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Special Eccentric Vertices for the Class of Chordal Graphs and Related Classes

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Abstract A vertex is simplicial if the vertices of its neighborhood are pairwise adjacent. It is known that, for every vertex v of a chordal graph, there exists a simplicial vertex among the vertices at maximum distance from v. Here we prove similar properties in other classes of graphs related to that of chordal graphs. Those properties will not be in terms of simplicial vertices, but in terms of other types of vertices that are used to characterize those classes.

Keywords Eccentric vertex · Chordal graph · Dually chordal graph · Power chordal graph · Doubly chordal graph · Strongly chordal graph

Mathematics Subject Classification (2000) 05C12

1 Introduction

Chordal graphs were originally introduced as those without chordless cycles of length at least four, but they proved later to have diverse characterizations. We focus on the one which states that a graph is chordal if and only if it has a perfect elimination ordering of its vertices [5]. The first element of such an ordering is a simplicial vertex and the following vertices are simplicial in certain induced subgraphs.

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Other types of orderings of vertices can be used to define many classes closely related to that of chordal graphs: dually chordal, power chordal, doubly chordal and strongly chordal graphs. These classes are the ones to be studied here.

Our starting point is going to be the fact that, given a vertex v of a chordal graph, there exists a simplicial vertex among the ones furthest from v [4,6]. Then, some basic metric properties, mostly involving chordal and dually chordal graphs, are reviewed. The concept of power of a graph is also used and is important for the development of the theory below.

At the end, we will have proved that properties similar to the one stated in the beginning of the previous paragraph are true for the other graphs classes we have mentioned. That is, we will prove that, given a vertex v of a graph G, we can find another vertex which is the first of a linear ordering characteristic to the graph class of G among the vertices furthest from v.

2 Basic Definitions

This paper only deals with graphs without loops or multiple edges. For a graph G, $V(G)$ denotes the set of its vertices and $E(G)$ denotes the set of its edges. A set $V' \subseteq V(G)$ is *complete* if its elements are pairwise adjacent in G. The subgraph *induced* by $A \subseteq V(G)$, $G[A]$, has A as vertex set and two vertices are adjacent in $G[A]$ if and only if they are adjacent in G. If $v_1 v_2 \dots v_n$ is an order of the vertices of G, then we define for every $1 \leq i \leq n$, $G_i = G[v_i, \dots, v_n]$.

Given two vertices v and w of G, the *distance* between v and w, or $d(v, w)$, is the length of a shortest path connecting v and w in G. If there is no path in G connecting v and w, then $d(v, w)$ is defined to be infinity. The *open neighborhood* of v, $N(v)$, is the set of all the vertices adjacent to v. The *closed neighborhood* of v, $N[v]$, is defined by the equality $N[v] = N(v) \cup \{v\}$. The vertex v is said to be a *universal vertex* when $N[v] = V(G)$. If v and w are such that $N[v] \subseteq N[w]$, then we say that w *dominates* v. The *disk* centered at vertex v with radius k is the set of vertices at distance at most k from v and it is denoted by $N^k[v]$. The *eccentricity* of v is $ecc(v) = max\{d(v, w),\ w \in V(G)\}$. We refer to w as an *eccentric* vertex of v if no vertex of G is further away from v than w, that is, if $ecc(v) = d(v, w)$. Another important concept related to the distance in graphs is the *diameter*, usually expressed as $diam(G)$. It is defined as the maximum possible distance between two vertices of the graph, i.e., $diam(G) = max\{d(v, w) :\ v, w \in V(G)\}$.

The *kth-power* of G, or G^k, is another graph which has the same vertices as G, two of them being adjacent in G^k if and only if the distance in G between them is at most k.

A *chord* of a cycle is an edge joining two nonconsecutive vertices of the cycle. *Chordal* graphs are defined as those without chordless cycles of length at least four.

A vertex v is *simplicial* if $N[v]$ is a complete set. Let $v_1 v_2 \dots v_n$ be an ordering of the vertices of the graph G. It is called a *perfect elimination ordering* if, for all $1 \leq i \leq n$, v_i is simplicial in G_i.

One of the most classical characterizations of chordal graphs states that a graph is chordal if and only if it has a perfect elimination ordering [5].

A vertex $w \in N[v]$ is a *maximum neighbor* of v if $N^2[v] \subseteq N[w]$. An ordering $v_1 \dots v_n$ of the vertices of G is a *maximum neighborhood ordering* if, for all $1 \leq i \leq n$, v_i has a maximum neighbor in G_i. *Dually chordal graphs* are defined as those possessing a maximum neighborhood ordering.

3 Eccentric Vertices

We can say from the previous definition that vertices with a maximum neighbor are to dually chordal graphs as simplicial vertices are to chordal graphs.

With regard to chordal graphs and simplicial vertices, the following property is known:

Theorem 1 [4,6] *Let G be a chordal graph and $v \in V(G)$. Then, v has an eccentric vertex which is simplicial.*

Theorem 1 is the cornerstone of this paper. It is our interest to ascertain whether we can find an analogous property about dually chordal graphs. More specifically, given a dually chordal graph G and $v \in V(G)$, we wonder whether v has an eccentric vertex with a maximum neighbor. The answer to this question is affirmative and we need some previous results before proving it.

Lemma 1 [1] *Let G be a dually chordal graph and A be a subset of $V(G)$ such that every pair of vertices of A is at a distance not greater than 2. Then, there exists a vertex w such that $A \subseteq N[w]$.*

Lemma 2 [1] *Let G be a dually chordal graph. Then, G^2 is chordal.*

Lemma 3 *Let G be a dually chordal graph and v be a simplicial vertex in G^2. Then, v has a maximum neighbor in G.*

Proof As v is simplicial in G^2, the distance in G between every pair of vertices of $N^2[v]$ is at most 2. Then, by Lemma 1, there exists a vertex w such that $N^2[v] \subseteq N[w]$. Therefore, w is a maximum neighbor of v. □

The first major result can now be proved. From now on, it will always be assumed that G is a connected graph.

Theorem 2 *Let G be a dually chordal graph and v be a vertex of G. Then, there exists an eccentric vertex of v with a maximum neighbor.*

Proof Suppose first that $ecc_G(v)$ is odd and let v' be an eccentric vertex of v in G. As G^2 is chordal, Theorem 1 implies that there exists a vertex w which is simplicial in G^2 and is eccentric of v in G^2. Hence, by Lemma 3, w has a maximum neighbor in G. We now prove that w is also eccentric of v in G.

Note first that, by the definition of G^2, if two vertices are at distance k in G, then their distance in G^2 is $\frac{k}{2}$ if k is even or $\frac{k+1}{2}$ if k is odd. Then, as $ecc_G(v)$ is odd, $d_{G^2}(v, v') = \frac{ecc_G(v)+1}{2}$. Furthermore, every eccentric vertex of v in G is also eccentric in G^2. Thus, v' is eccentric of v in G^2 and the eccentricity of v in G^2 equals $\frac{ecc_G(v)+1}{2}$. Since w is also eccentric of v in G^2, we infer that $d_{G^2}(v, w) = \frac{ecc_G(v)+1}{2}$.

By using the definition of G^2 again and that $d_{G^2}(v, w) = \frac{ecc_G(v)+1}{2}$, we get two possible values for $d_G(v, w)$, namely, $ecc_G(v)$ and $ecc_G(v) + 1$. Thus, $d_G(v, w) \geq ecc_G(v)$. Furthermore, the definition of eccentricity implies that $d_G(v, w) \leq ecc_G(v)$. Therefore, $d_G(v, w) = ecc_G(v)$ and w is the required vertex.

If $ecc_G(v)$ is even, let G' be the graph obtained from G by adding a new vertex v^* and making it adjacent to v. Then, v is a maximum neighbor of v^* in G'. Furthermore, for every maximum neighborhood ordering $v_1 \dots v_n$ of G, $v^* v_1 \dots v_n$ is a maximum neighborhood ordering of G'. Thus, G' is dually chordal. It holds that $ecc_{G'}(v^*)$ is odd and, by the previous case, there exists a vertex u with a maximum neighbor in G' such that $d(v^*, u) = ecc_{G'}(v^*)$. It is clear that u is eccentric of v in G. Now we show that u has a maximum neighbor in G.

Let w be a maximum neighbor of u in G'. If $w = v^*$, then $u = v$. As v^* is a maximum neighbor of v, v^* dominates v. By the construction of G', v dominates v^* as well. Thus, $N_{G'}[v] = N_{G'}[v^*] = \{v, v^*\}$. We infer that v is adjacent to no other vertex of G and hence is a maximum neighbor of itself in G.

If $w \neq v^*$, then $w \in V(G)$. As w is a maximum neighbor of u in G', $N^2_{G'}[u] \subseteq N_{G'}[w]$. Furthermore, $N^2_G[u] = N^2_{G'}[u] \setminus \{v^*\}$ and $N_G[w] = N_{G'}[w] \setminus \{v^*\}$. It follows that $N^2_G[u] \subseteq N_G[w]$. Therefore, w is a maximum neighbor of u in G. □

Corollary 1 *Let G be a nontrivial, i.e., not composed of just one vertex, dually chordal graph. Then, there are two vertices v_1 and v_2, each with a maximum neighbor, such that $d(v_1, v_2) = diam(G)$.*

Proof Let $k = diam(G)$ and x, y be two vertices such that $d(x, y) = k$. Then, by Theorem 2, there exists a vertex v_1 with a maximum neighbor which is eccentric of x. Thus, $d(x, v_1) = k$. Likewise, there exists a vertex v_2 with a maximum neighbor which is eccentric of v_1. Consequently, $d(v_1, v_2) = k$. □

As an example of application of Theorem 2, we present the proof of another metric property.

Lemma 4 *Let G be a graph and u, v, w, x be vertices of G such that w is different from the other three and w is dominated by x. Then, $d_{G-w}(u, v) = d_G(u, v)$.*

Proof Let P be a path in G from u to v of minimum length. If w is not a vertex of P, then P is also a path in $G - w$ and hence $d_{G-w}(u, v) = d_G(u, v)$.

If w is a vertex of P, let $y_1 \dots y_i w y_{i+1} \dots y_j$ be the sequence of vertices of P, with $y_1 = u$ and $y_j = v$. Now we prove that x is not a vertex of P. As w is dominated by x, we have that $y_i, y_{i+1} \in N[x]$. If there exists k such that $k \leq i$ and $y_k = x$, then $y_1 \dots y_k y_{i+1} \dots y_j$ is a path from u to v shorter than P, which is a contradiction. Therefore, $y_k \neq x$, $1 \leq k \leq i$. Similarly, $y_k \neq x$, $i + 1 \leq k \leq j$.

We infer that $y_1 \dots y_i x y_{i+1} \dots y_j$ is a path in $G - w$ from u to v with the same length as P. Therefore, the equality $d_{G-w}(u, v) = d_G(u, v)$ follows. □

Proposition 1 *Let G be a dually chordal graph. Let A be a nonempty set of vertices of G contained in the closed neighborhood of some vertex x, and w be a vertex such that $w \notin A$ and $d(y, w)$ has the same value for all $y \in A$. Then, there exists $v \in V(G) \setminus A$ such that $vy \in E(G)$ for all $y \in A$.*

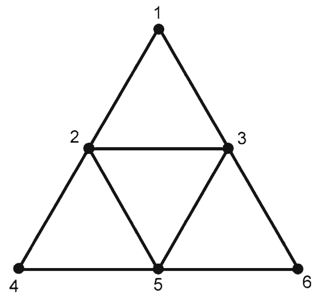

Fig. 1 A chordal graph for which Proposition 1 fails to be true

Proof It will be proved by induction on $|V(G)|$.

If $|V(G)| = 2$, then $|A| = 1$ and w is the only vertex not in A. Let u be the only element of A. Then, $d(u, w) = 1$ and we can set $v = w$.

For the remaining cases, the proof is direct if $x \notin A$ (set $v = x$) or w is universal (set $v = w$), so let us assume that $x \in A$ and that w is not universal.

Suppose now that, given $n \geq 2$, the proposition is true for every dually chordal graph with n vertices (Inductive hypothesis) and that G has $n + 1$ vertices.

Let u be an eccentric vertex of x such that it has a maximum neighbor. If $d(u, x) = 1$, then x is universal and hence $d(x, w) = 1$. As $x \in A$, the definition of w implies that $d(y, w) = 1$ for all $y \in A$. Therefore, w is the desired vertex.

If $d(u, x) > 1$, then u is not in A and we have the following cases:

Case 1: $u \neq w$.

Since u is dominated by any of its maximum neighbors, Lemma 4 can be applied and it implies that $d_{G-u}(y, w) = d_G(y, w)$ for all $y \in A$. We can get the desired vertex v by applying the inductive hypothesis on $G - u$.

Case 2: $u = w$.

Let u' be a maximum neighbor of u. Then, $u' \neq u$ because w is not a universal vertex. Thus, $d(u, u') = 1$.

If $u' \in A$, then the definition of w implies that $d(y, u) = 1$ for all $y \in A$. Therefore, u is the desired vertex.

If $u' \notin A$, then it is not difficult to see that $d(y, u') = d(y, u) - 1$ for all $y \in A$. Thus, $d(y, u')$ has the same value for all $y \in A$. By Lemma 4, the same is true when the distance is taken on $G - u$. We can get the desired vertex v by applying the inductive hypothesis on $G - u$. $\square$

It is interesting to note that Proposition 1 is not necessarily true when G is a chordal graph. For example, consider the chordal graph in the figure and set $A = \{4, 5, 6\}$ and $w = 1$. Then, $A \subseteq N[5]$ and $d(y, w) = 2$ for every vertex $y \in A$. However, there is no vertex $v \notin A$ such that $A \subseteq N[v]$.

Nevertheless, more restrictions can be added to A to make the proposition true for chordal graphs.

Proposition 2 *Let G be a chordal graph, A be a complete set of vertices of G and w be a vertex such that $w \notin A$ and $d(y, w)$ has the same value for all $y \in A$. Then, there exists $v \in V(G) \setminus A$ such that $vy \in E(G)$ for all $y \in A$.*

Proof It will be proved by induction on $|V(G)|$.

If $|V(G)| = 2$, then $|A| = 1$ and w is the only vertex not in A. Let u be the only element of A. Then, $d(u, w) = 1$ and we can set $v = w$.

Suppose now that, given $n \geq 2$, the proposition is true for every chordal graph with n vertices (Inductive hypothesis) and that G has $n + 1$ vertices.

If G is complete, we can set $v = w$. If G is not complete, then G has two simplicial vertices which are not adjacent [2].

Suppose that one of those two vertices, call it u, is in A. Let v be a vertex in $N[u]$ such that $d(v, w) = d(u, w) - 1$. Then, $v \notin A$, and v is adjacent to all the vertices of A because u is simplicial.

Otherwise, if the two simplicial vertices are not in A, one of them must be different from w, call it x. By Lemma 4, $d_{G-x}(y, w) = d_G(y, w)$ for every $y \in A$. Then, we can apply the inductive hypothesis on $G - x$ to find a vertex $v \notin A$ which is adjacent to all the vertices of A in $G - x$. Thus, v is the desired vertex. □

3.1 Power Chordal and Doubly Chordal Graphs

At this moment, it is interesting to determine if similar results are valid for more specific types of graphs. That is the case for power chordal and doubly chordal graphs, which are the graphs to be considered in this section.

A graph G is said to be *power chordal* if all of its powers are chordal. It is true that a graph is power chordal if and only if G and G^2 are chordal [1].

G is *doubly chordal* if it is chordal and dually chordal.

A vertex is *doubly simplicial* if it is simplicial and has a maximum neighbor. An ordering $v_1 v_2 \ldots v_n$ of the vertices of G is *doubly perfect* if, for all $1 \leq i \leq n$, v_i is doubly simplicial in G_i. It holds that a graph is doubly chordal if and only if it has a doubly perfect ordering [1].

It is known that a power chordal graph G is complete or there are two nonadjacent vertices which are simplicial in both G and G^2. The demonstration can be seen in [1]. A similar technique enables to prove the following result:

Theorem 3 *Let G be a power chordal graph and $v \in V(G)$. Then, there exists a vertex w eccentric of v in G^2 which is simplicial in both G and G^2.*

Proof The proof is direct if G^2 is complete. Assume that G^2 is not complete. Since G^2 is chordal, we can apply Theorem 1 to get a vertex u which is simplicial in G^2 and is eccentric of v in G^2. If u is also simplicial in G, then we are done. Otherwise, let x and y be two nonadjacent neighbors of u and S be a minimal xy-separator in G. Then, S is complete due to the chordality of G [2] and $u \in S$.

Let $G[A]$ and $G[B]$ be the connected components of $G - S$ containing x and y, respectively. We can assume without loss of generality that $v \notin A$. Our next step is to prove that $G[A \cup S]$ is power chordal.

It holds that $G[A \cup S]$ is chordal because it is an induced subgraph of the chordal graph G.

In order to prove the chordality of $(G[A \cup S])^2$, we show that $(G[A \cup S])^2 = G^2[A \cup S]$. It can be done by verifying that these two graphs have the same set of edges.

Let $w_1 w_2 \in E(G[A \cup S]^2)$. Then, the distance between w_1 and w_2 in $G[A \cup S]$ is not greater than 2. As $G[A \cup S]$ is a subgraph of G, the distance between w_1 and w_2 in G is not greater than 2, either. Thus, $w_1 w_2 \in E(G^2)$. Since $w_1, w_2 \in A \cup S$, $w_1 w_2 \in E(G^2[A \cup S])$ as well.

Conversely, let $w_1 w_2 \in E(G^2[A \cup S])$. Then, the distance between w_1 and w_2 in G is not greater than 2. Let P be a path in G from w_1 to w_2 of minimum length. If the length of P is 1, then P is clearly a path in $G[A \cup S]$, making w_1 and w_2 adjacent in $G[A \cup S]^2$.

If the length of P is 2, let w_3 be the vertex between w_1 and w_2 in P. If $w_3 \notin A \cup S$, then $w_1, w_2 \in S$. As S is complete, we infer that w_1 and w_2 are adjacent, which contradicts that P is a minimum length path from w_1 to w_2. Thus, $w_3 \in A \cup S$ and P is a path in $G[A \cup S]$. Therefore, w_1 and w_2 are adjacent in $G[A \cup S]^2$.

Thus, the equality $(G[A \cup S])^2 = G^2[A \cup S]$ is proven. We conclude that $(G[A \cup S])^2$ is chordal because it is an induced subgraph of G^2, which is chordal. Therefore, $G[A \cup S]$ is power chordal.

Since $G[A \cup S]$ is power chordal, we have two possibilities: either $G[A \cup S]$ is complete or it contains two nonadjacent vertices that are simplicial in both $G[A \cup S]$ and $(G[A \cup S])^2$ [1]. In either case, we conclude that the set A contains a vertex w which is simplicial in $G[A \cup S]$ and $G^2[A \cup S]$. It is evident that w is also simplicial in G. Now, it will be demonstrated that w is simplicial in G^2 as well.

The proof is straightforward if $N^2[w] \subseteq A \cup S$.

Otherwise, w is at a distance not greater than 2 from a vertex not in $A \cup S$, which is only possible if w is adjacent to a vertex w' in S.

Let $z \in N^2[w]$. Now we prove that $z \in N^2[u]$. The proof is divided into two cases.

Case 1: $z \in N^2[w] \cap (A \cup S)$.

Note that $u \in N^2[w]$ because $w \in N[w']$ and $w' \in N[u]$. Then, both u and z are adjacent to w in $G^2[A \cup S]$. Since w is simplicial in $G^2[A \cup S]$, $z \in N_G^2[u]$.

Case 2: $z \in N^2[w] \setminus (A \cup S)$.

In this case, z and w are in different connected components of $G - S$. Let wtz be a path in G from w to z of length two. Then, $t \in S$. As S is complete, $t \in N[u]$. Combine this with $z \in N[t]$ to conclude that $z \in N^2[u]$.

Therefore, $N^2[w] \subseteq N^2[u]$. Consequently, the fact that u is simplicial in G^2 implies that so is w.

Since v and w are in different connected components of $G-S$, any path joining them must have a vertex in S, and hence in $N[u]$. We infer that $d_G(v, u) \leq d_G(v, w)$. Then, $d_{G^2}(v, u) \leq d_{G^2}(v, w)$. As u is eccentric of v in G^2, w is also eccentric. Therefore, w has all the required properties. □

Theorem 4 *Let G be a power chordal graph and $v \in V(G)$. Then, there exists an eccentric vertex of v in G which is simplicial in G and G^2.*

Proof The proof is very similar to that of Theorem 2, so we just give a sketch of it.

If $ecc(v)$ is odd, then the vertex given by Theorem 3 has the required characteristics.

If $ecc(v)$ is even, construct G' as it was done in the proof of Theorem 2. G' is also power chordal and any eccentric vertex of v^* in G', simplicial both in G' and G'^2, has the required characteristics. □

Corollary 2 *Let G be a doubly chordal graph and $v \in V(G)$. Then, there exists an eccentric vertex of v which is doubly simplicial.*

Proof As G is dually chordal, we can use Lemma 2 to infer that G^2 is chordal. Thus, G is power chordal. By Theorem 4, there exists a vertex w which is simplicial in G and G^2 and is eccentric of v in G. By Lemma 3, w has a maximum neighbor in G, so it is doubly simplicial. □

3.2 Strongly Chordal Graphs

So far, it was possible to prove the existence of vertices which are eccentric and characteristic to every class that has been discussed. One that was not considered yet is that of *strongly chordal* graphs. We finish this paper by showing a similar property about this class.

Strongly chordal graphs are defined as those chordal graphs for which every cycle whose length is even and at least 6 has a chord joining two vertices at an odd distance in the cycle. Strongly chordal graphs can also be characterized in terms of elimination orderings.

A vertex v of a graph G is *simple* if the set $\{N[u] : u \in N[v]\}$ is totally ordered by inclusion. We infer from this definition that simple vertices are doubly simplicial. To prove it, let u_1 and u_2 be two vertices in $N[v]$. We can suppose without loss of generality that $N[u_1] \subseteq N[u_2]$. Then, $u_1 \in N[u_2]$. Therefore, $N[v]$ is a complete set and hence v is simplicial.

The definition also implies that there exists a vertex w such that $N[u] \subseteq N[w]$ for all $u \in N[v]$. Thus, $N^2[v] \subseteq N[w]$ and hence w is a maximum neighbor of v.

An ordering $v_1 v_2 \ldots v_n$ of $V(G)$ is called a *simple elimination ordering* if, for all $1 \leq i \leq n$, v_i is simple in G_i. It holds that a graph is strongly chordal if and only if it has a simple elimination ordering [3].

We can infer from the definition that the class of strongly chordal graphs is hereditary. In fact, being a strongly chordal graph is equivalent to being a hereditary dually chordal graph [1].

In connection with eccentric vertices, we have the following:

Lemma 5 *Let G be a graph and u, v and w be three vertices of G such that w is a maximum neighbor of v, $ecc(w) > 1$ and $d(u, v) \geq 2$. Then, $d(u, v) = d(u, w) + 1$ and the set of eccentric vertices of w is equal to the set of eccentric vertices of v.*

Proof If $d(u, v) = 2$, then $d(u, w) = 1$ because of the definition of maximum neighbor. Suppose now that $d(u, v) > 2$. By the triangle inequality, $d(u, v) \leq d(u, w) + d(w, v)$, that is, $d(u, v) \leq d(u, w) + 1$.

Let $vv_1v_2 \ldots u$ be a shortest path from v to u. Then, $wv_2 \ldots u$ is a path from w to u of length $d(u, v) - 1$. Thus, $d(u, v) - 1 \geq d(u, w)$ and hence $d(u, v) \geq d(u, w) + 1$. Therefore, the equality $d(u, v) = d(u, w) + 1$ holds. Combine this with the inequality

$ecc(w) > 1$ to deduce that every eccentric vertex of v is at distance greater than or equal to 3 of v and consequently,

$$\begin{aligned} d(v,u) = ecc(v) &\Leftrightarrow d(v,u) = max\{d(v,x) : \ x \in V(G)\} \Leftrightarrow d(v,u) \\ &= max\{d(v,x) : \ x \in V(G), \ d(v,x) \geq 3\} \Leftrightarrow d(w,u) + 1 \\ &= max\{d(w,x) + 1 : \ x \in V(G), \ d(w,x) \geq 2\} \Leftrightarrow d(w,u) \\ &= max\{d(w,x) : \ x \in V(G), \ d(w,x) \geq 2\} \Leftrightarrow d(w,u) \\ &= max\{d(w,x) : \ x \in V(G)\} \Leftrightarrow d(w,u) = ecc(w) \end{aligned}$$

Theorem 5 *Let G be a strongly chordal graph and $v \in V(G)$. Then, there exists an eccentric vertex of v which is simple.*

Proof The proof will be by induction on $n = |V(G)|$. The statement of the theorem is obviously valid when $n = 1$. Suppose now that it is always true when $n = k$, where $k \geq 1$, and that G is a strongly chordal graph with $k + 1$ vertices. The proof will be divided into cases.

Case 1: G has at least one universal vertex.

Let w be a universal vertex of G. If w is simple, then G is complete because simple vertices are simplicial. Thus, the existence of an eccentric simple vertex is evident.

If w is not simple and $v = w$, then the fact that w is universal implies that any simple vertex of G is an eccentric vertex of v.

Now assume that $v \neq w$ and that w is not simple. Then, we consider the strongly chordal graph $G - w$. In case that $G - w$ is not connected, any simple vertex of $G - w$ located in a connected component different from that of v is an eccentric simple vertex for v in G. If $G - w$ is connected, applying the inductive hypothesis to it yields an eccentric simple vertex u for v in $G - w$. It is not difficult to see that u is simple and eccentric of v in G.

Case 2: G has no universal vertices.

Case 2a: v is simple.

Let v' be a maximum neighbor of v. Then, $v \neq v'$ because v is not universal. By the inductive hypothesis, there exists a vertex w which is simple and eccentric of v' in $G-v$. Note that $d(v', w) \geq 2$ because otherwise v' would be universal in G. As v' is a maximum neighbor of v in G, and hence $N^2[v] \subseteq N[v']$, we conclude that $d(v, w) \geq 3$. Thus, the closed neighborhoods of vertices in $N[w]$ are coincident in G and $G - v$, which implies that w is also simple in G. By Lemma 5, w is also eccentric of v in G.

Case 2b: v is not simple and there is a simple vertex which is not adjacent to v.

Let w be a simple vertex not adjacent to v. If it is eccentric of v, then we are done. If not, consider the strongly chordal graph $G - w$ which, by inductive hypothesis, possesses a simple vertex w' eccentric of v. Then, as a consequence of Lemma 4, w' is also eccentric of v in G, so it suffices to prove that w' is simple in G.

If w' is not simple in G, we first prove that $w \in N^2[w']$. As w' is not simple, there are vertices u_1 and u_2 in $N[w']$ such that $N[u_1] \nsubseteq N[u_2]$ and $N[u_2] \nsubseteq N[u_1]$. If $u_1 = w$ or $u_2 = w$, then w is in $N[w']$ and so is in $N^2[w']$.

If $u_1 \neq w$ and $u_2 \neq w$, then u_1 and u_2 are in the closed neighborhood of w' in $G - w$. Since w' is simple in $G - w$, we can suppose without loss of generality that

$N_{G-w}[u_1] \subseteq N_{G-w}[u_2]$. If $w \notin N[u_1]$, then $N[u_1] \subseteq N[u_2]$, which contradicts our previous assumption. Therefore, $w \in N[u_1]$. Add this to the fact that $u_1 \in N[w']$ to conclude that $w \in N^2[w']$.

Let u be a maximum neighbor of w in G. Then, $u \in N[w']$ and hence $d(v, w') \leq d(v, u) + 1$. Combine this with Lemma 5 to get that $d(v, w') \leq d(v, w)$, thus contradicting that w is not an eccentric vertex of v.

Therefore, w' is necessarily simple in G.

Case 2c: v is not simple and v is adjacent to all the simple vertices of G.

We prove that $diam(G) \leq 2$. Suppose on the contrary that $diam(G) \geq 3$. Let x and y be vertices such that $d(x, y) = diam(G)$. Thus, $\{x, y\} \nsubseteq N[v]$, so we can assume without loss of generality that $x \notin N[v]$. Since all the simple vertices are simplicial and adjacent to v, we conclude that none of them is adjacent to x. Then, by case 2b, x has a simple eccentric vertex x' and hence $d(x, x') = diam(G)$. By case 2a, we know that x' has a simple eccentric vertex x'', so $d(x', x'') = diam(G)$. Since both x' and x'' are adjacent to v, we conclude that $d(x', x'') \leq 2$, contradicting that $diam(G) \geq 3$. Therefore, $diam(G) \leq 2$.

Since G is dually chordal, we can apply Lemma 1 to conclude that G has a universal vertex, contradicting the initial assumption of case 2. Therefore, it is not possible that G has no universal vertices, that v is not simple and that v is adjacent to all the simple vertices of G.

As all the possible cases have been considered, the proof is complete.

We have the following as corollaries of the last sections:

Corollary 3
- *Let G be a nontrivial power chordal graph. Then, there are two vertices v_1 and v_2 such that they are simplicial in both G and G^2 and $d(v_1, v_2) = diam(G)$.*
- *Let G be a nontrivial doubly/strongly chordal graph. Then, there are two doubly simplicial/simple vertices v_1 and v_2 such that $d(v_1, v_2) = diam(G)$.*

As a final remark, note that the proofs of Theorems 2, 4, 5 and their corollaries are trivial for nonconnected graphs. For the case of a nonconnected strongly chordal graph G, each of its connected components is also strongly chordal. If $v \in V(G)$, then any simple vertex in a connected component different from that of v is eccentric. Furthermore, if we consider two simple vertices in two different connected components of G, their distance equals the diameter of the graph.

The reasonings are analogous for the other classes.

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