Hartree approximation in curved spacetimes revisited II: The semiclassical Einstein equations and de Sitter self-consistent solutions

Diana L. López Nacir [1,2], Francisco D. Mazzitelli[3], and Leonardo G. Trombetta[2]

[1] *Abdus Salam International Centre for Theoretical*
Physics Strada Costiera 11, 34151, Trieste, Italy

[2] *Departamento de Física and IFIBA, FCEyN UBA,*
Facultad de Ciencias Exactas y Naturales, Ciudad Universitaria,
Pabellón I, 1428 Buenos Aires, Argentina and

[3] *Centro Atómico Bariloche Comisión Nacional de*
Energía Atómica, R8402AGP Bariloche, Argentina

(Dated: January 24, 2014)

Abstract

We consider the semiclassical Einstein equations (SEE) in the presence of a quantum scalar field with self-interaction $\lambda\phi^4$. Working in the Hartree truncation of the two-particle irreducible (2PI) effective action, we compute the vacuum expectation value of the energy-momentum tensor of the scalar field, which act as a source of the SEE. We obtain the renormalized SEE by implementing a consistent renormalization procedure. We apply our results to find self-consistent de Sitter solutions to the SEE in situations with or without spontaneous breaking of the Z_2-symmetry.

I. INTRODUCTION

Quantum field theory in curved spacetimes [1–4] is the natural framework for the study of quantum phenomena in situations where the gravitation itself can be treated classically. Of special interest is quantum field theory in de Sitter spacetime. In fact, de Sitter spacetime plays a central role in most of inflationary models of the early Universe [5–7], where the energy density and pressure of the inflaton field act approximately as a cosmological constant. Moreover, the amplification of quantum fluctuation during an inflationary period with an approximately de Sitter background metric, gives a natural mechanism for generating nearly scale-invariant spectrum of primordial inhomogeneities, which can successfully explain the observed CMB anisotropies [8, 9]. De Sitter spacetime is also potentially important for understanding the final fate of the Universe if the current accelerated expansion is due to a small cosmological constant, which nowadays is a possibility that is compatible with observations [9–12]. On the other hand, previous studies of interacting quantum scalar fields in de Sitter spacetime have revealed that the standard perturbative expansion gives rise to corrections that secularly grow with time and/or infrared divergences [13–20], signaling a possible deficiency of the perturbative approach. This has motivated several authors to consider alternative techniques (see for instance [18–27]) and in particular, to use nonperturbative resummation schemes [28–34].

In the above situations, it is important to study not only test fields evolving on a fixed background, but also to take into account the backreaction of the quantum fields on the dynamics of the spacetime geometry. The backreaction problem has been explored by a number of authors in the context of semiclassical gravity (see for instance [35–39]), where the dynamics of the classical metric is governed by the so-called Semiclasical Einstein Equations (SEE). The SEE are a generalization of the Einstein equations that contain as a source the expectation value of the energy-momentum tensor of the quantum matter fields, $\langle T_{\mu\nu}\rangle$ [1–4]. Self-consistent de Sitter solutions have been found for the case of free quantum fields [40–44]. The influence of the initial state of the quantum field on the semiclassical solutions has been studied in Refs. [45, 46].

Since $\langle T_{\mu\nu}\rangle$ is formally a divergent quantity, in order to address the backreaction problem it is necessary to analyze the renormalization process. For free and interacting quantum fields in the one-loop approximation, there are well known covariant renormalization meth-

ods [1–4]. Our main goal in this work is to improve the current understanding of these methods in the case in which the quantum effects are taken into account nonperturbatively. For this, we consider a quantum self-interacting scalar field in the Hartree approximation, which corresponds to the simplest nonperturbative truncation to the two-particle irreducible effective action (2PI EA), introduced by Cornwall, Jackiw and Tomboulis [47]. The Hartree (or Gaussian) approximation involves the resummation of a particular type of Feynman diagrams which are called superdaisy (see for instance [48]) to an infinite perturbative order. This approximation can also be introduced by means of a variational principle [49, 50]. However, the use of the 2PI EA is advantageous for at least two reasons. First, it provides a framework for resumming classes of diagrams that can be systematically improved. Second, for any truncation of the EA, it implies certain consistency relations between different counterterms that allow a renormalization procedure that is consistent with the standard perturbative (loop-by-loop) renormalization of the bare coupling constants [51]. The latter is crucial for the consistent renormalization procedure developed in Ref. [51] for Minkowski spacetime, which in [34] (from now on paper I), using the same model considered here, we have extended to general curved background metrics.

The renormalization problem of the SEE in the Hartree approximation has been considered previously in [32, 52]. However, it has not been analyzed using the consistent renormalization procedure [51] that we extended to curved spacetimes in paper I in order to renormalize the field and gap equations. Our focus in this paper is to prove that the same set of renormalized parameters leads to SEE that can be made finite, and independent on the arbitrary scale introduced by the regularization scheme (which for the field and gap equations was explicitly shown in paper I), by suitable renormalizations of the bare gravitational constants.

The paper is organized as follows. In Sec. II we introduce the 2PI EA in curved spacetimes. In Sec. III we present our model and summarize the main relevant results of paper I for the renormalization of the mass and coupling constant of the field. The reader acquainted with paper I may skip this section. In Sec. IV we show that the same counterterms that make finite the field and gap equations can also be used to absorb the non-geometric divergences in the SEE, extending the consistent renormalization procedure to the gravitational sector. The geometric divergences can be absorbed into the usual gravitational counterterms. In Sec. V we analyze the field, gap and SEE in de Sitter spacetimes. The high

symmetry of these spacetimes allows us to compute explicitly the two point function and the energy-momentum tensor, to end with a set of algebraic equations that determine self-consistently the mean value of the field and the de Sitter curvature. We will present some numerical solutions to these equations. In Sec. VI we include our conclusions. Throughout the paper we set $c = \hbar = 1$ and adopt the mostly plus sign convention.

II. THE 2PI EFFECTIVE ACTION

A detailed description to the 2PI EA formalism can be found in several papers and textbooks, such as [47, 53, 54]. In this section, in order to make this work as self-contained as possible and to set the notation, we briefly summarize the main relevant aspects of the formalism applied to a self-interacting scalar field ϕ in a general curved spacetime.

The 2PI generating functional can be written as [51]

$$\Gamma_{2PI}[\phi_0, G, g^{\mu\nu}] = S_0[\phi_0, g^{\mu\nu}] + \frac{i}{2}\text{Tr}\ln(G^{-1}) + \frac{i}{2}\text{Tr}(G_0^{-1}G) + \Gamma_{int}[\phi_0, G, g^{\mu\nu}], \qquad (1)$$

where S_0 is quadratic part of the classical action S without any counterterms,

$$iG_0^{ab}(x, x') = \frac{1}{\sqrt{-g}}\frac{\delta^2 S_0[\phi_0, g^{\mu\nu}]}{\delta\phi_a(x)\delta\phi_b(x')}\frac{1}{\sqrt{-g'}}, \qquad (2)$$

and

$$\Gamma_{int}[\phi_0, G, g^{\mu\nu}] = S_{int}[\phi_0, g^{\mu\nu}] + \frac{1}{2}\text{Tr}\left[\frac{\delta^2 S_{int}}{\delta\phi_0\delta\phi_0}G\right] + \Gamma_2[\phi_0, G, g^{\mu\nu}], \qquad (3)$$

where the functional Γ_2 is $-i$ times the sum of all two-particle-irreducible vacuum-to-vacuum diagrams with lines given by G and vertices obtained from the shifted action S_{int}^F, which comes from expanding $S_{int}[\phi_0 + \varphi]$ and collecting all terms higher than quadratic in the fluctuating field φ. Here a, b are time branch indices (with index set $\{+, -\}$ in the usual notation) corresponding to the ordering on the contour in the "closed-time-path"(CTP) or Schwinger-Keldysh [53] formalism.

The equations of motion for the field and propagator are obtained by

$$\frac{\delta\Gamma_{2PI}}{\delta\phi_0}\bigg|_{\phi_+=\phi_-=\phi; g_+^{\mu\nu}=g_-^{\mu\nu}=g^{\mu\nu}} = 0, \qquad (4a)$$

$$\frac{\delta\Gamma_{2PI}}{\delta G}\bigg|_{\phi_+=\phi_-=\phi; g_+^{\mu\nu}=g_-^{\mu\nu}=g^{\mu\nu}} = 0. \qquad (4b)$$
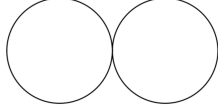
Figure 1: 2PI "double-bubble" diagram .

To arrive at the SEE we extremize the combination $S_g[g^{\mu\nu}] + \Gamma_{2PI}[\phi_0, G, g^{\mu\nu}]$ with respect to the metric,

$$\frac{\delta \left(S_g[g^{\mu\nu}] + \Gamma_{2PI}[\phi_0, G, g^{\mu\nu}]\right)}{\delta g^{\mu\nu}}\Bigg|_{\phi_+=\phi_-=\phi; g_+^{\mu\nu}=g_-^{\mu\nu}=g^{\mu\nu}} = 0, \tag{5}$$

where S_g is the gravitational action. As it is well known [1–3], this equation is formally divergent, with the divergences contained in the vacuum expectation value of the energy-momentum tensor $\langle T_{\mu\nu} \rangle$, defined by

$$\langle T_{\mu\nu} \rangle = -\frac{2}{\sqrt{-g}} \frac{\delta \Gamma_{2PI}[\phi_0, G, g^{\mu\nu}]}{\delta g^{\mu\nu}}\Bigg|_{\phi_+=\phi_-=\phi; g_+^{\mu\nu}=g_-^{\mu\nu}=g^{\mu\nu}}. \tag{6}$$

It is also well known [1–3] that the renormalization procedure requires the inclusion of terms quadratic in the curvature in the gravitational action, so that

$$S_g = \frac{1}{2} \int d^4x \sqrt{-g} \left\{ \kappa_B^{-1}(R - 2\Lambda_B) - \alpha_{1B}R^2 - \alpha_{2B}R_{\mu\nu}R^{\mu\nu} - \alpha_{3B}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \right\}, \tag{7}$$

where $R_{\mu\nu\rho\sigma}$ is the curvature tensor, $R_{\mu\nu} = R^\rho_{\mu\rho\nu}$, and $\kappa_B = 8\pi G_N^B$, Λ_B, α_{iB} $(i = 1, 2, 3)$ are bare parameters which are to be appropriately chosen to cancel the divergences in $\langle T_{\mu\nu} \rangle$.

III. $\lambda\phi^4$ THEORY IN THE HARTREE APPROXIMATION: RENORMALIZATION OF THE FIELD AND GAP EQUATIONS

We consider a nonminimally coupled scalar field with quartic self-coupling in a curved background with metric $g_{\mu\nu}$. The corresponding classical action reads

$$S_m[\phi, g^{\mu\nu}] = -\int d^4x \sqrt{-g} \left[\frac{1}{2}\phi \left(-\Box + m_B^2 + \xi_B R\right)\phi + \frac{1}{4!}\lambda_B \phi^4 \right], \tag{8}$$

where $\Box = \frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu\right)$, $g \equiv \det(g_{\mu\nu})$. In the Hartree approximation, which corresponds to the inclusion of only the double-bubble diagram shown in Fig. 1, the 2PI effective

action is given by

$$\Gamma_{2PI}[\phi_0, G, g_{\mu\nu}] = -\int d^4x \sqrt{-g} \left[\frac{1}{2}\phi_0 \left(-\Box + m_{B2}^2 + \xi_{B2}R \right) \phi_0 + \frac{1}{4!}\lambda_{B4}\phi_0^4 \right] + \frac{i}{2}\operatorname{Tr}\ln(G^{-1})$$
$$- \frac{1}{2}\int d^4x \sqrt{-g} \left[-\Box + m_{B0}^2 + \xi_{B0}R + \frac{1}{2}\lambda_{B2}\phi_0^2 \right] G(x,x) \qquad (9)$$
$$- \frac{\lambda_{B0}}{8}\int d^4x \sqrt{-g} \, G^2(x,x),$$

where, for the sake of simplicity, we drop the time branch indices, since for the Hartree approximation it is known that the CTP formalism gives the same equations of motion than the usual *in-out* formalism [54].

Taking the variation with respect to ϕ_0 and G we obtain equations of motion for the mean field and the propagator:

$$\left(-\Box + m_{B2}^2 + \xi_{B2}R + \frac{\lambda_{B4}}{6}\phi_0^2 + \frac{\lambda_{B2}}{2}[G] \right)\phi_0(x) = 0, \qquad (10)$$

$$\left(-\Box + m_{B0}^2 + \xi_{B0}R + \frac{\lambda_{B2}}{2}\phi_0^2 + \frac{\lambda_{B0}}{2}[G] \right)G(x,x') = -i\frac{\delta(x-x')}{\sqrt{-g'}}, \qquad (11)$$

with $[G]$ the coincidence limit of the propagator $G(x,x')$.

It is important to note that here we are taking into account the possibility of having different counterterms for a given parameter of the classical action Eq. (8). These are denoted using different subscripts in the bare parameters that refer to the power of ϕ_0 in the corresponding term of the action. In the Hartree approximation, this point turns out to be crucial for the implementation of the consistent renormalization procedure described in [51] . Indeed, as shown in [51] (see also Appendix A of paper I), there are various possible n-point functions that can be obtained from functionally differentiating $\Gamma_{2PI}[\phi_0, G, g^{\mu\nu}]$ with respect to ϕ_a and G_{ab}, which in the exact theory must satisfy certain consistency conditions. On the other hand, for any truncation of the 2PI EA, the validity of such consistency conditions is not guarantee. However, one can find a relation between the different counterterms by imposing the consistency conditions at a given renormalization point. Doing this, any possible deviation of the consistency conditions is finite and under perturbative control. In other words, had we not allowed for different counterterms, the diagrams contributing to the consistency conditions could contain perturbative divergent contributions which could not be absorbed anywhere.

In our case, the consistency conditions for the two- and four-point functions, evaluated at $\phi_0 = 0$, are given by

$$\left.\frac{\delta^2\Gamma_{int}}{\delta\phi_1\delta\phi_2}\right|_{\phi=0} = 2\left.\frac{\delta\Gamma_{int}}{\delta G_{12}}\right|_{\phi=0}, \tag{12}$$

and

$$\left.\frac{\delta^4\Gamma_{1PI}[\phi_0]}{\delta\phi_1\delta\phi_2\delta\phi_3\delta\phi_4}\right|_{\phi_0=0} = 2\left[\left.\frac{\delta^2\Gamma_{int}}{\delta G_{12}\delta G_{34}}\right|_{\bar{G},\phi_0=0} + perms(2,3,4)\right] - \frac{1}{2}\left.\frac{\delta^4\Gamma_{int}}{\delta\phi_1\delta\phi_2\delta\phi_3\delta\phi_4}\right|_{\bar{G},\phi_0=0}, \tag{13}$$

where

$$\Gamma_{1PI}[\phi_0, g^{\mu\nu}] = \Gamma_{2PI}[\phi_0, \bar{G}[\phi_0], g^{\mu\nu}]. \tag{14}$$

In what follows we consider two different parametrizations of the bare couplings:

$$m_{Bi}^2 = m^2 + \delta m_i^2 = m_R^2 + \delta\tilde{m}_i^2 \ (i = 0, 2), \tag{15a}$$

$$\xi_{Bi} = \xi + \delta\xi_i = \xi_R + \delta\tilde{\xi}_i \ (i = 0, 2), \tag{15b}$$

$$\lambda_{Bi} = \lambda + \delta\lambda_i = \lambda_R + \delta\tilde{\lambda}_i, \ (i = 0, 2, 4). \tag{15c}$$

The first separation corresponds to the MS scheme (i.e., the counterterms δm_i^2, $\delta\xi_i$ and $\delta\lambda_j$ ($i = 0, 2, j = 0, 2, 4$) contain only divergences and no finite part), while in the second separation m_R^2, ξ_R and λ_R are chosen to be the renormalized parameters as defined from the effective potential (see below).

By imposing the conditions (12) and (13), one can obtain the following relation between the different counterterms [34]:

$$\delta m_0^2 = \delta m_2^2 \equiv \delta m^2, \tag{16a}$$

$$\delta\xi_0 = \delta\xi_2 \equiv \delta\xi, \tag{16b}$$

$$\delta\lambda_0 = \delta\lambda_2, \tag{16c}$$

$$\delta\lambda_4 - 3\delta\lambda_2 = 2(\lambda - \lambda_R), \tag{16d}$$

with

$$\left.\frac{\delta^4\Gamma_{1PI}[\phi_0]}{\delta\phi_1\delta\phi_2\delta\phi_3\delta\phi_4}\right|_{\phi_0=0} = -\lambda_R\delta_{12}\delta_{13}\delta_{14}, \tag{17}$$

where we used $\phi_i \equiv \phi_0(x_i)$ as a notational shorthand. Recalling that the effective potential is proportional to the effective action at a constant value of ϕ_0, the renormalized self-interaction coupling λ_R can be also written as

$$\lambda_R = \left.\frac{d^4V_{eff}}{d\phi_0^4}\right|_0. \tag{18}$$

7

With the use of these relations, one can recast Eqs. (10) and (11) as

$$\left(-\Box + m_{ph}^2 + \xi_R R - \frac{1}{3}\lambda_R \phi_0^2\right)\phi_0(x) = 0, \tag{19}$$

$$\left(-\Box + m_{ph}^2 + \xi_R R\right) G_1(x, x') = 0, \tag{20}$$

where m_{ph}^2 is identified with the physical mass of the fluctuations and satisfies a self-consistent equation (i.e., the gap equation) that reads

$$m_{ph}^2 + \xi_R R = m^2 + \delta m^2 + (\xi + \delta\xi)R + \frac{1}{2}(\lambda + \delta\lambda_2)\phi_0^2 + \frac{1}{4}(\lambda + \delta\lambda_2)[G_1]. \tag{21}$$

A point that is worth emphasizing here is that these relations cannot be imposed in an arbitrary spacetime metric, since the renormalized parameters must be constant, while the fourth derivative of 1PI EA in Eq. (17) might not. However, in order to define the renormalized parameters, one can choose a particular fixed background metric with constant curvature invariants as the renormalization point at which the consistency conditions are imposed. In paper I we considered both Minkowski and de Sitter spacetimes. Here, for the sake of generality, we will also consider both renormalization points. Therefore, we define the renormalized parameters as those derived from the effective potential and evaluated for a fixed de Sitter spacetime with $R = R_0$,

$$\mathcal{M}_R^2 \equiv \left.\frac{d^2 V_{eff}}{d\phi_0^2}\right|_{\phi_0=0, R=R_0} = \mathcal{M}_{ph}^2(\phi_0 = 0, R = R_0), \tag{22a}$$

$$\xi_R \equiv \left.\frac{d^3 V_{eff}}{dR\, d\phi_0^2}\right|_{\phi_0=0, R=R_0} = \left.\frac{d\mathcal{M}_{ph}^2}{dR}\right|_{\phi_0=0, R=R_0}, \tag{22b}$$

$$\lambda_R \equiv \left.\frac{d^4 V_{eff}}{d\phi_0^4}\right|_{\phi_0=0, R=R_0} = 3\left.\frac{d^2 \mathcal{M}_{ph}^2}{d\phi_0^2}\right|_{\phi_0=0, R=R_0} - 2\lambda_R, \tag{22c}$$

where we are using the notation $\mathcal{M}_R^2 = m_R^2 + \xi_R R$. In particular, the limit $R_0 \to 0$ could be taken to recover the usual renormalized parameters defined in Minkowski spacetime.

In order to obtain the renormalized gap equation it is useful to consider the adiabatic expansion of the propagator at the coincidence limit:

$$[G_1] = \frac{1}{8\pi^2}\left(\frac{m_{ph}^2}{\mu^2}\right)^{\epsilon/2} \sum_{j\geq 0}[\Omega_j](m_{ph}^2)^{1-j}\,\Gamma\left(j - 1 - \frac{\epsilon}{2}\right)$$

$$\equiv \frac{1}{4\pi^2\epsilon}\left[m_{ph}^2 + \left(\xi_R - \frac{1}{6}\right)R\right] + 2T_F(m_{ph}^2, \xi_R, R, \tilde{\mu}), \tag{23}$$

8

where $\epsilon = n - 4$, $\Gamma(x)$ is the Gamma function, and the Schwinger-DeWitt coefficients $[\Omega_j]$ are scalars of adiabatic order $2j$ built from the metric and its derivatives and satisfy certain recurrence relations. In the second line, we have used the explicit expressions for the coefficients $[\Omega_0] = 1$ and $[\Omega_1] = -(\xi_R - 1/6)R$, given in [55], we have expanded for $\epsilon \to 0$ and we have redefined $\mu \to \tilde{\mu}$ to absorb some constant terms, defining

$$T_F(m_{ph}^2, \xi_R, R, \tilde{\mu}) = \frac{1}{16\pi^2} \left\{ \left[m_{ph}^2 + \left(\xi_R - \frac{1}{6} \right) R \right] \ln \left(\frac{m_{ph}^2}{\tilde{\mu}^2} \right) + \left(\xi_R - \frac{1}{6} \right) R \right.$$
$$\left. - 2F(m_{ph}^2, \{R\}) \right\}, \tag{24}$$

where the function $F(m_{ph}^2, \{R\})$ contains the adiabatic orders higher than two, is independent of ϵ and μ, and satisfies the following properties:

$$\left. F(m_{ph}^2, \{R\}) \right|_{R_{\mu\nu\rho\sigma}=0} = 0, \tag{25a}$$

$$\left. \frac{dF(m_{ph}^2, \{R\})}{dm_{ph}^2} \right|_{R_{\mu\nu\rho\sigma}=0} = 0, \tag{25b}$$

$$\left. \frac{dF(m_{ph}^2, \{R\})}{dR} \right|_{R_{\mu\nu\rho\sigma}=0,\phi_0=0} = 0. \tag{25c}$$

Taking into account the relations in Eq. (16) between the counterterms, the gap equation can be made finite with the use of the following MS counterterms:

$$\delta m^2 = -\frac{\lambda}{16\pi^2\epsilon} \frac{m^2}{1 + \frac{\lambda}{16\pi^2\epsilon}}, \tag{26a}$$

$$\delta\xi = -\frac{\lambda}{16\pi^2\epsilon} \frac{\left(\xi - \frac{1}{6} \right)}{1 + \frac{\lambda}{16\pi^2\epsilon}}, \tag{26b}$$

$$\delta\lambda_2 = -\frac{\lambda}{16\pi^2\epsilon} \frac{\lambda}{1 + \frac{\lambda}{16\pi^2\epsilon}}. \tag{26c}$$

Once made finite and written in terms of the MS parameters, it reads

$$m_{ph}^2 + \xi_R R = m^2 + \xi R + \frac{1}{2}\lambda\phi_0^2 + \frac{\lambda}{32\pi^2} \left\{ \left[m_{ph}^2 + \left(\xi_R - \frac{1}{6} \right) R \right] \ln \left(\frac{m_{ph}^2}{\tilde{\mu}^2} \right) \right.$$
$$\left. + \left(\xi_R - \frac{1}{6} \right) R - 2F(m_{ph}^2, \{R\}) \right\}. \tag{27}$$

Here, the explicit dependence on the renormalization scale $\tilde{\mu}$ should be compensated with an implicit $\tilde{\mu}$-dependence on the finite MS parameters $m^2(\tilde{\mu})$, $\xi(\tilde{\mu})$ and $\lambda(\tilde{\mu})$. Indeed, the

invariance of this equation under changes of $\tilde{\mu}$ becomes manifest when we express it in terms of the renormalized quantities m_R^2, ξ_R and λ_R. The latter are related to the former ones by

$$m_R^2 = \frac{m^2 + \frac{\lambda}{16\pi^2}\left[R_0 \left.\frac{dF_{dS}}{dR}\right|_{m_R^2,R_0} - F_{dS}(m_R^2,R_0)\right]}{\left[1 - \frac{\lambda}{32\pi^2}\ln\left(\frac{m_R^2}{\tilde{\mu}^2}\right)\right]}, \tag{28a}$$

$$\left(\xi_R - \frac{1}{6}\right) = \frac{\left(\xi - \frac{1}{6}\right) - \frac{\lambda}{16\pi^2}\left.\frac{dF_{dS}}{dR}\right|_{m_R^2,R_0}}{\left[1 - \frac{\lambda}{32\pi^2} - \frac{\lambda}{32\pi^2}\ln\left(\frac{m_R^2}{\tilde{\mu}^2}\right)\right]}, \tag{28b}$$

$$\lambda_R = \frac{\lambda}{\left[1 - \frac{\lambda}{32\pi^2} - \frac{\lambda}{32\pi^2}\ln\left(\frac{m_R^2}{\tilde{\mu}^2}\right) - \frac{\lambda}{32\pi^2}\left(\frac{(\xi_R - \frac{1}{6})R_0}{m_R^2} - 2\left.\frac{dF_{dS}}{dm_{ph}^2}\right|_{m_R^2,R_0}\right)\right]}. \tag{28c}$$

Two useful $\tilde{\mu}$-independent combinations follow immediately from these relations:

$$\frac{m_B^2}{\lambda_{B2}} = \frac{m^2}{\lambda} = \frac{m_R^2}{\lambda_R^*} + \frac{\left(\xi_R - \frac{1}{6}\right)R_0}{32\pi^2} \tag{29}$$

and

$$\frac{\left(\xi_B - \frac{1}{6}\right)}{\lambda_B} = \frac{\left(\xi - \frac{1}{6}\right)}{\lambda} \tag{30}$$

$$= \frac{\left(\xi_R - \frac{1}{6}\right)}{\lambda_R} + \frac{\left(\xi_R - \frac{1}{6}\right)}{32\pi^2}\left[\left(\xi_R - \frac{1}{6}\right)\frac{R_0}{m_R^2} - 2\left.\frac{dF_{dS}}{dm_{ph}^2}\right|_{m_R^2,R_0}\right] + \frac{1}{16\pi^2}\left.\frac{dF_{dS}}{dR}\right|_{m_R^2,R_0}$$

$$\equiv \frac{\left(\xi_R - \frac{1}{6}\right)}{\lambda_R} + J(R_0, m_R^2, \xi_R).$$

where λ_R^* is defined by

$$\frac{1}{\lambda_R^*} \equiv \frac{1}{\lambda_R} + \frac{1}{32\pi^2}. \tag{31}$$

Using these parameters, the self-consistent equation for m_{ph}^2 can be written as

$$m_{ph}^2 = m_R^2 + \frac{\lambda_R^*}{2}\phi_0^2 + \frac{\lambda_R^*}{32\pi^2}\left\{\left[m_{ph}^2 + \left(\xi_R - \frac{1}{6}\right)R\right]\ln\left(\frac{m_{ph}^2}{m_R^2}\right)\right.$$

$$+ \left(m_{ph}^2 - m_R^2\right)\left[2\left.\frac{dF_{dS}}{dm_{ph}^2}\right|_{m_R^2,R_0} - \frac{(\xi_R - \frac{1}{6})R_0}{m_R^2}\right] \tag{32}$$

$$\left. + 2\left[F_{dS}(m_R^2,R_0) + \left.\frac{dF_{dS}}{dR}\right|_{m_R^2,R_0}(R - R_0) - F(m_{ph}^2,R)\right]\right\}.$$

Finally, as will be needed for the renormalization of the energy-momentum tensor in next section, we write the results for the counterterms associated to the non-MS renormalized

parameters defined in Eq. (15):

$$\delta \tilde{m}^2 \equiv m_B^2 - m_R^2 = -\frac{m_B^2}{32\pi^2} \frac{m_R^2}{\left(\frac{m_R^2}{\lambda_R^*} + \frac{(\xi_R - \frac{1}{6})R_0}{32\pi^2}\right)} \left[\frac{2}{\epsilon} + \ln\left(\frac{m_R^2}{\tilde{\mu}^2}\right) - 2\frac{dF_{dS}}{dm_{ph}^2}\Big|_{m_R^2, R_0}\right], \quad (33)$$

$$\delta \tilde{\xi} \equiv \xi_B - \xi_R = -\frac{\left(\xi_B - \frac{1}{6}\right)}{32\pi^2} \frac{\left\{\left(\xi_R - \frac{1}{6}\right)\left[\frac{2}{\epsilon} + 1 + \ln\left(\frac{m_R^2}{\tilde{\mu}^2}\right)\right] + 2\frac{dF_{dS}}{dR}\Big|_{m_R^2, R_0}\right\}}{\left[\frac{(\xi_R - \frac{1}{6})}{\lambda_R} + J\right]}, \quad (34)$$

$$\delta \tilde{\lambda} \equiv \lambda_{B2} - \lambda_R = -\frac{\lambda_{B2}\lambda_R}{32\pi^2}\left[\frac{2}{\epsilon} + 1 + \ln\left(\frac{m_R^2}{\tilde{\mu}^2}\right) + \frac{(\xi_R - \frac{1}{6})R_0}{m_R^2} - 2\frac{dF_{dS}}{dm_{ph}^2}\Big|_{m_R^2, R_0}\right]. \quad (35)$$

Note that the well known one-loop results can be recovered from these expressions, making the replacements $m_B^2 \to m_R^2$, $\xi_B \to \xi_R$, $\lambda_{B2} \to \lambda_R$, and $R_0 \to 0$ on the right-hand-sides.

IV. RENORMALIZATION OF THE SEMICLASSICAL EINSTEIN EQUATIONS

So far we have dealt with Eqs. (19) and (20), that give the dynamics of ϕ_0 and G for a given choice of metric $g_{\mu\nu}$. However these equations do not take into account the effect of the quantum field on the background geometry. In order to assess whether this backreaction is important or not, we must deal with the SEE, obtained from the stationarity condition given in Eq. (5) with the gravitational action Eq. (7) and the definition of the vacuum expectation value of the energy-momentum tensor given in Eq. (6). The resulting equations are

$$\kappa_B^{-1} G_{\mu\nu} + \Lambda_B \kappa_B^{-1} g_{\mu\nu} + \alpha_{1B}\,^{(1)}H_{\mu\nu} + \alpha_{2B}\,^{(2)}H_{\mu\nu} + \alpha_{3B}\,H_{\mu\nu} = \langle T_{\mu\nu}\rangle, \quad (36)$$

where $\kappa_B = 8\pi G_B$. An explicit expression for the tensors $^{(1,2)}H_{\mu\nu}$ and $H_{\mu\nu}$ can be found for instance in [55].

The renormalization procedure then involves the calculation of $\langle T_{\mu\nu}\rangle$ and the regularization of its divergences. The divergences can be of either one of two types, independent of the field ϕ_0 and therefore only geometrical, or otherwise ϕ_0-dependent either explicitly or implicitly through $m_{ph}^2(\phi_0)$. The SEE are renormalizable if, with the same choice of counterterms as for the field and gap equations, the non-geometrical divergences can be completely dealt with. In order to absorb the geometrical divergences in the renormalization of the parameters of the gravitational part of the action, κ_B^{-1}, Λ_B and α_{iB}, these divergences must be

proportional to the tensors that appear on the left-hand side of Eq. (36) (note that in four spacetime dimensions the tensors $^{(1,2)}H_{\mu\nu}$ and $H_{\mu\nu}$ are not all independent).

We will follow the usual procedure and define the renormalized energy-momentum tensor as

$$\langle T_{\mu\nu}\rangle_{ren} = \langle T_{\mu\nu}\rangle - \langle T_{\mu\nu}\rangle_{ad4}\,, \tag{37}$$

where the fourth adiabatic order is understood as the expansion containing up to four derivatives of the metric and up to two derivatives of the mean field [55]. Our goal in this section is to show that with the choice of the counterterms for the field and gap equations, $\langle T_{\mu\nu}\rangle_{ad4}$ only contains geometric divergences, that can be absorbed into the bare gravitational constants.

The expectation value $\langle T_{\mu\nu}\rangle$ can be formally computed from the definition Eq. (6). One can show that [54]

$$\langle T_{\mu\nu}\rangle = T_{\mu\nu}(\phi_0) + \langle T^f_{\mu\nu}\rangle + \frac{\lambda_{B2}}{32}[G_1]^2 g_{\mu\nu}, \tag{38}$$

where the first term is the classical energy-momentum tensor evaluated at ϕ_0

$$T_{\mu\nu}(\phi_0) = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} = (1 - 2\xi_B)\phi_{0,\mu}\phi_{0,\nu} - 2\xi_B\phi_{0;\mu\nu}\phi_0 + 2\xi_B g_{\mu\nu}\phi_0\Box\phi_0 + \xi_B\phi_0^2 G_{\mu\nu}$$
$$+ \left(2\xi_B - \frac{1}{2}\right)g_{\mu\nu}\phi_0^{;\lambda}\phi_{0,\lambda} - \frac{m_B^2}{2}g_{\mu\nu}\phi_0^2 - \frac{\lambda_{B4}}{4!}g_{\mu\nu}\phi_0^4. \tag{39}$$

The second term is formally the mean value of the energy-momentum tensor of a free field, constructed with the two-point function G_1. More explicitly, it can be written as [55, 56]

$$\langle T^f_{\mu\nu}\rangle = -\frac{1}{2}[G_{1;\mu\nu}] + \frac{(1 - 2\xi_B)}{4}[G_1]_{;\mu\nu} + \left(\xi_B - \frac{1}{4}\right)\frac{g_{\mu\nu}}{2}\Box[G_1] + \xi_B R_{\mu\nu}\frac{[G_1]}{2}. \tag{40}$$

As a side point, we mention that one could also derive Eq. (38) using a different approach: take the classical energy-momentum tensor for the action Eq. (8), evaluate for $\phi = \phi_0 + \varphi$ and then expand on the fluctuation φ. Afterwards take the expectation value $\langle\ldots\rangle$ and recall that in the Hartree approximation one can write the expectation values of products of fields in terms of ϕ_0 and $\langle\varphi^2\rangle = [G_1]/2$ (and derivatives), using that

$$\langle\varphi^3\rangle = 0, \tag{41a}$$
$$\langle\varphi^4\rangle = \frac{3}{4}[G_1]^2. \tag{41b}$$

For the renormalization it is useful to separate, in the expressions for $T_{\mu\nu}(\phi_0)$ and $\langle T^f_{\mu\nu}\rangle$, the bare couplings into the corresponding renormalized parts and the nonminimal subtraction counterterms

$$T_{\mu\nu}(\phi_0) = T_{\mu\nu}(\phi_0)\Big|_{B=R} + \delta\tilde{\xi}\left(-\phi^2_{0;\mu\nu} + g_{\mu\nu}\Box\phi^2_0 + \phi^2_0 G_{\mu\nu}\right) - \frac{\delta\tilde{m}^2}{2}\phi^2_0 g_{\mu\nu} \tag{42}$$

$$\langle T^f_{\mu\nu}\rangle = \langle T^f_{\mu\nu}\rangle\Big|_{B=R} + \frac{\delta\tilde{\xi}}{2}\left(-[G_1]_{;\mu\nu} + g_{\mu\nu}\Box[G_1] + R_{\mu\nu}[G_1]\right), \tag{43}$$

where $B = R$ is a notational shorthand to indicate a replacement of the bare couplings with the renormalized ones. It will be also useful to write separately the interaction term in the classical energy momentum tensor

$$T_{\mu\nu}(\phi_0)\Big|_{B=R} = T_{\mu\nu}(\phi_0)\Big|_{B=R,free} - \frac{\lambda_{B4}}{4!}\phi^4_0 g_{\mu\nu}. \tag{44}$$

Note that while there are no divergences in $T_{\mu\nu}(\phi_0)|_{B=R,free}$, the quantity $\langle T^f_{\mu\nu}\rangle|_{B=R}$ still has divergences that arise from the coincidence limit of G_1 and of its derivatives. Recall Eq. (20), which implies that in our case the two-point function is that of a field of mass m^2_{ph} and curvature coupling ξ_R.

We are now ready to show that the counterterms already chosen to renormalize the mean field and gap equations also cancel the non-geometrical divergences in $\langle T_{\mu\nu}\rangle$. The third term of Eq. (38) as well as the terms that were isolated in Eq. (43) involve $[G_1]$ and its derivatives, and therefore they can be expressed in terms of m^2_{ph} and the bare couplings by using that the physical mass is defined by the equality of Eqs. (11) and (20), which in a more convenient form reads

$$\frac{\lambda_{B2}}{4}[G_1] = m^2_{ph} - \delta\tilde{\xi}R - m^2_B - \frac{\lambda_{B2}}{2}\phi^2_0. \tag{45}$$

With this replacement we have

$$\langle T_{\mu\nu}\rangle = T_{\mu\nu}(\phi_0)\Big|_{B=R,free} + \langle T^f_{\mu\nu}\rangle\Big|_{B=R} + \frac{(3\lambda_{B2} - \lambda_{B4})}{4!}\phi^4_0 g_{\mu\nu}$$

$$+ \frac{2\delta\tilde{\xi}}{\lambda_{B2}}\left[-m^2_{ph;\mu\nu} + g_{\mu\nu}\Box m^2_{ph} + G_{\mu\nu}m^2_{ph}\right] + \frac{m^4_{ph}}{2\lambda_{B2}}g_{\mu\nu} - m^2_{ph}\frac{m^2_B}{\lambda_B}g_{\mu\nu}$$

$$+ \frac{\delta\tilde{\xi}^2}{\lambda_{B2}}{}^{(1)}H_{\mu\nu} - 2\delta\tilde{\xi}\frac{m^2_B}{\lambda_{B2}}G_{\mu\nu} + \frac{m^2_B}{2}\frac{m^2_B}{\lambda_B}g_{\mu\nu}$$

$$+ (m^2_R - m^2_{ph})\frac{\phi^2_0}{2}g_{\mu\nu}. \tag{46}$$

13

Here the term proportional to ϕ_0^4 is already finite because of the relation Eq. (16d) between the counterterms, and thus equal to $\lambda_R \phi_0^4 g_{\mu\nu}/12$. The fourth, fifth and sixth terms contain the non-geometrical divergences that will have to be cancelled by those from $\langle T_{\mu\nu}^f \rangle|_{B=R}$. The remaining terms contain purely geometrical divergences.

It is worth to emphasize that the divergences in Eq. (46) are proportional to simple poles in ϵ. Indeed, from the definition of $\delta\tilde{\xi} = \xi_B - \xi_R$ and the relations (31) it is straightforward to see that

$$\frac{\delta\tilde{\xi}}{\lambda_{B2}} = \left(\frac{1}{\lambda_R} - \frac{1}{\lambda_{B2}} \right) \left(\xi_R - \frac{1}{6} \right) + J, \tag{47a}$$

$$\frac{\delta\tilde{\xi}^2}{\lambda_{B2}} = \lambda_{B2} \left[\frac{\left(\xi_R - \frac{1}{6} \right)}{\lambda_R} + J \right]^2 - 2 \left(\xi_R - \frac{1}{6} \right) \left[\frac{\left(\xi_R - \frac{1}{6} \right)}{\lambda_R} + J \right]$$

$$+ \frac{\left(\xi_R - \frac{1}{6} \right)^2}{\lambda_{B2}}, \tag{47b}$$

which are exact expressions. Note that λ_{B2}^{-1} contains just a simple pole,

$$\frac{1}{\lambda_{B2}} = \frac{1}{\lambda} + \frac{1}{16\pi^2\epsilon}. \tag{48}$$

We now expand $\langle T_{\mu\nu} \rangle$ up to the fourth adiabatic order. We will use the explicit expressions for the coincidence limit of G_1 and its derivatives that are given in Ref. [55]. The fourth adiabatic order expansion for $\langle \tilde{T}_{\mu\nu} \rangle \equiv \langle T_{\mu\nu}^f \rangle|_{B=R}$ is

$$\langle \tilde{T}_{\mu\nu} \rangle_{ad4} = \frac{1}{16\pi^2} \left(\frac{m_{ph}^2}{\mu^2} \right)^{\epsilon/2} \left[\frac{1}{2} m_{ph}^4 \, g_{\mu\nu} \, \Gamma \left(-2 - \frac{\epsilon}{2} \right) + m_{ph}^2 \left\{ \frac{1}{2} [\Omega_1] g_{\mu\nu} + \left(\xi_R - \frac{1}{6} \right) R_{\mu\nu} \right\} \right.$$

$$\times \Gamma \left(-1 - \frac{\epsilon}{2} \right) + \left\{ \frac{1}{2} [\Omega_2] g_{\mu\nu} + \left(\xi_R - \frac{1}{6} \right) R_{\mu\nu}[\Omega_1] - [\Omega_{1;\mu\nu}] \right.$$

$$\left. + \left(\frac{1}{2} - \xi_R \right) [\Omega_1]_{;\mu\nu} + \left(\xi_R - \frac{1}{4} \right) g_{\mu\nu} \Box[\Omega_1] \right\} \Gamma \left(-\frac{\epsilon}{2} \right) \right], \tag{49}$$

where the expressions for $[\Omega_1]$, $[\Omega_2]$ and $[\Omega_{1;\mu\nu}]$ can be found in the Appendix A of [55]. Notice however that here these contributions are expressed in terms of ξ_R instead of ξ_B. Expanding for $\epsilon \to 0$, regrouping the geometric terms to form the appropriate tensors and

separating the divergent part one arrives at

$$
\begin{aligned}
\langle \tilde{T}_{\mu\nu} \rangle_{ad4} = \frac{1}{16\pi^2 \epsilon} \Bigg\{ &-\frac{1}{2} m_{ph}^4 g_{\mu\nu} + 2m_{ph}^2 \left(\xi_R - \frac{1}{6} \right) G_{\mu\nu} + \frac{1}{90} \left[{}^{(2)}H_{\mu\nu} - H_{\mu\nu} \right] \\
&- \left(\xi_R - \frac{1}{6} \right)^2 {}^{(1)}H_{\mu\nu} + 2 \left(\xi_R - \frac{1}{6} \right) \left(g_{\mu\nu} \Box m_{ph}^2 - m_{ph;\mu\nu}^2 \right) \Bigg\} \\
&+ \frac{m_{ph}^4}{64\pi^2} g_{\mu\nu} \left[\frac{1}{2} - \ln \left(\frac{m_{ph}^2}{\tilde{\mu}^2} \right) \right] + \frac{m_{ph}^2}{16\pi^2} \left(\xi_R - \frac{1}{6} \right) G_{\mu\nu} \ln \left(\frac{m_{ph}^2}{\tilde{\mu}^2} \right) \\
&+ \frac{1}{32\pi^2} \Bigg[\frac{1}{90} \left({}^{(2)}H_{\mu\nu} - H_{\mu\nu} \right) - \left(\xi_R - \frac{1}{6} \right)^2 {}^{(1)}H_{\mu\nu} \\
&+ 2 \left(\xi_R - \frac{1}{6} \right) \left(g_{\mu\nu} \Box m_{ph}^2 - m_{ph;\mu\nu}^2 \right) \Bigg] \left[1 + \ln \left(\frac{m_{ph}^2}{\tilde{\mu}^2} \right) \right].
\end{aligned}
\tag{50}
$$

Replacing Eq. (50) into Eq. (46) one can verify that the non-geometrical divergences in Eq. (46) cancel out. This result shows the renormalizability of the SEE within the consistent renormalization approach.

In order to complete the analysis, we write the full expression for the fourth adiabatic order, which we separate in its divergent and a convergent parts:

$$
\langle T_{\mu\nu} \rangle_{ad4} = \langle T_{\mu\nu} \rangle_{ad4}^{div} + \langle T_{\mu\nu} \rangle_{ad4}^{con},
\tag{51}
$$

with

$$
\begin{aligned}
\langle T_{\mu\nu} \rangle_{ad4}^{div} = \frac{1}{90} \frac{1}{32\pi^2} &\left[\frac{2}{\epsilon} + 1 + \ln \left(\frac{m_R^2}{\tilde{\mu}^2} \right) \right] \left({}^{(2)}H_{\mu\nu} - H_{\mu\nu} \right) - 2\delta\tilde{\xi} \left[\frac{m_R^2}{\lambda_R^*} + \frac{\left(\xi_R - \frac{1}{6} \right) R_0}{32\pi^2} \right] G_{\mu\nu} \\
&+ \delta\tilde{\xi} \left[\frac{\left(\xi_R - \frac{1}{6} \right)}{\lambda_R} + J \right] {}^{(1)}H_{\mu\nu} + \frac{\delta\tilde{m}}{2} \left[\frac{m_R^2}{\lambda_R^*} + \frac{\left(\xi_R - \frac{1}{6} \right) R_0}{32\pi^2} \right] g_{\mu\nu} - \frac{m_R^4}{64\pi^2} g_{\mu\nu},
\end{aligned}
\tag{52}
$$

15

and

$$\langle T_{\mu\nu}\rangle^{con}_{ad4} = T_{\mu\nu}(\phi_0)\Big|_{B=R,free} + \frac{\lambda_R}{12}\phi_0^4 g_{\mu\nu} + \left(\frac{m_R^2}{2} - m_{ph}^2\right)\left[\frac{m_R^2}{\lambda_R^*} + \frac{(\xi_R - \frac{1}{6})R_0}{32\pi^2}\right]g_{\mu\nu}$$

$$+ \frac{m_{ph}^4}{64\pi^2}\left[\frac{32\pi^2}{\lambda_R^*} + \frac{1}{2} + \left(\xi_R - \frac{1}{6}\right)\frac{R_0}{m_R^2} - 2\frac{dF_{dS}}{dm_{ph}^2}\Big|_{m_R^2,R_0}\right]g_{\mu\nu}$$

$$+ \frac{1}{16\pi^2}\left[2m_{ph}^2 G_{\mu\nu} - \left(\xi_R - \frac{1}{6}\right){}^{(1)}H_{\mu\nu} + 2g_{\mu\nu}\Box m_{ph}^2 - 2m_{ph;\mu\nu}^2\right]\frac{dF_{dS}}{dR}\Big|_{m_R^2,R_0}$$

$$+ \frac{1}{32\pi^2}\left\{-\frac{m_{ph}^4}{2}g_{\mu\nu} + 2m_{ph}^2\left(\xi_R - \frac{1}{6}\right)G_{\mu\nu} + \frac{1}{90}\left({}^{(2)}H_{\mu\nu} - H_{\mu\nu}\right)\right.$$

$$\left. - \left(\xi_R - \frac{1}{6}\right)^2 {}^{(1)}H_{\mu\nu} + 2\left(\xi_R - \frac{1}{6}\right)\left(g_{\mu\nu}\Box m_{ph}^2 - m_{ph;\mu\nu}^2\right)\right\}\ln\left(\frac{m_{ph}^2}{m_R^2}\right)$$

$$- \frac{m_{ph}^2}{16\pi^2}\left(\xi_R - \frac{1}{6}\right)G_{\mu\nu} + \left(m_R^2 - m_{ph}^2\right)\frac{\phi_0^2}{2}g_{\mu\nu} + \frac{m_R^4}{64\pi^2}g_{\mu\nu}. \tag{53}$$

As anticipated, the divergent part contains purely geometric divergences. The convergent part is field dependent, finite, and written in terms of the renormalized parameters (therefore independent of $\tilde{\mu}$). To ensure the correct one-loop limit of the cosmological constant counterterm, we included the finite contribution $-\frac{m_R^4}{64\pi^2}g_{\mu\nu}$ in $\langle T_{\mu\nu}\rangle^{div}_{ad4}$.

Now we can add and subtract $\langle T_{\mu\nu}\rangle_{ad4}$ in the right-hand side of the SEE

$$\kappa_B^{-1}(G_{\mu\nu} + \Lambda_B g_{\mu\nu}) + \alpha_{1B}{}^{(1)}H_{\mu\nu} + \alpha_{2B}{}^{(2)}H_{\mu\nu} + \alpha_{3B}H_{\mu\nu} =$$

$$[\langle T_{\mu\nu}\rangle - \langle T_{\mu\nu}\rangle_{ad4}] + \langle T_{\mu\nu}\rangle^{div}_{ad4} + \langle T_{\mu\nu}\rangle^{con}_{ad4}, \tag{54}$$

where the quantity between square brackets on the right-hand side is defined as $\langle T_{\mu\nu}\rangle_{ren}$. Renormalization is completed by absorbing $\langle T_{\mu\nu}\rangle^{div}_{ad4}$ into a redefinition of the bare gravitational constants of the left-hand side. Then the renormalized gravitational parameters read

$$\kappa_B^{-1} = \kappa_R^{-1} + \frac{m_B^2}{8\pi^2}\left\{\left(\xi_R - \frac{1}{6}\right)\left[\frac{1}{\epsilon} + \frac{1}{2} + \frac{1}{2}\ln\left(\frac{m_R^2}{\tilde{\mu}^2}\right)\right] - \frac{dF_{dS}}{dR}\Big|_{m_R^2,R_0}\right\}, \tag{55a}$$

$$\Lambda_B\kappa_B^{-1} = \Lambda_R\kappa_R^{-1} - \frac{m_B^2 m_R^2}{32\pi^2}\left[\frac{1}{\epsilon} + \frac{1}{2}\ln\left(\frac{m_R^2}{\tilde{\mu}^2}\right) - \frac{dF_{dS}}{dm_{ph}^2}\Big|_{m_R^2,R_0}\right] - \frac{m_R^4}{64\pi^2}, \tag{55b}$$

$$\alpha_{1B} = \alpha_{1R} - \frac{(\xi_B - \frac{1}{6})}{16\pi^2}\left\{\left(\xi_R - \frac{1}{6}\right)\left[\frac{1}{\epsilon} + \frac{1}{2} + \frac{1}{2}\ln\left(\frac{m_R^2}{\tilde{\mu}^2}\right)\right] - \frac{dF_{dS}}{dR}\Big|_{m_R^2,R_0}\right\}, \tag{55c}$$

$$\alpha_{2B} = \alpha_{2R} + \frac{1}{1440\pi^2}\left[\frac{1}{\epsilon} + \frac{1}{2} + \frac{1}{2}\ln\left(\frac{m_R^2}{\tilde{\mu}^2}\right)\right], \tag{55d}$$

$$\alpha_{3B} = \alpha_{3R} - \frac{1}{1440\pi^2}\left[\frac{1}{\epsilon} + \frac{1}{2} + \frac{1}{2}\ln\left(\frac{m_R^2}{\tilde{\mu}^2}\right)\right]. \tag{55e}$$

These are consistent with the well known one-loop results when replacing the bare parameters in the right-hand side (in the counterterms) by the renormalized ones and setting $R_0 \to 0$, thus justifying the choice of $\langle T_{\mu\nu} \rangle_{ad4}^{div}$ in Eq. (52). As it happens for the field parameters, the relation between the bare and renormalized expressions is $\tilde{\mu}$-dependent.

Finally, the renormalized SEE are

$$\kappa_R^{-1} G_{\mu\nu} + \Lambda_R \kappa_R^{-1} g_{\mu\nu} + \alpha_{1R} \,^{(1)}H_{\mu\nu} + \alpha_{2R} \,^{(2)}H_{\mu\nu} + \alpha_{3R} \, H_{\mu\nu} = \langle T_{\mu\nu} \rangle_{ren} + \langle T_{\mu\nu} \rangle_{ad4}^{con}, \qquad (56)$$

which, as expected, are expressed only in terms of renormalized parameters.

V.   INTERACTING FIELDS IN DE SITTER SPACETIME

In this section we apply the previous results to de Sitter spacetime with $ds^2 = -dt^2 + e^{2Ht} d\vec{x}^2$, and compute explicitly the renormalized energy-momentum tensor, and the SEE. We then consider both the field equation and the SEE to analyze the existence of self-consistent solutions.

A.   Gap and semiclassical Einstein equations

In de Sitter spacetime, the solution of the Eq. (20) for the propagator, which is the one of a free field with mass m_{ph}^2, is known exactly for an arbitrary number of dimensions n. The expression for the coincidence limit $[G_1]$ is

$$[G_1] = \frac{2H^{n-2}}{(4\pi\mu^2)^{n/2}} \Gamma\left(1 - \frac{n}{2}\right) \frac{\Gamma\left(\frac{n-1}{2} + \nu_n\right) \Gamma\left(\frac{n-1}{2} - \nu_n\right)}{\Gamma\left(\frac{1}{2} + \nu_n\right) \Gamma\left(\frac{1}{2} - \nu_n\right)}, \qquad (57)$$

where $\nu_n^2 = \frac{(n-1)^2}{4} - \frac{m_{ph}^2}{H^2} - \xi_R n(n-1)$ and $R = n(n-1)H^2$.

To make use of the results of previous sections we need to extract the function $F_{dS}(m_{ph}^2, R)$, defined in Eq. (23), from this exact expression. For this, we set $n = 4 + \epsilon$ and expand for $\epsilon \to 0$, holding R fixed. Doing this, as shown in detail in paper I, we obtain the following expression for the function $F(m_{ph}^2, \{R\})$ in de Sitter spacetime

$$F_{dS}(m_{ph}^2, R) = R \, f(m_{ph}^2/R) = -\frac{R}{2} \left\{ \left(\frac{m_{ph}^2}{R} + \xi_R - \frac{1}{6} \right) \left[\ln\left(\frac{R}{12 m_{ph}^2} \right) + g\left(m_{ph}^2/R + \xi_R \right) \right] \right.$$
$$\left. - \left(\xi_R - \frac{1}{6} \right) - \frac{1}{18} \right\} \qquad (58)$$

with

$$g(y) \equiv \psi_+ + \psi_- = \psi \left(\frac{3}{2} + \nu_4(y) \right) + \psi \left(\frac{3}{2} - \nu_4(y) \right),$$ (59)

and $R = 12H^2$, $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function and $\nu_4(y) = \sqrt{9/4 - 12y}$. From this equation one can check that this function has all the expected properties: it is written only in terms of renormalized parameters, it is independent of ϵ and $\tilde{\mu}$, and it satisfies the correct limits Eqs. (25a), (25b) and (25c).

Therefore, the renormalized equation for the physical mass m_{ph}^2 we are going to solve self-consistently together with the SEE we calculate below, can be written as:

$$m_{ph}^2 = m_R^2 + \frac{\lambda_R^*}{2} \phi_0^2 + \frac{\lambda_R^*}{32\pi^2} \left[m_{ph}^2 + \left(\xi_R - \frac{1}{6} \right) R \right] \left[\ln \left(\frac{R}{12m_R^2} \right) + g \left(m_{ph}^2/R + \xi_R \right) \right]$$
$$- \frac{\lambda_R^*}{32\pi^2} \left(\xi_R - \frac{1}{9} \right) R.$$ (60)

In de Sitter spacetime all geometrical quantities can be written in terms of only R and $g_{\mu\nu}$. In n dimensions they are:

$$R_{\mu\nu} = \frac{R}{n} g_{\mu\nu},$$ (61a)

$$G_{\mu\nu} = \left(\frac{1}{n} - \frac{1}{2} \right) R \, g_{\mu\nu},$$ (61b)

$$^{(1)}H_{\mu\nu} = \frac{1}{2} \left(1 - \frac{4}{n} \right) R^2 \, g_{\mu\nu},$$ (61c)

$$^{(2)}H_{\mu\nu} = \frac{1}{2n} \left(1 - \frac{4}{n} \right) R^2 \, g_{\mu\nu},$$ (61d)

$$H_{\mu\nu} = \frac{1}{n(n-1)} \left(1 - \frac{4}{n} \right) R^2 \, g_{\mu\nu}.$$ (61e)

In fact, any 2nd-rank tensor is proportional to the metric, so that

$$[G_{1;\mu\nu}] = \frac{1}{n} [\Box G_1] \, g_{\mu\nu}.$$ (62)

De Sitter invariance also implies that any scalar function has vanishing derivative, and in particular that $[G_1]$ is independent of spacetime coordinates. The energy-momentum tensor will also be proportional to $g_{\mu\nu}$. Indeed, from the general expression Eq. (38) together with Eqs. (39) and (40), and using Eq. (61), we obtain

$$\langle T_{\mu\nu} \rangle = \left[-\frac{m_B^2}{2} \phi_0^2 - \frac{\lambda_{B4}}{4!} \phi_0^4 + \xi_B \phi_0^2 \left(\frac{1}{n} - \frac{1}{2} \right) R - \frac{1}{2n} [\Box G_1] - \frac{m_B^2}{4} [G_1] \right.$$
$$\left. + \frac{1}{4} [\Box G_1] + \xi_B \frac{[G_1]}{2} \left(\frac{1}{n} - \frac{1}{2} \right) R - \frac{\lambda_{B2}}{8} \phi_0^2 [G_1] - \frac{\lambda_{B2}}{32} [G_1]^2 \right] g_{\mu\nu}.$$ (63)

18

Once again we use Eq. (45) to make previous expression simpler, and we put $n = 4 + \epsilon$,

$$\langle T_{\mu\nu} \rangle = \left\{ -\frac{m_B^2}{2}\phi_0^2 - \frac{\xi_B}{4}\phi_0^2 R - \frac{\lambda_{B4}}{4!}\phi_0^4 - \frac{1}{8}\left[m_B^2 + \frac{\lambda_{B2}}{2}\phi_0^2\right][G_1] \right.$$
$$\left. + \frac{1}{4}\left(\frac{4}{4+\epsilon} - 1\right)\left[\xi_B\phi_0^2 R - \frac{1}{2}\left(m_{ph}^2 - \delta\tilde{\xi}R\right)[G_1]\right] \right\} g_{\mu\nu}. \tag{64}$$

Here we cannot set $\epsilon = 0$ in the denominator yet, as it is multiplied by both the bare parameters and $[G_1]$, that contain poles in ϵ that could give finite terms. After some manipulations and dropping terms that vanish for $\epsilon \to 0$, it reads

$$\langle T_{\mu\nu} \rangle = \left\{ \frac{1}{2}\left[\delta\tilde{m}^2 + \left(1 + \frac{\epsilon}{4+\epsilon}\right)\delta\tilde{\xi}R\right]\left(\frac{m_R^2}{\lambda_R^*} + \frac{\left(\xi_R - \frac{1}{6}\right)R_0}{32\pi^2}\right) \right. \tag{65}$$
$$+ \left(\frac{4}{4+\epsilon}\right)\frac{\epsilon\,\delta\tilde{\xi}}{8}\left(\frac{\left(\xi_R - \frac{1}{6}\right)}{\lambda_R} + J\right)R^2 + \frac{1}{2}\left(\frac{m_R^2}{\lambda_R^*} + \frac{\left(\xi_R - \frac{1}{6}\right)R_0}{32\pi^2}\right)\left(m_R^2 - m_{ph}^2\right)$$
$$\left. - \frac{1}{4}(m_{ph}^2 + \xi_R R)\phi_0^2 + \frac{\lambda_R}{12}\phi_0^4 + \frac{1}{128\pi^2}\left[m_{ph}^2 + \left(\xi_R - \frac{1}{6}\right)R\right]^2 \right\} g_{\mu\nu}.$$

To compute the renormalized expectation value, $\langle T_{\mu\nu}\rangle_{ren} = \langle T_{\mu\nu}\rangle - \langle T_{\mu\nu}\rangle_{ad4}$, we evaluate $\langle T_{\mu\nu}\rangle_{ad4}$ (given in Eq. (51)) in de Sitter spacetime, using the n-dimensional geometrical expressions Eq. (61). Separating the result again in $\langle T_{\mu\nu}\rangle_{ad4} = \langle T_{\mu\nu}\rangle_{ad4}^{div} + \langle T_{\mu\nu}\rangle_{ad4}^{con}$, up to order ϵ these two terms read

$$\langle T_{\mu\nu}\rangle_{ad4}^{div} = \left\{ \frac{1}{64\pi^2}\frac{R^2}{2160} + \frac{1}{2}\left[\delta\tilde{m}^2 + \left(1 + \frac{\epsilon}{4+\epsilon}\right)\delta\tilde{\xi}R\right]\left(\frac{m_R^2}{\lambda_R^*} + \frac{\left(\xi_R - \frac{1}{6}\right)R_0}{32\pi^2}\right) \right.$$
$$\left. + \left(\frac{4}{4+\epsilon}\right)\frac{\epsilon\,\delta\tilde{\xi}}{8}\left(\frac{\left(\xi_R - \frac{1}{6}\right)}{\lambda_R} + J\right)R^2 - \frac{m_R^4}{64\pi^2} \right\} g_{\mu\nu}, \tag{66}$$
$$\langle T_{\mu\nu}\rangle_{ad4}^{con} = \left\{ \frac{m_R^2}{2}\left[\frac{m_R^2}{\lambda_R^*} + \frac{\left(\xi_R - \frac{1}{6}\right)R_0}{32\pi^2} + \frac{m_R^2}{32\pi^2}\right] + \frac{m_{ph}^2}{64\pi^2}\left(\xi_R - \frac{1}{6}\right)R \right.$$
$$+ \frac{m_{ph}^4}{64\pi^2}\left[\frac{32\pi^2}{\lambda_R^*} + \frac{1}{2} + \frac{\left(\xi_R - \frac{1}{6}\right)R_0}{m_R^2} - 2\frac{dF_{dS}}{dm_{ph}^2}\Big|_{m_R^2, R_0}\right]$$
$$- \frac{m_{ph}^2}{64\pi^2}\left[m_{ph}^2 + \left(\xi_R - \frac{1}{6}\right)R\right]\ln\left(\frac{m_{ph}^2}{m_R^2}\right) - \frac{m_{ph}^2 R}{32\pi^2}\frac{dF_{dS}}{dR}\Big|_{m_R^2, R_0}$$
$$\left. - m_{ph}^2\left[\frac{m_R^2}{\lambda_R^*} + \frac{\left(\xi_R - \frac{1}{6}\right)R_0}{32\pi^2}\right] - \left(m_{ph}^2 + \frac{\xi_R}{2}R\right)\frac{\phi_0^2}{2} + \frac{\lambda_R}{12}\phi_0^4 \right\} g_{\mu\nu}. \tag{67}$$

19

The first term of Eq. (66) is finite and is the source of the trace anomaly [1]. Then we have

$$
\langle T_{\mu\nu}\rangle_{ren} = -\frac{1}{64\pi^2}\left\{ m_{ph}^2\left[\left(\frac{32\pi^2}{\lambda_R^*} + \frac{(\xi_R - \frac{1}{6})R_0}{m_R^2} - 2\frac{dF_{dS}}{dm_{ph}^2}\Big|_{m_R^2,R_0}\right)(m_{ph}^2 - m_R^2) - 16\pi^2\phi_0^2\right.\right.
$$
$$
\left. - \left[m_{ph}^2 + \left(\xi_R - \frac{1}{6}\right)R\right]\ln\left(\frac{m_{ph}^2}{m_R^2}\right) - 2R\frac{dF_{dS}}{dR}\Big|_{m_R^2,R_0} - 2m_R^2\frac{dF_{dS}}{dm_{ph}^2}\Big|_{m_R^2,R_0}\right]
$$
$$
\left. - \frac{1}{2}\left(\xi_R - \frac{1}{6}\right)^2 R^2 + \frac{R^2}{2160}\right\}g_{\mu\nu}. \tag{68}
$$

To make contact with the known free and one-loop expressions, we use Eq. (32) to arrive at a more familiar result

$$
\langle T_{\mu\nu}\rangle_{ren} = -\frac{1}{64\pi^2}\left\{ 2m_{ph}^2\left[F_{dS}(m_R^2,R_0) - R_0\frac{dF_{dS}}{dR}\Big|_{m_R^2,R_0} - m_R^2\frac{dF_{dS}}{dm_{ph}^2}\Big|_{m_R^2,R_0} - F(m_{ph}^2,R)\right]\right.
$$
$$
\left. - \frac{1}{2}\left(\xi_R - \frac{1}{6}\right)^2 R^2 + \frac{R^2}{2160}\right\}g_{\mu\nu}. \tag{69}
$$

Setting $R_0 \to 0$ and using Eq. (58) for F_{dS}, gives an expression that is exactly the same as in the one-loop calculation [55], provided $m_{ph}^2 = m_R^2 + \lambda_R\phi_0^2/2$ instead of being the solution of the self-consistent Eq. (60). Furthermore, it is straightforward that the usual free field limit [1] is satisfied, as $m_{ph}^2 \to m_R^2$ when $\lambda_R \to 0$.

Turning finally to the SEE, on the right-hand side we have

$$
\langle T_{\mu\nu}\rangle_{ren} + \langle T_{\mu\nu}\rangle_{con} = \left[-\frac{1}{4}(m_{ph}^2 + \xi_R R)\phi_0^2 + \frac{\lambda_R}{12}\phi_0^4\right]g_{\mu\nu} \tag{70}
$$
$$
- \frac{1}{64\pi^2}\left\{32\pi^2\left(\frac{m_R^2}{\lambda_R^*} + \frac{\left(\xi_R - \frac{1}{6}\right)R_0}{32\pi^2}\right)(m_{ph}^2 - m_R^2)\right.
$$
$$
\left. - m_R^4 + \frac{R^2}{2160} - \frac{1}{2}\left[m_{ph}^2 + \left(\xi_R - \frac{1}{6}\right)R\right]^2\right\}g_{\mu\nu},
$$

while on the left-hand side we have $G_{\mu\nu} + \Lambda_R g_{\mu\nu} = (-R/4 + \Lambda_R)g_{\mu\nu}$, as the quadratic tensors $^{(1)}H_{\mu\nu}$, $^{(2)}H_{\mu\nu}$ and $H_{\mu\nu}$ vanish for $n = 4$. Then, canceling the $g_{\mu\nu}$ that appears on both sides, we have:

$$
M_{pl}^2\left(-\frac{R}{4} + \Lambda_R\right) = -\frac{1}{8\pi}\left\{\frac{R^2}{2160} + 32\pi^2\left(\frac{m_R^2}{\lambda_R^*} + \frac{\left(\xi_R - \frac{1}{6}\right)R_0}{32\pi^2}\right)(m_{ph}^2 - m_R^2) - m_R^4 \right. \tag{71}
$$
$$
\left. + 16\pi^2(m_{ph}^2 + \xi_R R)\phi_0^2 - 64\pi^2\frac{\lambda_R}{12}\phi_0^4 - \frac{1}{2}\left[m_{ph}^2 + \left(\xi_R - \frac{1}{6}\right)R\right]^2\right\},
$$

where M_{pl} is Planck's mass, and $\kappa_R = 8\pi/M_{pl}^2$.

B. Self-consistent de Sitter solutions

The back-reaction problem consists in solving simultaneously the mean field Eq. (19), the m_{ph}^2 Eq. (60) and the SEE (71) self-consistently for the mean field ϕ_0, the physical mass m_{ph}^2 and the scalar curvature of de Sitter spacetime R. This is a closed system of equations for a given set of parameters m_R^2, ξ_R, λ_R and Λ_R, whose physically interesting solutions in a cosmological scenario are those with both R and $\mathcal{M}_{ph}^2 = m_{ph}^2 + \xi_R R$ positive. The second condition comes from the fact that $\mathcal{M}_{ph}^2$ is the mass of the propagator, and it is a well known fact that the equation

$$\Box G_1(x, x') = 0, \tag{72}$$

has no de Sitter invariant solutions.

The gap Eq. (60) is in itself a self-consistent equation for $m_{ph}^2(\phi_0, R)$, at fixed ϕ_0 and R. Following paper I, in the small mass approximation ($y \equiv \mathcal{M}_{ph}^2/R \ll 1$) we have $g(y) \simeq -1/4y + 11/6 - 2\gamma_E + 49y/9$ an thus the gap equation becomes a quadratic equation for y,

$$A_{dS}\, y^2 + \left[B_{dS} - \frac{\lambda_R \phi_0^2}{2R} \right] y + C_{dS} = 0 \tag{73}$$

where the coefficients are

$$A_{dS} = 1 - \frac{\lambda_R}{32\pi^2} \left[a\left(\frac{R}{R_0}\right) - g(y_0) - \left(y_0 - \frac{1}{6}\right) g'(y_0) - \frac{49}{54} \right], \tag{74a}$$

$$
\begin{aligned}
B_{dS} = &-\left[\frac{R_0}{R} y_0 + \xi_R \left(1 - \frac{R_0}{R}\right) \right] \\
&+ \frac{\lambda_R}{32\pi^2} \left\{ \frac{1}{4} + \frac{1}{6} \left[a\left(\frac{R}{R_0}\right) - g(y_0) - \left(y_0 - \frac{1}{6}\right) g'(y_0) \right] \right. \\
&\left. + \left(1 - \frac{R_0}{R}\right)\left(y_0 - \frac{1}{6}\right) - \left(y_0 - \frac{1}{6}\right)^2 g'(y_0) \right\},
\end{aligned}
\tag{74b}
$$

$$C_{dS} = -\frac{\lambda_R}{768\pi^2}, \tag{74c}$$

with

$$a(x) \equiv 11/6 - 2\gamma_E + \ln(x), \tag{75}$$

and $y_0 = y(\phi_0 = 0, R = R_0) = m_R^2/R_0 + \xi_R$. The solution can be expressed analytically

$$\mathcal{M}_{ph}^2(\phi_0, R) = \frac{-(R\, B_{dS} - \frac{\lambda_R \phi_0^2}{2}) + \sqrt{\left[R B_{dS} - \frac{\lambda_R \phi_0^2}{2}\right]^2 - 4R^2 A_{dS} C_{dS}}}{2A_{dS}}. \tag{76}$$

Here the "plus" branch was selected as the only real and positive solution (under the assumption that both $A_{dS} > 0$ and $B_{dS} > 0$, see paper I). This solution shall then be inserted into the mean field Eq. (19), which in de Sitter spacetime reads

$$\frac{dV_{eff}}{d\phi_0}\bigg|_{\bar{\phi}_0} = \left(\mathcal{M}_{ph}^2(\bar{\phi}_0, R) - \frac{1}{3}\lambda_R \bar{\phi}_0^2 \right) \bar{\phi}_0 = 0. \tag{77}$$

This equation admits both symmetric solutions with $\bar{\phi}_0 = 0$ and solutions that spontaneously break the Z_2 symmetry,

$$\bar{\phi}_0^2 = \frac{3}{\lambda_R} \mathcal{M}_{ph}^2(\bar{\phi}_0, R). \tag{78}$$

In other words, the effective potential $V_{eff}(\phi_0, R)$ may have other extrema besides the one in $\phi_0 = 0$. The analysis of the effective potential has been done in paper I.

Studying the full backreaction problem by including the SEE (71) brings a new parameter into play, namely the cosmological constant Λ_R, as well as a new mass scale M_{pl}^2. In paper I, R was considered fixed (i.e. as a parameter) and the effective potential and its minima were studied in order to find values of the remaining parameters m_R^2, ξ_R, λ_R and R_0 at which both symmetric and broken phase solutions exist. In the small mass approximation, this amounts to analyzing constrains on combinations of the coefficients A_{dS}, B_{dS} and C_{dS} as functions of the parameters. Considering R to be fixed makes sense under the assumption that the effect of the quantum field on the background curvature is small, and therefore it is possible to decouple the SEE from the field and gap equations. If this is indeed the case, the value of R becomes effectively independent of ϕ_0 and m_{ph}^2, and is simply given by the parameter Λ_R.

The aim of this section is to find some examples of self-consistent solutions involving all three equations and all three degrees of freedom. To this end, we take as starting point some sets of values of the parameters m_R^2, ξ_R, λ_R and R_0 that were already shown in paper I to allow both symmetric and broken phase solutions. Then we look for solutions of ϕ_0, m_{ph}^2 and R for various values of Λ_R and analyze how these differ from the classical solution. If this difference is small, then the backreaction can be indeed ignored, otherwise it should be taken into account.

One further point of discussion is whether the parameters R_0 and Λ_R should be related or not. If this were to be the case, a sensible way of fixing one given the other would be to use the classical solution $R_0 = 4\Lambda_R$.

22

1. Symmetric Phase

As mentioned above, the effective potential always has an extreme in $\phi_0 = 0$. Furthermore, it is easily shown that it must be a minimum as a consequence of both the restriction given by the Hartree approximation that $\mathcal{M}_{ph}^2 > 0$, and the fact that

$$\left. \frac{d^2 V_{eff}}{d\phi_0^2} \right|_{\bar{\phi}_0 = 0} = \mathcal{M}_{ph}^2(\bar{\phi}_0 = 0, R) > 0. \tag{79}$$

We solve the system of equations by setting $\phi_0 = 0$ in Eq. (76) to obtain $\mathcal{M}_{ph}^2$ as a function only of R and then substituing into the SEE (71) to obtain an equation of the form

$$\Lambda_R = I_s(R). \tag{80}$$

where I_s depends also on the parameters m_R^2, ξ_R, λ_R and R_0. The subindex s stands for symmetric.

2. Broken Phase

In this phase, the solution given in Eq. (78) to the field equation already implies $\mathcal{M}_{ph}^2 > 0$. It is important to note that the reason why the $\bar{\phi}_0 \neq 0$ solutions are allowed is the presence of the λ_R term in Eq. (77), which comes as a consequence of imposing the 2PI consistency relations. Otherwise, the absence of such term would require that for $\bar{\phi}_0 \neq 0$ we had $\mathcal{M}_{ph}^2 = 0$, and as mentioned before for that case there is no de Sitter invariant vacuum [52].

Replacing the non vanishing solution to the field Eq. (78) into the gap equation in its quadratic form Eq. (73) (small mass approximation), we obtain a new quadratic equation for the non symmetric extrema of the potential $\bar{\phi}_0^2(R)$, namely

$$\frac{\lambda_R}{3} \bar{\phi}_0^2(R) = \frac{-(R\,B_{dS} - \frac{\lambda_R \bar{\phi}_0^2}{2}) \pm \sqrt{\left[RB_{dS} - \frac{\lambda_R \bar{\phi}_0^2}{2} \right]^2 - 4R^2\,A_{dS}C_{dS}}}{2A_{dS}}. \tag{81}$$

Both branches give a solution with $\mathcal{M}_{ph}^2 > 0$, the smaller being the maximum and the larger the minimum of the effective potential. Following the analysis described in paper I, one can show that the condition for the existence of symmetry breaking solutions is

$$B_{dS} - 2\sqrt{\left(\frac{3}{2} - A_{dS} \right) |C_{dS}|} > 0. \tag{82}$$

Once again, replacing $\phi_0(R)$ and $\mathcal{M}_{ph}^2(R)$ into the SEE gives an equation of the form

$$\Lambda_R = I_b(R). \tag{83}$$

The subindex b stands for broken. Note that in general $I_b(R)$ will be different from $I_s(R)$.

3. Results

In what follows we present the results in terms of the relative deviation $(R - R_{cl})/R_{cl}$ of the backreaction solutions R with respect to the classical solution $R_{cl} = 4\Lambda_R$ as a function of Λ_R, for both the symmetric and broken phases, when they exist.
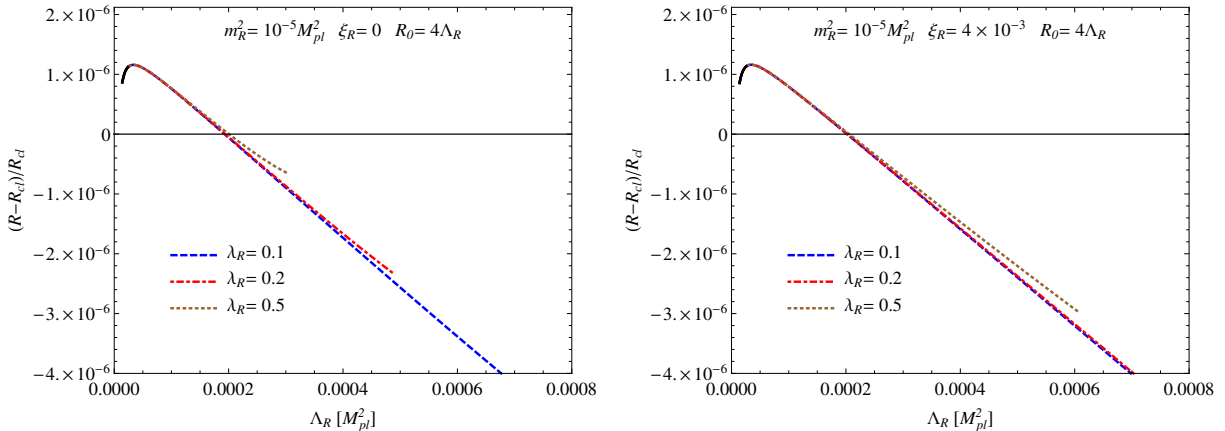

Figure 2: Relative deviation of the backreaction solution for the de Sitter spacetime curvature with respect to the classical solution, $(R - R_{cl})/R_{cl}$ as a function of Λ_R for different values of the coupling constant λ_R. The fixed parameters are $R_0 = 4\Lambda_R$, $m_R^2 = 10^{-5} M_{pl}^2$. The left panel corresponds to $\xi_R = 0$ and the right panel to $\xi_R = 4 \times 10^{-3}$. All curves correspond to the symmetric phase (which is the only possible phase when $R_0 = 4\Lambda_R$). From bottom up: $\lambda_R = 0.1$ (blue dashed line), $\lambda_R = 0.2$ (red dotted-dashed line), $\lambda_R = 0.5$ (brown dotted line). Notice that for small enough values of Λ_R the curves are continued by black solid lines, indicating the regions where
$$\mathcal{M}_{ph}^2 \geq R/10.$$

Let us first analyze the case where $R_0 = 4\Lambda_R$. This means that the renomalized parameters are defined at the value of scalar curvature of the background de Sitter spacetime that the theory would have had in the absence of backreaction. It is remarkable that in this

case no broken phase solutions exist. As an example, in Fig. 2 we have plotted the relative deviation for different values of the coupling constant λ_R, from bottom up: $\lambda_R = 0.1, 0.2$ and 0.5, with all curves corresponding to the symmetric phase and $m_R^2 = 10^{-5}M_{pl}^2$. On the left panel the coupling to the curvature is minimal $\xi_R = 0$, while on the right panel $\xi_R = 4 \times 10^{-3}$. It is interesting to see that, due to the quantum corrections, the curvature R can be both larger or smaller than the classical one depending on the value of Λ_R. Notice that solutions do not exist for all values of Λ_R. On the one hand, it can be seen that the approximation $\mathcal{M}_{ph}^2 \ll R$ breaks down for small enough values of Λ_R. In order to make this explicit, in Fig. 2 and in the following, black solid lines are used whenever $\mathcal{M}_{ph}^2 \geq R/10$. On the other hand, since we are considering only cases where the effective potential for ϕ_0 is well defined, there is a (λ_R-dependent) lower bound for the sum $m_R^2/R + \xi_R$ [57], which will be violated for large enough values of Λ_R.
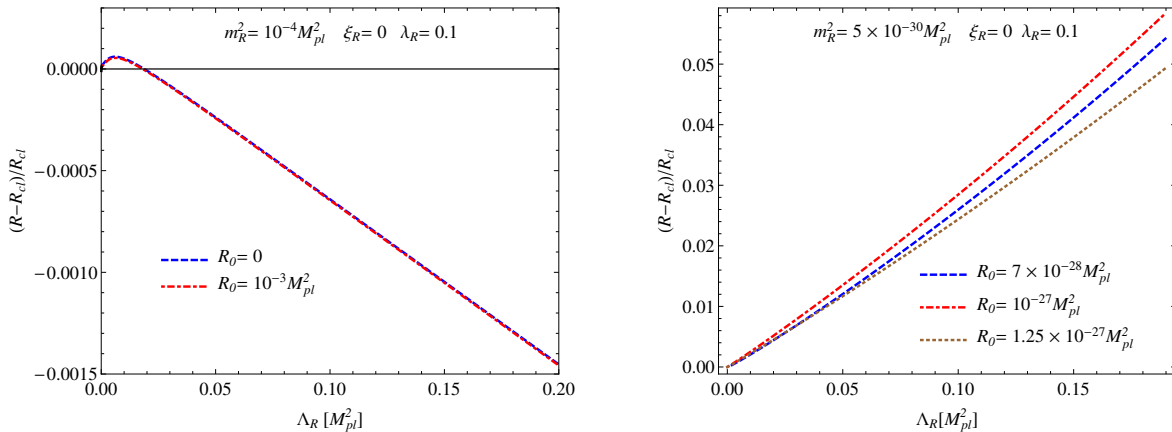

Figure 3: The same as in figure 2, but for different values of the curvature R_0 associated to the renormalization point. Left panel: symmetric phase solutions for $R_0 = 0$ (blue dashed line) and $R_0 = 10^{-3}M_{pl}^2$ (red dotted-dashed line) with fixed parameters $m_R^2 = 10^{-4}M_{pl}^2$, $\xi_R = 0$, and $\lambda_R = 0.1$. The curves are practically indistinguishable, illustrating the solutions are quite independent on R_0. Right panel: broken symmetry solutions for $R_0 = 7 \times 10^{-28}M_{pl}^2$ (blue dashed line), $R_0 = 10^{-27}M_{pl}^2$ (red dotted-dashed line), and $R_0 = 1.25 \times 10^{-27}M_{pl}^2$ (brown dotted line) where the fixed parameters are $m_R^2 = 5 \times 10^{-30}M_{pl}^2$, $\xi_R = 0$, and $\lambda_R = 0.1$. In this case, the values of R_0 where chosen to be in the small range where a broken phase solution exists.

Let us now analyze cases where R_0 is considered to be fixed and independent of Λ_R.

In Fig. 3, the left panel corresponds to the symmetric phase, while the right panel to the broken one. It can be seen that the backreaction is more significant in the broken phase (e.g. the deviation is about 1% for $\Lambda_R \simeq 0.04\, M_{pl}^2$, $R_0 \simeq 10^{-27} M_{pl}^2$ and $m_R^2 = 5 \times 10^{-30}\, M_{pl}^2$), while in the symmetric phase the solution stays closer to the classical one. The difference between the backreaction and classical solutions may become important for larger values of the cosmological constant (not shown in the figure). Indeed, it can be shown that the backreaction solution for the curvature R vanishes in the large (superplanckian) Λ_R limit. However, adopting an effective field theory perspective, here we are restricting the parameter space to subplanckian values.

As in general the broken phase solution is possible only for a suitable choice of the parameters [34], in the right panel, the values of R_0 had to be carefully chosen to be in the narrow window where broken phase solutions exist, and they disappear below a small parameter-dependent value of Λ_R (under $10^{-3} M_{pl}^2$ in the shown examples). One can verify that, depending on the values of the parameters, the approximation $\mathcal{M}_{ph}^2 \ll R$ may break down. For the values considered in left panel of Fig. 3 this happens for small enough values of Λ_R, while for the ones in the right panel the approximation remains valid.
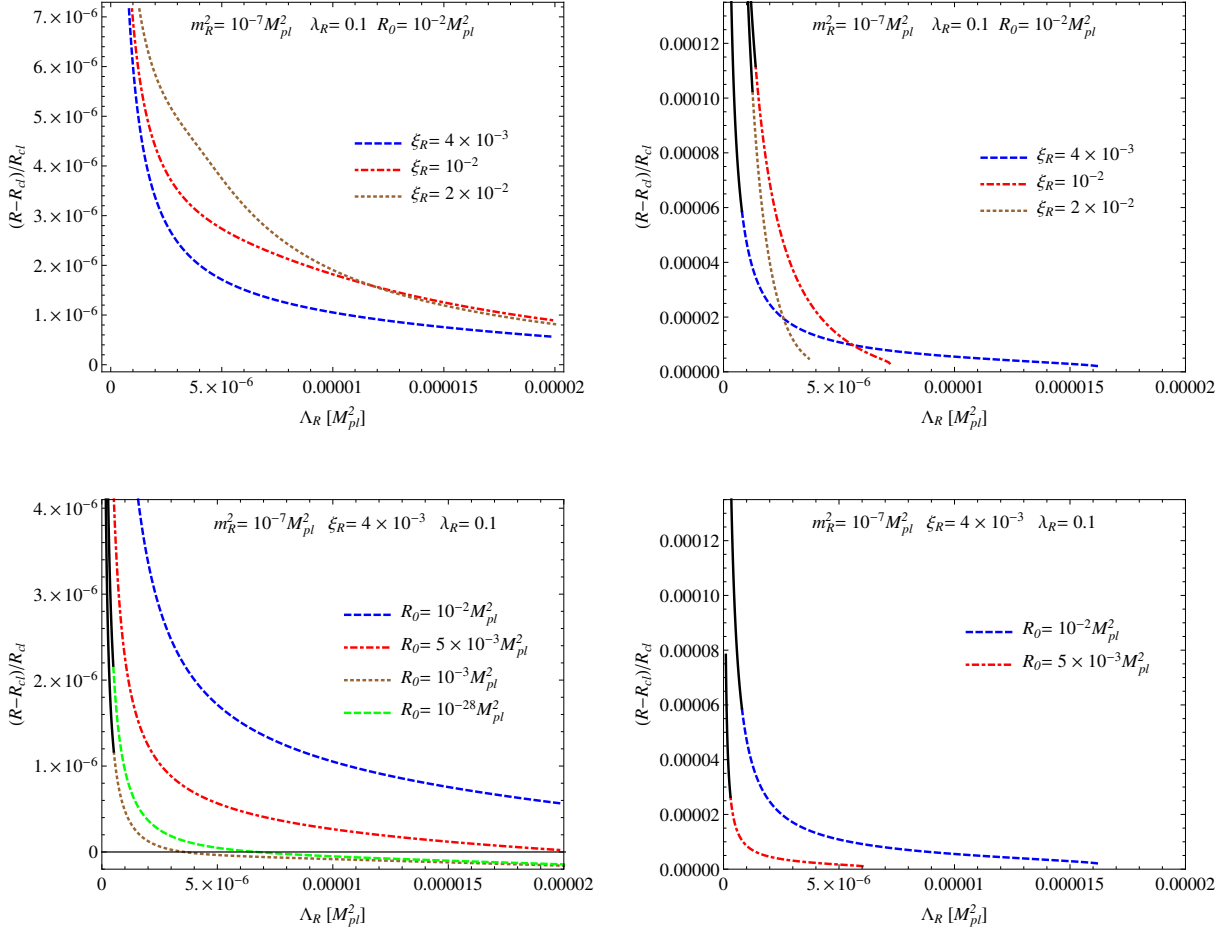
Figure 4: The deviation $(R - R_{cl})/R_{cl}$ vs. Λ_R for the backreaction solutions corresponding to the symmetric (on the left) and broken (to the right) phases. Upper panels: three curves corresponding to different values of the coupling to the curvature: $\xi_R = 4 \times 10^{-3}$ (blue dashed line), $\xi_R = 10^{-2}$ (red dotted-dashed line), and $\xi_R = 2 \times 10^{-2}$ (brown dotted line), where the fixed parameters are $m_R^2 = 10^{-7} M_{pl}^2$, $\lambda_R = 0.1$ and $R_0 = 10^{-2} M_{pl}^2$. Lower panels: four different curves illustrating the dependence on the value of R_0 for $m_R^2 = 10^{-7} M_{pl}^2$, $\xi_R = 4 \times 10^{-3}$, and $\lambda_R = 0.1$: $R_0 = 10^{-2} M_{pl}^2$ (blue dashed line), $R_0 = 5 \times 10^{-3} M_{pl}^2$ (red dotted-dashed line), $R_0 = 10^{-3} M_{pl}^2$ (brown dotted line), and $R_0 = 10^{-28} M_{pl}^2$ (green dashed line). Notice that for the last two values of R_0 no broken phase solutions exist.

The backreaction for the case of a nonminimal coupling to the curvature is illustrated in Fig. 4, where the left (right) panels correspond to the symmetric (broken) phase solutions. The upper panels illustrate the dependence of the solutions on the coupling to the curvature

ξ_R, while in the lower panels the coupling ξ_R is fixed and different values for R_0 are considered. In particular, from the figure on the bottom left, it can be seen that in the symmetric case, the effect of the quantum corrections may both increase or decrease the value of the de Sitter spacetime curvature R with respect to the classical one, depending on the value of Λ_R. In the symmetric phase there are self consistent solutions for large values of Λ_R, while in the broken phase they exist only for Λ_R below a (parameter-dependent) upper bound. Notice that there is also an upper bound for R_0 bellow which, under our approximations, no broken phase solution exist regardless the value of Λ_R. On the other hand, one can verify that the approximation $\mathcal{M}_{ph}^2 \ll R$ breaks down for small enough values of Λ_R in the broken phase, and also in the symmetric case but only when R_0 is smaller than a (parameter-dependent) critical value. However, as it can be seen from the examples considered in the two figures on the left panels, for larger values of R_0, there are symmetric phase solutions where the approximation breaks down for large values of Λ_R instead, while remaining valid all the way to $\Lambda_R \to 0$. In these latter cases, we can conclude that there is a divergence of the relative deviation in this limit, which indicates that as $R_{cl} \to 0$, the curvature R goes to a finite positive value. Therefore, for this set of parameters the backreaction is crucial to determine the spacetime curvature.

VI. CONCLUSIONS

In this paper we have considered a self-interacting scalar field with Z_2 symmetry in a general curved spacetime. In order to include some nonperturbative quantum effects of the scalar field, we have worked within the Hartree (or Gaussian) approximation to the 2PI EA.

Our first goal has been to show that in this approximation the "consistent renormalization procedure" described in [51] for flat spacetime can be extended to curved spacetimes to make finite not only the mean field and gap equations of the matter sector of the theory (which has been shown in paper I), but also the SEE, which also involve the gravitational sector. That is, we have shown that the same set of counterterms can be used to renormalize the SEE (along with the usual gravitational counterterms that are needed even for free fields). In order to maintain the covariance of the regularized theory, we have used dimensional regularization.

In Sec. V, we have applied our results to de Sitter spacetimes. We have considered

the explicit form of the mean value and gap equations, computed in paper I, together with the SEE for these particular spacetimes, and we have found some self-consistent de Sitter solutions. The simultaneous solution of the resulting algebraic equations allowed us to discuss the occurrence of spontaneous symmetry breaking and, at the same time, to assess the effect of quantum fluctuations on the classical metric. An important conclusion of our analysis is that the importance of the backreaction depends strongly on the value of the curvature at the renormalization point R_0. We have found both self-consistent solutions where the backreaction is important and solutions where it is not, depending on the values of the parameters. In particular, we have found self-consistent de Sitter solutions for vanishing cosmological constant $\Lambda_R = 0$, where the quantum effects play a crucial role.

I would be interesting to analyze the spontaneous symmetry breaking and existence of self-consistent solutions beyond the Hartree approximation, including the setting-sun diagram in the calculation of the effective action. We hope to address these issues in a future work.

[1] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).

[2] R. M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (University of Chicago Press, Chicago, 1994).

[3] S. M. Fulling, *Aspects of Quantum Field Theory in Curved Spacetime* (Cambridge University Press, Cambridge, 1989).

[4] L.E. Parker and D.J. Toms, *Quantum Field Theory in Curved Spacetime* (Cambridge University Press, Cambridge, 2009).

[5] A. H. Guth, Phys. Rev. D **23**, 347 (1981).

[6] A. D. Linde, Phys. Lett. B **108**, 389 (1982).

[7] A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982).

[8] P. A. R. Ade, N. Aghanim et al. [PLANCK Collaboration], arXiv:1303.5082 [astro-ph].

[9] D. N. Spergel et al. [WMAP Collaboration], Astrophys. J. Suppl. 148, **175** (2003) [arXiv:astro-ph/0302209]; ibidem, Astrophys. J. Suppl. **170**, 377 (2007) [arXiv:astro-ph/0603449].

[10] P. A. R. Ade, N. Aghanim et al. [PLANCK Collaboration], arXiv:1303.5076 [astro-ph].

[11] A. G. Riess et al. [Supernova Search Team Collaboration], Astron. J. 116, 1009 (1998) [arXiv:astro-ph/9805201].

[12] S. Perlmutter et al. [Supernova Cosmology Project Collaboration], Astrophys. J. 517, 565 (1999) [arXiv:astroph/9812133].

[13] S. Weinberg, Phys. Rev. D **72**, 043514 (2005) [arXiv:hep-th/0506236]; ibidem Phys. Rev. D **74**, 023508 (2006) [arXiv:hep-th/0605244].

[14] M. van der Meulen and J. Smit, JCAP **0711**, 023 (2007) [arXiv:0707.0842 [hep-th]].

[15] D. Seery, JCAP **0711**, 025 (2007) [arXiv:0707.3377v3 [astro-ph]]; ibidem JCAP **0802**, 006 (2008) [arXiv:0707.3378 [astro-ph]].

[16] D. Seery, Class. Quantum Grav. **27**, 124005 (2010) [arXiv:1005.1649 [astro-ph.CO]].

[17] T. Tanaka and Y. Urakawa, Class. Quantum Grav. **30**, 233001 (2013) [arXiv:1306.4461 [hep-th]].

[18] A. A. Starobinsky and J. Yokoyama, Phys. Rev. D **50**, 6357 (1994).

[19] N. C. Tsamis and R. P. Woodard, Nucl. Phys. **B724**, 295 (2005) [arXiv:gr-qc/0505115].

[20] C.P. Burgess, L. Leblond, R. Holman, and S. Shandera, JCAP **1003**, 033 (2010) [arXiv:0912.1608 [hep-th]]; ibidem JCAP **1010**, 017 (2010) [arXiv:1005.3551 [hep-th]].

[21] G. Lazzari and T. Prokopec, arXiv:1304.0404 [hep-th]; T. Prokopec, JCAP **1212**, 023 (2012)[arXiv:1110.3187v2 [gr-qc]].

[22] A. Rajaraman, Phys. Rev. D **82**, 123522 (2010) [arXiv:1008.1271 [hep-th]].

[23] S. Hollands, Ann. H. Poincaré **13**, 1039 (2012) [arXiv:1105.1996 [gr-qc]].

[24] M. Beneke and P. Moch, Phys. Rev. D **87**, 064018 (2013) [arXiv:1212.3058 [hep-th]].

[25] A. Riotto and M. S. Sloth, JCAP **0804**, 030 (2008) [arXiv:0801.1845 [hep-ph]].

[26] D. Boyanovsky, Phys. Rev. D **85**, 123525 (2012) [arXiv:1203.3903 [hep-ph]].

[27] E. T. Akhmedov, JHEP **1201**, 066 (2012) [arXiv:1110.2257 [hep-th]]; E. T. Akhmedov amd Ph. Burda, Phys. Rev. D **86**, 044031 (2012) [arXiv:1202.1202 [hep-th]]; E. T. Akhmedov, F. K. Popov, and V. M. Slepukhin, Phys. Rev. D **88**, 024021 (2013) [arXiv:1303.1068 [hep-th]].

[28] B. Garbrecht and G. Rigopoulos, Phys. Rev. D **84**, 063516 (2011) [arXiv:1105.0418 [hep-th]].

[29] R. Parentani and J. Serreau, Phys. Rev. D **87**, 045020 (2013) [arXiv:1212.6077 [hep-th]]; ibidem, Phys. Rev. D **87**, 085012 (2013) [arXiv:1302.3262 [hep-th]].

[30] J. Serreau, Phys. Rev. Lett. **107**, 191103 (2011) [arXiv:1105.4539 [hep-th]].

[31] A. Youssef and D. Kreimer, arXiv:1301.3205 [gr-qc].

[32] T. Arai, Phys.Rev. D **86**, 104064 (2012).

[33] B. Garbrecht, G. Rigopoulos, and Y. Zhu, arXiv:1310.0367 [hep-th].

[34] D. L. López Nacir, F. D. Mazzitelli, and L. G. Trombetta, Phys. Rev. D. **89**, 024006 (2014) [arXiv 1309.0864 [hep-th]].

[35] J. B. Hartle and B. L. Hu, Phys. Rev. D **20**, 1772 (1979); ibidem, Phys. Rev. D **21**, 2756 (1980).

[36] P. R. Anderson, C. Molina-París, and E. Mottola, Phys. Rev. D **80**, 084005 (2009).

[37] M. B. Fröb, D. B. Papadopoulos, A. Roura, and E. Verdaguer, Phys. Rev. D. **87**, 064019 (2013) [arXiv:1301.5261 [gr-qc]].

[38] J. Z. Simon, Phys. Rev. D **45**, 1953 (1992).

[39] E. E. Flanagan and R. M. Wald, Phys. Rev. D **54**, 6233 (1996) [gr-qc/9602052].

[40] A. A. Starobinsky, Phys. Lett. B **91**, 99 (1980).

[41] A. Vilenkin, Phys. Rev. D **32**, 2511 (1985).

[42] M. A. Castagnino, D. D. Harari, and J. P. Paz, Class. Quantum Grav. **3**, 569 (1986).

[43] D. L. López Nacir and F. D. Mazzitelli, Phys. Rev. D **76**, 024013 (2007).

[44] S. Wada, T. Azuma, Phys. Lett. B **132**, 313 (1983).

[45] P. R. Anderson, W. Eaker, S. Habib, C. Molina-Paris and E. Mottola, Int. J. Theor. Phys. **40**, 2217 (2001).

[46] G. Perez-Nadal, A. Roura and E. Verdaguer, Class. Quant. Grav. **25**, 154013 (2008) [arXiv:0806.2634 [gr-qc]].

[47] J. M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. D **10**, 2428 (1974).

[48] G. Amelino-Camelia, S. -Y. Pi, Phys. Rev. D **47**, 2356 (1993).

[49] P. M. Stevenson, Phys. Rev. D **32**, 1389 (1985).

[50] P. M. Stevenson and R. Tarrach, Phys. Lett. **B176**, 436 (1986); P. M. Stevenson, B. Alles, and R. Tarrach, Phys. Rev. D **35**, 2407 (1987); P. M. Stevenson, Z. Phys. C **35**, 467 (1987).

[51] J. Berges, Sz. Borsanyi, U. Reinosa, and J. Serreau, Annals Phys. **320**, 344 (2005).

[52] F. D. Mazzitelli and J. P. Paz, Phys. Rev. D **39**, 2234 (1989).

[53] E. Calzetta and B. L. Hu, "Nonequilibrium Quantum Field Theory," Cambridge University Press (2008).

[54] S. A. Ramsey and B. L. Hu, Phys. Rev. D **56**, 661 (1997) [arXiv:gr-qc/9706001].

[55] J. P. Paz and F. D. Mazzitelli, Phys. Rev. D **37**, 2170 (1988).

[56] S. M. Chistensen, Phys. Rev. D **14**, 2490 (1976).

[57] See Ref.[34], Fig. 4.