FRACTIONAL INTEGRALS AND RIESZ TRANSFORMS ACTING ON CERTAIN LIPSCHITZ SPACES

M. RAMSEYER, O. SALINAS AND B. VIVIANI

Abstract. We make a unifying approach to the study of mapping properties of fractional integrals and Riesz transforms acting on spaces of functions f verifying

$$\sup_B \left(\frac{1}{w(a,r)} \left(\frac{1}{|B|} \int_B |f - m_B f|^q \right)^{1/q} \right) < \infty\,,$$

where w is a non negative functional defined on the family of balls $B \subset \mathbb{R}^n$ with center a and radius r. So, at the same time, we are able to treat such cases as BMO, Lipschitz spaces and spaces of functions with variable smoothness among others. Results about pointwise smoothness related to these spaces are included as well.

1. Introduction

Let $w : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function. For $1 \le q < \infty$ given we define the space $\mathrm{BMO}_{w,q}$ as the set of locally integrable functions f on $\mathbb{R}^n$ such that

$$\frac{1}{w(a,r)} \left(\frac{1}{|B|} \int_B |f(x) - m_B f|^q \, dx \right)^{\frac{1}{q}} \le C\,, \tag{1.1}$$

for some $C > 0$ and for every ball $B \subset \mathbb{R}^n$ with center a and radius r, where $m_B f$ is the average of f over B, namely $m_B f = |B|^{-1} \int_B f(y) dy$. As it can be easily seen the expression

$$\|f\|_{w,q} = \sup_{B \subset \mathbb{R}^n} \left\{ \frac{1}{w(a,r)} \left(\frac{1}{|B|} \int_B |f(x) - m_B f|^q \, dx \right)^{\frac{1}{q}} \right\}\,,$$

turns out to be a seminorm for this space. Then, $\mathrm{BMO}_{w,q}$ modulo constants is a Banach space. The space $\mathrm{BMO}_{w,1}$ was introduced by E. Nakai and K. Yabuta ([20], although a version defined on the $n-$dimensional torus had already appeared in [15], due to S. Janson) in connection with the identification of pointwise multipliers of the space of functions with mean oscillation controlled by a positive, non-decreasing function φ, i.e. BMO_φ (see [24]). The general case $\mathrm{BMO}_{w,q}$, $1 \le q < \infty$ was introduced in [21], where a complete study on their pointwise multiplier is done.

As in [21], the following properties will be supposed on w. We assume that there exists a positive constant C such that

$$w(x, t_1) \le C\, w(x, t_2), \qquad \forall\, x \in \mathbb{R}^n, \quad \forall\, t_1 < t_2\,. \tag{1.2}$$

2010 *Mathematics Subject Classification.* Primary 42B35.

Key words and phrases. Bounded Mean Oscillation, Fractional Integral, Variable Exponent.

This research is partially supported by grants from Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET) and Facultad de Ingeniería Química Universidad Nacional del Litoral (UNL), Argentina.

$$w(x,2t) \le C\, w(x,t), \qquad \forall\, x \in \mathbb{R}^n, \quad \forall\, t>0\,. \tag{1.3}$$

$$|x-y|<t \quad \Rightarrow \quad w(x,t) \le C\, w(y,t), \qquad \forall\, x,y \in \mathbb{R}^n, \quad \forall\, t>0\,. \tag{1.4}$$

As a first remark, since $w(x,t)$ satisfies (1.2) and (1.3) the definition of $\mathrm{BMO}_{w,q}$ through inequality (1.1) over cubes with center a and sidelength r instead balls is clearly equivalent. On the other hand, we say that the one variable function $w(x,\cdot)$ satisfies the doubling condition if (1.3) holds for each x.

The spaces $\mathrm{BMO}_{w,q}$ provide an adequate setting to make a unifying approach to the study of several well known spaces. For instance, particular case of (1.1) can be found in [18], where the authors prove a weighted extension of the result that the Hilbert Transform is a bounded map of L^∞ into BMO. Also, taking $q=1$ and $w(x,t)=\Phi(t)\,t^{-n}\,\int_{B(x,t)} v(y)dy$, with a positive and locally integrable function v and assuming certain properties on $\Phi:\mathbb{R}^+\to\mathbb{R}^+$ we get the $\mathrm{BMO}_\Phi(v)$ of [14]. For $\Phi(t)=t^n$ we recover the weighted BMO space of Muckenhoupt and Wheeden ([19]). When $v\equiv 1$ we get the classical BMO ($\Phi\equiv 1$), the Lipschitz integral spaces ($\Phi(t)=t^\beta\,,\beta>0$) and, for a more general ϕ, the spaces BMO_Φ considered by Spanne in [24]. (See [3] in addition).

The case $w(x,t)=t^{\alpha-n}\|\chi_{B(x,t)}\|_{p'(\cdot)}$, where $0<\alpha<n$, $p'(\cdot)=p(\cdot)/(p(\cdot)-1)$ and $\|\cdot\|_{p(\cdot)}$ denotes the norm in the variable Lebesgue space $L^{p(\cdot)}$ (see [16]), has a special interest since, the spaces $\mathrm{BMO}_{w,q}$ are the spaces $\mathfrak{L}^q_{\alpha,p(\cdot)}$ introduced in [23]), which, under natural condition on $p(\cdot)$, they turn out to coincide with $\mathfrak{L}_{\alpha,p(\cdot)}$. (See Corollary 2.38 below). And, in turn, the space $\mathfrak{L}_{\alpha,p(\cdot)}$ has been identified (see Theorems 1.11 and 1.13 in [23]) as the suitable target space for the fractional integral operator acting on certain $L^{p(\cdot)}$.

Some particular cases of the spaces (1.1) are useful in the study of regularity of solutions of elliptic PDEs (see for instance [1], [2] and [23]).

The main purpose of our article is to make a unifying approach to the study of mapping properties of fractional integrals as well as Riesz transforms in relation to the spaces $\mathrm{BMO}_{w,q}$ so that this approach includes all the afore mentioned particular cases. In addition we proved some properties of these spaces such as, for instance, a pointwise characterization.

The structure of the article is as follows. Section 2 contains properties of the $\mathrm{BMO}_{w,q}$ spaces in general (section 2.1) and the particular case $\mathfrak{L}_{\alpha,p(\cdot)}$ (section 2.2). In section 3 are our main results related to the boundedness of the fractional integral. Finally section 4 is devoted to the boundedness of Riesz transforms.

2. Properties of the spaces $\mathrm{BMO}_{w,q}$ and $\mathfrak{L}^q_{\alpha,p(\cdot)}$.

In this section we prove some useful properties of the spaces involved. We start by recalling some definitions and properties related to real functions. They will be important tools in our results.

Definition 2.1. *Let $h:\mathbb{R}^+_0\to\mathbb{R}^+_0$ be a function. We say that h is of upper type $\beta>0$, if there exists a positive constant c such that*

$$h(st) \le c\, s^\beta h(t)\,,$$

for every $s \geq 1$ and for every $t > 0$. We also say that h is of lower type $\beta > 0$ if the last inequality holds for every $0 < s \leq 1$ and for every $t > 0$. We mean that h satisfies a doubling condition if there exists a constant c such that $h(2t) \leq c\,h(t)$ for all $t > 0$.

Definition 2.2. *We say that h is quasi decreasing if there exists a constant c such that $h(t_2) \leq c\,h(t_1)$ whenever $t_1 < t_2$.*

The proofs of the following lemmas are easy and left to the reader.

Lemma 2.3. *Let h be a function of upper type β with $0 < \beta \leq 1$, then h satisfies a doubling condition. Moreover, $h(t)/t$ is quasi decreasing.*

Lemma 2.4. *Consider a function h such that $h(t)/t^\beta$ is quasi decreasing for some $0 < \beta \leq 1$, then h is of upper type β. Moreover, h satisfies a doubling condition.*

2.1. The space $\mathrm{BMO}_{w,q}$.

Now we study conditions on the function $w : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ under which the functions belonging to $\mathrm{BMO}_{w,q}$ satisfy some kind of pointwise smoothness. Conversely, we also see that under certain hypothesis on w this smoothness implies the function belongs to $\mathrm{BMO}_{w,q}$.

Proposition 2.5 (Pointwise Condition)**.** *Let $1 \leq q < \infty$ and let $w(x,t)$ be a function satisfying (1.2). Then, for every $f \in BMO_{w,q}$, we have*

$$|f(x) - f(y)| \leq C\ \|f\|_{w,q} \int_0^{4|x-y|} \Big(w(x,t) + w(y,t) \Big) \frac{dt}{t}\,, \tag{2.6}$$

for some constant $C > 0$ and for almost every $x, y \in \mathbb{R}^n$.

Proof. Let x, y be Lebesgue points of f in $\mathbb{R}^n$. Taking $B = B(x, |x-y|)$ and $B' = B(y, |x-y|)$ we have

$$|f(x) - f(y)| \leq |f(x) - m_B f| + |f(y) - m_{B'} f| + |m_{B'} f - m_B f|\,.$$

We only estimate the first term on the right-hand side, since the second is analogous. Letting $B_i = B(x, 2^{-i}|x-y|)$ for each integer i, and using the hypothesis on w, we get

$$\begin{aligned}
|f(x) - m_B f| &\leq \lim_{k\to\infty} \left(|f(x) - m_{B_k} f| + \sum_{i=0}^{k-1} |m_{B_{i+1}} f - m_{B_i} f| \right) \\
&\leq C \sum_{i=0}^{\infty} \left(|B_i|^{-1} \int_{B_i} |f(z) - m_{B_i} f|^q\, dz \right)^{\frac{1}{q}} \\
&\leq C\ \|f\|_{w,q} \sum_{i=0}^{\infty} w(x, 2^{-i}|x-y|) \\
&= C\ \|f\|_{w,q} \sum_{i=0}^{\infty} \int_{2^{-i}|x-y|}^{2^{-i+1}|x-y|} w(x, 2^{-i}|x-y|)\, \frac{dt}{t} \qquad (2.7) \\
&\leq C\ \|f\|_{w,q} \int_0^{2|x-y|} w(x,t)\, \frac{dt}{t}\,.
\end{aligned}$$

Finally, denoting $2B = B(x, 2|x-y|)$, we have

$$\begin{aligned} |m_{B'}f - m_B f| &\leq |m_{B'}f - m_{2B}f| + |m_{2B}f - m_B f| \\ &\leq C\ \|f\|_{w,q} \int_0^{4|x-y|} w(x,t)\, \frac{dt}{t}\,. \end{aligned}$$

This completes the proof. □

Remark 2.8. If in addition $w(x,t)$ satisfies the the doubling condition (1.3) we can obtain

$$|f(x) - f(y)| \leq C\ \|f\|_{w,q} \int_0^{|x-y|} \Big(w(x,t) + w(y,t) \Big) \frac{dt}{t}\,, \tag{2.9}$$

for almost every $x, y \in \mathbb{R}^n$.

Proposition 2.10. *Let us consider w be a measurable function satisfying* (1.3)*. Suppose that for some $1 \leq q < \infty$*

$$\Psi_q(x,r) \doteq \left(\frac{1}{r^n} \int_{B(x,r)} \left(\int_0^r w(z,t)\, \frac{dt}{t} \right)^q dz \right)^{1/q}\,,$$

is finite for all $x \in \mathbb{R}^n$ and every $r > 0$. If a measurable function f satisfies a pointwise condition given by (2.6)*, then $f \in BMO_{\Psi_q,q}$. Moreover, if there exists a constant $C > 0$ such that*

$$\Psi_q(x,r) \leq C\ w(x,r)\,, \tag{2.11}$$

independent of x and r, then $f \in BMO_{w,q}$.

Proof. The finiteness of $\Psi_q(x,r)$ implies that the right-hand side of (2.6) is finite a.e. Moreover, it is not difficult to see that f is locally integrable. In order to prove that $f \in \mathrm{BMO}_{\Psi_q,q}$, we will prove that

$$\int_B |f(y) - m_B f|^q\, dy \leq C\, \Psi_q(x,r)^q\, |B|\,, \tag{2.12}$$

for every ball $B = B(x,r)$. In fact

$$\begin{aligned} &\int_B |f(y) - m_B f|^q\, dy \\ &\leq C \int_B \left(\frac{1}{|B|} \int_B \left| \int_0^{4|y-z|} \Big(w(y,t) + w(z,t) \Big) \frac{dt}{t} \right| dz \right)^q dy \\ &\leq C \int_B \left(\int_0^{8r} w(y,t)\, \frac{dt}{t} + \frac{1}{|B|} \int_B \int_0^{8r} w(z,t)\, \frac{dt}{t}\, dz \right)^q dy \\ &\leq C \int_B \left(\int_0^{8r} w(y,t)\, \frac{dt}{t} \right)^q dy + C\ |B|^{1-q} \left(\int_B \int_0^{8r} w(z,t)\, \frac{dt}{t}\, dz \right)^q \\ &\leq C \int_B \left(\int_0^{8r} \frac{w(z,t)}{t}\, dt \right)^q dz\,, \end{aligned}$$

where in the last step the Hölder's inequality was applied. Thus, using the doubling condition on w we have (2.12). Moreover, if (2.11) holds then it is clear that $f \in \mathrm{BMO}_{w,q}$ and the proposition is proved. □

The propositions above allow us to get the following theorem.

Theorem 2.13. *Let w be a measurable function satisfying* (1.2) *and* (1.3)*. Moreover, suppose* (2.11) *holds for some $1 \le q < \infty$, then $BMO_{w,1} = BMO_{w,s}$ for every $1 \le s \le q$.*

Proof. By Hölder's inequality it is clear that $\mathrm{BMO}_{w,s} \subset \mathrm{BMO}_{w,1}$. On the other hand, if $f \in \mathrm{BMO}_{w,1}$, from Proposition 2.5, f satisfies (2.6). In view of Proposition 2.10 we have that $f \in \mathrm{BMO}_{w,q}$. Hence, using Hölder's inequality again, the theorem follows. □

The next proposition gives sufficient conditions on w so that inequality (2.11) holds.

Proposition 2.14. *Let w be a measurable function. If w is of lower type $\beta > 0$ on the second variable, then inequality* (2.11) *holds for every $1 \le q < \infty$.*

Proof. Let $B = B(x,r)$, then by a change of variable we have

$$\begin{aligned}\int_B \left(\int_0^r \frac{w(z,u)}{u}\, du \right)^q dz &= \int_B \left(\int_0^1 \frac{w(z,rt)}{t}\, dt \right)^q dz \\ &\le C \int_B \Big(\int_0^1 t^{\beta-1}\, dt \Big)^q w(z,r)^q\, dz \\ &\le C \int_B w(x,r)^q\, dz \quad \le \quad C\, w(x,r)^q\, r^n .\end{aligned}$$

This completes the proof. □

We note that Theorem 2.13 makes use the pointwise condition (2.6) and the hypothesis (2.11). However, in the case $w \equiv 1$ neither of them is valid and it is well known that $\mathrm{BMO}_{1,1} = \mathrm{BMO}_{1,q}$ for every $1 \le q < \infty$. So it is natural to wonder what other properties of w can assure the same coincidence of spaces. In order to give an answer we first state the following result, which are due to Franchi, Pérez and Wheeden (see [11]).

Definition 2.15. *For a number $1 \le t < \infty$, we say that a function w satisfies the D_t-condition if there exists a positive constant c such that for each ball $B = B(x,r)$ and any family $\{B_i\}$ of pairwise disjoint subballs of B*

$$\sum_i w(x_i, r_i)^t\, r_i^n \le c^t\; w(x,r)^t\, r^n \,, \tag{2.16}$$

where x_i and r_i are the center and the radious of B_i, respectively. We denote the smallest constant c for which (2.16) *holds by $\|w\|$.*

It is not difficult to see that the D_t-condition implies the D_s-condition for every $1 \le s < t$.

Theorem 2.17 ([11], Theorem 2.3)**.** *Let $B_0 = B(x_0, r_0)$ be a ball in $\mathbb{R}^n$. Suppose that w satisfies the D_t-condition for some $1 \le t < \infty$. Let f be a measurable function defined on $17B_0$ and such that*

$$\frac{1}{|B|} \int_B |f(y) - f_B|\, dy \le \|f\|_{w,1}\, w(x,r) \,. \tag{2.18}$$

for every ball $B = B(x,r) \subset 17B_0$. Then

$$\sup_{\lambda > 0} \lambda \left(\frac{|\,\{x \in B_0 :\ |f - f_{B_0}| > \lambda\}\,|}{|B_0|} \right)^{1/r} \le C\ \|w\|\, \|f\|_{w,1}\, w(x_0, 17r_0)\,, \tag{2.19}$$

where the constant C is independent of f and B_0.

Corollary 2.20. *Let $1 < t < \infty$. Under the same hypotheses of the Theorem 2.17, we have*

$$\left(\frac{1}{|B_0|}\int_{B_0} |f(y) - f_{B_0}|^q \, dy\right)^{1/q} \leq C \; \|w\| \, \|f\|_{w,1} \, w(x_0, 17r_0)\,, \tag{2.21}$$

for every $1 < q < t$, where the constant C is independent of f and B_0.

Now, in view of the above theorem and its corollary, we are able to prove the following result.

Theorem 2.22. *Let w be a measurable function satisfying* (1.2), (1.3) *and* (1.4). *Then, the spaces $BMO_{w,q}$ coincide for all $1 \leq q < \infty$.*

Proof. It is easily seen that $\mathrm{BMO}_{w,q} \subset \mathrm{BMO}_w$. On the other hand, if $f \in \mathrm{BMO}_w$, by Corollary 2.20, we have to see that w satisfies the D_q-condition for every $1 \leq q < \infty$. In fact, let B be a ball and $\{B_i\}$ a family of pairwise disjoint subballs of B. Then, from the hypotheses on w we have

$$\begin{aligned}\sum_i w(x_i, r_i)^q \, r_i^n \quad &\leq \quad C \sum_i w(x_i, r)^q \, r_i^n \quad \leq \quad C \sum_i w(x, r)^q \, r_i^n \\ &\leq \quad C \; w(x, r)^q \sum_i |B_i| \quad \leq \quad C \; w(x, r)^q |B| \quad = \quad C \; w(x, r)^q \, r^n\,.\end{aligned}$$

Then, the spaces coincide. □

2.2. **The space $\mathfrak{L}^q_{\alpha, p(\cdot)}$.**

Here we consider the spaces $\mathfrak{L}^q_{\alpha, p(\cdot)}$ defined in the previous section. We will denote $p_-(\Omega)$ and $p_+(\Omega)$ the infimum and supremum of $p(\cdot)$ over Ω, when Ω is a subset of $\mathbb{R}^n$. We only write p_- and p_+ in the case $\Omega = \mathbb{R}^n$. The following lemma shows very useful relations between the norm of a characteristic function of a ball and its Lebesgue measure.

Lemma 2.23. *Let $B = B(x_0, r)$ be a ball in $\mathbb{R}^n$.*

(a) *There exist positive constants a_1 and a_2 such that if $r < 1$ we get*

$$a_1 \, |B|^{\frac{1}{p_-(B)}} \leq \|\chi_B\|_{p(\cdot)} \leq a_2 \, |B|^{\frac{1}{p_+(B)}}\,.$$

(b) *There exist positive constants b_1 and b_2 such that if $r > 1$ we get*

$$b_1 \, |B|^{\frac{1}{p_+(B)}} \leq \|\chi_B\|_{p(\cdot)} \leq b_2 \, |B|^{\frac{1}{p_-(B)}}\,.$$

In the setting of variable exponent Lebesgue spaces it is common to assume the following Log–Hölder conditions on the exponent functions $p(\cdot)$.

$$LH_0 : \qquad \exists\, c_0 > 0 \,/ \qquad |p(x) - p(y)| \leq \frac{c_0}{\log(e + \frac{1}{|x-y|})}\,, \qquad \forall\, x, y \in \mathbb{R}^n\,;$$

$$LH_\infty : \qquad \exists\, p_\infty, c_1 > 0 \,/ \qquad |p(x) - p_\infty| \leq \frac{c_1}{\log(e + |x|)}\,, \qquad \forall\, x \in \mathbb{R}^n\,.$$

Remark 2.24. In [5] (Proposition 4.57) it is proved that LH_0 and LH_∞ imply that there exists a constant $C>0$ such that

$$\|\chi_B\|_{p(\cdot)}\,\|\chi_B\|_{p'(\cdot)} \le C\,|B|\,, \tag{2.25}$$

for every ball $B\subset\mathbb{R}^n$. Considering this inequality and applying Hölder's inequality it is easy to see that $w(x,\cdot)=\left\|\chi_{B(x,\cdot)}\right\|_{p(\cdot)}$ verifies the following doubling condition

$$\left\|\chi_{B(x,2t)}\right\|_{p(\cdot)} \le C\;\left\|\chi_{B(x,t)}\right\|_{p(\cdot)}\,, \tag{2.26}$$

where the constant C is independent of x and t. Obviously, $p'(\cdot)$ has the same property.

Definition 2.27. *If inequality* (2.26) *holds we say that the exponent function* $p(\cdot)$ *satisfies a doubling condition.*

In connection with these Log–Hölder continuity properties we state two important lemmas whose proofs can be found in several articles, see for instance [10, 7, 12, 4, 8].

Lemma 2.28. *Let* $p_+<\infty$. *Then the following conditions are equivalent:*

(a) *The function* $p(\cdot)$ *satisfies* LH_0.

(b) *There exists a constant* C *such that*

$$|B|^{p_-(B)-p_+(B)}\le C\,,$$

for every ball $B\subset\mathbb{R}^n$.

Lemma 2.29. *Let* $p(\cdot)$ *be an exponent function satisfying* LH_∞. *Then, there exists a constant* c *such that*

$$c^{-1}|B|^{1/p_\infty}\le \|\chi_B\|_{p(\cdot)}\le c\,|B|^{1/p_\infty}\,,$$

for every ball B *with radius greater than or equal to* $1/4$.

The following three technical lemmas give some properties of the particular function $w(x,t)=t^{\alpha-n}\left\|\chi_{B(x,t)}\right\|_{p'(\cdot)}$.

Lemma 2.30. *Let* $p(\cdot)$ *be an exponent function such that* $p_-\ge\frac{n}{\alpha}$ *and satisfying* LH_0 *and* LH_∞. *Then* $w(x,t)=t^{\alpha-n}\left\|\chi_{B(x,t)}\right\|_{p'(\cdot)}$ *is quasi–increasing as a function of* t.

Proof. We will see that there exists a constant $C>0$ such that, given $0<s<t$, we get

$$t^{\alpha-n}\left\|\chi_{B(x,t)}\right\|_{p'(\cdot)}\le C\;s^{\alpha-n}\left\|\chi_{B(x,s)}\right\|_{p'(\cdot)}\,, \tag{2.31}$$

where C does not depent on x. For this, we divide the proof in three parts.

(a) If $1<t<s$, by Lemma 2.29 and the hypotheses on $p(\cdot)$, we have

$$\begin{aligned} w(x,t) &= t^{\alpha-n}\left\|\chi_{B(x,t)}\right\|_{p'(\cdot)} \le C\;t^{\alpha-n}\,t^{n-\frac{n}{p_\infty}} = C\;t^{\alpha-\frac{n}{p_\infty}}\\ &\le C\;s^{\alpha-\frac{n}{p_\infty}} \le C\;s^{\alpha-n}\left\|\chi_{B(x,s)}\right\|_{p'(\cdot)} = C\;w(x,s)\,.\end{aligned}$$

(b) Now, if $t<s<1$, by Lemma 2.23 with $p'(\cdot)$ instead of $p(\cdot)$, Lemma 2.28 and taking into account that $p_-(B(x,t))\ge p_-(B(x,s))$, we have

$$\begin{aligned} w(x,t) &= t^{\alpha-n}\left\|\chi_{B(x,t)}\right\|_{p'(\cdot)}\\ &\le C\;t^{\alpha-n}\;t^{n-\frac{n}{p_-(B(x,t))}}\end{aligned}$$

$$\begin{aligned}
&\leq \quad C\ s^{\alpha-\frac{n}{p_-(B(x,t))}} \\
&\leq \quad C\ s^{\alpha-\frac{n}{p_+(B(x,s))}} \left(s^{\frac{n}{p_+(B(x,s))}-\frac{n}{p_-(B(x,s))}}\right) \\
&\leq \quad C\ s^{\alpha-n} \left\|\chi_{B(x,s)}\right\|_{p'(\cdot)} \quad = \quad C\ w(x,s)\,.
\end{aligned}$$

(c) Finally, suppose that $t < 1 < s$. By Lemmas 2.23 and 2.29 we get that

$$w(x,t) \quad \leq \quad C\ t^{\alpha-\frac{n}{p_-(B(x,t))}} \quad \leq \quad C \quad \leq \quad C\ s^{\alpha-\frac{n}{p_\infty}} \quad \leq \quad C\ w(x,s)\,.$$

The proof is complete. □

Lemma 2.32. *Let $p(\cdot)$ be an exponent function satisfying LH_0 and LH_∞. Then there exists $C > 0$ such that*

$$\left\|\chi_{B(x,t)}\right\|_{p'(\cdot)} \leq C\ \left\|\chi_{B(y,t)}\right\|_{p'(\cdot)}\ ,$$

whenever $|x-y| < t$, for all $t > 0$.

Proof. It is not difficult to see that $p'(\cdot)$ satisfies LH_0 and LH_∞ whenever $p(\cdot)$ does. Moreover $1/p'_\infty = 1 - 1/p_\infty$. So, in order to prove the lemma, we are going to consider two cases. First, suppose $t > 1$. By Lemma 2.29 we have

$$\left\|\chi_{B(x,t)}\right\|_{p'(\cdot)} \leq C\ |B(x,t)|^{1-\frac{1}{p_\infty}} = C\ |B(y,t)|^{1-\frac{1}{p_\infty}} \leq C\ \left\|\chi_{B(y,t)}\right\|_{p'(\cdot)}\ .$$

Now, suppose $t \leq 1$. Since that $B(y,t) \subset B(x,2t)$ if $|x-y| < t$, by Lemmas 2.23 and 2.28 we have

$$\begin{aligned}
\left\|\chi_{B(x,t)}\right\|_{p'(\cdot)} \quad &\leq \quad C\ |B(x,t)|^{1-\frac{1}{p_-(B(x,t))}} \\
&= \quad C\ |B(x,t)|^{1-\frac{1}{p_+(B(y,t))}}\, |B(x,t)|^{\frac{1}{p_+(B(y,t))}-\frac{1}{p_-(B(x,t))}} \\
&\leq \quad C\ |B(y,t)|^{1-\frac{1}{p_+(B(y,t))}} \left(|B(x,t)|^{p_-(B(x,2t))-p_+(B(x,2t))}\right)^{1/p_-^2} \\
&\leq \quad C\ \left\|\chi_{B(y,t)}\right\|_{p'(\cdot)}\ ,
\end{aligned}$$

and the lemma is proved. □

Lemma 2.33. *Let $p(\cdot)$ be an exponent function such that $p_- > \frac{n}{\alpha}$. If $p(\cdot)$ satisfies LH_0 and LH_∞ then*

$$\int_0^r \frac{\left\|\chi_{B(x,t)}\right\|_{p'(\cdot)}}{t^{n-\alpha}}\,\frac{dt}{t} \leq C\ \frac{\left\|\chi_{B(x,r)}\right\|_{p'(\cdot)}}{r^{n-\alpha}}\,. \tag{2.34}$$

for every $B = B(x,r)$, where C is independent of B.

Proof. Let $x \in \mathbb{R}^n$ fixed. First we suppose $r \leq 1$. By Lemma 2.23 we have

$$\begin{aligned}
\int_0^r \frac{\left\|\chi_{B(x,t)}\right\|_{p'(\cdot)}}{t^{n-\alpha}}\,\frac{dt}{t} \quad &\leq \quad C\ \int_0^r \frac{|B(x,t)|^{1-\frac{1}{p_-(B(x,t))}}}{t^{n-\alpha}}\,\frac{dt}{t} \\
&\leq \quad C\ \int_0^r t^{\alpha-\frac{n}{p_-(B(x,r))}-1}\,dt \quad = \quad C\ r^{\alpha-\frac{n}{p_-(B(x,r))}}\ ,
\end{aligned}$$

Now, from Lemma 2.28 and Lemma 2.23 again we get

$$\begin{aligned}
\int_0^r \frac{\left\|\chi_{B(x,t)}\right\|_{p'(\cdot)}}{t^{n-\alpha}}\,\frac{dt}{t} \quad &\leq \quad C\ r^{\alpha-\frac{n}{p_+(B(x,r))}} \left(r^{\frac{n}{p_+(B(x,r))}-\frac{n}{p_-(B(x,r))}}\right) \\
&\leq \quad C\ \frac{\left\|\chi_{B(x,r)}\right\|_{p'(\cdot)}}{r^{n-\alpha}}\,.
\end{aligned}$$

On the other hand, if $r > 1$, we write

$$\int_0^r \frac{\left\|\chi_{B(x,t)}\right\|_{p'(\cdot)}}{t^{n-\alpha}} \frac{dt}{t} = \int_0^1 \frac{\left\|\chi_{B(x,t)}\right\|_{p'(\cdot)}}{t^{n-\alpha}} \frac{dt}{t} + \int_1^r \frac{\left\|\chi_{B(x,t)}\right\|_{p'(\cdot)}}{t^{n-\alpha}} \frac{dt}{t}\,. \tag{2.35}$$

The previous estimate allows to obtain

$$\int_0^1 \frac{\left\|\chi_{B(x,t)}\right\|_{p'(\cdot)}}{t^{n-\alpha}} \frac{dt}{t} \quad \leq \quad C\,.$$

Now, since $\alpha p_- - n > 0$ and $(p_-)' = (p')_+$ it is not difficult to see that $(\alpha-n)(p_-)' + n \geq 1$. Then, we get

$$\begin{aligned} 1 \quad < \quad r^{(\alpha-n)(p_-)'+n} \quad &= \quad C\,|B(x,r)|\,r^{(\alpha-n)(p_-)'} \quad = \quad C\,|B(x,r)|\,r^{(\alpha-n)(p')_+} \\ &< \quad C \int_{B(x,r)} r^{(\alpha-n)p'(y)}\,dy \quad = \quad C \int_{\mathbb{R}^n} \left(\frac{\chi_{B(x,r)}}{r^{(n-\alpha)}}\right)^{p'(y)} dy\,. \end{aligned}$$

So it follows that $r^{(n-\alpha)}$ is a lower bound for $\left\|\chi_{B(x,r)}\right\|_{p'(\cdot)}$. Then

$$\int_0^1 \frac{\left\|\chi_{B(x,t)}\right\|_{p'(\cdot)}}{t^{n-\alpha}} \frac{dt}{t} \quad \leq \quad C\, \frac{\left\|\chi_{B(x,r)}\right\|_{p'(\cdot)}}{r^{n-\alpha}}\,. \tag{2.36}$$

For the second term, by Lemma 2.29, the estimate is clear. In fact

$$\begin{aligned} \int_1^r \frac{\left\|\chi_{B(x,t)}\right\|_{p'(\cdot)}}{t^{n-\alpha}} \frac{dt}{t} \quad &\leq \quad C \int_1^r \frac{|B_t|^{1-\frac{1}{p_\infty}}}{t^{n-\alpha}} \frac{dt}{t} \quad \leq \quad C \int_1^r t^{\alpha-\frac{n}{p_\infty}-1}\,dt \\ &\leq \quad C\, r^{\alpha-\frac{n}{p_\infty}} \quad \leq \quad C\, \frac{\left\|\chi_{B(x,r)}\right\|_{p'(\cdot)}}{r^{n-\alpha}}\,. \end{aligned} \tag{2.37}$$

Finally, note that inequalities (2.35), (2.36) and (2.37) imply (2.34). □

Corollary 2.38. *Let $0 < \alpha < n$ and let $p(\cdot)$ be an exponent function with $p_- > \frac{n}{\alpha}$ such that the conditions LH_0 and LH_∞ hold. Then $\mathfrak{L}_{\alpha,p(\cdot)} = \mathfrak{L}^q_{\alpha,p(\cdot)}$ for $1 \leq q < \infty$.*

Proof. By Hölder's inequality, it is clear that $\mathfrak{L}^q_{\alpha,p(\cdot)} \subset \mathfrak{L}_{\alpha,p(\cdot)}$. On the other hand, from Lemma 2.30, the hypotheses on $p(\cdot)$ and Proposition 2.5, every $f \in \mathfrak{L}_{\alpha,p(\cdot)}$ satisfies the pointwise inequality (2.6). Moreover, the right-hand side of this inequality is finite.
Now, for $1 < q < \infty$ fixed, from Lemmas 2.33 and 2.32 we have

$$\begin{aligned} \int_{B(x,r)} \left(\int_0^r \frac{\left\|\chi_{B(z,u)}\right\|_{p'(\cdot)}}{u^{n-\alpha}} \frac{du}{u}\right)^q dz \quad &\leq \quad C \int_{B(x,r)} \frac{\left\|\chi_{B(z,r)}\right\|^q_{p'(\cdot)}}{r^{(n-\alpha)q}}\,dz \\ &\leq \quad C \left(r^{\alpha-n} \left\|\chi_{B(x,r)}\right\|_{p'(\cdot)}\right)^q r^n\,, \end{aligned}$$

which states that (2.11) holds for $w(x,r) = r^{\alpha-n}\left\|\chi_{B(x,r)}\right\|_{p'(\cdot)}$. Then, Remark 2.24 and Proposition 2.10 ensures that $f \in \mathfrak{L}^q_{\alpha,p(\cdot)}$, which finishes the proof. □

Now, if we consider an exponent function $p(\cdot)$ that does not necessarily verify the Log–Hölder conditions LH_0 and LH_∞, a different approach can be adopted.

However, a smaller range of q is obtained. In order to do this, we first recall (see [16]) that we can write the representation of the norm given by

$$\|f\|_{p(\cdot)} \approx \sup_{g:\ \|g\|_{p'(\cdot)}\leq 1} \int_{\mathbb{R}^n} f(x)\, g(x)\, dx\,. \tag{2.39}$$

In order to prove our previous statement, we need the following variable version of Minkowski's integral inequality. This result can be found in [C-U,F], however for the sake of completeness we include the proof here.

Proposition 2.40. *Let $p(\cdot)$ be an exponent function and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a measurable function.*

(a) Suppose that $f(\cdot, y) \in L^{p(\cdot)}$ a.e. $y \in \mathbb{R}^n$ and the mapping $y \to \|f(\cdot, y)\|_{p(\cdot)}$ is in L^1, then

$$\left\| \int_{\mathbb{R}^n} f(\cdot, y)\, dy \right\|_{p(\cdot)} \leq C \int_{\mathbb{R}^n} \|f(\cdot, y)\|_{p(\cdot)}\, dy\,,$$

where C is only dependent on the bounds of $p(\cdot)$.

(b) Moreover, for $1 < q < p_-$ we get

$$\left\| \left(\int_{\mathbb{R}^n} |f(\cdot, y)|^q\, dy \right)^{\frac{1}{q}} \right\|_{p(\cdot)} \leq C \left(\int_{\mathbb{R}^n} \|f(\cdot, y)\|_{p(\cdot)}^q\, dy \right)^{\frac{1}{q}}.$$

Proof. By Hölder's inequality and (2.39), we have

$$\begin{aligned} \left\| \int_{\mathbb{R}^n} f(\cdot, y)\, dy \right\|_{p(\cdot)} &\leq C \sup_{g:\ \|g\|_{p'(\cdot)}\leq 1} \left[\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x, y)\, dy \right) |g(x)|\, dx \right] \\ &\leq C \sup_{g:\ \|g\|_{p'(\cdot)}\leq 1} \left[\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x, y) |g(x)|\, dx\, dy \right] \\ &\leq C \int_{\mathbb{R}^n} \|f(\cdot, y)\|_{p(\cdot)}\, dy\,. \end{aligned}$$

Now to prove *(b)* we observe that $\frac{p(\cdot)}{q}$ is an exponent function with $\left(\frac{p(\cdot)}{q}\right)_- > 1$ and $\|f\|_{p(\cdot)}^q = \|f^q\|_{\frac{p(\cdot)}{q}}$ whenever $f \in L^{p(\cdot)}$. Then, from *(a)*, we get

$$\begin{aligned} \left\| \left(\int_{\mathbb{R}^n} |f(\cdot, y)|^q\, dy \right)^{\frac{1}{q}} \right\|_{p(\cdot)}^q &= \left\| \int_{\mathbb{R}^n} |f(\cdot, y)|^q\, dy \right\|_{\frac{p(\cdot)}{q}} \\ &\leq C \int_{\mathbb{R}^n} \|\, |f(\cdot, y)|^q \,\|_{\frac{p(\cdot)}{q}}\, dy \\ &= C \int_{\mathbb{R}^n} \|\, |f(\cdot, y)|\, \|_{p(\cdot)}^q\, dy\,, \end{aligned}$$

and so we get the desired result. □

Proposition 2.41. *Let $0 < \alpha < n$ and let $p(\cdot)$ be an exponent function with $p_- > \frac{n}{\alpha}$ such that $p'(\cdot)$ satisfies a doubling condition. If a measurable function f satisfies the following pointwise condition*

$$|f(x) - f(y)| \leq C \int_0^{2|x-y|} \frac{\left\|\chi_{B(x,t)}\right\|_{p'(\cdot)} + \left\|\chi_{B(y,t)}\right\|_{p'(\cdot)}}{t^{n-\alpha}} \frac{dt}{t}\,, \tag{2.42}$$

for almost every $x, y \in \mathbb{R}^n$, then $f \in \mathfrak{L}_{\alpha, p(\cdot)}$.

Proof. Given $r > 0$ and $x_0 \in \mathbb{R}^n$ we consider the ball $B = B(x_0, r)$. In order to prove that $f \in \mathfrak{L}_{\alpha,p(\cdot)}$ we will see that

$$\int_B |f(x) - m_B f|\,dx \le C\,|B|^{\frac{\alpha}{n}}\,\|\chi_{2B}\|_{p'(\cdot)}\,. \tag{2.43}$$

In fact, as in the proof of Proposition 2.10, from the hypothesis on f and the fact that $p'(\cdot)$ satisfies a doubling condition we get

$$\begin{aligned}\int_B |f(x) - m_B f|\,dx &\le C\int_B\int_0^{4r}\frac{\|\chi_{B(x,2t)}\|_{p'(\cdot)}}{t^{n-\alpha}}\,\frac{dt}{t}\,dx\\ &\le C\int_0^r\int_B\|\chi_{B(x,t)}\|_{p'(\cdot)}\,dx\,\frac{dt}{t^{n-\alpha+1}}\,.\end{aligned}$$

Now, let $q > 1$ to be determined later. Applying the Hölder's inequality we have

$$\begin{aligned}\int_B |f(x) - m_B f|\,dx &\le C\,|B|^{\frac{1}{q'}}\int_0^r\Big(\int_B\|\chi_{B(x,t)}\|^q_{p'(\cdot)}\,dx\Big)^{\frac{1}{q}}\frac{dt}{t^{n-\alpha+1}}\\ &\le C\,|B|^{\frac{1}{q'}}\int_0^r\Big(\int_{\mathbb{R}^n}\|\chi_{B(x_0,r)}(x)\chi_{B(x,t)}(\cdot)\|^q_{p'(\cdot)}\,dx\Big)^{\frac{1}{q}}\frac{dt}{t^{n-\alpha+1}}\,.\end{aligned}$$

We claim that by taking q such that $\frac{n}{\alpha} < q' < p_-$ we have

$$\Big(\int_{\mathbb{R}^n}\|\chi_{B(x_0,r)}(x)\chi_{B(x,t)}(\cdot)\|^q_{p'(\cdot)}\,dx\Big)^{\frac{1}{q}} \le C\,t^{\frac{n}{q}}\,\|\chi_{2B}\|_{p'(\cdot)} < \infty\,, \tag{2.44}$$

for every $0 < t < r$. Thus,

$$\begin{aligned}\int_B |f(x) - m_B f|\,dx &\le C\,\|\chi_{2B}\|_{p'(\cdot)}\,|B|^{\frac{1}{q'}}\int_0^r t^{\frac{n}{q}-n+\alpha-1}\,dt\\ &= C\,\|\chi_{2B}\|_{p'(\cdot)}\,|B|^{\frac{1}{q'}}\,r^{\alpha-\frac{n}{q'}} = C\,\|\chi_{2B}\|_{p'(\cdot)}\,|B|^{\frac{\alpha}{n}}\,,\end{aligned}$$

and so we get our result.

Now, it only remains to prove the claim. Using the theory of integration for vector-valued functions our claim says that $\chi_{B(x_0,r)}(x)\chi_{B(x,t)}(z)$ belongs to the Bochner–Lebesgue space $L^q_{L^{p'(\cdot)}}$ (see [9], chapter V), whose topological dual space is $L^{q'}_{L^{p(\cdot)}}$. Then by duality we can write

$$\begin{aligned}&\|\chi_{B(x_0,r)}(x)\chi_{B(x,t)}(\cdot)\|_{L^q_{L^{p'(\cdot)}}}\\ &\quad\le C\sup_{\|g_t(x,z)\|_{L^{q'}_{L^{p(\cdot)}}}\le 1}\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\chi_{B(x_0,r)}(x)\chi_{B(x,t)}(z)g_t(x,z)\,dx\,dz\Big)\\ &\quad\le C\sup_{\|g_t(x,z)\|_{L^{q'}_{L^{p(\cdot)}}}\le 1}\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\chi_{B(x_0,r)}(x)\chi_{B(z,t)}(x)g_t(x,z)\,dx\,dz\Big)\,,\end{aligned}$$

where in the last expression we use that $\chi_{B(x,t)}(z) = \chi_{B(z,t)}(x)$. Now, taking into account that for every fixed $0 < t < r$ we have $\chi_{B(x_0,2r)}(z) = 1$ for all $z \in B(x,t)$ whenever $x \in B(x_0, r)$, we have

$$\begin{aligned}&\|\chi_{B(x_0,r)}(x)\chi_{B(x,t)}(z)\|_{L^q_{L^{p'(\cdot)}}}\\ &\quad\le C\sup_{\|g_t(x,z)\|_{L^{q'}_{L^{p(\cdot)}}}\le 1}\Big(\int_{\mathbb{R}^n}\chi_{B(x_0,2r)}(z)\int_{\mathbb{R}^n}\chi_{B(z,t)}(x)|g_t(x,z)|\,dx\,dz\Big)\end{aligned}$$

$$\leq \quad C\, t^{\frac{n}{q}} \sup_{\|g_t(x,z)\|_{L^{q'}_{L^{p(\cdot)}}} \leq 1} \int_{2B} \|g_t(\cdot,z)\|_{q'}\, dz$$

$$\leq \quad C\, t^{\frac{n}{q}} \|\chi_{2B}\|_{p'(\cdot)} \sup_{\|g_t(x,z)\|_{L^{q'}_{L^{p(\cdot)}}} \leq 1} \|g_t(x,z)\|_{L^{p(\cdot)}_{L^{q'}}}\ .$$

Finally, recalling that $\frac{n}{\alpha} < q' < p_-$, by Proposition 2.40 we conclude that

$$\begin{aligned} &\left\|\chi_{B(x_0,r)}(x)\chi_{B(x,t)}(z)\right\|_{L^q_{L^{p'(\cdot)}}} \\ &\quad \leq \quad C\, t^{\frac{n}{q}} \|\chi_{B_{2R}}\|_{p'(\cdot)} \sup_{\|g_t(x,z)\|_{L^{q'}_{L^{p(\cdot)}}} \leq 1} \|g_t(x,z)\|_{L^{q'}_{L^{p(\cdot)}}} \\ &\quad \leq \quad C\, t^{\frac{n}{q}} \|\chi_{B_{2R}}\|_{p'(\cdot)} \quad < \quad \infty\,. \end{aligned}$$

□

We note that if in Proposition 2.41 $p(\cdot)$ is assumed to be a constant in the interval $(n/\alpha, n/(\alpha-1)^+)$, then the pointwise condition (2.42) implies that f belongs to the Lipschitz space of order $0 < \beta = \alpha - n/p < 1$ (See, in addition, [13] and [24]). In view of Proposition 2.10, we get the following pointwise characterization of $\mathfrak{L}_{\alpha,p(\cdot)}$ without Log–Hölder hypotheses on $p(\cdot)$.

Theorem 2.45. *Let $0 < \alpha < n$ and $p(\cdot)$ be an exponent function such that $p_- > \frac{n}{\alpha}$ and $p'(\cdot)$ satisfies a doubling condition. The following conditions are equivalent*

(1) $f \in \mathfrak{L}_{\alpha,p(\cdot)}$.
(2) f *satisfies* (2.42).

Proof. From Proposition 2.41, clearly *(2)* implies *(1)*. In order to prove the converse we proceed in the same way as in Proposition 2.5 but this time considering in (2.7) the properties of $w(x,t) = t^{\alpha-n}\left\|\chi_{B(x,t)}\right\|_{p'(\cdot)}$, i.e.: $p'(\cdot)$ satisfies a doubling condition and $t^{\alpha-n}$ is decreasing. □

Theorem 2.46. *Let $0 < \alpha < n$ and let $p(\cdot)$ be an exponent function such that $p_- > \frac{n}{\alpha}$. If $p'(\cdot)$ satisfies a doubling condition. then $\mathfrak{L}_{\alpha,p(\cdot)} = \mathfrak{L}^s_{\alpha,p(\cdot)}$ for every s, with $1 \leq s < \frac{n}{n-\alpha}$.*

Proof. By Hölder's inequality, clearly $\mathfrak{L}^s_{\alpha,p(\cdot)} \subset \mathfrak{L}_{\alpha,p(\cdot)}$. On the other hand, if $f \in \mathfrak{L}_{\alpha,p(\cdot)}$ by Corollary 2.45 f satisfies the pointwise estimate (2.42). In view of Proposition 2.10 with $w(x,t) = t^{\alpha-n}\left\|\chi_{B(x,t)}\right\|_{p'(\cdot)}$, we only have to prove (2.11). Taking s in $(1, \frac{n}{n-\alpha})$ we get

$$\begin{aligned} &\int_B \left(\int_0^r t^{\alpha-n-1}\left\|\chi_{B(x,t)}\right\|_{p'(\cdot)} dt\right)^s dx \\ &\quad \leq \quad \left(\int_0^r \left(\int_B t^{(\alpha-n-1)s}\left\|\chi_{B(x,t)}\right\|^s_{p'(\cdot)} dx\right)^{\frac{1}{s}} dt\right)^s \\ &\quad = \quad \left(\int_0^r t^{\alpha-n-1}\left\|\chi_{B(x_0,r)}(x)\chi_{B(x,t)}(z)\right\|_{L^s_{L^{p'(\cdot)}}} dt\right)^s \\ &\quad \leq \quad C\left(\int_0^r t^{\alpha-n-1}t^{\frac{n}{s}}\left\|\chi_{B(x_0,r)}\right\|_{p'(\cdot)} dt\right)^s \end{aligned}$$

$$
\begin{aligned}
&= C\ \left\|\chi_{B(x_0,r)}\right\|_{p'(\cdot)}^s \left(\int_0^r t^{\alpha-n-1+\frac{n}{s}}\,dt\right)^s \\
&= C\ \left\|\chi_{B(x_0,r)}\right\|_{p'(\cdot)}^s r^{(\alpha-n)s} r^n \\
&= C\ \left(\left\|\chi_B\right\|_{p'(\cdot)} r^{(\alpha-n)}\right)^s r^n\,,
\end{aligned}
$$

where $\alpha - n + \frac{n}{s} > 0$. Then $\mathfrak{L}_{\alpha,p(\cdot)} = \mathfrak{L}^s_{\alpha,p(\cdot)}$ for every $1 \leq s < \frac{n}{n-\alpha}$ as we wanted to prove. □

3. Fractional integrals on $\mathrm{BMO}_{w,q}$ spaces.

In this section, we prove boundedness results for the fractional integral operator. In order to do this, we consider the following definition.

Definition 3.1. *Let $w(x,t)$ be a measurable function. We say that $w \in \mathcal{W}_\infty$ if there exists a constant $C > 0$ such that*

$$
\int_r^\infty \frac{w(x,t)}{t}\frac{dt}{t} \leq C\ \frac{w(x,r)}{r}\,, \tag{3.2}
$$

for all $x \in \mathbb{R}^n$ and every $r > 0$.

It should be noticed that condition (3.2) appears in the literature in different contexts. See, for instance [13], [6] and, in the particular case $w(x,t) = t^{\alpha-n}\left\|\chi_{B(x,t)}\right\|_{p'(\cdot)}$, in [23] by the authors of the present paper.
Now, we prove a technical lemma that will be useful in order to get one of our main results.

Lemma 3.3. *Let α be a real number and let w be a measurable function satisfying (1.2) and (1.3). If a function f belongs to $BMO_{w,q}$ for a some $1 \leq q < \infty$, then we have*

$$
\int_B \frac{|f(y) - m_B f|}{|x-y|^{n-\alpha}}\,dy \quad \leq \quad C\ \|f\|_{w,q} \int_0^r \frac{t^\alpha\, w(x,t)}{t}\,dt\,, \tag{3.4}
$$

where $B = B(x,r)$ and C is not dependent on B.

Proof. Let $x \in \mathbb{R}^n$ and $r > 0$. We consider the ball $B = B(x,r)$ and denote $B_k = B(x, 2^{-k}r)$, $k \in \mathbb{N}_0$. Then, we estimate

$$
\begin{aligned}
\int_B \frac{|f(y) - m_B f|}{|x-y|^{n-\alpha}}\,dy &= \sum_{k=0}^\infty \int_{B_k - B_{k+1}} \frac{|f(y) - m_B f|}{|x-y|^{n-\alpha}}\,dy \\
&\leq C \sum_{k=0}^\infty (2^{-k}r)^\alpha\, |B_k|^{-1} \int_{B_k} |f(y) - m_B f|\,dy \\
&\leq C \sum_{k=0}^\infty (2^{-k}r)^\alpha \sum_{j=0}^k \left(|B_j|^{-1} \int_{B_j} \left|f(y) - m_{B_j} f\right|^q\,dy\right)^{\frac{1}{q}} \\
&\leq C\ \|f\|_{w,q} \sum_{k=0}^\infty (2^{-k}r)^\alpha \sum_{j=0}^k w(x, 2^{-j}r)
\end{aligned}
$$

$$
\begin{aligned}
&= C\ \|f\|_{w,q} \sum_{j=0}^{\infty} (2^{-j}r)^{\alpha}\, w(x, 2^{-j}r) \\
&\le C\ \|f\|_{w,q} \sum_{j=0}^{\infty} \int_{2^{-j-1}r}^{2^{-j}r} t^{\alpha}\, w(x,t)\, \frac{dt}{t} \\
&\le C\ \|f\|_{w,q} \int_0^r \frac{t^{\alpha}\, w(x,t)}{t}\, dt\,,
\end{aligned}
$$

as we wanted to prove. □

Remark 3.5. Note that the conditions (1.2) and (1.3) are only applied in Lemma 3.3 to get the integral expression involving the function $w(x,t)$. For the particular case $w(x,t) = \Phi(t)t^{-n} \int_{B(x,t)} v(y)dy$ a similar expression can be proved without assuming those hypothesis on the whole $w(x,t)$. In that case we only need to consider the properties of each factor.

Now, for a number $\sigma \ge 0$, we denote $w_\sigma(x,t) = t^\sigma\, w(x,t)$ (clearly $w_0 = w$). We will prove the main theorem of this section.

Theorem 3.6. *Let $0 < \alpha < n$ and w be a non-negative measurable function satisfying conditions* (1.2), (1.3) *and* (1.4). *Now, if $w_\alpha \in \mathcal{W}_\infty$ then the fractional integral I_α can be extended to a linear bounded operator from $BMO_{w,q}$ in to $BMO_{w_\alpha,q}$, with $1 \le q < \infty$ as follows*

$$
\widetilde{I}_\alpha f(x) = \int_{\mathbb{R}^n} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|y|^{n-\alpha}} \right) f(y)\, dy\,, \tag{3.7}
$$

so, that $\widetilde{I}_\alpha$ is well defined on $BMO_{w,q}$.

Proof. First we prove that the extension of I_α to $\widetilde{I}_\alpha$ is well defined. For this, we take $f \in \mathrm{BMO}_{w,q}$, $x \in \mathbb{R}^n$, $r > |x|$ and the ball $B = B(0,r)$. We need to show that $|\widetilde{I}_\alpha f(x)| < \infty$. Since the expression in brackets in (3.7) has null integral over $\mathbb{R}^n$ as a function of y, we get

$$
\widetilde{I}_\alpha f(x) \;=\; \int_{\mathbb{R}^n} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|y|^{n-\alpha}} \right) (f(y) - m_{2B} f)\ dy \;=\; I_1(x) + I_2(x)\,,
$$

where I_1 and I_2 are the integrals over $B(0,2r)$ and $\mathbb{R}^n - B(0,2r)$, respectively. For I_1, by Lemma 3.3 and the condition (1.2) on w, we have

$$
\begin{aligned}
|I_1(x)| &\le \int_{B(0,2r)} \frac{\left|f(y) - m_{B(0,2r)} f\right|}{|y|^{n-\alpha}}\, dy + \int_{B(0,2r)} \frac{\left|f(y) - m_{B(0,2r)} f\right|}{|x-y|^{n-\alpha}}\, dy \\
&\le \int_{B(0,2r)} \frac{\left|f(y) - m_{B(0,2r)} f\right|}{|y|^{n-\alpha}}\, dy + \int_{B(x,4r)} \frac{\left|f(y) - m_{B(x,4r)} f\right|}{|x-y|^{n-\alpha}}\, dy \\
&\quad + \ |m_{B(x,4r)} f - m_{B(0,2r)} f| \int_{B(0,2r)} \frac{dy}{|x-y|^{n-\alpha}} \qquad (3.8) \\
&\le C\ \|f\|_{w,q} \int_0^{2r} \frac{t^{\alpha}\, w(0,t)}{t}\, dt + \ C\ \|f\|_{w,q} \int_0^{4r} \frac{t^{\alpha}\, w(x,t)}{t}\, dt
\end{aligned}
$$

$$
\begin{aligned}
&+ \; C \; \|f\|_{w,q}\, w(x,4r)\; r^{\alpha} \\
\leq \;& C \; \|f\|_{w,q} \; r^{\alpha} \; \big(w(0,2r) + w(x,4r)\big) \;\; < \;\; \infty \,.
\end{aligned}
$$

Now, let us estimate $I_2(x)$ for each $x \in B(0,r)$. Applying the mean value theorem, we get

$$
|I_2(x)| \leq C \; |B|^{\frac{1}{n}} \int\limits_{\mathbb{R}^n - B(0,2r)} \frac{|f(y) - m_B f|}{|y|^{n-\alpha+1}}\, dy \,.
$$

Then, letting $B_k = 2^k B = B(0, 2^k r),\; k \in \mathbb{N}$, we have

$$
\begin{aligned}
|I_2(x)| \;\; &\leq \;\; C\; r \sum_{k=1}^{\infty} \int\limits_{B_{k+1}-B_k} \frac{|f(y) - m_B f|}{|y|^{n-\alpha+1}}\, dy \\
&\leq \;\; C\; r \sum_{k=1}^{\infty} (2^k r)^{\alpha-1} |B_{k+1}|^{-1} \int\limits_{B_{k+1}} |f(y) - m_B f| \; dy \\
&\leq \;\; C\; r \sum_{k=1}^{\infty} (2^k r)^{\alpha-1} \sum_{j=1}^{k+1} \Big(|B_j|^{-1} \int\limits_{B_j} \big| f(y) - m_{B_j} f\big|^q \; dy \Big)^{\frac{1}{q}} \\
&\leq \;\; C \;\; \|f\|_{w,q}\, r \sum_{k=1}^{\infty} (2^k r)^{\alpha-1} \sum_{j=1}^{k+1} w(0, 2^j r) \\
&\leq \;\; C \;\; \|f\|_{w,q}\, r \sum_{j=1}^{\infty} (2^j r)^{\alpha-1} w(0, 2^j r) \\
&\leq \;\; C \;\; \|f\|_{w,q}\, r \sum_{j=1}^{\infty} \int_{2^j r}^{2^{j+1} r} \frac{t^{\alpha} w(0,t)}{t}\, \frac{dt}{t} \\
&\leq \;\; C \;\; \|f\|_{w,q}\, r \int_{r}^{\infty} \frac{t^{\alpha} w(0,t)}{t}\, \frac{dt}{t} \,.
\end{aligned}
$$

Since $w_\alpha \in \mathcal{W}_\infty$ we can conclude that

$$
|I_2(x)| \;\; \leq \;\; C \;\; \|f\|_{w,q} \;\; r \, \frac{r^{\alpha} w(0,r)}{r} \quad = \quad C \;\; \|f\|_{w,q} \;\; r^{\alpha} w(0,r) \,. \tag{3.9}
$$

Finally, from (3.8) and (3.9) we have that $|\widetilde{I}_\alpha f(x)| < \infty$ for all $x \in \mathbb{R}^n$.
Let us show the boundedness of the operator $\widetilde{I}_\alpha$. To this aim, we observe that from the properties of w, for $x \in \mathbb{R}^n$, $r > 0$ and $1 \leq q < \infty$, we have

$$
\begin{aligned}
\int_{B(x,r)} \Big(\int_0^r \frac{t^{\alpha}\, w(z,t)}{t}\, dt \Big)^q dz \;\; &\leq \;\; \int_{B(x,r)} w(z,r)^q \Big(\int_0^r t^{\alpha-1}\, dt \Big)^q dz \\
&\leq \;\; C \; w(x,r)^q \int_{B(x,r)} r^{\alpha\, q}\, dz \\
&\leq \;\; C \; \big(r^{\alpha} w(x,r) \big)^q \; r^n \,.
\end{aligned}
$$

So, this estimate proves that w_α satisfies (2.11). Then, by Proposition 2.10, the proof of the theorem will be complete as soon as we prove that for every function f in $\mathrm{BMO}_{w,q}$, $\widetilde{I}_\alpha f$ satisfies a pointwise inequality like (2.6) with w_α instead of

w. In fact, for such a function and given x_1, x_2 points in $\mathbb{R}^n$, considering the ball $B = B(x_1, 2|x_1 - x_2|)$ we have

$$\begin{aligned} |\widetilde{I}_\alpha f(x_1) - \widetilde{I}_\alpha f(x_2)| &\leq \int\limits_{\mathbb{R}^n} \left| \frac{1}{|x_1 - y|^{n-\alpha}} - \frac{1}{|x_2 - y|^{n-\alpha}} \right| |f(y) - m_B f| \, dy \\ &= \int_B + \int_{\mathbb{R}^n - B} = I_1 + I_2 \, . \end{aligned}$$

Proceeding in a similar way as in (3.8) and (3.9), we get that

$$|\widetilde{I}_\alpha f(x_1) - \widetilde{I}_\alpha f(x_2)| \leq C \, \|f\|_{w,1} \int_0^{2|x_1 - x_2|} \frac{t^\alpha w(x_1, t) + t^\alpha w(x_2, t)}{t} \, dt \, , \tag{3.10}$$

as we wanted to prove. □

Corollary 3.11. *Let $\alpha, \beta \in \mathbb{R}^+$ be such that $0 < \alpha + \beta < 1$ and let $w(x,t)$ be a non-negative measurable function satisfying conditions* (1.2), (1.3) *and* (1.4)*. If $w_{\alpha+\beta} \in \mathcal{W}_\infty$ then the fractional integral I_α can be extended to a linear bounded operator from $BMO_{w_\beta, q}$ to $BMO_{w_{\alpha+\beta}, q}$, with $1 \leq q < \infty$, as in* (3.7).

Proof. It is clear that $w_\beta(x,t)$ satisfies properties (1.2), (1.3) and (1.4) if $w(x,t)$ does. Then, applying Theorem 3.6 with w_β we get the result. □

Remark 3.12. It is obvious that $w \equiv 1$ satisfies the hypotheses of the previous corollary, then the well known classical results

$$\begin{aligned} I_\alpha &: \text{BMO} \to \text{Lip}(\alpha) \, , \\ I_\alpha &: \text{Lip}(\beta) \to \text{Lip}(\alpha + \beta) \, , \end{aligned}$$

for $\beta > 0$ such that $0 < \alpha + \beta < 1$ are included.
Also, Theorem 3.6 and Corollary 3.11 recover the following results contained in [13] (see Theorem 2.9 and Corollary 2.12):

$$\begin{aligned} I_\alpha : \text{BMO}(v) &\to \text{BMO}_\alpha(v) \qquad \text{whenever } v \in H(\alpha, \infty) \, , \\ I_\alpha : \text{BMO}_\beta(v) &\to \text{BMO}_{\alpha+\beta}(v) \qquad \text{whenever } v \in H(\alpha + \beta, \infty) \, , \end{aligned}$$

where $H(\alpha, \infty)$ is defined by

$$|B|^{1/n - \alpha/n} \int_{\mathbb{R}^n - B} \frac{v(y)}{|x_B - y|^{n - \alpha + 1}} \, dy \leq C \, \frac{1}{|B|} \int_B v(y) \, dy \, .$$

In fact, it is easy to see that $v \in H(\alpha, \infty)$ implies that $w(x,t) = t^{\alpha - n} v(B(x,t))$ satisfies (1.3)(see [13]), (1.4) and $\mathcal{W}_\infty$ condition. As we note in Remark 3.5, although the condition (1.2) does not necessarily holds for this particular $w(x,t)$, we get the integral expression appearing in (3.10), that is

$$|\widetilde{I}_\alpha f(x_1) - \widetilde{I}_\alpha f(x_2)| \leq C \, \|f\|_{w,1} \int_0^{2|x_1 - x_2|} \frac{t^{\alpha - n} v(B(x_1, t)) + t^{\alpha - n} v(B(x_2, t))}{t} \, dt \, ,$$

which in view of (1.3) (that is the doubling condition of the weight v), the following inequality holds

$$|\widetilde{I}_\alpha f(x_1) - \widetilde{I}_\alpha f(x_2)| \leq C \, \|f\|_{w,1} \left(\int\limits_{|z - x_1| < 2|x_1 - x_2|} \frac{v(z)}{|z - x_1|^{n - \alpha}} \, dz \right.$$

$$+ \int\limits_{|z-x_2|<2|x_1-x_2|} \frac{v(z)}{|z-x_2|^{n-\alpha}}\,dz\Bigg)$$

This clearly implies that $\widetilde{I}_\alpha f \in BMO_\alpha(v)$ taking double average over B.

4. The Riesz Transforms.

Let f be a locally integrable function. Recall that for each $j \in \{1, \dots, n\}$ the Riesz Transform operator is given by

$$\mathrm{R}_j f(x) = \lim_{\varepsilon\to 0^+} \int\limits_{|x-y|>\varepsilon} \frac{x_j - y_j}{|x-y|^{n+1}}\, f(y)\,dy\,.$$

It is well known that these operators are bounded in weighted L^p spaces (see [9] for instance). Moreover, in [18], M. Morvidone proved the boundedness of the Hilbert transform as an operator between certain weighted spaces of functions with mean oscillation controlled by a function φ, which generalized results due to Muckenhoupt and Wheeden ([19]) and Peetre ([22]).
It is important to note that the hypotheses assumed by Morvidone are not contained in ours because that author takes advantage of a better knowledge of the structure of $w(x,t)$, since just a particular case is considered.

Our next theorem gives an analogous result for general functions $w(x,t)$.

Theorem 4.1. *Let $1 \le q < \infty$ and w be a measurable function satisfying* (1.2), (1.3) *and* (1.4). *Suppose that $w \in \mathcal{W}_\infty$, then R_j can be extended to a linear bounded operator $\mathcal{R}_j f(x)$ on $BMO_{w,q}$ as follows*

$$\mathcal{R}_j f(x) = \lim_{\varepsilon\to 0^+} \int\limits_{|x-y|>\varepsilon} \left[\frac{x_j - y_j}{|x-y|^{n+1}} + \frac{y_j \Gamma(y)}{|y|^{n+1}}\right] f(y)\,dy\,. \tag{4.2}$$

Here $\Gamma(y)$ is the characteristic function of $|y| > 1$.

Proof. In view of Theorem 2.22 we have to prove the result only for $1 < q < \infty$. Now, let $1 < q < \infty$ and $f \in \mathrm{BMO}_{w,q}$. First we prove that $\mathcal{R}_j f(x)$ is well defined over $\mathrm{BMO}_{w,q}$. It is not difficult to see that $\mathcal{R}_j 1 = 0$. Using this, for each $x \in B = B(0,r)$, we get

$$\begin{aligned}
\mathcal{R}_j f(x) &= \mathcal{R}_j(f - m_B f)(x) \\
&= \lim_{\varepsilon\to 0^+} \int\limits_{\substack{|x-y|>\varepsilon \\ |y|<2r}} \left[\frac{x_j - y_j}{|x-y|^{n+1}} + \frac{y_j \Gamma(y)}{|y|^{n+1}}\right] (f(y) - m_B f)\,dy \\
&\quad + \lim_{\varepsilon\to 0^+} \int\limits_{\substack{|x-y|>\varepsilon \\ |y|>2r}} \left[\frac{x_j - y_j}{|x-y|^{n+1}} + \frac{y_j \Gamma(y)}{|y|^{n+1}}\right] (f(y) - m_B f)\,dy \\
&= \mathrm{T}_1(x) \quad + \quad \mathrm{T}_2(x)\,.
\end{aligned} \tag{4.3}$$

For T_1, by the definition of the operator

$$|\mathrm{T}_1(x)| \leq \left| \lim_{\varepsilon\to 0^+} \int\limits_{\substack{|x-y|>\varepsilon \\ |y|<2r}} \frac{x_j-y_j}{|x-y|^{n+1}}(f(y)-m_B f)\,dy \right| + \lim_{\varepsilon\to 0^+} \int\limits_{\substack{|x-y|>\varepsilon \\ 1<|y|<2r}} \frac{|f(y)-m_B f|}{|y|^n}\,dy < \infty .$$

Since $(f(y)-m_B f)\chi_{2B} \in L^q$, the finiteness a.e. of the first term is consequence of the boundedness of R_j. For the second one, Lebesgue's dominated convergence theorem is applied to get the conclusion.

On the other hand, taking $\varepsilon < r$ in T_2, applying the mean value theorem and considering the increasing sequence of balls $B_k = B(0,2^k r)$, with $k=1,2,\dots$ we have

$$\begin{aligned}
|\mathrm{T}_2(x)| &\leq C\, r \int\limits_{|y|>2r} \frac{|f(y)-m_B f|}{|x-y|^{n+1}}\,dy \\
&\leq C\, r \sum_{k=1}^{\infty} \int\limits_{2^k r<|y|<2^{k+1}r} \frac{|f(y)-m_B f|}{|y|^{n+1}}\,dy \\
&\leq C \sum_{k=1}^{\infty} \frac{1}{2^k}\frac{1}{|B_{k+1}|}\int_{B_{k+1}} |f(y)-m_B f|\,dy \\
&\leq C \sum_{k=1}^{\infty}\sum_{j=1}^{k+1} \frac{1}{2^k}\left(\frac{1}{|B_j|}\int_{B_j} |f(y)-m_{B_j} f|^q\,dy\right)^{\frac{1}{q}} \\
&= C\, \|f\|_{w,q} \sum_{j=1}^{\infty} w(0,2^j r)\int_{2^j r}^{2^{j+1}r} \frac{1}{2^j}\frac{dt}{t} \\
&\leq C\, \|f\|_{w,q}\, r \int_r^{\infty} \frac{w(0,t)}{t}\frac{dt}{t} \\
&\leq C\, \|f\|_{w,q}\, w(0,r) < \infty ,
\end{aligned} \tag{4.4}$$

where the last inequality holds because $w \in \mathcal{W}_\infty$. Finally, taking a sequence $\{B_n\}$ of balls such that $B_n \nearrow \mathbb{R}^n$ and applying the above reasoning for each B_n we get that $\mathcal{R}_j f$ is finite a.e.

Now, we prove the boundedness of the operator acting on $\mathrm{BMO}_{w,q}$. As for (4.3), we consider $g = f - m_B f$, $g_1 = g\chi_{2B}$ and $g_2 = g - g_1$, with $B = B(x_0,r)$ given. First we study $R_j g_1$. From the boundedness of classical Riesz Transform, we have

$$\begin{aligned}
\int_B |\mathcal{R}_j g_1(x) - m_B(\mathcal{R}_j g_1)|^q\,dx &\leq C \int_{2B} |f(x)-m_B f|^q\,dx \\
&\leq C\, \|f\|_{w,q}^q\, w(x_0,2r)^q |B| ,
\end{aligned}$$

This and (1.3) allow us to conclude that

$$\frac{1}{w(x_0,r)}\left(\frac{1}{|B|}\int_B |\mathcal{R}_j g_1(x) - m_B(\mathcal{R}_j g_1)|^q\,dx\right)^{\frac{1}{q}} \leq C\, \|f\|_{w,q} .$$

On the other hand, for R_jg_2 taking $\varepsilon < r$ and then applying the mean value theorem, the same reasoning used for (4.4) yields

$$\begin{aligned} |\mathcal{R}_jg_2(x) - \mathcal{R}_jg_2(z)| &\leq \int_{\mathbb{R}^n-2B} \left| \frac{x_j - y_j}{|x-y|^{n+1}} + \frac{z_j - y_j}{|z-y|^{n+1}} \right| |f(y) - m_Bf| \, dy \\ &\leq C \; r \int_{\mathbb{R}^n-2B} \frac{|f(y) - m_Bf|}{|x_0 - y|^{n+1}} \, dy \\ &\leq C \; \|f\|_{w,q} \; r \int_r^\infty \frac{w(x_0,t)}{t} \frac{dt}{t} \\ &\leq C \; \|f\|_{w,q} \; w(x_0,r) \quad < \quad \infty \, . \end{aligned}$$

Finally, by Hölder's inequality we can write

$$\begin{aligned} \int_B |\mathcal{R}_jg_2(x) - m_B(\mathcal{R}_jg_2)|^q \, dx &\leq \frac{1}{|B|} \int_B \int_B |\mathcal{R}_jg_2(x) - \mathcal{R}_jg_2(z)|^q \, dz \, dx \\ &\leq C \; \|f\|^q_{w,q} \; \frac{1}{|B|} \int_B \int_B w(x_0,r)^q \, dz \, dx \\ &= C \; \|f\|^q_{w,q} \; |B| \, w(x_0,r)^q \, . \end{aligned}$$

So the theorem is proved. □

Remark 4.5. For $w(x,t) = \phi(t)$, the condition $\phi \in \mathcal{W}_\infty$ implies that ϕ is of upper type β with $\beta < 1$, as it is proved in Lemma (3.3) of [13], which in particular establishes that ϕ satisfies (1.3). Hence we can prove that it ϕ is a non negative and non decreasing function such that $\phi \in \mathcal{W}_\infty$, then $\mathcal{R}_j$ can be extended to a bounded linear operator on BMO_ϕ. This last result is contained in [22].

References

[1] Acquistapace, P. *"On BMO regularity for lineal elliptic systems."*. Annali di Matematica pura ed applicata (IV), Vol CLXI, (1992), 231–269.

[2] Bramanti, M. and Brandolini, L. *"Estimates of BMO type for singular integrals on spaces of homogeneous type and applications to hypoelliptic PDEs."*. Rev. Iberoamericana 21, (2005), 511–556.

[3] Campanato, S. *"Propietà di hölderianità di alcune classi di funzioni"*. Ann. Sc. Normale. Sup. Pisa 17, (1963), 175–188.

[4] Cruz-Uribe, D.; Fiorenza, A.; Neugebauer, C. J. *"Weighted norm inequalities for the maximal operator on variable Lebesgue spaces"*. J. Math. Anal. Appl. 394(2), (2012), 744–760.

[5] Cruz-Uribe, D.; Fiorenza, A. *"Variable Lebesgue spaces. Foundations and harmonic analysis"*. Birkhäuser/Springer, Heidelberg, (2013), x+312.

[6] Cruz-Uribe, D.; Forzani, L. and Maldonado, D. *"The structure of increasing weights on the real line"*. Houston J. Math. 34(3), (2008), 951–983.

[7] Diening, L.; Harjulehto, P.; Hästö, P. and Růžička, M. *"Lebesgue and Sobolev spaces with variable exponents"*. Lecture Notes in Mathematics, 2017. Springer, Heidelberg, 2011.

[8] Diening, L. and Hästö, P. *"Muckenhoupt weights in variable exponent spaces"*. 2000 MSC: 42B25;46E30. Preprint. Available in http://www.helsinki.fĩhasto/pp/p75_submit.pdf

[9] García-Cuerva, J. and Rubio de Francia, J. L. *"Weighted norm inequalities and related topics"*. North-Holland Mathematics Studies 116 (1985), x+604.

[10] Fan, X. and Zhao, D. *"On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$"*. J. Math. Anal. Appl. 263(2), (2001), 424–446.

[11] Franchi, B.; Pérez, C. and Wheeden, R. L. *"Self-improving properties of John-Nirenberg and Poincaré inequalities on spaces of homogeneous type"*. J. Funct. Anal. 153(1), (1998), 108–146.

[12] Harjulehto, P.; Hästö, P. and Pere, M. *"Variable exponent Lebesgue spaces on metric spaces: the Hardy-Littlewood maximal operator"*. Real Anal. Exchange 30(1) (2004/2005) 87–103.

[13] Harboure, E.; Salinas, O. and Viviani, B. *"Boundedness of the fractional integral on weighted Lebesgue and Lipschitz spaces"*. Trans. Amer. Math. Soc. 349(1), (1997), 235–255.

[14] Harboure, E.; Salinas, O. and Viviani, B. *"Relations between weighted Orlicz and BMO_Φ spaces through fractional integrals"*. Comment. Math. Univ. Carolin. 40, (1999), 53–69.

[15] Janson, S. *"On functions with conditions on the mean oscillation"*. Ark. Math., 14, (1976), 189–196.

[16] Kováčik, O. and Rákosník, J. *"On spaces $L^{p(x)}$ and $W^{k,p(x)}$"*. Czechoslovak Math. J. 41(116)(4) (1991), 592–618.

[17] Meyer, G. N. *"Mean oscillation over cubes and Hölder continuity"*. Proc. Am. Math. Soc. 15, (1964), 717–721.

[18] Morvidone, M. *"Weighted BMO_ϕ spaces and the Hilbert transform"*. Rev. Un. Mat. Argentina 44(1), (2003), 1–16.

[19] Muckenhoupt, B. and Wheeden, R.L. *"Weighted bounded mean oscillation and the Hilbert transform"*. Studia Math. T. LIV., (1976).

[20] Nakai, E. and Yabuta, K. *"Pointwise multipliers for functions of bounded mean oscillation"*. J. Math. Soc. Japan 37, (1985), 207–218.

[21] Nakai, E. *"Pointwise multipliers for functions of weighted bounded mean oscillation"*. Studia Math. 105, (1993), 105–119.

[22] Peetre, J. *"On the theory of $\mathfrak{L}_{p,\lambda}$ Spaces"*. J. Funct. Anal. 4 (1969), 71–87.

[23] Ramseyer, M.; Salinas, O. and Viviani, B. *"Lipschitz type smoothness of the fractional integral on variable exponent spaces"*. J. Math. Anal. Appl. 403(1) (2013), 95–106.

[24] Spanne, S. *"Some function spaces defined using the mean oscillation over cubes"*. Ann. Sc. Normale. Sup. Pisa 19 (1965), 593–607.

Instituto de Matemática Aplicada del Litoral CONICET-UNL and Facultad de Ingeniería Química, Universidad Nacional del Litoral, Santa Fe, Argentina.

E-mail address: mramseyer@santafe-conicet.gov.ar
E-mail address: salinas@santafe-conicet.gov.ar
E-mail address: viviani@santafe-conicet.gov.ar