\Omega_i \times \Omega_i}$.

Next we consider $r = d - i$ or $s = n - d - i$. Write $\mu'_{i,r} = p_{d-i}^{(d,d-i)}(r)$ and $\lambda'_{i,s} = p_i^{(n-d,i)}(s)$. Since $(A_d)_{|\Omega_i\times\Omega_i} = A_{d-i}^{(d,d-i)} \otimes A_i^{(n-d,i)}$,

$$\begin{aligned}(A_d + aI)_{|\Omega_i\times\Omega_i} &- \sum_{r=0}^{d-i-1}\sum_{s=0}^{n-d-i-1}(\mu'_{i,r}\lambda'_{i,s} + a)E_r^{(d,d-i)} \otimes E_s^{(n-d,i)}\\ &= \sum_{s=0}^{n-d-i}(\mu'_{i,d-i}\ \lambda'_{i,s} + a)E_{d-i}^{(d,d-i)} \otimes E_s^{(n-d,i)} + \sum_{r=0}^{d-i-1}(\mu'_{i,r}\ \lambda'_{i,n-d-i} + a)E_r^{(d,d-i)} \otimes E_{n-d-i}^{(n-d,i)}.\end{aligned}$$

It follows that the right hand side of the equality belongs to $\mathcal{T}_{|\Omega_i\times\Omega_i}$. By (5), observe that $\mu'_{i,r}\ \lambda'_{i,n-d-i} + a$ is not equal to any other coefficient. Then $E_r^{(d,d-i)} \otimes E_{n-d-i}^{(n-d,i)} \in \mathcal{T}_{|\Omega_i\times\Omega_i}$ for $0 \le r \le d - i - 1$.

By (6),

$$\sum_{s=0}^{n-d-i}(\mu_{i,d-i} + \lambda_{i,s} + a)E_{d-i}^{(d,d-i)} \otimes E_s^{(n-d,i)}$$

belongs to $\mathcal{T}_{|\Omega_i\times\Omega_i}$. Moreover, its coefficients are pairwise distinct, so $E_{d-i}^{(d,d-i)} \otimes E_s^{(n-d,i)}$ belongs to $\mathcal{T}_{|\Omega_i\times\Omega_i}$ for $0 \le s \le n-d-i$. Therefore, (8) holds.

Case 3. $\lceil\frac{d}{2}\rceil \le i \le \lceil\frac{n-d}{2}\rceil - 1$.

Note that in this case $\min(d - i, i) = d - i$ and $\min(n - d - i, i) = i$.

Similarly, by Lemma 3.5 we have that $E_r^{(d,d-i)} \otimes E_s^{(n-d,i)}$ belongs to $\mathcal{T}_{|\Omega_i\times\Omega_i}$ for $0 \le r \le d - i - 1$ and $0 \le s \le i$.

By (6) again, the matrix

$$\sum_{s=0}^{i}(\mu_{i,d-i} + \lambda_{i,s} + a)E_{d-i}^{(d,d-i)} \otimes E_s^{(n-d,i)}$$

belongs to $\mathcal{T}_{|\Omega_i\times\Omega_i}$. Moreover, its coefficients are pairwise distinct, so (8) holds.

Case 4. $0 \le i \le \lceil\frac{d}{2}\rceil - 1$.

Note that in this case $\min(d - i, i) = i$ and $\min(n - d - i, i) = i$.

By Lemma 3.5 again, (8) holds. □

Next we shall decompose $\mathcal{T}$ as a direct sum of some simple ideals.

For $0 \le r \le \lfloor\frac{d}{2}\rfloor$ and $0 \le s \le \lfloor\frac{n-d}{2}\rfloor$, define

$$e_{r,s} = \min\{i \mid 0 \ne E_r^{(d,d-i)} \otimes E_s^{(n-d,i)} \in M_{i,i}^{(n,d)}\},$$
$$d_{r,s} = |\{i \mid 0 \ne E_r^{(d,d-i)} \otimes E_s^{(n-d,i)} \in M_{i,i}^{(n,d)}\}| - 1.$$

Note that $e_{r,s} = \max(r, s)$ and $e_{r,s} + d_{r,s} = \min(d - r, n - d - s)$.

For $0 \le r \le \min(d - i, i)$ and $0 \le s \le \min(n - d - i, i)$, define

$$^{rs}T_{ij} = (E_r^{(d,d-i)}H_{d-i,d-j}(d)) \otimes (E_s^{(n-d,i)}H_{i,j}(n-d)), \tag{9}$$
$$^{rs}\mathcal{T} = \langle\{L(^{rs}T_{i,j})\}_{0\le i,j\le d}\rangle. \tag{10}$$

Proposition 4.2. *Let* $^{rs}\mathcal{T}$ *be as in* (10)*. Then* $^{rs}\mathcal{T}$ *is an ideal of* $\mathcal{T}$*.*

Proof. It suffices to show that $L(^{rs}T_{ij})L(^{pq}T_{lm}) \in {}^{rs}\mathcal{T}$ and $L(^{pq}T_{lm})L(^{rs}T_{ij}) \in {}^{rs}\mathcal{T}$.

If $j \ne l$, then $L(^{rs}T_{ij})L(^{pq}T_{lm}) = 0$. Suppose $j = l$. Since $H_{d-i,d-j}(d) \otimes H_{i,j}(n-d) \in M_{i,j}^{(n,d)}$ and $H_{d-j,d-m}(d) \otimes H_{j,m}(n-d) \in M_{j,m}^{(n,d)}$, we obtain $(H_{d-i,d-j}(d)H_{d-j,d-m}(d)) \otimes (H_{i,j}(n-d)H_{j,m}(n-d)) \in M_{i,m}^{(n,d)}$. It follows that there exist scalars $\beta_{u,v}$ such that

$$\begin{aligned}&(H_{d-i,d-j}(d)H_{d-j,d-m}(d)) \otimes (H_{i,j}(n-d)H_{j,m}(n-d))\\ &\quad= \sum_{u=0}^{\min(i,d-i)}\sum_{v=0}^{\min(i,n-d-i)}\beta_{u,v}(E_u^{(d,d-i)}H_{d-i,d-m}(d)) \otimes (E_v^{(n-d,i)}H_{i,m}(n-d)).\end{aligned}$$

By Corollary 3.6,

$$\begin{aligned}^{rs}T_{ij}\ ^{pq}T_{jm} &= \delta_{r,p}\ \delta_{s,q}(H_{d-i,d-j}(d)E_r^{(d,d-j)}H_{d-j,d-m}(d)) \otimes (H_{i,j}(n-d)E_s^{(d,j)}H_{j,m}(n-d))\\ &= \delta_{r,p}\ \delta_{s,q}(E_r^{(d,d-i)} \otimes E_s^{(d,i)})((H_{d-i,d-j}(d)H_{d-j,d-m}(d)) \otimes (H_{i,j}(n-d)H_{j,m}(n-d)))\\ &= \delta_{r,p}\ \delta_{s,q}\ \beta_{r,s}(E_r^{(d,d-i)}H_{d-i,d-m}(d)) \otimes (E_s^{(d,i)}H_{i,m}(n-d)),\end{aligned}$$

where $\delta_{r,p}$ is the Kronecker delta; and so

$$^{rs}T_{ij}\ ^{pq}T_{jm} = \delta_{r,p}\ \delta_{s,q}\ \beta_{r,s}\ ^{rs}T_{im}. \tag{11}$$

It follows that $L(^{rs}T_{ij})L(^{pq}T_{jm}) \in {}^{rs}\mathcal{T}$. Similarly $L(^{pq}T_{lm})L(^{rs}T_{ij}) \in {}^{rs}\mathcal{T}$. □

By (11) we observe that $^{rs}\mathcal{T}\,^{pq}\mathcal{T} = \{0\}$ if and only if $(r, s) \neq (p, q)$. From the construction of $\mathcal{M}^{(n,d)}$, we have

$$\mathcal{T} = \bigoplus_{r=0}^{\lfloor d/2\rfloor} \bigoplus_{s=0}^{\lfloor (n-d)/2\rfloor} {}^{rs}\mathcal{T}.$$

Lemma 4.3. *Let $^{rs}T_{ij}$ be as in (9). Then $^{rs}T_{ij} \neq 0$ if and only if $i, j \in \{e_{r,s},\ e_{r,s}+1, \ldots,\ e_{r,s}+d_{r,s}\}$.*

Proof. Note that $i, j \in \{\max(r, s), \ldots, \min(d-r, n-d-s)\}$ if and only if $0 \le r \le \min(i, j, d-i, d-j)$ and $0 \le s \le \min(i, j, n-d-i, n-d-j)$. If r or s does not belong to the above ranges, then $^{rs}T_{ij} = 0$ by Corollary 3.6. Since

$$H^r_{d-i,d-j}(d) \otimes H^s_{i,j}(n-d), \quad r \in R(d, d-i, d-j), s \in R(n-d, i, j)$$

is a basis of $M^{(n,d)}_{i,j}$, we have

$$\dim(M^{(n,d)}_{i,j}) = (\min(i, j, d-i, d-j) + 1) \times (\min(i, j, n-d-i, n-d-j) + 1).$$

The set of all matrices $^{rs}T_{ij}$ generate $M^{(n,d)}_{i,j}$, so the desired result follows. □

For $i, j \in \{e_{r,s},\ e_{r,s}+1, \ldots,\ e_{r,s}+d_{r,s}\}$, write

$$n^r_{ij} = \begin{cases} \displaystyle\sum_{m=0}^{j-i} \binom{d-j-m}{d-i} p^{(d,d-i)}_m(r), & i \le j, \\ \displaystyle\sum_{m=0}^{i-j} \binom{i-m}{j} p^{(d,d-i)}_m(r), & i \ge j; \end{cases}$$

$$n^{ij}_s = \begin{cases} \displaystyle\sum_{m=0}^{j-i} \binom{n-d-i-m}{j-i-m} p^{(n-d,i)}_m(s), & i \le j, \\ \displaystyle\sum_{m=0}^{i-j} \binom{i-m}{j} p^{(n-d,i)}_m(s), & i \ge j. \end{cases}$$

By Lemma 2.2 we have $(^{rs}T_{ij})(^{rs}T_{ij})^{\mathrm{T}} = n^r_{ij}\, n^{ij}_s E^{(d,d-i)}_r \otimes E^{(n-d,i)}_s \neq 0$. By computing the trace of this matrix, one gets $n^r_{ij} > 0$ and $n^{ij}_s > 0$. By (11), we may assume that $^{rs}T_{ij}\ ^{rs}T_{jl} = \beta_{r,s}(i, j, l)\ ^{rs}T_{il}$. Then $\beta_{r,s}(i, j, i) = n^r_{ij}\ n^{ij}_s > 0$. Taking the transpose on both sides of above equation, we obtain $\beta_{r,s}(i, j, l) = \beta_{r,s}(l, j, i)$. By Lemma 2.2(i) and Corollary 3.6, we have $\beta_{r,s}(i, j, l) > 0$ if $i \ge j \ge l$. Note that

$$^{rs}T_{ij}\ ^{rs}T_{jl}\ ^{rs}T_{li} = \beta_{r,s}(i, j, l)\beta_{r,s}(i, l, i)\ ^{rs}T_{ii} = \beta_{r,s}(j, l, i)\beta_{r,s}(i, j, i)\ ^{rs}T_{ii}.$$

$$^{rs}T_{li}\ ^{rs}T_{ij}\ ^{rs}T_{jl} = \beta_{r,s}(l, i, j)\beta_{r,s}(l, j, l)\ ^{rs}T_{ll} = \beta_{r,s}(i, j, l)\beta_{r,s}(l, i, l)\ ^{rs}T_{ll}.$$

Hence, we have $\beta_{r,s}(i, j, l) > 0$ for any $i, j, l \in \{\max(r, s), \ldots, \min(d-r, n-d-s)\}$.

By Lemma 2.2 again,

$$(^{rs}T_{ij}\ ^{rs}T_{jl})(^{rs}T_{ij}\ ^{rs}T_{jl})^{\mathrm{T}} = \frac{n^r_{ij}\ n^{ij}_s\ n^r_{jl}\ n^{jl}_s}{n^r_{il}\ n^{il}_s}\ ^{rs}T_{il}(^{rs}T_{il})^{\mathrm{T}}.$$

By (11), we have

$$^{rs}T_{ij}\ ^{rs}T_{jl} = \sqrt{\frac{n^r_{ij}\ n^{ij}_s\ n^r_{jl}\ n^{jl}_s}{n^r_{il}\ n^{il}_s}}\ ^{rs}T_{il}. \tag{12}$$

Let $\mathrm{Mat}_{d_{r,s}+1}(\mathbb{C})$ be the algebra consisting of all matrices whose rows and columns are indexed by $\{e_{r,s},\ e_{r,s}+1, \ldots,\ e_{r,s}+d_{r,s}\}$. Let $E_{i,j}$ be the matrix in $\mathrm{Mat}_{d_{r,s}+1}(\mathbb{C})$ whose (i, j)-entry is 1 and others are 0.

Theorem 4.4. *Suppose $2d < n < 3d$. Let $\mathcal{T}$ be the Terwilliger algebra of the Johnson scheme $J(n, d)$. Then*

$$\mathcal{T} \simeq \bigoplus_{r=0}^{\lfloor d/2\rfloor} \bigoplus_{s=0}^{\lfloor (n-d)/2\rfloor} \mathrm{Mat}_{d_{r,s}+1}(\mathbb{C}).$$

Proof. It suffices to prove that ${}^{rs}\mathcal{T} \simeq \mathrm{Mat}_{d_{r,s}+1}(\mathbb{C})$. Define the linear mapping ϕ from ${}^{rs}\mathcal{T}$ to $\mathrm{Mat}_{d_{r,s}+1}(\mathbb{C})$ such that $\phi(L({}^{rs}T_{ij})) = \sqrt{n_{ij}^{r}\, n_{s}^{ij}}E_{i,j}$. By (12), we have ${}^{rs}\mathcal{T} \simeq \mathrm{Mat}_{d_{r,s}+1}(\mathbb{C})$. □

5. $\mathcal{T}$-algebra of $J(2d, d)$

Let $\mathcal{N}$ be as in (3). In this section we shall prove that $\mathcal{N}$ is the Terwilliger algebra $\mathcal{T}$ of $J(2d, d)$.
Write $H^{r}_{d-i,d-j} := H^{r}_{d-i,d-j}(d)$ and $H^{s}_{i,j} := H^{s}_{i,j}(d)$ for simplicity.

Theorem 5.1. *Let $\mathcal{T}$ be the Terwilliger algebra of $J(2d, d)$ and $\mathcal{N}$ be the algebra as in (3). Then $\mathcal{T} = \mathcal{N}$.*

Proof. Since $L(N_{i,j}) = L(N_{i,i}(H_{d-i,d-j} \otimes H_{i,j}))$, by [10, Proposition 5.2] it is sufficient to prove that, for $0 \le r \le s \le \min(d, d-i)$,

$$E_r^{(d,d-i)} \otimes E_s^{(d,i)} + E_s^{(d,d-i)} \otimes E_r^{(d,i)} \in \mathcal{T}_{|\Omega_i\times\Omega_i}. \tag{13}$$

We shall prove (13) by induction on i (i decreases from d to 0). For $i = d$, it is trivial.
Case 1. $\lceil \frac{d}{2} \rceil \le i \le d-1$.
For $0 \le s \le d-i-1$ and $0 \le r \le s$, by Lemma 3.5,

$$E_r^{(d,d-i)} \otimes E_s^{(d,i)} + E_s^{(d,d-i)} \otimes E_r^{(d,i)} = l^{-1}_{d,d-i-1,r}p^{-1}_{d,i+1,s}\mathcal{P}_{i+1}(E_r^{(d,d-i-1)} \otimes E_s^{(d,i+1)} + E_s^{(d,d-i-1)} \otimes E_r^{(d,i+1)}),$$

which implies that $E_r^{(d,d-i)} \otimes E_s^{(d,i)} + E_s^{(d,d-i)} \otimes E_r^{(d,i)} \in \mathcal{T}_{|\Omega_i\times\Omega_i}$. Write $\lambda_r = p_1^{(d,i)}(r)$. By Lemma 3.4, we have

$$\begin{aligned}(A+aI)_{|\Omega_i\times\Omega_i} &- \sum_{s=0}^{d-i-1}\sum_{r=0}^{d-i-1}\frac{1}{2}(\lambda_r+\lambda_s+a)(E_r^{(d,d-i)} \otimes E_s^{(d,i)} + E_s^{(d,d-i)} \otimes E_r^{(d,i)})\\ &= \sum_{q=0}^{d-i-1}(\lambda_{d-i}+\lambda_q+a)(E_{d-i}^{(d,d-i)} \otimes E_q^{(d,i)} + E_q^{(d,d-i)} \otimes E_{d-i}^{(d,i)}) + (2\lambda_{d-i}+a)E_{d-i}^{(d,d-i)} \otimes E_{d-i}^{(d,i)}.\end{aligned}$$

Then the right hand side of the equality belongs to $\mathcal{T}_{|\Omega_i\times\Omega_i}$. Moreover, its coefficients are pairwise distinct, so (13) holds.
Case 2. $0 \le i \le \lceil \frac{d}{2} \rceil - 1$.
By Lemma 3.5 again, (13) holds. □

Next we shall decompose $\mathcal{T}$ as a direct sum of some simple ideals.
For $0 \le r \le \frac{d}{2}$ and $0 \le s \le \frac{d}{2}$, define

$$\begin{aligned}e_{r,s} &= \min\{i \mid 0 \ne E_r^{(d,d-i)} \otimes E_s^{(d,i)} + E_s^{(d,d-i)} \otimes E_r^{(d,i)} \in N_{i,i}\},\\ d_{r,s} &= |\{i \mid 0 \ne E_r^{(d,d-i)} \otimes E_s^{(d,i)} + E_s^{(d,d-i)} \otimes E_r^{(d,i)} \in N_{i,i}\}| - 1.\end{aligned}$$

Note that $e_{r,s} = \max(r, s)$ and $e_{r,s} + d_{r,s} = \min(d-r, d-s)$.
For $r, s \in \{0, 1, \dots, \min(d-i, i)\}$, define

$${}^{rs}T_{ij} = (E_r^{(d,d-i)}H_{d-i,d-j}) \otimes (E_s^{(d,i)}H_{i,j}) + (E_s^{(d,d-i)}H_{d-i,d-j}) \otimes (E_r^{(d,i)}H_{i,j}), \tag{14}$$

$${}^{rs}\mathcal{T} = \langle\{L({}^{rs}T_{ij})\}_{0\le i,j\le d}\rangle. \tag{15}$$

Proposition 5.2. *Let ${}^{rs}\mathcal{T}$ be as in (15). Then ${}^{rs}\mathcal{T}$ is an ideal of $\mathcal{T}$.*

Proof. It suffices to show that $L({}^{rs}T_{ij})L({}^{pq}T_{lm}) \in {}^{rs}\mathcal{T}$ and $L({}^{pq}T_{lm})L({}^{rs}T_{ij}) \in {}^{rs}\mathcal{T}$.
If $j \ne l$, then $L({}^{rs}T_{ij})L({}^{pq}T_{lm}) = 0$. Suppose $j = l$. Since $H_{d-i,d-j} \otimes H_{i,j} \in N_{i,j}$ and $H_{d-j,d-m} \otimes H_{j,m} \in N_{j,m}$, we obtain $(H_{d-i,d-j}H_{d-j,d-m}) \otimes (H_{i,j}H_{j,m}) \in N_{i,m}$. It follows that there exist scalars $\beta_{u,v}$ such that

$$(H_{d-i,d-j}H_{d-j,d-m}) \otimes (H_{i,j}H_{j,m}) = \sum_{v=0}^{\min(i,d-i)}\sum_{u=0}^{v}\beta_{u,v}(E_u^{(d,d-i)} \otimes E_v^{(d,i)} + E_v^{(d,d-i)} \otimes E_u^{(d,i)})(H_{d-i,d-m} \otimes H_{i,m}).$$

By Corollary 3.6,

$$\begin{aligned}{}^{rs}T_{ij}\,{}^{pq}T_{jm} &= \delta_{r,p}\,\delta_{s,q}(E_r^{(d,d-i)} \otimes E_s^{(d,i)} + E_s^{(d,d-i)} \otimes E_r^{(d,i)})((H_{d-i,d-j}H_{d-j,d-m}) \otimes (H_{i,j}H_{j,m}))\\ &= \delta_{r,p}\,\delta_{s,q}\,\beta_{r,s}((E_r^{(d,d-i)}H_{d-i,d-m}) \otimes (E_s^{(d,i)}H_{i,m}) + (E_s^{(d,d-i)}H_{d-i,d-m}) \otimes (E_r^{(d,i)}H_{i,m})),\end{aligned}$$

so we have

$${}^{rs}T_{ij}\,{}^{pq}T_{jm} = \delta_{r,p}\,\delta_{s,q}\,\beta_{r,s}\,{}^{rs}T_{im}. \tag{16}$$

It follows that $L({}^{rs}T_{ij})L({}^{pq}T_{jm}) \in {}^{rs}\mathcal{T}$. Similarly $L({}^{pq}T_{lm})L({}^{rs}T_{ij}) \in {}^{rs}\mathcal{T}$. □

By (16) we observe that ${}^{rs}\mathcal{T}\ {}^{pq}\mathcal{T} = \{0\}$ if and only if $(r, s) \neq (p, q)$. From the construction of $\mathcal{N}$, we have

$$\mathcal{T} = \bigoplus_{s=0}^{\lfloor d/2 \rfloor} \bigoplus_{r=0}^{s} {}^{rs}\mathcal{T}.$$

Lemma 5.3. *Let ${}^{rs}T_{ij}$ be as in (14). Then ${}^{rs}T_{ij} \neq 0$ if and only if $i, j \in \{e_{r,s}, \ldots, e_{r,s} + d_{r,s}\}$.*

Proof. The proof is similar to that of Lemma 4.3 and will be omitted. □

For $i, j \in \{\max(r, s), \ldots, \min(d - r, d - s)\}$, write

$$n_s^{ij} = \begin{cases} \sum_{m=0}^{j-i} \binom{d-i-m}{j-i-m} p_m^{(d,i)}(s), & i \leq j, \\ \sum_{m=0}^{i-j} \binom{i-m}{j} p_m^{(d,i)}(s), & i \geq j. \end{cases}$$

Similarly to the proof of (12), we have

$${}^{rs}T_{ij}\ {}^{rs}T_{jl} = \sqrt{\frac{n_r^{ij}\ n_s^{ij}\ n_r^{jl}\ n_s^{jl}}{n_r^{il}\ n_s^{il}}}\ {}^{rs}T_{il}. \tag{17}$$

Let ϕ be the linear mapping from ${}^{rs}\mathcal{T}$ to $\mathrm{Mat}_{d_{r,s}+1}(\mathbb{C})$ satisfying $\phi(L({}^{rs}T_{ij})) = \sqrt{n_r^{ij}\ n_s^{ij}}E_{i,j}$. By (17), we obtain the following result.

Theorem 5.4. *Let $\mathcal{T}$ be the Terwilliger algebra of the Johnson scheme $J(2d, d)$. Then*

$$\mathcal{T} \simeq \bigoplus_{s=0}^{\lfloor d/2 \rfloor} \bigoplus_{r=0}^{s} \mathrm{Mat}_{d_{r,s}+1}(\mathbb{C}).$$

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