On the category of Nelson paraconsistent lattices

STEFANO AGUZZOLI, *DI, University of Milano, Italy.*
E-mail: aguzzoli@di.unimi.it

MANUELA BUSANICHE, *Instituto de Matemática Aplicada del Litoral, UNL, CONICET, FIQ, Argentina.*
E-mail: mbusaniche@santafe-conicet.gov.ar

BRUNELLA GERLA, *DiSTA, University of Insubria, Varese, Italy.*
E-mail: brunella.gerla@uninsubria.it

MIGUEL ANDRÉS MARCOS, *Instituto de Matemática Aplicada del Litoral, UNL, CONICET, FIQ, Argentina.*
E-mail: mmarcos@santafe-conicet.gov.ar

Abstract

We present an equivalence between the category of Nelson Paraconsistent lattices (NPc-lattices) and a category of pairs of Brouwerian algebras and regular filters. Specializing such category of pairs to Gödel hoops, we get the subvariety of Gödel NPc-lattices and, using the dual equivalence of finite Gödel hoops with finite trees, we obtain a duality for finite Gödel NPc-lattices. This duality is used to describe finitely generated free Gödel NPc-lattices..

Keywords: Nelson paraconsistent lattices, Brouwerian algebras, Gödel hoops, dual equivalences, free algebras.

1 Introduction

Nelson's paraconsistent logic **N4** is the paraconsistent variant of Nelson's system [26]. We recall that Paraconsistent logics are those logics that admit inconsistent but non-trivial theories and Nelson's system (constructive logic with strong negation, [3, 23]) is an expansion of intuitionistic logic by a new negation symbol that behaves as an involutive negation.

It turns out that **N4** is algebraizable and the corresponding algebraic structures are N4-lattices, which were studied and analysed by Odintsov in [24, 26].

Following some of the ideas of [28, 29] and [6], in [5] a class of residuated lattices with involution is defined, called Nelson paraconsistent lattices (NPc-lattices for short). There it is proved that NPc-lattices and *e*N4-lattices (an extension of N4-lattices by a constant *e*) are termwise equivalent. This situates Nelson's paraconsistent logic within the framework of substructural logics [16], providing an alternative semantics in terms of well-known algebraic structures.

The most interesting property of NPc-lattices is that they can be represented by twist-products of Brouwerian algebras, sometimes also known as generalized Heyting algebras, which are bottom-free reducts of Heyting algebras. By a *twist-product* of a lattice **L** we mean a suitably defined sublattice

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of the cartesian product of $\mathbf{L}$ with its order-dual $\mathbf{L}^\partial$ equipped with the natural order involution $(x,y)\mapsto(y,x)$ for all $(x,y)\in L\times L^\partial$.

The idea of considering this kind of construction to deal with order involutions on lattices goes back to Kalman's 1958 paper [20], and it has been used widely to represent many involutive lattices with additional operations (see [5, 6, 9, 10, 14, 21, 25–27, 30, 31]).

In the present article the fact that NPc-lattices are representable by twist-products of Brouwerian algebras is exploited to obtain some results about these residuated lattices. To begin with, we give a categorical equivalence between the category of NPc-lattices and morphisms and a category whose objects are pairs consisting of a Brouwerian algebra and a regular filter of it. The equivalence follows the ideas given by Sendlewski [27] and by Odsintov [26], but we rephrase them in the context of residuated lattices.

Then we focus our attention on Gödel NPc-lattices. These structures form the proper subvariety of NPc-lattices that can be represented by twist-products of Gödel hoops (prelinear Brouwerian algebras). As is well known, Esakia duality [13] can be specialized to a duality between finite prelinear Heyting algebras and finite forests with order preserving open maps [12, 19]. In [1, 2] the latter duality is adapted to Gödel hoops: finite Gödel hoops are dually equivalent with finite trees and order preserving open maps. In particular, each finite Gödel hoop arises as the set of all non-empty downward closed subsets of a tree, equipped with suitably defined operations. Based on this duality, we present a duality for finite Gödel NPc-lattices and we use it to describe finitely generated free algebras in this subvariety.

We refer to [22] for all results and notions of Category Theory needed along the paper.

2 Brouwerian algebras and NPc-lattices

By a *commutative residuated lattice* we mean a *residuated lattice-ordered commutative monoid*, that is, an algebra $\mathbf{A}=(A,\vee,\wedge,*,\Rightarrow,e)$ of type $(2,2,2,2,0)$ such that $(A,\vee,\wedge)$ is a lattice, $(A,*,e)$ is a *commutative* monoid and the following residuation condition is satisfied:

$$x*y\leq z \text{ if and only if } x\leq y\Rightarrow z, \tag{1}$$

where x,y,z denote arbitrary elements of A and $\leq$ is the order given by the lattice structure.

It is well known that commutative residuated lattices form a variety that we shall denote by $\mathbb{CRL}$ (see, for instance, [4, 16, 18]).

A commutative residuated lattice $\mathbf{A}$ is called *integral* if $x\leq e$ for all $x\in A$. The *negative cone* of $\mathbf{A}\in\mathbb{CRL}$ is the set $A^-=\{x\in A : x\leq e\}$. It is easy to see that A^- is closed under the operations $\vee,\wedge,*$, and if the binary operation $\Rightarrow_e$ is defined as

$$x\Rightarrow_e y=(x\Rightarrow y)\wedge e, \tag{2}$$

then $\mathbf{A}^-=(A^-,\vee,\wedge,*,\Rightarrow_e,e)$ is an integral commutative residuated lattice. An integral commutative residuated lattice is a *Brouwerian algebra* [16, Chapter 2] (also a *generalized Heyting algebra* or an *implicative lattice*) if it satisfies the equation $x*x=x^2=x$.

2.1 Regular filters on Brouwerian algebras

Let $\mathbf{L}$ be a Brouwerian algebra (also known as implicative lattice). In Brouwerian algebras both products $*$ and $\wedge$ coincide and the neutral element of the product e is also the greatest element of the algebra. We say an element $x\in L$ is **dense** if it is of the form $x=w\vee(w\Rightarrow z)$, with $w,z\in L$.

PROPOSITION 2.1
The set F_d of dense elements of L is a (lattice) filter.

PROOF. Assume first L has a minimum element $\bot$. Then an element x is dense iff $x \Rightarrow \bot = \bot$. In details, if $x \Rightarrow \bot = \bot$ then x is clearly dense as $x = x \vee (x \Rightarrow \bot)$. Conversely, if x is dense then $x = w \vee (w \Rightarrow z)$ and

$$\begin{aligned} x \Rightarrow \bot &= (w \vee (w \Rightarrow z)) \Rightarrow \bot = (w \Rightarrow \bot) \wedge ((w \Rightarrow z) \Rightarrow \bot) \\ &\leq (w \Rightarrow z) \wedge ((w \Rightarrow z) \Rightarrow \bot) \leq \bot. \end{aligned}$$

In this case F_d is a filter. Now consider the case L unbounded. Take $\langle F_d \rangle$, the filter generated by F_d and let $x \in \langle F_d \rangle$. Then x is of the form

$$x \geq \bigwedge_{i=1}^{n} w_i \vee (w_i \Rightarrow z_i)$$

for some $w_i, z_i \in L$, and take $m = \bigwedge_{i=1}^{n} (w_i \wedge z_i)$, so $L_m = \{y : y \geq m\}$ is a subalgebra of L with $x, w_i, z_i \in L_m$ and minimum element m. Then x is dense in L_m (as it is greater than or equal to the infimum of finitely many dense elements of L_m) and we have $x \Rightarrow m = m$ (in L_m but also in L as the former is a subalgebra of the latter) and therefore $x = x \vee (x \Rightarrow m)$, obtaining $x \in F_d$. ■

Observe that if L is a chain, we have $x \Rightarrow y = \top$ if $x \leq y$ and $x \Rightarrow y = y$ if $x > y$, then every non-bottom element (in case it exists) will be dense, as given $x \in L$ if there exists y with $x > y$, we will have $x = x \vee (x \Rightarrow y)$.

We will work with filters containing the filter F_d, which we call *regular*. It turns out that they have a specific structure.

LEMMA 2.2
If the filter F is an intersection of maximal filters, then it is regular.

PROOF. Assume first F maximal and take $a, b \in L$. If $a \in F$ then $a \vee (a \Rightarrow b) \in F$ and we are done. If $a \notin F$, then $\langle F \cup \{a\} \rangle = L$, being F maximal, and therefore $b \in \langle F \cup \{a\} \rangle$. Then there will exist $c \in F$ such that $b \geq a \wedge c$. But this is equivalent to $c \leq a \Rightarrow b$, so $(a \Rightarrow b) \in F$ and $a \vee (a \Rightarrow b) \in F$. This way $F_d \subseteq F$ for F maximal.

If F is an intersection of maximal filters, clearly $F_d \subseteq F$, as it is contained in each one of them. ■

LEMMA 2.3
If L bounded, then every regular proper filter is an intersection of maximal filters.

PROOF. Take $\bot$ to be the minimum of L and let F be a proper regular filter. If $F \subseteq P$ with P a prime filter, then P must be maximal. Indeed, if not there would exist M maximal (and proper) such that $P \subsetneq M$ and given $a \in M \setminus P$, as $a \vee (a \Rightarrow \bot) \in F \subseteq P$ with P prime and $a \notin P$, it should be $a \Rightarrow \bot \in P$, then $a, a \Rightarrow \bot \in M$ and therefore $\bot = a \wedge (a \Rightarrow \bot) \in M$, absurd as M is proper. Then every prime filter containing F must be maximal.

As every proper filter is the intersection of every prime filter containing it, this last result implies F is an intersection of maximal filters. ■

COROLLARY 2.4
If L is bounded, then regular proper filters are exactly intersections of maximal filters.

2.2 *NPc-lattices*

An *involution* on $\mathbf{A}\in\mathbb{CRL}$ is a unary operation $\sim$ satisfying the equations $\sim\sim x = x$ and $x\Rightarrow \sim y = y\Rightarrow \sim x$. If $f := \sim e$, then $\sim x = x\Rightarrow f$ and f satisfies the equation

$$(x\Rightarrow f)\Rightarrow f = x. \tag{3}$$

The element f in Equation (3) is called a *dualizing element*.

Conversely, if $f\in A$ is a dualizing element and we define $\sim x = x\Rightarrow f$ for all $x\in A$, then $\sim$ is an involution on $\mathbf{A}$ and $\sim e = f$. Hence there is a bijective correspondence between involutions on $\mathbf{A}$ and dualizing elements in A (see [15, 30] for details).

Taking $f = e$ in (3) we obtain an equation in the language of residuated lattices that determines a subvariety $\mathbb{I}_e\mathbb{CRL}$ of $\mathbb{CRL}$. We call the elements of this subvariety *e-involutive commutative residuated lattices* or *e-lattices* for short (they were called *residuated lattices with involution* in [6, 7]). It is easy to see that the involution $\sim$ given by the prescription $\sim x = x\Rightarrow e$ for all $x\in A$, satisfies the following properties:

(1) $\sim\sim x = x$,
(2) $\sim(x\vee y) = \sim x\wedge\sim y$,
(3) $\sim(x\wedge y) = \sim x\vee\sim y$,
(4) $\sim(x*y) = x\Rightarrow\sim y$.

Moreover, we have that $\sim e = e$.

Lattice-ordered abelian groups with $x*y = x+y$, $x\to y = y-x$ and $e = 0$ are examples of e-lattices. Other examples of e-lattices are given by twist structures, which will be defined in the next section.

DEFINITION 2.5
(see Definition 2.1 in [7]) A *Nelson Paraconsistent residuated lattice* (*NPc-lattice* for short), is a distributive e-lattice $\mathbf{A} = (A,\vee,\wedge,*,\Rightarrow,e)$ satisfying the following equations:

$$(x*y)\wedge e = (x\wedge e)*(y\wedge e), \tag{4}$$

$$(x\wedge e)^2 = x\wedge e, \tag{5}$$

$$((x\wedge e)\Rightarrow y)\wedge((\sim y\wedge e)\Rightarrow\sim x) = x\Rightarrow y. \tag{6}$$

The reader can check that $\mathbf{B}^-$ with the implication as defined in 2 is a Brouwerian algebra. It is also well known and easy to verify that NPc-lattices satisfy the quasiequation:

$$\text{if } x\wedge e = y\wedge e \text{ and } \sim x\wedge e = \sim y\wedge e, \text{ then } x = y. \tag{7}$$

3 Representation of NPc-lattices

By a full *twist-product* of a lattice $\mathbf{L}$ we mean the cartesian product of $\mathbf{L}$ with its order-dual $\mathbf{L}^\partial$ equipped with the natural order involution $(x,y)\mapsto(y,x)$ for all $(x,y)\in L\times L^\partial$. As far as we know the idea of considering this kind of construction to handle order involutions on lattices goes back to Kalman's 1958 paper [20], but the denomination 'twist' appeared thirty years later on Kracht's paper [21]. The following result is a particular case of [30, Corollary 3.6].

THEOREM 3.1
Let $\mathbf{L} = (L, *, \Rightarrow, \vee, \wedge, e)$ be an integral commutative residuated lattice. Then

$$\mathbf{K}(\mathbf{L}) = (L \times L, \sqcup, \sqcap, *, \rightarrow, (e,e))$$

with the operations $\sqcup, \sqcap, *, \rightarrow$ given by

$$(a,b) \sqcup (c,d) = (a \vee c, b \wedge d) \tag{8}$$

$$(a,b) \sqcap (c,d) = (a \wedge c, b \vee d) \tag{9}$$

$$(a,b) * (c,d) = (a * c, (a \Rightarrow d) \wedge (c \Rightarrow b)) \tag{10}$$

$$(a,b) \rightarrow (c,d) = ((a \Rightarrow c) \wedge (d \Rightarrow b), a * d) \tag{11}$$

is an e-lattice. Moreover, the correspondence

$$(a,e) \mapsto a$$

defines an isomorphism from $(\mathbf{K}(\mathbf{L}))^-$ onto $\mathbf{L}$.

We refer to $\mathbf{K}(\mathbf{L})$ as the *full twist-product* obtained from $\mathbf{L}$, and every subalgebra $\mathbf{A}$ of $\mathbf{K}(\mathbf{L})$ containing the set $\{(a,e) : a \in L\}$ is called *a twist-product* obtained from $\mathbf{L}$. Thus if $\mathbf{A}$ is a twist-product obtained from $\mathbf{L}$ its negative cone is isomorphic to $\mathbf{L}$.

K-lattices, introduced in [8], are e-lattices satisfying equations (4), (6) and the distributive law of lattices when one of the variables is the neutral e. Thus NPc-lattices form a proper subvariety of the variety of K-lattices. But K-lattices are exactly those e-lattices that are isomorphic to a twist-product of their negative cone [8, Theorem 3.7]. As a particular case one can verify the following result:

THEOREM 3.2
If $\mathbf{L}$ is a Brouwerian algebra, then $\mathbf{K}(\mathbf{L})$ is an NPc-lattice. Moreover, for every NPc-lattice $\mathbf{B}$, the application $\phi_{\mathbf{B}} : \mathbf{B} \rightarrow \mathbf{K}(\mathbf{B}^-)$ given by

$$x \mapsto (x \wedge e, \sim x \wedge e)$$

is an injective morphism.

As it is clear from the definition of the operations in the twist-products, each term γ in the language of NPc-lattices, with variables $x_1, \ldots x_n$, can be uniquely identified with a couple of terms (γ^1, γ^2) in the language of Brouwerian algebras. A simple proof by induction on the complexity of γ yields the pair of terms. In details, let γ be a term in the language of NPc-lattices and assume that $\mathbf{A}$ is an NPc-lattice, that by Theorem 3.2 can be identified with a subalgebra of $\mathbf{K}(\mathbf{A}^-)$. Let $\gamma_{\mathbf{A}}$ be the corresponding term function from $\mathbf{A}^n$ to $\mathbf{A}$. If $\phi = \phi_{\mathbf{A}} : \mathbf{A} \rightarrow \mathbf{K}(\mathbf{A}^-)$ as in Theorem 3.2, for each $(a_1, a_2, \ldots, a_n) \in A^n$ if $\phi(a_i) = (b_i, c_i)$ for every $i = 1, 2, \ldots, n$, we get

$$\begin{aligned} \phi((\gamma_{\mathbf{A}})(a_1, \ldots, a_n)) &= \gamma_{\mathbf{K}(\mathbf{A}^-)}(\phi(a_1), \ldots, \phi(a_n))) \\ &= \gamma_{\mathbf{K}(\mathbf{A}^-)}((b_1, c_1), \ldots, (b_n, c_n)) \\ &= (\gamma^1_{\mathbf{A}^-}(b_1, c_1, \ldots, b_n, c_n), \gamma^2_{\mathbf{A}^-}(b_1, c_1, \ldots, b_n, c_n)). \end{aligned}$$

We now proceed to prove a categorical equivalence between the category of NPc-lattices and residuated lattices morphims and a category whose objects are pairs of Brouwerian algebras and regular filters. The idea is to reformulate the characterization of N4-lattices given by Odintsov [26] in terms of residuated lattices. In Section 6 of [8] it is proved that some varieties of e-lattices can be represented by pairs formed by a bounded integral residuated lattices and a lattice filter of it. But those ideas cannot be applied directly to the present case, since the lower bound of the residuated lattice plays a crucial role. Following Odintsov's notation [26], in the sequel we shall often denote with ∇ the regular filter of a Brouwerian algebra $\mathbf{L}$ used to build a twist-product.

THEOREM 3.3
Let $\mathbf{L}$ be a Brouwerian algebra and ∇ a regular filter of $\mathbf{L}$. Then the subset

$$Tw(L,\nabla)=\{(a,b)\in L\times L : a\vee b\in\nabla\},$$

of the NPc-lattice $\mathbf{K}(\mathbf{L})$ is a twist-product obtained from $\mathbf{L}$, whose negative cone is isomorphic with $\mathbf{L}$.

Moreover, if $\mathbf{L}'$ is another Brouwerian algebra and ∇' a regular filter in $\mathbf{L}'$, for each morphism $f:\mathbf{L}\to\mathbf{L}'$ satisfying $f(\nabla)\subseteq\nabla'$ we obtain an NPc-lattice morphism

$$\mathbf{f}:\mathbf{Tw}(\mathbf{L},\nabla)\to\mathbf{Tw}(\mathbf{L}',\nabla')$$

given by $\mathbf{f}((a,b))=(f(a),f(b))$.

PROOF. For the first part we prove that $B=Tw(L,\nabla)$ is the universe of a subalgebra of $\mathbf{K}(\mathbf{L})$ whose negative cone is isomorphic to $\mathbf{L}$, i.e., the operations are closed in B and $(a,e)\in B$ for each $a\in L$. Take $(a,b),(c,d)\in B$, then

- $(a,b)\sqcap(c,d)\in B$, as $(a,b)\sqcap(c,d)=(a\wedge c,b\vee d)$ and therefore $(a\wedge c)\vee(b\vee d)=(a\vee b\vee d)\wedge(c\vee d\vee b)\geq(a\vee b)\wedge(c\vee d)\in\nabla$.
- $(a,b)\sqcup(c,d)\in B$, as $(a,b)\sqcup(c,d)=(a\vee c,b\wedge d)$ and therefore $(a\vee c)\vee(b\wedge d)=(a\vee b\vee c)\wedge(a\vee c\vee d)\geq(a\vee b)\wedge(c\vee d)\in\nabla$.
- $(a,b)\cdot(c,d)\in B$, as $(a,b)\cdot(c,d)=(a\wedge c,(a\Rightarrow d)\wedge(c\Rightarrow b))$ and therefore

$$\begin{aligned}(a\wedge c)\vee((a\Rightarrow d)\wedge(c\Rightarrow b))&=\\ &=(a\vee(a\Rightarrow d))\wedge(c\vee(a\Rightarrow d))\wedge(a\vee(c\Rightarrow b))\wedge(c\vee(c\Rightarrow b))\\ &\geq(a\vee(a\Rightarrow d))\wedge(c\vee d)\wedge(a\vee b)\wedge(c\vee(c\Rightarrow b))\in\nabla.\end{aligned}$$

- $\sim(a,b)\in B$, this is immediate as $\sim(a,b)=(b,a)$ and $b\vee a=a\vee b\in\nabla$.
- $(a,b)\to(c,d)\in B$, as $x\to y=\sim(x\cdot\sim y)$ in e-lattices.
- $(a,e)\in B$ for each $a\in L$, as $a\vee e=e\in\nabla$ (in particular $(e,e)\in B$).

Finally, assume $\mathbf{L}'$ is another Brouwerian algebra with ∇' a regular filter in it, and take a morphism $f:\mathbf{L}\to\mathbf{L}'$ satisfying $f(\nabla)\subseteq\nabla'$. We will show that $\mathbf{f}(a,b)=(f(a),f(b))$ is well defined and is a morphism from $\mathbf{Tw}(\mathbf{L},\nabla)$ to $\mathbf{Tw}(\mathbf{L}',\nabla')$. The condition $f(\nabla)\subseteq\nabla'$ guarantees that if $a\vee b\in\nabla$, then $f(a)\vee f(b)=f(a\vee b)\in\nabla'$, then $\mathbf{f}$ is well defined. From the fact that f is a morphism and the definition of the operations for $\mathbf{Tw}(\mathbf{L},\nabla)$ and $\mathbf{Tw}(\mathbf{L}',\nabla')$, we obtain that $\mathbf{f}$ is an NPc-lattice morphism. ■

Now we will assign to each NPc-lattice $\mathbf{B}$ a pair composed by a Brouwerian algebra $\mathbf{L}$ and a regular filter ∇ such that $\mathbf{B}\cong\mathbf{Tw}(\mathbf{L},\nabla)$. This is achieved by gluing the result of Theorem 3.2 and the following theorem:

THEOREM 3.4
Given a twist-product $\mathbf{B}$ obtained from $\mathbf{L}$, the set $\nabla=\{a\vee b:(a,b)\in B\}$ is a regular filter in $\mathbf{L}$, and

$$\mathbf{B}=\mathbf{Tw}(\mathbf{L},\nabla).$$

Moreover, let $\mathbf{L}'$ be another Brouwerian algebra and $\mathbf{B}'$ be a twist-product obtained from $\mathbf{L}'$. Let further $\pi_1: \mathbf{B}' \to \mathbf{L}'$ be the projection on the first coordinate, and $\nabla' = \{c \vee d : (c,d) \in B'\}$. Then for each NPc-lattice morphism $\mathbf{f}: \mathbf{B} \to \mathbf{B}'$ we obtain a Brouwerian morphism $f: \mathbf{L} \to \mathbf{L}'$ given by

$$f(a) = \pi_1(\mathbf{f}((a,e)))$$

that satisfies $f(\nabla) \subseteq \nabla'$.

Proof. We first observe that if $a \in \nabla$, then there exists $b \leq a$ such that $(a,b) \in B$. Indeed, if $a \in \nabla$ there exists $(c,d) \in B$ such that $a = c \vee d$, then $(c \vee d, c \wedge d) = (c,d) \sqcup \sim(c,d) \in B$ and taking $b = c \wedge d$ we obtain $b \leq a$ and $(a,b) \in B$.

Now we show that ∇ is a regular filter.

- $e \in \nabla$, as $(e,e) \in B$ and $e = e \vee e$.
- if $a,c \in \nabla$, then $a \wedge c \in \nabla$. In fact, by the observation above there exist $b,d \in L$ such that $b \leq a$, $d \leq c$ and $(a,b),(c,d) \in B$. Then since $(b,e),(d,e)$ are also in B, $(b \wedge d, e) \in B$ and

$$\begin{aligned}(a,b) \sqcap ((a,b) \to (b \wedge d, e)) &= (a,b) \sqcap ((a \Rightarrow (b \wedge d)) \wedge b, a)\\ &= (a,b) \sqcap (b \wedge (a \Rightarrow d), a)\\ &= (b \wedge d, a),\end{aligned}$$

  we have $(b \wedge d, a) \in B$, and similarly $(b \wedge d, c) \in B$. Finally $(b \wedge d, a \wedge c) = (b \wedge d, a) \sqcup (b \wedge d, c) \in B$ and as $b \wedge d \leq a \wedge c$ we obtain $a \wedge c \in \nabla$.
- if $a \in \nabla$ and $c \geq a$, again from the observation there exists $b \leq a$ such that $(a,b) \in B$, and as we also have $(c,e) \in B$, we obtain $(c,b) = (a,b) \sqcup (c,e) \in B$, and as $b \leq a \leq c$, we get $c = c \vee b \in \nabla$.
- if $a,b \in L$, then $a \vee (a \Rightarrow b) \in \nabla$, as $(a,e),(b,e) \in B$ and $(a \Rightarrow b, a) = (a,e) \to (b,e) \in B$.

For the next part, observe that if $\tilde{B} = \{(a,b) \in L \times L : a \vee b \in \nabla\}$, then it is clear that $B \subseteq \tilde{B}$. For the other inclusion take $(a,b) \in \tilde{B}$ with $a,b \in L$. Since B is an algebra that contains all the elements of the form (x,e) with $x \in L$ we have that (e,b) and (e,a) are in B. Then the element $(a \Rightarrow b, a) = (e,b) \to (e,a)$ is also in B. From the definition of ∇ there exists $(c,d) \in B$ such that $a \vee b = c \vee d$. Hence $(c,d) \sqcap (d,c) = (c \vee d, c \wedge d) = (a \vee b, c \wedge d)$ is also in B. Then

$$\begin{aligned}(a \vee b, c \wedge d) \sqcap (a \Rightarrow b, a) \sqcap (e,b) &= ((a \wedge (a \Rightarrow b)) \vee (b \wedge (a \Rightarrow b)), a \vee b)\\ &= (b, a \vee b),\end{aligned}$$

so $(b, a \vee b) \in B$ and similarly $(a, a \vee b) \in B$. From this we obtain $(a \wedge b, a \vee b) \in B$, and as $(b,a) = (a \vee b, a \wedge b) \sqcap (a \Rightarrow b, a) \in B$, we get what we wanted.

For the last part, take $\mathbf{L}'$ another Brouwerian algebra, $\mathbf{B}'$ a twist-product obtained from $\mathbf{L}'$ and $\mathbf{f}: \mathbf{B} \to \mathbf{B}'$ an NPc-lattice morphism. As $\mathbf{f}$ sends negative cones to negative cones, f is well defined from L to L', and it is also clear that it is a lattice morphism and $f(e) = e$. We now check that it also preserves implication, define $c = f(a), d = f(b)$, then

$$\begin{aligned}f(a \Rightarrow b) &= \pi_1(\mathbf{f}(a \Rightarrow b, e)) = \pi_1(\mathbf{f}(((a,e) \to (b,e)) \sqcap (e,e)))\\ &= \pi_1((\mathbf{f}(a,e) \to \mathbf{f}(b,e)) \sqcap \mathbf{f}(e,e))\\ &= \pi_1(((c,e) \to (d,e)) \sqcap (e,e)) = \pi_1(c \Rightarrow d, e)\\ &= f(a) \Rightarrow f(b).\end{aligned}$$

Finally, if $\nabla' = \{c \vee d : (c,d) \in B'\}$, taking $a \vee b \in \nabla$ define $(c,d) = \mathbf{f}(a,b) \in B'$ and observe that

$$\begin{aligned}\mathbf{f}(a \vee b, e) &= \mathbf{f}(((a,b) \sqcup \sim(a,b)) \sqcap (e,e)) \\ &= (\mathbf{f}(a,b) \sqcup \sim \mathbf{f}(a,b)) \sqcap \mathbf{f}(e,e) \\ &= ((c,d) \sqcup (d,c)) \sqcap (e,e) \\ &= (c \vee d, e),\end{aligned}$$

so $c \vee d = \pi_1(\mathbf{f}(a \vee b, e)) = f(a \vee b)$, and thus $f(\nabla) \subseteq \nabla'$. ■

THEOREM 3.5
Let $\mathbf{B}$ be an NPc-lattice. Then the set $\nabla = \{(x \vee \sim x) \wedge e : x \in B\}$ is a regular filter in $\mathbf{B}^-$, and

$$\mathbf{B} \cong \mathbf{Tw}(\mathbf{B}^-, \nabla).$$

Moreover, if $\mathbf{B}'$ is another NPc-lattice, for each NPc-lattice morphism $\mathbf{f} : \mathbf{B} \to \mathbf{B}'$ we obtain a Brouwerian morphism $f : \mathbf{B}^- \to (\mathbf{B}')^-$ given by $f = \mathbf{f}|_{\mathbf{B}^-}$, that satisfies $f(\nabla) \subseteq \nabla'$, where $\nabla' = \{(y \vee \sim y) \wedge e : y \in B'\}$.

PROOF. As $\mathbf{B} \cong \phi_{\mathbf{B}}(\mathbf{B})$, and the latter is a twist-product of $\mathbf{B}^-$ (and $\mathbf{B}^-$ is a Brouwerian algebra), the set

$$\begin{aligned}\nabla &= \{\pi_1(\phi_{\mathbf{B}}(x)) \vee \pi_2(\phi_{\mathbf{B}}(x)) : x \in B\} \\ &= \{(x \wedge e) \vee (\sim x \wedge e) : x \in B\} \\ &= \{(x \vee \sim x) \wedge e : x \in B\}\end{aligned}$$

is a regular filter in $\mathbf{B}^-$ and

$$\phi_{\mathbf{B}}(\mathbf{B}) = \mathbf{Tw}(\mathbf{B}^-, \nabla).$$

For the second part, if $\mathbf{f} : \mathbf{B} \to \mathbf{B}'$ is an NPc-lattice morphism, it maps negative cones into negative cones, so f is well defined. To check that it is a Brouwerian algebra morphism only need to see that $f(x \Rightarrow_e y) = f(x) \Rightarrow_e f(y)$. To see this, let $x, y \in B^-$,

$$\begin{aligned}f(x \Rightarrow_e y) &= \mathbf{f}(x \Rightarrow_e y) = \mathbf{f}((x \Rightarrow y) \wedge e) \\ &= (\mathbf{f}(x) \Rightarrow \mathbf{f}(y)) \wedge e = \mathbf{f}(x) \Rightarrow_e \mathbf{f}(y) \\ &= f(x) \Rightarrow_e f(y).\end{aligned}$$

Finally, to check that $f(\nabla) \subseteq \nabla'$, if $(x \vee \sim x) \wedge e \in \nabla$, then it is clear that if $y = \mathbf{f}(x) \in B'$,

$$\begin{aligned}f((x \vee \sim x) \wedge e) &= \mathbf{f}((x \vee \sim x) \wedge e) \\ &= (\mathbf{f}(x) \vee \sim \mathbf{f}(x)) \wedge e = (y \vee \sim y) \wedge e \in \nabla'.\end{aligned}$$

■

We now obtain a categorical equivalence. Consider the category $\mathbb{NPC}$ of NPc-lattices together with NPc-lattice morphisms, and the category $\mathbb{BF}$ that has as objects pairs of the form $(\mathbf{L}, \nabla)$ where $\mathbf{L}$ is a Brouwerian algebra and $\nabla \subseteq L$ is a regular filter, and as arrows $f : (\mathbf{L}, \nabla) \to (\mathbf{L}', \nabla')$ such that $f : \mathbf{L} \to \mathbf{L}'$ is a Brouwerian morphism that satisfies $f(\nabla) \subseteq \nabla'$.

THEOREM 3.6
The functor $F:\mathbb{BF}\to\mathbb{NPC}$ that acts on objects as

$$F((\mathbf{L},\nabla))=\mathbf{Tw}(\mathbf{L},\nabla)$$

and on arrows, for $f:(\mathbf{L},\nabla)\to(\mathbf{L}',\nabla')$ obtaining $F(f):\mathbf{Tw}(\mathbf{L},\nabla)\to\mathbf{Tw}(\mathbf{L}',\nabla')$ given by

$$F(f)(x,y)=(f(x),f(y)),$$

gives an equivalence of categories.

PROOF. F is well defined from Theorems 3.3 and 3.4, and it is clearly functorial, as $F(\mathrm{id}_{(\mathbf{L},\nabla)})=\mathrm{id}_{F((\mathbf{L},\nabla))}$, and for arrows $g:(\mathbf{L},\nabla)\to(\mathbf{L}',\nabla')$ and $f:(\mathbf{L}',\nabla')\to(\mathbf{L}'',\nabla'')$, if $(x,y)\in\mathbf{Tw}(\mathbf{L},\nabla)$,

$$\begin{aligned}F(f\circ g)(x,y)&=(f\circ g(x),f\circ g(y))\\&=F(f)(g(x),g(y))\\&=F(f)\circ F(g)(x,y).\end{aligned}$$

Now, to prove it is an equivalence of categories, we will prove that F is full, faithful and essentially surjective:

- **full.** Let $\mathbf{f}:\mathbf{Tw}(\mathbf{L},\nabla)\to\mathbf{Tw}(\mathbf{L}',\nabla')$ be an NPc-lattice morphism. Take $f(x)=\pi_1(\mathbf{f}(x,e))$, for $x\in L$. From Theorem 3.4, it is a morphism from $(\mathbf{L},\nabla)$ to $(\mathbf{L}',\nabla')$, let us see now that $\mathbf{f}=F(f)$. In the negative cone, it is clear that $\mathbf{f}(x,e)=(f(x),e)=F(f)(x,e)$. Then, as they are NPc-lattice morphisms, they must be equal everywhere. Indeed, if $\mathbf{g},\mathbf{h}:\mathbf{B}\to\mathbf{B}'$ are NPc-lattice morphisms such that $\mathbf{g}(x\wedge e)=\mathbf{h}(x\wedge e)$, for each $x\in B$, then if $y=\mathbf{g}(x)$ and $z=\mathbf{h}(x)$, from $y\wedge e=\mathbf{g}(x\wedge e)=\mathbf{h}(x\wedge e)=z\wedge e$ and $\sim y\wedge e=\mathbf{g}(\sim x\wedge e)=\mathbf{h}(\sim x\wedge e)=\sim z\wedge e$ we obtain $y=z$, as NPc-lattices satisfy the quasiequation (7).
- **faithful.** If $F(f):\mathbf{Tw}(\mathbf{L},\nabla)\to\mathbf{Tw}(\mathbf{L}',\nabla')$ and $F(g):\mathbf{Tw}(\mathbf{L},\nabla)\to\mathbf{Tw}(\mathbf{L}',\nabla')$ satisfy $F(f)=F(g)$, in particular they coincide on the negative cone, $(f(x),e)=F(f)(x,e)=F(g)(x,e)=(g(x),e)$ for all $x\in L$, so $f=g$.
- **essentially surjective.** From Theorem 3.5, every object $\mathbf{B}$ on $\mathbb{NPC}$ satisfies $\mathbf{B}\cong\mathbf{Tw}(\mathbf{B}^-,\nabla)$.

■

LEMMA 3.7
In the category $\mathbb{BF}$, finite products are given coordinatewise. That is, if $(\mathbf{L}_1,\nabla_1),\ldots,(\mathbf{L}_n,\nabla_n)$ are objects in $\mathbb{BF}$, then

$$\prod_{i=1}^{n}(\mathbf{L}_i,\nabla_i)\cong\left(\prod_{i=1}^{n}\mathbf{L}_i,\prod_{i=1}^{n}\nabla_i\right),$$

where $\prod_{i=1}^{n}L_i$ and $\prod_{i=1}^{n}\nabla_i$ are products in the category of Brouwerian algebras (filters are subalgebras, products are defined as set-products with operations defined pointwise), and projections coincide with the projections in $\prod_{i=1}^{n}\mathbf{L}_i$.

PROOF. It suffices to prove the result for $n=2$. Let $(L_1,\nabla_1),(L_2,\nabla_2)$ be objects in $\mathbb{BF}$, take $L=L_1\times L_2$, $\nabla=\nabla_1\times\nabla_2$ and $\pi_i=\pi_i^L$, where π_i^L is the projection from L onto L_i, for $i=1,2$. Clearly ∇ is a filter and contains all the dense elements, as operations are given coordinatewise. Then π_1,π_2 are clearly morphisms in $\mathbb{BF}$, as they are morphisms in the category $\mathbb{B}$ of Brouwerian algebras, and besides $\pi_i(\nabla)=\pi_i^L(\nabla_1\times\nabla_2)=\nabla_i$.

Let (L',∇') be another object in $\mathbb{BF}$ and take $f_i:(L',\nabla')\to(L_i,\nabla_i)$ morphisms. Define $f:L'\to L$ by $f(x')=(f_1(x'),f_2(x'))$ for $x'\in L'$, we will show that it is a morphism in $\mathbb{BF}$ and that $\pi_i\circ f=f_i$. The fact that it is a morphism in $\mathbb{B}$ and that $\pi_i\circ f=f_i$ follow from the fact that L is the product of L_1 and L_2 in the category of Brouwerian algebras, we only need to show that it is a morphism in $\mathbb{BF}$. To see this, observe that $f(\nabla')=\{(f_1(x'),f_2(x')):x'\in L'\}\subseteq f_1(\nabla')\times f_2(\nabla')$, but as $f_i(\nabla')\subseteq\nabla_i$, we obtain that $f(\nabla')\subseteq\nabla_1\times\nabla_2=\nabla$. ■

THEOREM 3.8
In the category $\mathbb{NPC}$, finite products are characterized as follows: let $\mathbf{B}_1,\ldots,\mathbf{B}_n$ be objects in $\mathbb{NPC}$ and for each i, let ∇_i be the regular filter in B_i^- such that $\mathbf{B}_i\cong\mathbf{Tw}(\mathbf{B}_i^-,\nabla_i)$. Then

$$\prod_{i=1}^{n}\mathbf{B}_i\cong\mathbf{Tw}\left(\prod_{i=1}^{n}\mathbf{B}_i^-,\prod_{i=1}^{n}\nabla_i\right).$$

PROOF. This follows from Lemma 3.7 and the fact that $\mathbb{NPC}$ and $\mathbb{BF}$ are categorically equivalent. ■

4 Gödel hoops and Gödel NPc-lattices

A Gödel hoop is a Brouwerian algebra satisfying the prelinearity equation $(x\Rightarrow y)\vee(y\Rightarrow x)=e$. Every linearly ordered set can be equipped with a structure of Gödel hoop in a unique way. We denote by $[0,1]_{\mathbf{G}}$ the Gödel hoop on $[0,1]$ and by $\mathbf{G}_n$ the finite linearly ordered Gödel hoop with n elements. Gödel hoops form a variety that is generated by $[0,1]_{\mathbf{G}}$. Given a Gödel hoop $\mathbf{G}=(G,\vee,\wedge,*,\Rightarrow,e)$ and a new element $\bot$, we extend operations of $\mathbf{G}$ on $G\cup\{\bot\}$ by setting $\bot$ smaller than all the elements of G and $x*\bot=\bot=\bot*\bot=\bot*x$, $x\Rightarrow\bot=\bot$, $\bot\Rightarrow x=e=\bot\Rightarrow\bot$ for every $x\in G$. Then $\mathbf{G}_\bot=(G\cup\{\bot\},\vee,\wedge,*,\Rightarrow,e)$ is a Gödel hoop which is lower bounded.

DEFINITION 4.1
A Gödel NPc-lattice is a NPc-lattice satisfying the equation

$$(((x\wedge e)\to y)\vee((y\wedge e)\to x))\wedge e=e.$$

Then, as a consequence of Theorem 3.6 we have the following.

THEOREM 4.2
The restriction of the functor F to the category $\mathbb{GHF}$ of pairs consisting of Gödel hoops and regular filters, gives an equivalence of categories between $\mathbb{GHF}$ and the full subcategory $\mathbb{GNPC}$ of $\mathbb{NPC}$ having Gödel NPc-lattices as objects.

4.1 Duality for Gödel hoops

In [1] it is shown that the category of finite Gödel hoops is dually equivalent to the category $\mathcal{T}_{fin}$ of finite trees and open maps. We recall here some details of such construction. A *forest* is a poset F such that $\downarrow x=\{y\in F\mid y\leq x\}$ is totally ordered for any $x\in F$. If P is a poset, by $P_\bot$ we denote the poset obtained by adding a new bottom element $\bot$ to P. A *tree* is a forest with a minimum element (the *root* of the tree), hence for each forest F, $F_\perp$ is a tree. We hence denote by $\emptyset_\perp$ the singleton tree only consisting of its root. Given a tree T we denote by $T^\uparrow$ the unique forest such that $T=(T^\uparrow)_\perp$.

A downset (i.e. a downward closed set) of a forest (tree) is itself a forest (tree), and we shall call it a *subforest* (*subtree*) of F.

Given two forests F and G, an order preserving map $f: F\to G$ is *open* if $x'\leq f(x)$ in G implies that there exists $y\leq x$ in F such that $f(y)=x'$. Open maps carry downsets to downsets.

We denote by $\mathcal{F}_{fin}$ and $\mathcal{T}_{fin}$ the category of finite forests and finite trees, respectively, with open maps.

In $\mathcal{F}_{fin}$ the coproduct, denoted by $+$ from here on, is just the disjoint union, whereas in $\mathcal{T}_{fin}$ it is given by

$$S\oplus T\cong\left(S^\uparrow+T^\uparrow\right)_\perp$$

(i.e. all roots merge in a single root). It is clear that $\emptyset_\perp$ is the neutral element of the coproduct (that is, the initial object) in $\mathcal{T}_{fin}$.

Given two trees S and T, their product in the category $\mathcal{T}_{fin}$ of finite trees coincide with the product in the category $\mathcal{F}_{fin}$ of finite forests, and it can be calculated by the following recursive laws [2]:

- $\emptyset_\perp\times T\cong T$ (i.e. $\emptyset_\perp$ is the neutral element of the product, being the terminal object, in both $\mathcal{T}_{fin}$ and $\mathcal{F}_{fin}$);
- $S\times T\cong(S^\uparrow\times T+S^\uparrow\times T^\uparrow+S\times T^\uparrow)_\perp$;
- If F,G,H are finite forests, $(F+G)\times H\cong(F\times H)+(G\times H)$.

Then the projection maps π_S and π_T are recursively defined as follows (we focus on π_S, the other projection being analogous): if $x\in S\times T$ then either x is the root of $S\times T$ and in this case we set $\pi_S(x)$ equal to the root of S, or $x\in S^\uparrow\times T+S^\uparrow\times T^\uparrow+S\times T^\uparrow$. In turns, if $x\in S^\uparrow\times T+S^\uparrow\times T^\uparrow$ then we set $\pi_S(x)=\iota_S(\pi_{S^\uparrow}(x))$, where ι_S is the inclusion function of $S^\uparrow$ in S and $\pi_{S^\uparrow}$ is the projection function of $S^\uparrow\times T$ or $S^\uparrow\times T^\uparrow$. If $x\in S\times T^\uparrow$ then $\pi_S(x)$ coincides with the projection function in S of the product $S\times T^\uparrow$.

Note that an atom x of $S\times T$ satisfies that either $\pi_S(x)$ is the root of S and $\pi_T(x)$ is an atom of T, or $\pi_S(x)$ is an atom of S and $\pi_T(x)$ is the root of T; or both $\pi_S(x)$ and $\pi_T(x)$ are atoms of S and T respectively.

THEOREM 4.3
[1] The category $\mathcal{T}_{fin}$ is dually equivalent to the category $\mathbb{GH}_{fin}$ of finite Gödel hoops and (Brouwerian) morphisms.

The duality is given by the functor Spec* that sends a Gödel hoop $\mathbf{L}$ to its prime filter tree $(\mathrm{Spec}(\mathbf{L}))_\perp$ (identifying L with the root of the tree, that is $\mathrm{Spec}^*(L)=\{\mathfrak{p}:\mathfrak{p}$ is a prime filter of L or $\mathfrak{p}=L\}$), and given a morphism $f:\mathbf{L}\to\mathbf{L}'$, its image under the functor is $f^{-1}:(\mathrm{Spec}(\mathbf{L}'))_\perp\to(\mathrm{Spec}(\mathbf{L}))_\perp$.

We recall from [1, Thm. 4.3.1] that the free Gödel hoop $\mathrm{Free}_{\mathbb{GH}}(n)$ over n generators is inductively defined as follows: $\mathrm{Free}_{\mathbb{GH}}(1)=\mathbf{G}_2$ and

$$\mathrm{Free}_{\mathbb{GH}}(n)=\prod_{i=0}^{n-1}\mathrm{Free}_{\mathbb{GH}}(i)_\perp^{\binom{n}{i}}. \tag{12}$$

Finally, from [1, Theorem 4.3.1] we have that the dual of the free Gödel hoop over n generators

$$H_n=\mathrm{Spec}^*(\mathrm{Free}_{\mathbb{GH}}(n))$$

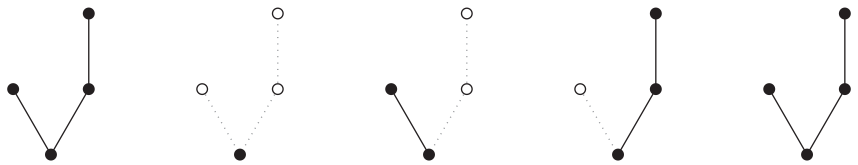

FIGURE 1. A tree and all of its atomic upward closed subtrees.

is given by $H_0 = \emptyset_\perp$ and

$$H_n = \left(\sum_{i=0}^{n-1} \binom{n}{i} H_i \right)_\perp ,$$

where the sum here is taken as the coproduct in forest (*i.e.* the disjoint union).

4.2 Duality for Gödel NPc-lattices

To establish a duality for Gödel NPc-lattices, we will introduce another category, consisting of pairs of trees, as follows.

DEFINITION 4.4
Given a finite tree T, a subtree t of T is an **atomic upward closed** subtree of T if t contains the root of T and whenever an atom a of T belongs to t and $b \in T$ with $b \geq a$, then $b \in t$.

We consider the following category denoted by $\mathcal{T}_{t,fin}$: objects are pairs (T,t) where T is a finite tree and t is an atomic upward closed subtree of T; arrows $\phi : (T,t) \to (T',t')$ are open maps $\phi : T \to T'$ such that $\phi(t) \subseteq t'$.

In contrast with general embeddings of subtrees, note that if T is a tree and t is another tree embeddable in T in such a way that its image is an atomic upward closed subtree of T, then this embedding is unique up to isomorphism. See Fig. 1 and Fig. 2 for examples. Notice further that given a tree T, the only atomic upward closed subtrees of $T_\perp$ are $\emptyset_\perp$ (that is the root of $T_\perp$) and $T_\perp$ itself.

THEOREM 4.5
$\mathcal{T}_{t,fin}$ is the dual of the category $\mathbb{GNPC}_{fin}$ of finite Gödel NPc-lattices.

PROOF. Since $\mathbb{GNPC}_{fin}$ is equivalent to the category $\mathbb{GHF}_{fin}$ of pairs of finite Gödel hoops and regular filters (Theorem 4.2), it is enough to see the duality of $\mathcal{T}_{t,fin}$ and $\mathbb{GHF}_{fin}$. As the functor Spec* gives the dual isomorphism with $\mathbb{GH}_{fin}$, we only need to check that it is well-behaved with respect to atomic upward closed subtrees and regular filters.

Given a regular filter ∇, define

$$t(\nabla) = \{\mathfrak{p} \in \mathrm{Spec}(L) : \exists \mathfrak{m} \in \mathrm{Spec}(L), \nabla \subseteq \mathfrak{m}, \mathfrak{p} \subseteq \mathfrak{m}\}_\perp ,$$

(observe that if $\nabla = L$, then $t(\nabla) = \emptyset_\perp = \{L\}$). Clearly $t(\nabla)$ is an atomic upward closed subtree of Spec$^*(L)$ with the order $\supseteq$ (the filters $\mathfrak{m}$ are the maximals of L or all of L, i.e. they are atoms or the root of Spec$^*(L)$). From Corollary 2.4, one can recover ∇ from $t(\nabla)$,

$$\nabla = \cap\{\mathfrak{m} \in t(\nabla) : \mathfrak{m} \text{ is the root or an atom of } t(\nabla)\}.$$

We now define

$$\mathrm{Spec}^*(\mathbf{L},\nabla)=(\mathrm{Spec}^*(\mathbf{L}),t(\nabla)).$$

We still need to check that it is well-behaved with respect to arrows. Let $f:L\to L'$ be a (Brouwerian) morphism and let ∇,∇' be regular filters in L and L', respectively. We will check that $f(\nabla)\subseteq\nabla'$ if and only if $f^{-1}(t(\nabla'))\subseteq t(\nabla)$, so Spec^* sends arrows in $\mathbb{GF}_{fin}$ into arrows in $\mathcal{T}_{t,fin}$, and vice-versa.

- If $f(\nabla)\subseteq\nabla'$, then $\nabla\subseteq f^{-1}(\nabla')$. Now if $\mathfrak{p}'\in t(\nabla')$, we should check that $f^{-1}(\mathfrak{p}')\in t(\nabla)$. This is clear if $\mathfrak{p}'$ is the root or an atom of $t(\nabla')$, as $\nabla'\subseteq\mathfrak{p}'$ so by hypothesis $f(\nabla)\subseteq\mathfrak{p}'$, which in turn gives $\nabla\subseteq f^{-1}(\mathfrak{p}')$ and therefore $f^{-1}(\mathfrak{p}')\in t(\nabla)$ (as f^{-1} is an open map, $f^{-1}(\mathfrak{p}')$ is the root or an atom of $\mathrm{Spec}^*(L')$). Now, if $\mathfrak{p}'$ is not the root or an atom, let $\mathfrak{m}'$ be the unique atom (maximal filter) such that $\mathfrak{p}'\subseteq\mathfrak{m}'$. As $\mathfrak{m}'\in t(\nabla')$ is an atom, we just proved that $f^{-1}(\mathfrak{m}')\in t(\nabla)$, but as $f^{-1}(\mathfrak{p}')\subseteq f^{-1}(\mathfrak{m}')$ the fact that $t(\nabla)$ is an atomic upward closed subtree gives us $f^{-1}(\mathfrak{p}')\in t(\nabla)$.
- If $f^{-1}(t(\nabla'))\subseteq t(\nabla)$, we need to check that $f(\nabla)\subseteq\nabla'$, or equivalently that $\nabla\subseteq f^{-1}(\nabla')$. As

$$\nabla'=\cap\{\mathfrak{m}'\in t(\nabla'):\mathfrak{m}' \text{ is the root or an atom of } t(\nabla')\},$$

  we have that

$$f^{-1}(\nabla')=\cap\{f^{-1}(\mathfrak{m}'):\mathfrak{m}' \text{ is the root or an atom of } t(\nabla')\}.$$

  By hypothesis, each of these $\mathfrak{m}'$ satisfies $f^{-1}(\mathfrak{m}')\in t(\nabla)$, and as they are the root or an atom of $t(\nabla)$ (f^{-1} being an open map), we have $\nabla\subseteq f^{-1}(\mathfrak{m}')$ and we conclude $\nabla\subseteq f^{-1}(\nabla')$.

The functor $S:\mathbb{GNPC}_{fin}\to\mathcal{T}_{t,fin}$ obtained as composition of $F^{-1}:\mathbb{GNPC}_{fin}\to\mathbb{GHF}_{fin}$ of Theorem 4.2 and $\mathrm{Spec}^*:\mathbb{GHF}_{fin}\to\mathcal{T}_{t,fin}$ is the desired duality.

■

In the category $\mathcal{T}_{t,fin}$, the coproduct is given coordinatewise, i.e.

$$(S,s)\oplus(T,t)\cong(S\oplus T,s\oplus t).$$

This fact can be easily proven directly, but it is also a consequence of Theorem 3.8.

To define the product in the category $\mathcal{T}_{t,fin}$, first observe that for any (S,s) in $\mathcal{T}_{t,fin}$

$$(S,s)\times(\emptyset_\perp,\emptyset_\perp)\cong(S,s)$$

as $(\emptyset_\perp,\emptyset_\perp)$ is the terminal object in $\mathcal{T}_{t,fin}$. Now set, for every other (T,t) in $\mathcal{T}_{t,fin}$,

$$r=\left(\left(s^\uparrow\times T\right)+(s^\uparrow\times t^\uparrow)+\left(S\times t^\uparrow\right)\right)_\perp$$

and we are going to prove that

$$(S,s)\times(T,t)\cong(S\times T,r).$$

Proposition 4.6

With the notation as before, r is an atomic upward closed subtree of $S\times T$.

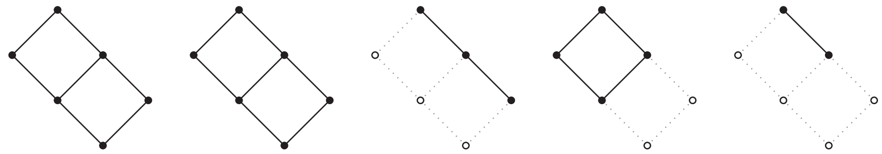

FIGURE 2. The dual in $\mathbb{GH}_{fin}$ of the tree in Figure 1 and all of its regular filters, in correspondence to its atomic upward closed subtrees.

PROOF. Clearly r is a subtree of $S\times T$ and the set of atoms of r is $\{a\in r \mid a \text{ is an atom of } S\times T\}$.

Let us denote by a^0 and b^0 the roots of S and T (hence of s and t) and by $a_1^1,\dots,a_n^1$ and $b_1^1,\dots,b_m^1$ the atoms of s and t, respectively. If x is an atom of $S\times T$ and $x\in r$, then x is the root of a tree in one of the forests $s^{\uparrow}\times T$ or $s^{\uparrow}\times t^{\uparrow}$ or $S\times t^{\uparrow}$. Suppose x is the root of a tree in $s^{\uparrow}\times T$ hence the root of a tree in $S^{\uparrow}\times T$. Then $\pi_T(x)=b^0$ while $\pi_S(x)=a_i^1$ for some $i\in\{1,\dots,n\}$. Now if $y\geq x$ and $y\in S\times T$, then it must be $\pi_T(y)\geq b^0$ and $\pi_S(y)\geq a_i^1$, hence $\pi_T(y)\in T$ and $\pi_S(y)\in s^{\uparrow}$ and so $y\in s^{\uparrow}\times T\subseteq r$. The other cases are similar, hence r is an atomic upward closed subtree of $S\times T$. ∎

THEOREM 4.7
$(S\times T,r)$ is the product of (S,s) and (T,t) in the category $\mathcal{T}_{t,fin}$.

PROOF. Note that the projection map $\pi_S: S\times T\to S$ is such that $\pi_S(r)\subseteq s$, hence it is a map in the category $\mathcal{T}_{t,fin}$ and we set $\pi_{(S,s)}=\pi_S$. Analogously, we set $\pi_{(T,t)}=\pi_T$.

The proof follows by the properties of product in the category $\mathcal{T}_{fin}$. ∎

5 Free GNPc-lattices

THEOREM 5.1
Let $[0,1]_{\mathbf{G}}$ denote the standard Gödel hoop over the real interval $[0,1]$. The variety $\mathbb{GNPC}$ of Gödel NPc-lattices is generated by the full twist product $\mathbf{K}([0,1]_{\mathbf{G}})$.

PROOF. We have to prove that given two terms τ,γ in the language of NPc-lattices, an equation $\tau=\gamma$ holds in $\mathbb{GNPC}$ if and only if it holds in $\mathbf{K}([0,1]_{\mathbf{G}})$. One direction is immediate, since $\mathbf{K}([0,1]_{\mathbf{G}})\in\mathbb{GNPC}$. For the other direction, recall that if $\tau(x_1,\dots,x_n)$ is a term in the language of NPc-lattices there are unique terms τ^1,τ^2 in the language of Gödel hoops such that if $\mathbf{A}\in\mathbb{GNPC}$, then replacing x_i by the pair of variables (y_i,z_i) we get

$$\tau_{\mathbf{K}(\mathbf{A}^-)}(x_1,\dots x_n)=\tau_{\mathbf{K}(\mathbf{A}^-)}((y_1,z_1),\dots(y_n,z_n))$$

and

$$\tau_{\mathbf{K}(\mathbf{A}^-)}((y_1,z_1),\dots(y_n,z_n))=(\tau^1_{\mathbf{A}^-}(y_1,z_1,\dots,y_n,z_n),\tau^2_{\mathbf{A}^-}(y_1,z_1,\dots,y_n,z_n)).$$

Now assume that $\tau=\gamma$ does not hold in $\mathbb{GNPC}$ and let $\tau^1,\tau^2,\gamma^1,\gamma^2$ be the corresponding terms in the language of Gödel hoops. Then there is an algebra $\mathbf{A}$ in $\mathbb{GNPC}$ and elements $a_1,\dots a_n\in A$ such that

$$\tau_{\mathbf{A}}(a_1,\dots,a_n)\neq\gamma_{\mathbf{A}}(a_1,\dots,a_n).$$

Since **A** can be identified with a subalgebra of the full twist-product $\mathbf{K}(\mathbf{A}^-)$ (see Theorem 3.2) there are elements $b_1, c_1, b_2, c_2, \ldots, b_n, c_n \in A^-$ such that if $a_i = (b_i, c_i)$ for each $i = 1, \ldots, n$ one of the equations

$$\tau^1_{\mathbf{A}^-}(b_1, c_1 \ldots, b_n, c_n) = \gamma^1_{\mathbf{A}^-}(b_1, c_1 \ldots, b_n, c_n)$$

or

$$\tau^2_{\mathbf{A}^-}(b_1, c_1 \ldots, b_n, c_n) = \gamma^2_{\mathbf{A}^-}(b_1, c_1 \ldots, b_n, c_n)$$

does not hold in $\mathbf{A}^-$. But since $\mathbf{A}^-$ is in the variety of Gödel hoops and this variety is generated by $[0,1]_{\mathbf{G}}$, we can assert that there are elements $f_1, g_1, \ldots, f_n, g_n$ in $[0,1]_{\mathbf{G}}$ such that either

$$\tau^1_{\mathbf{A}^-}(f_1, g_1, \ldots, f_n, g_n) \neq \gamma^1_{\mathbf{A}^-}(f_1, g_1, \ldots, f_n, g_n)$$

or

$$\tau^2_{\mathbf{A}^-}(f_1, g_1, \ldots, f_n, g_n) \neq \gamma^2_{\mathbf{A}^-}(f_1, g_1, \ldots, f_n, g_n).$$

Take $d_i = (f_i, g_i) \in ([0,1]_{\mathbf{G}})^2$ and $\mathbf{B} = \mathbf{K}([0,1]_{\mathbf{G}})$ and we get

$$\tau_{\mathbf{B}}(d_1, \ldots, d_n) \neq \gamma_{\mathbf{B}}(d_1, \ldots, d_n).$$

Therefore the equation $\tau = \gamma$ does not hold in $\mathbf{K}([0,1]_{\mathbf{G}})$. ■

The following is a well known result of universal algebra.

THEOREM 5.2
([11, Chapter IV, Theorem 3.13]) If a variety $\mathcal{V}$ of algebras is generated by an algebra **A**, then the free algebra in $\mathcal{V}$ with α generators is isomorphic to the subalgebra of functions $f : \mathbf{A}^{\alpha} \to \mathbf{A}$ generated by the projection functions.

5.1 The case of one generator

We intend to use Theorem 5.1 and Theorem 5.2 to describe the free Gödel NPc-lattice with one generator $\mathrm{Free}_{\mathbb{GNPC}}(1)$.

Now, the carrier of $\mathbf{K}([0,1]_{\mathbf{GH}})$ is just $[0,1]^2$, so we have to characterize exactly the class of functions $\{f : [0,1]^2 \to [0,1]^2\}$ generated, with the pointwise operations of $\mathbf{K}([0,1]_{\mathbf{GH}})$, by the identity function $(a,b) \mapsto (a,b)$. This is equivalent to the determination of all functions $f : [0,1]^2 \to [0,1]^2$ such that there is a term τ in one variable such that $f(a,b) = \tau(a,b)$ for all $(a,b) \in [0,1]^2$. We first prove some necessary results taking finite subalgebras of the Gödel hoop $[0,1]$:

LEMMA 5.3
Consider the three-element Gödel chain $\mathbf{G}_3 = \{a, b, 1\}$ with $a < b < 1$. Then the Gödel NPc-lattices respectively generated by the elements (a,b) or (b,a), *i.e.* the smallest subalgebras of the full-twist $\mathbf{K}(\mathbf{G}_3)$ respectively containing the elements (a,b) or (b,a), are in both cases $\mathbf{Tw}(\mathbf{G}_3, \{b,1\})$, whose carrier is $K(\mathbf{G}_3) \setminus \{(a,a)\}$. Moreover, they coincide with the Gödel NPc-lattice generated by the elements $(a,1)$ and $(b,1)$.

PROOF. First notice that the carrier of $\mathbf{Tw}(\mathbf{G}_3,\{b,1\})$ is clearly $K(\mathbf{G}_3)\setminus\{(a,a)\}$. Let us focus on (a,b) and let $\langle(a,b)\rangle$ be the subalgebra generated by (a,b). As $K(\mathbf{G}_3)\setminus\{(a,a)\}$ is a subalgebra and contains the element (a,b), for it is the twist-product $\mathbf{Tw}(\mathbf{G}_3,\nabla)$ with $\nabla=\{b,1\}$, it only remains to be shown that every element of $K(\mathbf{G}_3)$ different from (a,a) belongs to $\langle(a,b)\rangle$.

- $(1,1),(a,b)\in\langle(a,b)\rangle$ trivially.
- $(b,a)\in\langle(a,b)\rangle$, as $(b,a)={\sim}(a,b)$.
- $(a,1)\in\langle(a,b)\rangle$, as $(a,1)=(a,b)\sqcap(1,1)$.
- $(1,a)\in\langle(a,b)\rangle$, as $(1,a)={\sim}(a,1)$.
- $(b,1)\in\langle(a,b)\rangle$, as $(b,1)=(b,a)\sqcap(1,1)$.
- $(1,b)\in\langle(a,b)\rangle$, as $(1,b)={\sim}(b,1)$.
- $(b,b)\in\langle(a,b)\rangle$, as $(b,b)=(a,b)\sqcup(b,1)$.

For the other part, $(a,1),(b,1)\in\langle(a,b)\rangle$, and as $(b,1)\to(a,1)=(a,b)$, the result follows. The case $\langle(b,a)\rangle$ is promptly settled by noticing that $(a,b)={\sim}(b,a)$. ■

LEMMA 5.4
Consider the two-element Gödel chain $\mathbf{G}_2=\{a,1\}$ with $a<1$. Then:

(1) The Gödel NPc-lattice generated by the element (a,a) is $\mathbf{K}(\mathbf{G_2})$.
(2) The smallest subalgebras of the full-twist $\mathbf{K}(\mathbf{G_2})$ generated either by the element $(a,1)$ or by $(1,a)$, are both isomorphic with $\mathbf{Tw}(\mathbf{G}_2,\{1\})$ whose carrier is $K(\mathbf{G}_2)\setminus\{(a,a)\}$.

PROOF. 1) Just notice that $(a,a)\sqcap(1,1)=(a,1)$ and $(a,a)\sqcup(1,1)=(1,a)$.
2) As in Lemma 5.3, $(a,a)\notin\langle(a,1)\rangle$. The rest follows trivially by $(1,a)={\sim}(a,1)$. Clearly, the carrier of $\mathbf{Tw}(\mathbf{G}_2,\{1\})$ is $K(\mathbf{G}_2)\setminus\{(a,a)\}$. ■

We shall now determine the structure of the free Gödel NPc-lattice over one generator. The result hinges on the characterization given in [17] of the free prelinear Heyting algebras (or, *Gödel algebras*) as algebras of $[0,1]$-valued functions.

LEMMA 5.5
In the variety $\mathbb{GNPC}$, the algebra $\mathrm{Free}_{\mathbb{GNPC}}(1)$ embeds into the following product:

$$\mathbf{Tw}(\mathbf{G}_3,\mathbf{G}_2)\times\mathbf{Tw}(\mathbf{G}_2,\mathbf{G}_2)\times\mathbf{Tw}(\mathbf{G}_3,\mathbf{G}_2).$$

PROOF. Consider the following subsets of $[0,1]^2$: $A=\{(a,b)\in[0,1]^2, a<b\}$, $B=\{(a,b)\in[0,1]^2, a=b\}$, $C=\{(a,b)\in[0,1]^2, a>b\}$. Clearly, $\{A,B,C\}$ forms a partition of $[0,1]^2$.

Now, pick two distinct points $(a_1,b_1),(a_1',b_1')\in A$, with $b_1\neq 1\neq b_1'$. By Lemma 5.3, the algebras $\langle(a_1,b_1)\rangle$, $\langle(a_1',b_1')\rangle$ singly generated by these two points are isomorphic. Moreover, the function from $\langle(a_1,b_1)\rangle$ into $\langle(a_1,b_1)\rangle\times\langle(a_1',b_1')\rangle$ that maps (a_1,b_1) to $((a_1,b_1),(a_1',b_1'))$ yields an isomorphism

$$\langle(a_1,b_1)\rangle\cong\langle((a_1,b_1),(a_1',b_1'))\rangle,$$

and clearly $\langle((a_1,b_1),(a_1',b_1'))\rangle$ embeds into $\langle(a_1,b_1)\rangle\times\langle(a_1',b_1')\rangle$.

Pick now $(a_1',b_1')\in A$, with $b_1'=1$. By Lemma 5.4, $\langle(a_1',b_1')\rangle$ is isomorphic to the quotient of $\langle(a_1,b_1)\rangle$, given by the congruence θ generated by $((b_1,1),(1,1))$. Therefore

$$\langle(a_1,b_1)\rangle\cong\langle((a_1,b_1),(a_1',1))\rangle$$

via the maps $(a_1,b_1)\mapsto((a_1,b_1),(a_1,b_1)/\theta)\mapsto((a_1,b_1),(a_1',1))$. Repeating the argument above for each point in A, it turns out that the embedding from $\langle(a_1,b_1)\rangle$ into $\prod_{(a,b)\in A}\langle(a,b)\rangle$ given by

$$(a_1,b_1)\mapsto((a,b))_{(a,b)\in A}$$

is an isomorphism between $\langle(a_1,b_1)\rangle$ and $\langle((a,b))_{(a,b)\in A}\rangle$.

But $\langle((a,b))_{(a,b)\in A}\rangle$ is by its very definition the algebra of all functions $f: A\to[0,1]^2$ generated by the identity function $id_A: A\to A$. The latter, in turn, by Lemma 5.3 is isomorphic with $\mathbf{Tw}(\mathbf{G}_3,\mathbf{G}_2)$.

In a completely analogous fashion, one shows that the algebra of all functions $f: B\to[0,1]^2$ generated by the identity function over B is isomorphic to $\mathbf{K}(\mathbf{G}_2)\cong\mathbf{Tw}(\mathbf{G}_2,\mathbf{G}_2)$, and that the algebra of all functions $f: C\to[0,1]^2$ generated by the identity function over C is isomorphic to $\mathbf{Tw}(\mathbf{G}_3,\mathbf{G}_2)$.

To end the proof, notice that every element of $\mathrm{Free}_{\mathbb{GNPC}}(1)$ can be expressed as a triplet of functions (f,g,h), with $f: A\to[0,1]^2$, $g: B\to[0,1]^2$, and $h: C\to[0,1]^2$. Therefore the generator of $\mathrm{Free}_{\mathbb{GNPC}}(1)$ can be chosen as a triplet

$$((a_1,b_1),(a_2,b_2),(a_3,b_3)),$$

for some arbitrarily fixed choice of $a_1,b_1,a_2,b_2,a_3,b_3\in[0,1]$ such that $a_1<b_1<1$, $a_2=b_2<1$ and $b_3<a_3<1$. ■

Notice that we cannot drop any of the three factors in $\mathbf{Tw}(\mathbf{G}_3,\mathbf{G}_2)\times\mathbf{Tw}(\mathbf{G}_2,\mathbf{G}_2)\times\mathbf{Tw}(\mathbf{G}_3,\mathbf{G}_2)$ without losing the property that $\mathrm{Free}_{\mathbb{GNPC}}(1)$ embeds into the remaining algebra. As a matter of fact each of the maps $(a_i,b_i)\mapsto(a_j,b_j)$, for $i,j\in\{1,2,3\}$ and a_i,b_i,a_j,b_j being the corresponding elements forming the chosen generator triplet in Lemma 5.5, is an isomorphism iff $i=j$.

THEOREM 5.6
The following holds:

$$\begin{aligned}\mathrm{Free}_{\mathbb{GNPC}}(1)&\cong\mathbf{Tw}(\mathbf{G}_3,\mathbf{G}_2)\times\mathbf{Tw}(\mathbf{G}_2,\mathbf{G}_2)\times\mathbf{Tw}(\mathbf{G}_3,\mathbf{G}_2)\\&\cong\mathbf{Tw}(\mathbf{G}_3\times\mathbf{G}_2\times\mathbf{G}_3,\mathbf{G}_2\times\mathbf{G}_2\times\mathbf{G}_2)\\&\cong\mathbf{Tw}(\mathrm{Free}_{\mathbb{GH}}(2),\nabla),\end{aligned}$$

where $\nabla=\mathbf{G}_2\times\mathbf{G}_2\times\mathbf{G}_2$.

PROOF. We need to prove that for every triplet

$$((p_1,q_1),(p_2,q_2),(p_3,q_3))\in\mathbf{Tw}(\mathbf{G}_3,\mathbf{G}_2)\times\mathbf{Tw}(\mathbf{G}_2,\mathbf{G}_2)\times\mathbf{Tw}(\mathbf{G}_3,\mathbf{G}_2),$$

there is a one-variable term $t(x)$ in the language of NPc-lattices, such that

$$((p_1,q_1),(p_2,q_2),(p_3,q_3))=t(((a_1,b_1),(a_2,b_2),(a_3,b_3))),$$

where $((a_1,b_1),(a_2,b_2),(a_3,b_3))$ is the chosen triplet in Lemma 5.5.

We consider the terms: $\tau_1(x):=\sim((\sim x)*(\sim x))*\sim((\sim x)*(\sim x))$, $\tau_2(x):=\sim((x\leftrightarrow\sim x)*(x\leftrightarrow\sim x))$, and $\tau_3(x):=\sim(x*x)*\sim(x*x)$.

Notice that:

$$\tau_1(((a_1,b_1),(a_2,b_2),(a_3,b_3)))=((a_1,1),(1,a_2),(1,b_3)),$$

$$\tau_2(((a_1,b_1),(a_2,b_2),(a_3,b_3)))=((1,a_1),(a_2,1),(1,b_3)),$$

and

$$\tau_3(((a_1,b_1),(a_2,b_2),(a_3,b_3)))=((1,a_1),(1,a_2),(b_3,1)).$$

Now, by the proofs of Lemmas 5.3 and 5.4, we have that for each $i\in\{1,2,3\}$, there is a one-variable term

$$t_i(x)\in\{e,x,\sim x,x\wedge e,x\vee e,\sim x\vee e,\sim x\wedge e,x\vee(\sim x\wedge e),x\wedge(\sim x\vee e)\}$$

such that $\pi_i(t_i(((a_1,b_1),(a_2,b_2),(a_3,b_3))))=(p_i,q_i)$ where π_i is the i-th projection.
Observe then that

$$(t_1\vee\tau_1)(((a_1,b_1),(a_2,b_2),(a_3,b_3)))=((p_1,q_1),(1,a_2),(1,b_3)),$$

$$(t_2\vee\tau_2)(((a_1,b_1),(a_2,b_2),(a_3,b_3)))=((1,a_1),(p_2,q_2),(1,b_3)),$$

and

$$(t_3\vee\tau_3)(((a_1,b_1),(a_2,b_2),(a_3,b_3)))=((1,a_1),(1,a_2),(p_3,q_3)).$$

The proof is settled by checking that

$$\left(\bigwedge_{i=1}^{3}(t_i\vee\tau_i)\right)(((a_1,b_1),(a_2,b_2),(a_3,b_3)))=((p_1,q_1),(p_2,q_2),(p_3,q_3)).$$

Since the operator **Tw** commutes with direct products (Theorem 4.2), we equivalently have

$$\mathrm{Free}_{\mathbb{GNPC}}(1)\cong\mathbf{Tw}(\mathbf{G}_3\times\mathbf{G}_2\times\mathbf{G}_3,\mathbf{G}_2\times\mathbf{G}_2\times\mathbf{G}_2),$$

(see Fig. 3 for a display of the two components of the twist-product above) and the last isomorphism follows from (12) for $n=2$:

$$\mathrm{Free}_{\mathbb{GH}}(2)\cong\mathbf{G}_3\times\mathbf{G}_2\times\mathbf{G}_3.$$

■

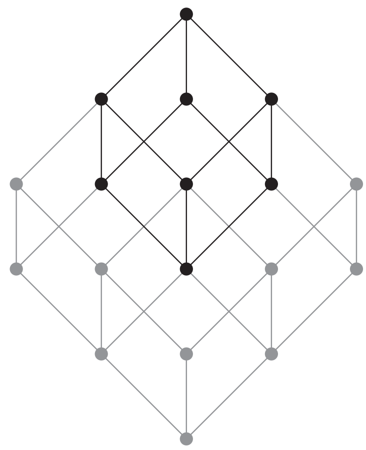

FIGURE 3. The Gödel hoop $\mathbf{G}_3 \times \mathbf{G}_2 \times \mathbf{G}_3$ together with its filter $\mathbf{G}_2 \times \mathbf{G}_2 \times \mathbf{G}_2$.

Notice that, for every finite Gödel hoop $\mathbf{A}$, with $\mathrm{Spec}^*(\mathbf{A}) \cong T$, it holds that $\mathrm{Spec}^*(\mathbf{A}_\perp, \mathbf{A}) \cong (T_\perp, T_\perp)$, since the only pair $(a,b) \in A_\perp^2$ such that $a \vee b \notin A$ is $(a,b) = (\perp, \perp)$. On the other hand, $\mathrm{Spec}^*(\mathbf{A}_\perp, \mathbf{A}_\perp) \cong (T_\perp, \emptyset_\perp)$. We recall that $S : \mathbb{GNPC} \to \mathcal{T}_{t,fin}$ is the functor realising the duality as in Theorem 4.5.

LEMMA 5.7
$S(\mathrm{Free}_{\mathbb{GNPC}}(1)) \cong (H_2, (2H_1)_\perp)$.

PROOF. By Theorem 5.6,

$$S(\mathrm{Free}_{\mathbb{GNPC}}(1)) \cong S(\mathbf{Tw}(\mathbf{G}_3 \times \mathbf{G}_2 \times \mathbf{G}_3, \mathbf{G}_2 \times \mathbf{G}_2 \times \mathbf{G}_2)).$$

Recall that $\mathbf{G}_3 \cong \mathrm{Free}_{\mathbb{GH}}(1)_\perp$ and $\mathbf{G}_2 \cong \mathrm{Free}_{\mathbb{GH}}(1) \cong \mathrm{Free}_{\mathbb{GH}}(0)_\perp$. So,

$$\begin{aligned}
S(\mathrm{Free}_{\mathbb{GNPC}}(1)) &\cong S(\mathbf{Tw}(\mathrm{Free}_{\mathbb{GH}}(1)_\perp \times \mathrm{Free}_{\mathbb{GH}}(0)_\perp \times \mathrm{Free}_{\mathbb{GH}}(1)_\perp, \\
&\qquad \mathrm{Free}_{\mathbb{GH}}(1) \times \mathrm{Free}_{\mathbb{GH}}(0)_\perp \times \mathrm{Free}_{\mathbb{GH}}(1))) \\
&\cong S(\mathbf{Tw}(\mathrm{Free}_{\mathbb{GH}}(1)_\perp, \mathrm{Free}_{\mathbb{GH}}(1))) \\
&\quad \oplus S(\mathbf{Tw}(\mathrm{Free}_{\mathbb{GH}}(0)_\perp, \mathrm{Free}_{\mathbb{GH}}(0)_\perp)) \\
&\quad \oplus S(\mathbf{Tw}(\mathrm{Free}_{\mathbb{GH}}(1)_\perp, \mathrm{Free}_{\mathbb{GH}}(1))) \\
&\cong (H_{1\perp}, H_{1\perp}) \oplus (H_{0\perp}, \emptyset_\perp) \oplus (H_{1\perp}, H_{1\perp}) \\
&\cong (H_{1\perp} \oplus H_{0\perp} \oplus H_{1\perp}, H_{1\perp} \oplus \emptyset_\perp \oplus H_{1\perp}) \\
&\cong (H_2, (2H_1)_\perp).
\end{aligned}$$

■

5.2 *The case of n generators*

We plan now to use the results from sections 4.1, 4.2 and 5.1 to obtain the free GNPc-lattice with n generators.

Since $H_n = \mathrm{Spec}^*(\mathrm{Free}_{\mathbb{GH}}(n))$, it immediately follows that

$$H_i \times H_j \cong \mathrm{Spec}^*(\mathrm{Free}_{\mathbb{GH}}(i) \amalg \mathrm{Free}_{\mathbb{GH}}(j)) \cong \mathrm{Spec}^*(\mathrm{Free}_{\mathbb{GH}}(i+j)) \cong H_{i+j},$$

where $\amalg$ is the coproduct in $\mathbb{GH}$.

Let now $T_n = S(\mathrm{Free}_{\mathbb{GNPC}}(n))$. Note that $T_n \cong T_{n-1} \times T_1$ and by Lemma 5.7:

$$T_1 \cong (H_2, (2H_1)_\perp).$$

Set, for $i = 0, \ldots, n-1$, $c_{i,n} = 0$ and for $i = n, \ldots, 2n$:

$$c_{i,n} = 2^{2n-i} \binom{n}{2n-i}.$$

LEMMA 5.8
For $i = n+2, \ldots, 2n$ it holds $c_{i,n+1} = c_{i-2,n} + 2c_{i-1,n}$.

PROOF. By definition $c_{i-1,n} = 2^{2n+1-i}\binom{n}{2n+1-i}$, $c_{i-2,n} = 2^{2n+2-i}\binom{n}{2n+2-i}$, and $c_{i,n+1} = 2^{2n+2-i}\binom{n+1}{2n+2-i}$. The claim follows by properties of binomial coefficients, since:

$$\binom{n+1}{2n+2-i} = \binom{n}{2n+1-i} + \binom{n}{2n+2-i}.$$

■

LEMMA 5.9
$T_n \cong (H_{2n}, t_n)$ where t_n is the uniquely determined (up to isomorphisms) subtree of H_{2n} given by

$$t_n = \left(\sum_{i=n}^{2n-1} c_{i,n} H_i \right)_\perp.$$

PROOF. As $T_1 \cong (H_2, (2H_1)_\perp)$, $T_{n+1} \cong T_n \times T_1$ and $(H_2)^n \cong H_{2n}$, we only need to check the subtree part. We proceed by induction on n.

Assume by induction hypothesis, that $T_n \cong (H_{2n}, t_n)$ with

$$t_n = \left(\sum_{i=n}^{2n-1} c_{i,n} H_i \right)_\perp.$$

We are going to prove that $T_{n+1} \cong (H_{2(n+1)}, t_{n+1})$ with

$$t_{n+1} = \left(\sum_{i=n+1}^{2n+1} c_{i,n+1} H_i \right)_\perp.$$

By definition of product

$$\begin{aligned} t_{n+1} &\cong \left(\left(t_n^\uparrow \times H_2 \right) + (t_n^\uparrow \times 2H_1) + (H_{2n} \times 2H_1) \right)_\perp \\ &\cong \left(\sum_{i=n}^{2n-1} c_{i,n} H_{i+2} + \sum_{i=n}^{2n-1} 2c_{i,n} H_{i+1} + 2H_{2n+1} \right)_\perp. \end{aligned}$$

Notice that, by index shifting,

$$\sum_{i=n}^{2n-1} c_{i,n} H_{i+2} \cong \sum_{i=n+2}^{2n+1} c_{i-2,n} H_i$$

and

$$\sum_{i=n}^{2n-1} 2c_{i,n} H_{i+1} \cong \sum_{i=n+1}^{2n} 2c_{i-1,n} H_i.$$

Hence, by Lemma 5.8,

$$\begin{aligned}
t_{n+1}^{\uparrow} &\cong \sum_{i=n}^{2n-1} c_{i,n} H_{i+2} + \sum_{i=n}^{2n-1} 2c_{i,n} H_{i+1} + 2H_{2n+1} \\
&\cong \sum_{i=n+1}^{2n} 2c_{i-1,n} H_i + \sum_{i=n+2}^{2n+1} c_{i-2,n} H_i + 2H_{2n+1} \\
&\cong 2c_{n,n} H_{n+1} + \sum_{i=n+2}^{2n} 2c_{i-1,n} H_i + \sum_{i=n+2}^{2n+1} c_{i-2,n} H_i + 2H_{2n+1} \\
&\cong 2c_{n,n} H_{n+1} + \sum_{i=n+2}^{2n} (c_{i-2,n} + 2c_{i-1,n}) H_i + c_{2n-1,n} H_{2n+1} + 2H_{2n+1} \\
&\cong 2c_{n,n} H_{n+1} + \sum_{i=n+2}^{2n} c_{i,n+1} H_i + (2 + c_{2n-1,n}) H_{2n+1}.
\end{aligned}$$

Since

$$2c_{n,n} = 2 \cdot 2^n = 2^{n+1} = 2^{n+1} \binom{n+1}{n+1} = c_{n+1,n+1},$$

$$2 + c_{2n-1,n} = 2 + 2n = 2\binom{n+1}{1} = c_{2n+1,n+1}$$

we have

$$t_{n+1} \cong \left(\sum_{i=n+1}^{2n+1} c_{i,n+1} H_i \right)_{\perp}$$

and the claim follows. ■

So we have that $T_n \cong (H_{2n}, t_n)$, with

$$H_{2n} = \left(\sum_{i=0}^{2n-1} \binom{2n}{i} H_i \right)_{\perp}, \quad t_n = \left(\sum_{i=n}^{2n-1} c_{i,n} H_i \right)_{\perp}.$$

Rewriting them using coproducts in the category of trees, we obtain

$$H_{2n} = \bigoplus_{i=0}^{2n-1} \binom{2n}{i} (H_i)_\perp, \quad t_n = \bigoplus_{i=n}^{2n-1} c_{i,n} (H_i)_\perp.$$

Combining the fact that coproducts in the category $\mathcal{T}_{t,fin}$ are given coordinatewise, that $\emptyset_\perp$ is both the terminal and the initial object in $\mathcal{T}_{fin}$, and that $c_{i,n} = 0$ for $i = 0, \ldots, n-1$, we have that

$$T_n \cong \bigoplus_{i=0}^{2n-1} \left(\binom{2n}{i} - c_{i,n} \right) ((H_i)_\perp, \emptyset_\perp) \oplus \bigoplus_{i=n}^{2n-1} c_{i,n} ((H_i)_\perp, (H_i)_\perp).$$

Notice now that the NPc-lattice dual of the pair $((H_i)_\perp, \emptyset_\perp)$ is the full twist-product $\mathbf{K}((\mathrm{Free}_{\mathbb{GH}}(i))_\perp)$ and that the NPc-lattice dual of the pair $((H_i)_\perp, (H_i)_\perp)$ is

$$\mathbf{Tw}\,((\mathrm{Free}_{\mathbb{GH}}(i))_\perp, \mathrm{Free}_{\mathbb{GH}}(i)).$$

Finally, recalling that the carrier of this algebra is $K((\mathrm{Free}_{\mathbb{GH}}(i))_\perp) \setminus \{(\perp, \perp)\}$, we conclude the following theorem.

THEOREM 5.10

$$\begin{aligned} &\mathrm{Free}_{\mathbb{GNPC}}(n) \cong \\ &\cong \prod_{i=0}^{2n-1} \mathbf{K}((\mathrm{Free}_{\mathbb{GH}}(i))_\perp)^{\left(\binom{2n}{i} - c_{i,n}\right)} \times \prod_{i=n}^{2n-1} \mathbf{Tw}\,((\mathrm{Free}_{\mathbb{GH}}(i))_\perp, \mathrm{Free}_{\mathbb{GH}}(i))^{c_{i,n}} \\ &\cong \mathbf{Tw}\,(\mathrm{Free}_{\mathbb{GH}}(2n), \nabla), \end{aligned}$$

where

$$\nabla = \prod_{i=0}^{2n-1} ((\mathrm{Free}_{\mathbb{GH}}(i))_\perp)^{\left(\binom{2n}{i} - c_{i,n}\right)} \times \prod_{i=n}^{2n-1} (\mathrm{Free}_{\mathbb{GH}}(i))^{c_{i,n}}.$$

PROOF. By Lemma 5.9. ■

COROLLARY 5.11
For each integer $n \geq 0$, the cardinality of $\mathrm{Free}_{\mathbb{GNPC}}(n)$ is given by the following recurrences:

$$|\mathrm{Free}_{\mathbb{GNPC}}(n)| = \prod_{i=0}^{2n-1} (h_i + 1)^{2\left(\binom{2n}{i} - c_{i,n}\right)} \cdot (h_i^2 + 2h_i)^{c_{i,n}},$$

where $h_0 = 1$ and, for all integers $k \geq 0$,

$$h_k = \prod_{i=0}^{k-1} (h_i + 1)^{\binom{k}{i}}.$$

PROOF. By [1, Theorem 4.3.1], the cardinality of $\mathrm{Free}_{\mathbb{GH}}(k)$ is h_k, for all integers $k \geq 0$. Then, clearly, the cardinality of $\mathbf{K}((\mathrm{Free}_{\mathbb{GH}}(i))_\perp)$ is $(h_i + 1)^2$ and the cardinality of $\mathbf{Tw}\,((\mathrm{Free}_{\mathbb{GH}}(i))_\perp, \mathrm{Free}_{\mathbb{GH}}(i))$ is $(h_i + 1)^2 - 1$. The claim follows by Theorem 5.10. ■

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