The sharp maximal function approach to L^p estimates for operators structured on Hörmander's vector fields[*]

Marco Bramanti, Marisa Toschi

July 4, 2016

Abstract

We consider a nonvariational degenerate elliptic operator of the kind

$$Lu \equiv \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j u$$

where $X_1, ..., X_q$ are a system of left invariant, 1-homogeneous, Hörmander's vector fields on a Carnot group in $\mathbb{R}^n$, the matrix $\{a_{ij}\}$ is symmetric, uniformly positive on a bounded domain $\Omega \subset \mathbb{R}^n$ and the coefficients satisfy

$$a_{ij} \in VMO_{loc}(\Omega) \cap L^{\infty}(\Omega).$$

We give a new proof of the interior $W_X^{2,p}$ estimates

$$\|X_i X_j u\|_{L^p(\Omega')} + \|X_i u\|_{L^p(\Omega')} \le c \left\{ \|Lu\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} \right\}$$

for $i,j = 1,2,...,q$, $u \in W_X^{2,p}(\Omega)$, $\Omega' \Subset \Omega$ and $p \in (1,\infty)$, first proved by Bramanti-Brandolini in [3], extending to this context Krylov' technique, introduced in [15], consisting in estimating the sharp maximal function of $X_i X_j u$.

1   Introduction

Let us consider a linear second order elliptic operator in nondivergence form:

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j}$$

1

with $\{a_{ij}\}$ symmetric matrix of bounded measurable functions defined on some domain $\Omega \subset \mathbb{R}^n$ and satisfying the uniform ellipticity condition

$$\mu \left| \xi \right|^2 \leq \sum_{i,j=1}^{n} a_{ij}\left(x\right) \xi_i \xi_j \leq \frac{1}{\mu} \left| \xi \right|^2$$

for some $\mu > 0$, every $\xi \in \mathbb{R}^n$, a.e. $x \in \Omega$. While the classical $W^{2,p}$-theory of elliptic equations, dating back to Agmon-Douglis-Nirenberg [1] and essentially exploiting the L^p theory of singular integrals due to Calderón-Zygmund [8] requires the uniform continuity of the coefficients $a_{ij}\left(x\right)$, in 1993 Chiarenza-Frasca-Longo [9] proved $W^{2,p}$ estimates under the mere assumption $a_{ij} \in L^\infty \cap VMO$, which allows for some kind of discontinuities in the coefficients. Their technique is based on representation formulas of $u_{x_i x_j}$ by means of singular integrals with variable kernels, and commutators of these singular integrals with BMO functions. Thanks to a deep real analysis theorem by Coifman-Rochberg-Weiss [10], these commutators have small operator norm on small balls, hence the old idea of seeing a variable coefficient operator as a small perturbation of the model operator with constant coefficients is ingeniously generalized to an operator with possibly discontinuous coefficients. This technique, by now classic, has been extended to several contexts, for instance parabolic operators (see [5]) and nonvariational operators structured on Hörmander's vector fields (see [3], [4]).

In 2007 Krylov [15] introduced a differerent technique to prove similar and more general results for elliptic and parabolic operators, based on the *pointwise* estimate of the *sharp maximal function of* $u_{x_i x_j}$, that is $\left(u_{x_i x_j}\right)^\#$. The idea is then again that of approximating the operator with variable coefficients with a model operator with constant coefficients; these constants in this case are not simply the original coefficients *frozen* at some point, but suitable integral averages of these functions. The theory of singular integrals is not *explicitly* used, but it is replaced by Fefferman-Stein maximal theorem, which allows to control the L^p norm of $u_{x_i x_j}$ by that of $\left(u_{x_i x_j}\right)^\#$. On the other hand, throughout the computation which is carried out on the model operator, many classical results are employed, implicitly involving also the classical Calderón-Zygmund theory.

The research started with this paper aims to investigate whether Krylov' technique can be extended also to the context of linear degenerate equations structured on Hörmander's vector fields, and if it can be used to get new results not easily obtainable with the techniques previously used. We give a partial positive answer to this question.

We are now going to describe the main results of this paper, postponing to section 2 the precise definitions of all the concepts that are involved. We consider the class of operators

$$Lu \equiv \sum_{i,j=1}^{q} a_{ij}(x) X_i X_j u,$$

where $X_1, ..., X_q$ are a system of left invariant and 1-homogeneous Hörmander's vector fields on a Carnot group in $\mathbb{R}^n$ (see section 2.1), the matrix $\{a_{ij}\}$ is symmetric, the coefficients satisfy

$$a_{ij} \in VMO_{loc}(\Omega) \cap L^\infty(\Omega) \tag{1}$$

on a bounded domain $\Omega \subset \mathbb{R}^n$ (see section 2.2), and the uniform positivity condition holds: there exists $\mu > 0$ such that

$$\mu|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \frac{1}{\mu}|\xi|^2 \tag{2}$$

for every $\xi \in \mathbb{R}^q$ and a.e. $x \in \Omega$.

In this context we prove a pointwise bound on the local sharp maximal function of X_iX_ju. This, combined with an extension of Fefferman-Stein's theorem to the context of locally homogeneous spaces, recently obtained in [6] (see Theorem 3 in this paper) allows to get the local estimates first proved by Bramanti-Brandolini in [3] with an approach that parallels that of Chiarenza-Frasca-Longo. More precisely, the main result that can be proved is the following:

Theorem 1 *Under the previous assumptions, for any $\Omega_m \Subset \Omega$ as in section 2.2 and $p \in (1, \infty)$ there exists a constant c such that*

$$\|X_iX_ju\|_{L^p(\Omega_m)} + \|X_iu\|_{L^p(\Omega_m)} \leq c\left\{\|Lu\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right\}$$

for $i, j = 1, 2, ..., q$ and any $u \in W_X^{2,p}(\Omega)$.

(See section 2.3 for the definition of $W_X^{2,p}(\Omega)$). What we will actually prove here is the basic step towards the above theorem, namely:

Theorem 2 *Under the previous assumptions, for any $\Omega_m \Subset \Omega$ and $p \in (1, \infty)$ the set Ω_m can be covered with a finite number of balls $B_R(x_i)$ such that for every $u \in C_0^\infty(B_R)$*

$$\sum_{i,j=1}^{q} \|X_iX_ju\|_{L^p(B_R)} \leq c\|Lu\|_{L^p(B_R)}$$

where the constant c depends on $\Omega, \Omega_m, p, \mu, \mathbb{G}$ and the function $a_{m,r}^\sharp$ (defined in (22)).

The proof of Theorem 2 is where the different real analysis approach of this paper with respect to [3] plays its role. Proving Theorem 1 starting with Theorem 2 is mainly a matter of cutoff functions and interpolation inequalities for Sobolev norms, which can be performed exactly like in [3] and therefore will not be repeated here.

As already explained, the main result in this paper is not original in itself; the novelty lies in the approach, which allows some simplification with respect

to that of [3]. The assumption of existence of an underlying Carnot group structure such that L is translation invariant and 2-homogeneous is quite natural in consideration of the important role of dilations in Krylov' approach. We hope to extend in the future the present approach to different classes of degenerate operators, getting some kind of new L^p estimate. Some natural candidates to test this technique are operators of Kolmogorov-Fokker-Planck type. However, the presence of a drift term in these operators poses substantial new difficulties, preventing us to use some of the tools that we exploit in this paper, and requiring some further new insight.

2 Preliminaries and known results

2.1 Carnot groups

We start recalling some standard terminology and known facts about Carnot groups. For more details and for the proofs of known results the reader is referred to [2, Chaps. 1, 2], [12], [18, Chap.XIII, §5].

We call *homogeneous group* the space $\mathbb{R}^n$ equipped with a Lie group structure, together with a family of dilations that are group automorphisms. Explicitly, assume that we are given a pair of mappings:

$$[(x, y) \mapsto x \circ y] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \text{ and } \left[x \mapsto x^{-1}\right] : \mathbb{R}^n \to \mathbb{R}^n$$

that are smooth and such that $\mathbb{R}^n$, together with these mappings, forms a group, for which the identity is the origin. We will think to the operation $\circ$ as a *translation*. Next, suppose that we are given an n-tuple of strictly positive integers $\alpha_1 \leq \alpha_2 \leq ... \leq \alpha_n$, such that the *dilations*

$$D(\lambda) : (x_1, ..., x_n) \to (\lambda^{\alpha_1} x_1, ..., \lambda^{\alpha_n} x_n) \tag{3}$$

are group automorphisms, for all $\lambda > 0$. We will denote by $\mathbb{G}$ the space $\mathbb{R}^n$ with this structure of homogeneous group, and we will say that a constant depend on $\mathbb{G}$ if it depends on the numbers $n, \alpha_1, ..., \alpha_n$ and the group law $\circ$.

We say that a differential operator Y on $\mathbb{G}$ is *homogeneous of degree* $\beta > 0$ if

$$Y(f(D(\lambda)x)) = \lambda^\beta (Yf)(D(\lambda)(x))$$

for every test function $f, \lambda > 0$, $x \in \mathbb{R}^n$. Also, we say that a function f is homogeneous of degree $\alpha \in \mathbb{R}$ if

$$f(D(\lambda)x) = \lambda^\alpha f(x) \text{ for every } \lambda > 0, x \in \mathbb{R}^n \setminus \{0\}.$$

4

Clearly, if Y is a differential operator homogeneous of degree β and f is a homogeneous function of degree α, then Yf is homogeneous of degree $\alpha - \beta$.

We say that a differential operator Y on $\mathbb{G}$ is *left invariant* if for every smooth function $f : \mathbb{G} \to \mathbb{R}$,

$$Y\left(f\left(L_y\left(x\right)\right)\right) = \left(Yf\right)\left(y \circ x\right) \text{ for every } x, y \in \mathbb{G},$$
$$\text{where } L_y\left(x\right) = y \circ x.$$

Let us now consider the Lie algebra ℓ associated to the group $\mathbb{G}$, that is, the Lie algebra of left invariant vector fields on $\mathbb{G}$, endowed with the Lie bracket given by the commutator of vector fields: $[X, Y] = XY - YX$. We can fix a basis $X_1, ..., X_N$ in ℓ choosing X_i as the (unique) left invariant vector field which agrees with $\frac{\partial}{\partial x_i}$ at the origin. It turns out that X_i is homogeneous of degree α_i. Then, we can extend the dilations $D(\lambda)$ to ℓ setting

$$D(\lambda)X_i = \lambda^{\alpha_i} X_i.$$

$D\left(\lambda\right)$ becomes a Lie algebra automorphism, i.e.,

$$D(\lambda)[X, Y] = [D(\lambda)X, D(\lambda)Y].$$

In this sense, ℓ is said to be a *homogeneous Lie algebra*; as a consequence, ℓ is nilpotent.

We will assume that the first q vector fields $X_1, ..., X_q$ are 1-homogeneous and generate ℓ as a Lie algebra. In other words, $X_1, ..., X_q$ are a system of Hörmander's vector fields in $\mathbb{R}^n$: there exists a positive integer s, called the step of the Lie algebra, such that $X_1, ..., X_q$, together with their iterated commutators of length $\leq s$ span $\mathbb{R}^n$ at every point. Under these assumptions we say that ℓ is a stratified homogeneous Lie algebra and that $\mathbb{G}$ is a stratified homogeneous group, or briefly a *Carnot group*.

As any system of Hörmander's vector fields, $X_1, ..., X_q$ induce in $\mathbb{R}^n$ a distance d called *the control distance*. The explicit definition of d will never be used, hence we do not recall it (see [17]). Since $\mathbb{G}$ is a Carnot group, d turns out to be left invariant and 1-homogeneous, that is

$$d\left(x, y\right) = d\left(z \circ x, z \circ y\right)$$
$$d\left(D\left(\lambda\right)x, D\left(\lambda\right)y\right) = \lambda d\left(x, y\right)$$

for any $x, y, z \in \mathbb{G}$ and $\lambda > 0$. Then, if we set

$$\|x\| = d\left(x, 0\right),$$

it turns out that $\|\cdot\|$ is a *homogeneous norm*, satisfying the following properties:

(i) $\|D(\lambda)x\| = \lambda\|x\|$ for every $x \in \mathbb{R}^n, \lambda > 0$;

(ii) the function $x \mapsto \|x\|$ is continuous;

(iii) for every $x, y \in \mathbb{R}^n$

$$\|x \circ y\| \leq \|x\| + \|y\| \text{ and } \|x^{-1}\| = \|x\|;$$

(iv) there exists a constant $c \geq 1$ such that

$$\frac{1}{c} |y| \leq \|y\| \leq c |y|^{1/s} \text{ if } \|y\| \leq 1,$$

where s is the step of the Lie algebra.

Note that from (iii) we have that

$$\|y^{-1} \circ x\| \geq \|y\| - \|x\|. \tag{4}$$

We also define the balls with respect to d as

$$B(x, r) \equiv B_r(x) \equiv \{y \in \mathbb{R}^n : d(x, y) < r\},$$

and denote $B_r = B(0, r)$.

Note that $B(0, r) = D(r)B(0, 1)$. It can be proved that the Lebesgue measure in $\mathbb{R}^n$ is the Haar measure of $\mathbb{G}$ and

$$|B(x, r)| = |B(0, 1)| \, r^Q, \tag{5}$$

for every $x \in \mathbb{R}^n$ and $r > 0$, where

$$Q = \alpha_1 + ... + \alpha_n$$

with α_i as in (3). We will call Q the *homogeneous dimension* of $\mathbb{G}$.

2.2 Real analysis tools

We start noting that (5) in particular implies that the Lebesgue measure dx is a *doubling measure* with respect to d, and therefore $(\mathbb{R}^n, d, dx)$ is a *space of homogenous type* in the sense of Coifman-Weiss (see [11]).

In this context, for a given locally integrable function f, the Hardy-Littlewood maximal operator is given by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy, \tag{6}$$

where the supremum is taken over all the d-balls (containing the point x). By the general theory of spaces of homogeneous type, it is known that for every $p \in (1, \infty)$ there exists a constant $c > 0$ such that

$$\|Mf\|_{L^p(\mathbb{R}^n)} \leq c \|f\|_{L^p(\mathbb{R}^n)}. \tag{7}$$

Since we will study a differential operator defined on a bounded domain $\Omega \subset \mathbb{R}^n$ and we will prove interior estimates in Ω, a natural framework for the

real analysis tools we need is that of *locally homogeneous spaces*, as developed in [7] and [6]. We are going to introduce the minimum amount of definitions in order to apply this abstract theory in our concrete context. So, for a fixed bounded domain $\Omega \subset \mathbb{R}^n$, fix a strictly increasing sequence $\{\Omega_m\}_{m=1}^{\infty}$ of bounded domains such that

$$\bigcup_{m=1}^{\infty} \Omega_m = \Omega$$

and such that for any m there exists $\varepsilon_m > 0$ such that

$$\{x \in \Omega : d(x,y) < 2\varepsilon_m \text{ for some } y \in \Omega_m\} \subset \Omega_{m+1}$$

where d is, as above, the distance induced in $\mathbb{R}^n$ by the vector fields X_i. Then $(\Omega, \{\Omega_m\}_{m=1}^{\infty}, d, dx)$ (where dx stands for the Lebesgue measure) is a locally homogeneous space in the sense of [7].

With respect to this structure, we can define the *local sharp maximal operator*: for any function $f \in L^1_{loc}(\Omega_{m+1})$ and $x \in \Omega_m$, let

$$f^{\#}_{\Omega_m, \Omega_{m+1}}(x) = \sup_{\substack{B(\overline{x},r) \ni x \\ \overline{x} \in \Omega_m, r \leq \varepsilon_m}} \frac{1}{|B(\overline{x},r)|} \int_{B(\overline{x},r)} |f(y) - f_{B(\overline{x},r)}| \, dy, \qquad (8)$$

where $f_B = \frac{1}{|B|} \int_B f \, dx$.

Note that the supremum is taken over all the d-balls containing the point $x \in \Omega_m$ and having radius small enough so that the ball itself is contained in the larger set Ω_{m+1} where the function f is defined. Thus, we focus on the behavior of f on a bounded domain but on the other hand avoid the necessity of integrating over *restricted balls* $B(\overline{x}, r) \cap \Omega_{m+1}$. The continuity of the sharp maximal operator is contained in the next result:

Theorem 3 (Local Fefferman-Stein inequality, see [6, Corollary 3.9])
There exists $\delta \in (0,1)$ such that for any m and for every integer k large enough, the set Ω_m can be covered by a finite union of balls B_R of radii comparable to δ^k, such that for any such ball B_R and every f supported in B_R, with $f \in L^1(B_R)$, $\int_{B_R} f = 0$, and $f^{\#}_{\Omega_{m+2}, \Omega_{m+3}} \in L^p_{loc}(\Omega_{m+1})$ for some $p \in [1, \infty)$ one has

$$\|f\|_{L^p(B_R)} \leq c \left\| f^{\#}_{\Omega_{m+2}, \Omega_{m+3}} \right\|_{L^p(B_{\gamma R})}$$

with $\gamma > 1$ absolute constant and c only depending on p, the sets Ω_k and the constants ε_k for a finite number of indices k.

Let us also define the *local VMO spaces*. For a fixed Ω_m, $f \in L^1_{loc}(\Omega_{m+1})$ and $0 < r \leq \varepsilon_m$, let

$$\eta_{m,f}(r) = \sup_{x \in \Omega_m, \rho \leq r} \frac{1}{|B(x,\rho)|} \int_{B(x,\rho)} |f(y) - f_{B(x,\rho)}| \, dy.$$

We say that $f \in VMO_{loc}(\Omega_m, \Omega_{m+1})$ if $\eta_{m,f}(r) \to 0$ for $r \to 0^+$ and we say that a function $f \in L^1_{loc}(\Omega)$ belongs to $VMO_{loc}(\Omega)$ if

$$\eta_f(r) \equiv \sup_{x \in \Omega, \rho \leq r, B(x,\rho) \Subset \Omega} \frac{1}{|B(x,\rho)|} \int_{B(x,\rho)} |f(y) - f_{B(x,\rho)}| \, dy \to 0 \text{ for } r \to 0^+.$$

Note that the requirement $B(x,\rho) \Subset \Omega$ is meaningful because the distance d is define in the whole $\mathbb{R}^n$, not only in Ω. Observe that

$$VMO_{loc}(\Omega) \subset \bigcap_{m=1}^{\infty} VMO_{loc}(\Omega_m, \Omega_{m+1}). \tag{9}$$

2.3 Sobolev spaces and fundamental solutions

Let us introduce some useful notation. For $X_1, ..., X_q$ the vector fields as above and any multiindex $I = (i_1, ..., i_k)$ with $i_j \in \{1, 2, ..., q\}$ we set

$$X_I u = X_{i_1} X_{i_2} ... X_{i_k} u, \quad |I| = k.$$

We then define, for any positive integer k,

$$D^k u \equiv \sum_{|I|=k} |X_I u|.$$

(We will write Du instead of $D^1 u$). Here the X_i-derivatives are meant in classical or weak sense. For Ω a domain in $\mathbb{R}^n$ and $p \in [1, \infty]$ the space $W_X^{k,p}(\Omega)$ will consist of all $L^p(\Omega)$ functions such that

$$\|u\|_{W_X^{k,p}(\Omega)} = \sum_{h=0}^{k} \|D^h u\|_{L^p(\Omega)}$$

is finite (with $\|D^0 u\|_{L^p(\Omega)} = \|u\|_{L^p(\Omega)}$). We shall also denote by $W_{X,0}^{k,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W_X^{k,p}(\Omega)$. Note that the fields X_i, and therefore the definition of the above norms and spaces, are completely determined by the structure of $\mathbb{G}$.

A couple of standard facts about these Sobolev spaces on Carnot groups are the following:

Theorem 4 (Poincaré's inequality on stratified groups, see [13]) *Let $\mathbb{G}$ be a Carnot group with generators $X_1, ..., X_q$. For every $p \in [1, \infty)$ there exist constants $c > 0, \Lambda > 1$ such that for any ball $B = B(x_0, r)$ and any $u \in C^1(\overline{\Lambda B})$ (with $\Lambda B = B(x_0, \Lambda r)$) we have:*

$$\left(\frac{1}{|B|} \int_B |u(x) - u_B|^p \, dx\right)^{1/p} \leq cr \left(\frac{1}{|\Lambda B|} \int_{\Lambda B} |Du(x)|^p \, dx\right)^{1/p}.$$

Note that the constants c, Λ in the previous Poincaré's inequality are independent of r and x_0.

Proposition 5 (Interpolation inequality, see [3, Prop. 4.1]) *Let X be a left invariant vector field homogeneous of degree 1. Then for every $\varepsilon > 0$ and $u \in W_{X,0}^{2,p}(\mathbb{R}^n)$ with $p \in [1, \infty)$,*

$$\|Xu\|_{L^p} \leq \varepsilon \|X^2 u\|_{L^p} + \frac{2}{\varepsilon} \|u\|_{L^p}.$$

Let us now consider the class of *model operators*

$$\overline{L}u(x) = \sum_{i,j=1}^{q} \overline{a}_{ij} X_i X_j u(x) \tag{10}$$

where the matrix $\{\overline{a}_{ij}\}$ is constant, symmetric and satisfies the ellipticity condition: there exists $\mu > 0$ such that

$$\mu |\xi|^2 \leq \overline{a}_{ij} \xi_i \xi_j \leq \frac{1}{\mu} |\xi|^2 \tag{11}$$

for every $\xi \in \mathbb{R}^q$.

The operator $\overline{L}$ is a left invariant differential operator homogeneous of degree two on $\mathbb{G}$; it is easy to see that $\overline{L}$ can be rewritten in the form $\overline{L} = \sum_{i=1}^{q} Y_i^2$ where $Y_1, ..., Y_q$ are a different system of Hörmander' vector fields (for details, see [3, §2.4]); hence $\overline{L}$ is hypoelliptic, by Hörmander's theorem (see [16]). By general properties of Carnot groups, the formal transposed of X_i is $X_i^* = -X_i$; hence the transposed of $\overline{L}$ is still $\overline{L}$; in particular, both $\overline{L}$ and $\overline{L}^*$ are hypoelliptic. We can therefore apply the theory developed by Folland [12] about the fundamental solution of $\overline{L}$. The following theorem collects the properties we will need:

Theorem 6 (Homogeneous fundamental solution of $\overline{L}$) *The operator $\overline{L}$ has a unique global fundamental solution $\Gamma_{\overline{a}} \leq 0$ with pole at the origin wich is homogeneous of degree $2 - Q$ and such that:*

(a) $\Gamma_{\overline{a}} \in C^{\infty}(\mathbb{R}^n \setminus \{0\})$;

(b) for every $u \in C_0^{\infty}(\mathbb{R}^n)$ and every $x \in \mathbb{R}^n$,

$$u(x) = \overline{L}u * \Gamma_{\overline{a}}(x) = \int_{\mathbb{R}^n} \Gamma_{\overline{a}}(y^{-1} \circ x) \overline{L}u(y) dy;$$

(c) for every $f \in L^2(\mathbb{R}^n), f$ compactly supported, the function

$$u(x) = f * \Gamma_{\overline{a}}(x) = \int_{\mathbb{R}^n} \Gamma_{\overline{a}}(y^{-1} \circ x) f(y) dy$$

belongs to $W_X^{2,2}(\mathbb{R}^n)$ and solves the equation $\overline{L}u = f$ in $\mathbb{R}^n$.

We also need some uniform bound for $\Gamma_{\overline{a}}$, with respect to the constant matrix $\{\overline{a}_{ij}\}$ in a fixed ellipticity class. The next result is contained in [3, Thm. 12]:

Proposition 7 (Uniform estimate on $\Gamma_{\overline{a}}$) *There exists a positive constant, depending on $\{\overline{a}_{ij}\}$ only through the number μ, such that*

$$|\Gamma_{\overline{a}}(x)| \leq \frac{c}{\|x\|^{Q-2}} \text{ for every } x \in \mathbb{R}^n \setminus \{0\}.$$

Another key tool that we need from the general theory of Hörmander's operators is represented by the so-called subelliptic estimates. To formulate these, we need to recall the standard definition of (Euclidean, isotropic) fractional Sobolev spaces: for any $s \in \mathbb{R}$ the space H^s is defined as the set of functions (or tempered distributions) such that

$$\|u\|_{H^s}^2 = \int_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^s |\widehat{u}(\xi)|^2 \, d\xi$$

is finite, where $\widehat{u}(\xi)$ denotes the Fourier transform of u. Then:

Theorem 8 (Subelliptic estimates, see [14]) *There exists $\varepsilon > 0$, depending on the X_i and, for every $\eta, \eta_1 \in C_0^\infty(\mathbb{R}^n)$ with $\eta_1 = 1$ on $\operatorname{sprt} \eta$ and any $\sigma, \tau > 0$, there exists a constant c depending on $\sigma, \tau, \eta, \eta_1, X_i$ such that*

$$\|\eta u\|_{H^{\sigma+\varepsilon}} \leq c \left(\left\|\eta_1 \overline{L} u\right\|_{H^\sigma} + \|\eta_1 u\|_{H^{-\tau}}\right)$$

where $\overline{L}$ is like in (10). Moreover, the constant c depends on the coefficients $\overline{a}_{ij}$ only through the number μ.

Classical subelliptic estimates are proved for a *fixed* operator of Hörmander's type; however, the last statement about the dependence of c on the $\overline{a}_{ij}$ can be directly checked following the proof.

For the operator $\overline{L}$ we can give a standard definition of weak solution to a Dirichlet problem:

Definition 9 *Let Ω a bounded domain. Given two functions $f \in W_X^{1,2}(\Omega), g \in L^2(\Omega)$, we say that $u \in W_X^{1,2}(\Omega)$ is a weak solution to the Dirichlet problem*

$$\begin{cases} \overline{L}u = g & in \ \Omega \\ u = f & on \ \partial\Omega \end{cases} \tag{12}$$

if $u - f \in W_{X,0}^{1,2}(\Omega)$ and

$$-\int_\Omega \sum_{i,j=1}^q \overline{a}_{ij} X_j u X_i \varphi = \int_\Omega g\varphi \quad \forall \varphi \in C_0^\infty(\Omega).$$

The validity of Poincaré's inequality allows to prove in the standard way, by Lax-Milgram's Lemma, the *unique solvability of* (12). We stress the fact that, although the operator $\overline{L}$ is hypoelliptic, so that any distributional solution to $\overline{L}u = g$ is smooth in any open subset where g is smooth, the solvability of a Dirichlet problem in classical sense is not a trivial result for $\overline{L}$, but requires careful assumptions on the domain. Also, $W_X^{2,p}(\Omega)$ estimates up to the boundary are not known, so far, so that the Dirichlet problem is not even solvable in the sense of strong solutions. This is a major difference between the present context and that of elliptic and parabolic equations, in the application of Krylov' technique.

A maximum principle for weak solutions can be easily proved in the standard way. This requires some preliminary (standard) definition:

Definition 10 *For $u \in W_X^{1,2}(\Omega)$, we say that*

$$\overline{L}u \geq 0 \ \text{in} \ \Omega$$

in weak sense if

$$\int_\Omega \sum_{i,j=1}^q \overline{a}_{ij} X_j u X_i \varphi \leq 0 \quad \forall \varphi \in C_0^\infty(\Omega), \varphi \geq 0 \ \text{in} \ \Omega.$$

We say that

$$u \leq 0 \ \text{on} \ \partial\Omega$$

in weak sense if

$$\max(u,0) \in W_{X,0}^{1,2}(\Omega).$$

The following can be easily proved exactly like in the elliptic case:

Proposition 11 (Maximum Principle) *Let Ω an open set of $\mathbb{R}^n$. For any $u \in W_X^{1,2}(\Omega)$, if $\overline{L}u \geq 0$ in Ω and $u \leq 0$ on $\partial\Omega$ (in weak sense), then $u \leq 0$ in Ω a.e.*

3 Local estimates for the model operator

We start with several a priori estimates for the operator $\overline{L}$, defined as in (10) with *constant* $\{\overline{a}_{ij}\}$. The constants in our estimates will depend on this matrix only through the number μ. Recall that the operator $\overline{L}$, which in our context is the analog of the constant coefficient operator in the elliptic case, is hypoelliptic, 2-homogeneous and translation invariant on $\mathbb{G}$.

Lemma 12 *For any $u \in C^\infty(\mathbb{R}^n)$ and $R > 0$, let $h \in W_X^{1,2}(B_R)$ be the weak solution to*

$$\begin{cases} \overline{L}h = 0 & \text{in} \ B_R \\ h = u & \text{on} \ \partial B_R. \end{cases} \tag{13}$$

(Here B_R stands for $B_R(0)$). Then $h \in C^\infty(B_R)$ and if $R \geq 4\Lambda^2$, where from now on Λ is the constant appearing in Poincaré's inequality (Thm. 4), the following holds:

$$\sup_{B_1} |X_i X_j X_k h| \leq c \sum_{i,j=1}^q \|X_i X_j u\|_{L^1(B_R)} \tag{14}$$

for all $i, j, k = 1, ..., q$. The constant c only depends on $\mathbb{G}$ and μ. In particular it is independent of u.

Proof. Let $w \in W_X^{1,2}(B_R)$ be the unique weak solution to the Dirichlet problem

$$\begin{cases} \overline{L} w = -\overline{L} u & \text{in } B_R \\ w = 0 & \text{on } \partial B_R \end{cases}$$

and let $h = u + w$. Then h solves (13) and, since $\overline{L}$ is hypoelliptic in $\mathbb{R}^n$ and $-\overline{L} u \in C^\infty(B_R)$, $h \in C^\infty(B_R)$.

To prove (14), let us now assume $R \geq 4\Lambda^2$ (in particular, $R > 4$) and let us apply the subelliptic estimates (Thm. 8) with cutoff functions $\eta, \eta_1 \in C_0^\infty(B_2)$, $\eta_1 = 1$ in sprt η:

$$\|\eta h\|_{H^{\sigma+\epsilon}} \leq c \left\{ \|\eta_1 \overline{L} h\|_{H^\sigma} + \|\eta_1 h\|_{H^{-\tau}} \right\}.$$

Then since $\overline{L} h = 0$ in B_R, taking $\tau = 0$ and σ large enough we have

$$\sup_{B_1} |X_i X_j X_k h| \leq c \|\eta h\|_{H^{\sigma+\epsilon}} \leq c \|h\|_{L^2(B_2)},$$

where the first inequality follows by the classical Sobolev embedding theorems.

Then, it is enough to prove that

$$\|h\|_{L^2(B_2)} \leq c \sum_{i,j=1}^q \|X_i X_j u\|_{L^1(B_R)}. \tag{15}$$

Let $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\varphi(x) = 1$ if $\|x\| \geq 3.5$ and $\varphi(x) = 0$ if $\|x\| \leq 3$ and define

$$v = h - \varphi u.$$

Then $v \in C^\infty(B_R)$ and

$$\overline{L} v = \overline{L}(-\varphi u) = -\varphi \overline{L} u - u \overline{L} \varphi - 2 \sum_{i,j=1}^q \overline{a}_{ij} X_i \varphi \, X_j u =: -g.$$

Also, since $h - u \in W_{X,0}^{1,2}(B_R)$ and $\varphi = 1$ near ∂B_R, we have $v \in W_{X,0}^{1,2}(B_R)$.

On the other hand, for

$$f = \left(|\varphi \overline{L} u| + |u \overline{L} \varphi| + 2 \left| \sum_{i,j=1}^q \overline{a}_{ij} X_i \varphi \, X_j u \right| \right) \chi_{B_R}$$

12

defined in $\mathbb{R}^n$, and $\Gamma_{\overline{a}}$ the global homogeneous fundamental solution of $\overline{L}$, let

$$w(x) = -\int_{\mathbb{R}^n} \Gamma_{\overline{a}}(y^{-1} \circ x) f(y) dy.$$

Then $-\overline{L}w = f$ in strong sense (that is, $w \in W_X^{2,2}(B_R)$ and $-\overline{L}w(x) = f(x)$ for a.e. $x \in B_R$) and then also in the weak sense, and $w \geq 0$ in $\mathbb{R}^n$ (since both $-\Gamma_{\overline{a}}$ and f are nonnegative). Hence the functions v, w satisfy, in weak sense,

$$\begin{cases} \overline{L}(v - w) = f - g \geq 0 & \text{in } B_R \\ v - w \leq 0 & \text{on } \partial B_R \end{cases}$$

$$\begin{cases} \overline{L}(-v - w) = g + f \geq 0 & \text{in } B_R \\ -v - w \leq 0 & \text{on } \partial B_R \end{cases}$$

and since $|g| \leq f$, by the maximum principle (Proposition 11) we conclude $|v| \leq w$ in B_R.

Now for $x \in B_2$, since $\varphi(x) = 0$ if $\|x\| \leq 3$ and $f(x) \neq 0$ only for $3 \leq \|x\| \leq R$,

$$|h(x)| = |(v + \varphi u)(x)| = |v(x)| \leq w(x) = -\int_{\mathbb{R}^n} \Gamma_{\overline{a}}(y^{-1} \circ x) f(y) dy$$

$$= -\int_{B_R \setminus B_3} \Gamma_{\overline{a}}(y^{-1} \circ x) f(y) dy.$$

On the other hand, for $x \in B_2$ and $y \in B_R \setminus B_3$ the function $\Gamma_{\overline{a}}(y^{-1} \circ x)$ is bounded. Actually, by Proposition 7 and (4)

$$0 \leq -\Gamma_{\overline{a}}(y^{-1} \circ x) \leq \frac{c}{\|y^{-1} \circ x\|^{Q-2}} \leq \frac{c}{(\|y\| - \|x\|)^{Q-2}} \leq c.$$

Hence

$$|h(x)| \leq c\|f\|_1 \leq c \left\{ \|\overline{L}u\|_{L^1(B_R)} + \|u\|_{L^1(B_{3.5})} + \sum_{j=1}^q \|X_j u\|_{L^1(B_{3.5})} \right\}$$

which in particular gives

$$\|h\|_{L^1(B_2)} \leq c \left\{ \sum_{i,j=1}^q \|X_i X_j u\|_{L^1(B_R)} + \|u\|_{L^1(B_4)} + \sum_{j=1}^q \|X_j u\|_{L^1(B_4)} \right\}. \quad (16)$$

In order to prove (15) we should remove from the right-hand side of (16) the terms in u and $X_j u$. To this aim, let

$$\widetilde{u}(x) = u(x) + c_0 + \sum_{i=1}^q c_i x_i$$

13

for some constants c_i, $i = 0, 1, 2, ..., q$ that we can choose so that

$$\int_{B_4} \widetilde{u}(x)\, dx = 0$$

and

$$\int_{B_{4\Lambda}} X_i \widetilde{u}(x)\, dx = 0 \text{ for } i = 1, 2, ..., q.$$

Namely, since for $i = 1, 2, ..., q$ the vector fields X_i have the structure

$$X_i = \partial_{x_i} + \sum_{j=q+1}^{n} b_{ij}(x)\, \partial_{x_j},$$

so that $X_j \widetilde{u} = X_j u + c_j$, we can choose

$$c_i = -\frac{1}{|B_{4\Lambda}|} \int_{B_{4\Lambda}} X_i u(x)\, dx \text{ for } i = 1, 2, ..., q$$

and

$$c_0 = -\frac{1}{|B_4|} \left(\int_{B_4} u(x)\, dx + \sum_{i=1}^{q} c_i \int_{B_4} x_i dx \right).$$

For this choice of c_i, $i = 0, 1, 2, ..., q$ and $\widetilde{u}$, we can now repeat the above proof defining $\widetilde{h}$ as the solution to

$$\begin{cases} \overline{L}\widetilde{h} = 0 & \text{in } B_R \\ \widetilde{h} = \widetilde{u} & \text{on } \partial B_R \end{cases}$$

(with $R \geq 4\Lambda^2$ as before). Clearly, one simply has

$$\widetilde{h}(x) = h(x) + c_0 + \sum_{i=1}^{q} c_i x_i$$

and

$$\sup_{B_1} \left| X_i X_j X_k \widetilde{h} \right| \leq c \|\widetilde{h}\|_{L^2(B_2)}$$

$$\leq c \left\{ \sum_{i,j=1}^{q} \|X_i X_j \widetilde{u}\|_{L^1(B_R)} + \|\widetilde{u}\|_{L^1(B_4)} + \sum_{j=1}^{q} \|X_j \widetilde{u}\|_{L^1(B_4)} \right\}.$$

Next, note that $X_i X_j \widetilde{u} = X_i (X_j u + c_j) = X_i X_j u$ and by Poincaré's inequality (Thm. 4)

$$\|\widetilde{u}\|_{L^1(B_4)} + \sum_{j=1}^{q} \|X_j \widetilde{u}\|_{L^1(B_4)}$$

$$= \int_{B_4} |\widetilde{u}(x) - \widetilde{u}_{B_4}|\, dx + \sum_{j=1}^{q} \|X_j \widetilde{u}\|_{L^1(B_4)}$$

$$\leq c \sum_{i=1}^{q} \int_{B_{4\Lambda}} |X_i \widetilde{u}(x)|\, dx + \sum_{j=1}^{q} \|X_j \widetilde{u}\|_{L^1(B_{4\Lambda})}$$

14

$$= c \sum_{i=1}^{q} \int_{B_{4\Lambda}} |X_i \widetilde{u}(x) - X_i \widetilde{u}_{B_{4\Lambda}}| \, dx$$

$$\leq c \sum_{i,j=1}^{q} \int_{B_{4\Lambda^2}} |X_j X_i \widetilde{u}(x)| \, dx$$

$$= c \sum_{i,j=1}^{q} \int_{B_{4\Lambda^2}} |X_j X_i u(x)| \, dx.$$

Also, $X_i X_j X_k \widetilde{h} = X_i X_j X_k h$ hence

$$\sup_{B_1} |X_i X_j X_k h| \leq c \sum_{i,j=1}^{q} \|X_i X_j u\|_{L^1 \left(B_{4\Lambda^2} \right)} \leq c \sum_{i,j=1}^{q} \|X_i X_j u\|_{L^1(B_R)}$$

and we are done. ∎

Lemma 13 *For any $k \geq 4\Lambda^3$, $r > 0$, $u \in C^\infty(\mathbb{R}^n)$ and h the weak solution to*

$$\begin{cases} \overline{L}h = 0 & \text{in } B_{kr} \\ h = u & \text{on } \partial B_{kr} \end{cases}$$

we have that for $i, j = 1, 2, ..., q$

$$\frac{1}{|B_r|} \int_{B_r} |X_i X_j h(x) - (X_i X_j h)_{B_r}| dx \leq \frac{c}{k} \sum_{i,j=1}^{q} \frac{1}{|B_{kr}|} \int_{B_{kr}} |X_i X_j u(x)| dx, \quad (17)$$

where the constant c depends on $\mathbb{G}$ and μ, but is independent of k and r.

Proof. It is enough to prove the result for $r = 1$. Namely, if we define $\widetilde{h}(x) = h(D(r)(x))$ and $\widetilde{u}(x) = u(D(r)(x))$, using the 1-homogenety of X_i, by dilations we have

$$\frac{1}{|B_r|} \int_{B_r} |X_i X_j h(x)| dx = \frac{1}{r^Q |B_1|} \int_{B_1} |(X_i X_j h)(D_r(y))| r^Q dy$$

$$= \frac{1}{|B_1|} r^{-2} \int_{B_1} |X_i X_j \widetilde{h}(y)| dy$$

Analogously, we obtain

$$\frac{1}{|B_r|} \int_{B_r} |X_i X_j h(x) - (X_i X_j h)_{B_r}| dx = \frac{1}{|B_1|} r^{-2} \int_{B_1} |X_i X_j \widetilde{h}(y) - (X_i X_j \widetilde{h})_{B_1}| dy$$

and

$$\frac{1}{|B_{kr}|} \int_{B_{kr}} |X_i X_j u(x)| dx = \frac{1}{|B_k|} r^{-2} \int_{B_k} |X_i X_j \widetilde{u}(y)| dy$$

hence if the result holds for $r = 1$ it holds for every $r > 0$.

Now, for $k \geq 4\Lambda^3$, let $h \in W_X^{1,2}(B_k)$ satisfy

$$\begin{cases} \overline{L}h = 0 & \text{in } B_k \\ h = u & \text{on } \partial B_k. \end{cases} \tag{18}$$

Let us assume that for every $s, i, j = 1, ..., q$ and $x \in B_\Lambda$,

$$|X_s X_i X_j h(x)| \leq \frac{c}{k} \sum_{i,j}^q \frac{1}{|B_k|} \int_{B_k} |X_i X_j u(x)| dx \tag{19}$$

(with c independent of k) and let us prove (17) for $r = 1$.

By Theorem 4,

$$\frac{1}{|B_1|} \int_{B_1} |X_i X_j h(x) - (X_i X_j h)_{B_1}| dx \leq \frac{c}{|B_\Lambda|} \sum_{s=1}^q \int_{B_\Lambda} |X_s X_i X_j h(x)| dx$$

$$\leq c \sum_{s=1}^q \sup_{B_\Lambda} |X_s X_i X_j h|$$

$$\leq \frac{c}{k} \sum_{i,j}^q \frac{1}{|B_k|} \int_{B_k} |X_i X_j u(x)| dx,$$

which is the assertion for $r = 1$.

It remains to prove (19). To do that, for $x \in B_{4\Lambda^2}$ we define $\widetilde{h}(x) = h(D(k/4\Lambda^2)(x))$ and $\widetilde{u}(x) = u(D(k/4\Lambda^2)(x))$. Then $\overline{L}\widetilde{h} = 0$ in $B_{4\Lambda^2}$ with boundary condition $\widetilde{u}$ and we can apply Lemma 12, which jointly with dilations and homogenety gives for $x \in B_1$

$$\left(\frac{k}{4\Lambda^2}\right)^3 |(X_s X_i X_j h)(D(k/4\Lambda^2)(x))| = \left|X_s X_i X_j \widetilde{h}(x)\right|$$

$$\leq c \sum_{i,j=1}^q \int_{B_{4\Lambda^2}} |X_i X_j \widetilde{u}(x)| dx$$

$$= c |B_{4\Lambda^2}| \sum_{i,j=1}^q \frac{1}{|B_{4\Lambda^2}|} \int_{B_{4\Lambda^2}} |X_i X_j (u(D(k/4\Lambda^2)(x))| dx$$

$$= c \left(\frac{k}{4\Lambda^2}\right)^2 \sum_{i,j=1}^q \frac{1}{|B_{4\Lambda^2}|} \int_{B_{4\Lambda^2}} |(X_i X_j u)(D(k/4\Lambda^2)(x))| dx$$

$$= c \left(\frac{k}{4\Lambda^2}\right)^2 \sum_{i,j=1}^q \frac{1}{|B_k|} \int_{B_k} |X_i X_j u(y)| dy.$$

Hence, for $x \in B_1$,

$$\left|(X_s X_i X_j h)(D(k/4\Lambda^2)(x))\right| \leq \frac{c}{k} \sum_{i,j=1}^q \frac{1}{|B_k|} \int_{B_k} |X_i X_j u(x)| dx.$$

16

But, since x ranges in B_1, the point $y = D(k/4\Lambda^2)(x)$ ranges in $B_{k/4\Lambda^2} \supset B_\Lambda$ (because $k/4\Lambda^2 \geq \Lambda$) and the Lemma is proved. $\blacksquare$

The next Lemma can be of independent interest:

Lemma 14 *Let $p \in (1, \infty)$. There exists a constant c depending on $p, \mathbb{G}, \mu$ such that for any $r > 0, k \geq 2, v \in W^{1,2}_{X,0}(B_{kr})$ the following holds:*

$$\left\| D^2 v \right\|_{L^p(B_r)} \leq c k^2 \left\| \overline{L} v \right\|_{L^p(B_{kr})}.$$

Before proving this result, let us explain why it is not trivial. From the local estimates proved by Folland [12] it is known that for any $v \in W^{1,2}_X(B_{kr})$

$$\left\| D^2 v \right\|_{L^p(B_r)} \leq c \left(\left\| \overline{L} v \right\|_{L^p(B_{kr})} + \left\| D v \right\|_{L^p(B_{kr})} + \left\| v \right\|_{L^p(B_{kr})} \right).$$

Also, for $v \in C_0^\infty(B_r)$ one can prove

$$\left\| D^2 v \right\|_{L^p(B_r)} \leq c \left\| \overline{L} v \right\|_{L^p(B_r)}.$$

The nontrivial fact, in the subelliptic context (where L^p estimates up to the boundary are unknown), is removing the L^p norm of v from the right hand side under the weak vanishing condition $v \in W^{1,2}_0(B_{kr})$.

Proof. For any $\sigma \in (\frac{1}{2}, 1)$, we can construct (see [3] for details) a cutoff function $\varphi_\sigma \in C_0^\infty(\mathbb{R}^n)$ satisfying: $\varphi_\sigma = 1$ on $B_{\sigma r}$, sprt $\varphi_s \subset B_{\sigma' r}$, where $\sigma' = \frac{(1+\sigma)}{2}$,

$$|X_j \varphi_\sigma| \leq \frac{c}{(1-\sigma)r}$$

$$|X_i X_j \varphi_\sigma| \leq \frac{c}{(1-\sigma)^2 r^2}.$$

Let us define two cutoff functions φ_1, φ_2 corresponding to $\sigma_1 \in (\frac{1}{2}, 1), \sigma_2 = \sigma_1'$, and let $\sigma_3 = \sigma_2'$. We can apply Folland's local estimates for the model operator (see [12, Theorem 4.9]) to $v\varphi_1$, so that

$$\|X_i X_j(v\varphi_1)\|_{L^p(B_{\sigma_2 r})} \leq c \|\overline{L}(v\varphi_1)\|_{L^p(B_{\sigma_2 r})}.$$

Then, expanding the operator $\overline{L}(v\varphi_1)$, using the estimate for the derivatives of φ_1 and multiplying by $(1-\sigma_1)^2 r^2$ in both sides, we have

$$(1-\sigma_1)^2 r^2 \|X_i X_j v\|_{L^p(B_{\sigma_1 r})} \leq c r^2 \|\overline{L} v\|_{L^p(B_r)}$$
$$+ c(1-\sigma_1) r \|X_i v\|_{L^p(B_{\sigma_2 r})}$$
$$+ c \|v\|_{L^p(B_r)}. \tag{20}$$

In order to estimate $(1-\sigma_1) r \|X_i v\|_{L^p(B_{\sigma_2 r})}$, let us apply Proposition 5 to $v\varphi_2$. We have

$$\|X_i v\|_{L^p(B_{\sigma_2 r})} \leq \varepsilon \left\{ \|X_i^2 v\|_{L^p(B_{\sigma_3 r})} + \frac{1}{(1-\sigma_2) r} \|X_i v\|_{L^p(B_{\sigma_3 r})} \right.$$
$$\left. + \frac{1}{(1-\sigma_2)^2 r^2} \|v\|_{L^p(B_{\sigma_3 r})} \right\} + \frac{2}{\varepsilon} \|v\|_{L^p(B_{\sigma_3 r})}.$$

Now, taking $\varepsilon = (1 - \sigma_2)r\delta$ for some δ and using the fact that $\frac{1-\sigma}{1-\sigma'} = \frac{1}{2}$ we obtain

$$(1 - \sigma_2)r\|X_i v\|_{L^p(B_{\sigma_2 r})} \leq c\delta(1 - \sigma_3)^2 r^2 \|X_i^2 v\|_{L^p(B_{\sigma_3 r})}$$
$$+ c\delta(1 - \sigma_3)r\|X_i v\|_{L^p(B_{\sigma_3 r})}$$
$$+ c\delta\|v\|_{L^p(B_{\sigma_3 r})} + \frac{2}{\delta}\|v\|_{L^p(B_{\sigma_3 r})},$$

which, letting

$$\phi_k = \sup_{\sigma \in (\frac{1}{2}, 1)} (1 - \sigma)^k r^k \|D^k v\|_{L^p(B_{\sigma r})}$$

implies that

$$\phi_1 \leq c\delta(\phi_2 + \phi_1 + \|v\|_{L^p(B_r)}) + \frac{c}{\delta}\|v\|_{L^p(B_r)},$$

and taking δ small enough we have

$$\phi_1 \leq c\delta\phi_2 + C\|v\|_{L^p(B_r)}.$$

Finally, inserting this in (20) and taking the supremum on σ_1 we have

$$\phi_2 \leq cr^2\|\overline{L}v\|_{L^p(B_r)} + c\|v\|_{L^p(B_r)},$$

which can be read as

$$r^2\|X_i X_j v\|_{L^p(B_r)} \leq cr^2\|\overline{L}v\|_{L^p(B_{kr})} + c\|v\|_{L^p(B_{kr})} \qquad (21)$$

for $r > 0$, $k > 2$ and for some c depending on $p, \mathbb{G}, \mu$.

On the other hand, the function

$$w(x) = -\int_{B_{kr}} \Gamma_{\overline{a}}\left(x^{-1} \circ y\right) |f(y)| \, dy$$

solves

$$\begin{cases} \overline{L}w = -|f| \text{ in } B_{kr} \\ w \geq 0 \text{ on } \partial B_{kr} \end{cases}$$

and taking $f = \overline{L}v \cdot \chi_{B_{kr}}$, by the same reasoning of the proof of Lemma 12, the maximum principle implies $|v| \leq w$ in B_{kr}. Then, by Proposition 7

$$|v(x)| \leq w(x) \leq c \int_{B_{kr}} \frac{1}{\|x^{-1} \circ y\|^{Q-2}} |f(y)| dy$$
$$\leq c \sum_{s=0}^{\infty} \int_{\frac{2kr}{2^{s+1}} \leq \|x^{-1} \circ y\| \leq \frac{2kr}{2^s}} \frac{1}{\|x^{-1} \circ y\|^{Q-2}} |f(y)| dy$$
$$\leq c \sum_{s=0}^{\infty} \left(\frac{2^{s+1}}{2kr}\right)^{Q-2} \int_{\|x^{-1} \circ y\| \leq \frac{2kr}{2^s}} |f(y)| dy$$
$$\leq c(kr)^2 \sum_{s=0}^{\infty} \frac{1}{2^{2s}} Mf(x),$$

18

and by (7)
$$\|v\|_{L^p(B_{kr})} \le (kr)^2 \|\overline{L}v\|_{L^p(B_{kr})}$$
which inserted in (21) gives us the result. ∎

Lemma 15 *Let $p \in (1, \infty)$. Then there exists a constant c depending on $p, \mathbb{G}, \mu$ such that for $k \ge 4\Lambda^3$, $r > 0$ and $u \in C^\infty(\mathbb{R}^n)$*

$$\frac{1}{|B_r|} \int_{B_r} |X_i X_j u(x) - (X_i X_j u)_{B_r}| dx$$

$$\le \frac{c}{k} \sum_{i,j=1}^q \frac{1}{|B_{kr}|} \int_{B_{kr}} |X_i X_j u(x)| dx + ck^{2+Q/p} \left(\frac{1}{|B_{kr}|} \int_{B_{kr}} |\overline{L}u(x)|^p dx \right)^{1/p}.$$

Proof. For u and k as in the statement, let h be the solution to

$$\begin{cases} \overline{L}h = 0 & \text{in } B_{kr} \\ h = u & \text{on } \partial B_{kr}, \end{cases}$$

then

$$\frac{1}{|B_r|} \int_{B_r} |X_i X_j u(x) - (X_i X_j u)_{B_r}| dx \le \frac{1}{|B_r|} \int_{B_r} |X_i X_j u(x) - X_i X_j h(x)| dx$$

$$+ \frac{1}{|B_r|} \int_{B_r} |X_i X_j h(x) - (X_i X_j h)_{B_r}| dx$$

$$+ \frac{1}{|B_r|} \int_{B_r} |(X_i X_j h)_{B_r} - (X_i X_j u)_{B_r}| dx$$

$$\equiv A + B + C.$$

By Lemma 13 we have

$$B \le \frac{c}{k} \sum_{i,j=1}^q \frac{1}{|B_{kr}|} \int_{B_{kr}} |X_i X_j u(x)| dx.$$

As to C, since $(X_i X_j h)_{B_r} - (X_i X_j u)_{B_r} = (X_i X_j h - X_i X_j u)_{B_r}$ it is enough to estimate the term A.

Applying Lemma 14 to the weak solution v of the problem

$$\begin{cases} \overline{L}v = \overline{L}u & \text{in } B_{kr} \\ v = 0 & \text{on } \partial B_{kr} \end{cases}$$

we have

$$\|X_i X_j v\|_{L^p(B_r)} \le ck^2 \|\overline{L}v\|_{L^p(B_{kr})}.$$

Then, by Hölder inequality we obtain

$$\frac{1}{|B_r|}\int_{B_r}|X_iX_jv(x)|dx \le \left(\frac{1}{|B_r|}\int_{B_r}|X_iX_jv(x)|^pdx\right)^{1/p}$$

$$\le ck^2|B_r|^{-1/p}\left(\int_{B_{kr}}|\overline{L}v(x)|^pdx\right)^{1/p}$$

$$= ck^2|B_r|^{-1/p}\left(\int_{B_{kr}}|\overline{L}u(x)|^pdx\right)^{1/p}$$

$$= ck^{2+Q/p}\left(\frac{1}{|B_{kr}|}\int_{B_{kr}}|\overline{L}u(x)|^pdx\right)^{1/p}$$

and we are done. ■

4 Local estimates for operators with variable coefficients

Let us now come to study the operator L with variable $VMO_{loc}(\Omega)$ coefficients. The next theorem contains the key local estimate involving L.

For a fixed domain $\Omega_m \Subset \Omega_{m+1}$, let us cover Ω_m with a finite number of balls B_R with R small enough (R to be chosen later). In the following theorem B_R is one of these balls. The maximal operator and the local sharp maximal operator which appear in the statement are defined in (6) and (8) respectively. By the assumption (1) and the inclusion (9), if we define the VMO modulus of the coefficients a_{ij} as

$$a_{m,r}^\sharp = \sum_{i,j=1}^q \eta_{m,a_{ij}}(r), \tag{22}$$

we have

$$\sup_{r\le \varepsilon_m} a_{m,r}^\sharp < \infty \text{ and } \lim_{r\to 0^+} a_{m,r}^\sharp = 0.$$

Theorem 16 *Let $p, \alpha, \beta \in (1,\infty)$ with $\alpha^{-1} + \beta^{-1} = 1$ and $R \in (0,\infty)$. Then there exists a constant c depending on $p, \alpha, \mathbb{G}, \mu$ such that for any $u \in C_0^\infty(B_R)$ and $k \ge 4\Lambda^3$ we have*

$$(X_iX_ju)_{\Omega_{m+2},\Omega_{m+3}}^\#(x) \le \frac{c}{k}\sum_{i,j=1}^q M(X_iX_ju)(x) + ck^{2+Q/p}\left(M(|Lu|^p)(x)\right)^{1/p}$$

$$+ ck^{2+Q/p}\left(a_{m+2,R}^\sharp\right)^{1/\beta p}\left(M(|X_iX_ju|^{p\alpha})(x)\right)^{1/\alpha p}$$

for every $x \in B_R$, $R < \varepsilon_{m+2}$.

The choice of bounding the local sharp maximal function relative to the domains $\Omega_{m+2}, \Omega_{m+3}$ is just for consistence with Theorem 3. As will be apparent

from the proof, we could bound $(X_iX_ju)^{\#}_{\Omega_k,\Omega_{k+1}}$ for any desired value of the integer k.

Proof. Fix $k \geq 4\Lambda^3$, $r \in (0, \varepsilon_{m+2})$ and $\overline{x} \in B_R$. Let B_r be a ball containing $\overline{x}$. Let $\overline{L}$ be a constant coefficients operator corresponding to a constant matrix $\{\overline{a}_{ij}\}$ which will be chosen later, depending on the values of r and k, in the class of matrices satisfying (11). By Lemma 15 we have that

$$\frac{1}{|B_r|} \int_{B_r} |X_iX_ju(x) - (X_iX_ju)_{B_r}|dx$$

$$\leq \frac{c}{k} \sum_{i,j=1}^{q} \frac{1}{|B_{kr}|} \int_{B_{kr}} |X_iX_ju(x)|dx + ck^{2+Q/p} \left(\frac{1}{|B_{kr}|} \int_{B_{kr}} |\overline{L}u(x)|^p dx \right)^{1/p}$$

$$\equiv A + B. \tag{23}$$

To handle the term B, let us write

$$\left(\int_{B_{kr}} |\overline{L}u(x)|^p dx \right)^{1/p} \leq \left(\int_{B_{kr}} |\overline{L}u(x) - Lu(x)|^p dx \right)^{1/p} + \left(\int_{B_{kr}} |Lu(x)|^p dx \right)^{1/p} \tag{24}$$

with

$$\int_{B_{kr}} |\overline{L}u(x) - Lu(x)|^p dx \leq c \sum_{i,j=1}^{q} \int_{B_{kr} \cap B_R} |\overline{a}_{ij} - a_{ij}(x)|^p |X_iX_ju(x)|^p dx$$

$$\leq c \sum_{i,j=1}^{q} \left(\int_{B_{kr} \cap B_R} |\overline{a}_{ij} - a_{ij}(x)|^{p\beta} dx \right)^{1/\beta} \left(\int_{B_{kr} \cap B_R} |X_iX_ju(x)|^{p\alpha} dx \right)^{1/\alpha}$$

$$\equiv c \sum_{i,j=1}^{q} J_2^{1/\beta} \, J_1^{1/\alpha}. \tag{25}$$

We have

$$J_1 \leq \int_{B_{kr}} |X_iX_ju(x)|^{p\alpha} dx = c(kr)^Q \frac{1}{|B_{kr}|} \int_{B_{kr}} |X_iX_ju(x)|^{p\alpha} dx \tag{26}$$

and since the coeficients $\overline{a}_{ij}, a_{ij}$ are bounded by $1/\mu$ we also have

$$J_2 \leq \mu^{-\beta p + 1} \int_{B_{kr} \cap B_R} |a_{ij}(x) - \overline{a}_{ij}| \, dx.$$

We now choose a particular constant matrix $\{\overline{a}_{ij}\}$, depending on the values of r, k, as follows

$$\overline{a}_{ij} = \left\{ \begin{array}{ll} (a_{ij})_{B_R} & \text{if } kr \geq R \\ (a_{ij})_{B_{kr}} & \text{if } kr \leq R. \end{array} \right.$$

Then, if $kr \geq R$

$$J_2 \leq c \int_{B_R} |a_{ij}(x) - (a_{ij})_{B_R}| dx \leq c|B_R| a_R^{\sharp} \leq cR^Q a_R^{\sharp} \leq c(kr)^Q a_R^{\sharp} \tag{27}$$

while if $kr \le R$

$$J_2 \le c \int_{B_{kr}} |a_{ij}(x) - (a_{ij})_{B_{kr}}| dx \le c|B_{kr}|a_{kr}^\sharp \le c(kr)^Q a_R^\sharp \qquad (28)$$

where, here and in the rest of the proof, we write $a_R^\sharp$ for $a_{m+2,R}^\sharp$.

In any case, by (25), (26), (27) and (28) we obtain

$$\int_{B_{kr}} |\overline{L}u(x) - Lu(x)|^p dx \le c \sum_{i,j=1}^q \left((kr)^Q a_R^\sharp \right)^{1/\beta} \left((kr)^Q (|X_i X_j u|^{p\alpha})_{B_{kr}} \right)^{1/\alpha}$$

$$= c(kr)^Q (a_R^\sharp)^{1/\beta} \sum_{i,j=1}^q \left((|X_i X_j u|^{p\alpha})_{B_{kr}} \right)^{1/\alpha}$$

which inserted in (24) gives

$$\left(\frac{1}{|B_{kr}|} \int_{B_{kr}} |\overline{L}u(x)|^p dx \right)^{1/p} \le \left(\frac{1}{|B_{kr}|} \int_{B_{kr}} |Lu(x)|^p dx \right)^{1/p}$$

$$+ c(a_R^\sharp)^{1/\beta p} \sum_{i,j=1}^q \left(\frac{1}{|B_{kr}|} \int_{B_{kr}} |X_i X_j u(x)|^{p\alpha} dx \right)^{1/\alpha p} .$$

In turn, inserting this estimate in (23) we get

$$\frac{1}{|B_r|} \int_{B_r} |X_i X_j u(x) - (X_i X_j u)_{B_r}| dx$$

$$\le \frac{c}{k} \sum_{i,j=1}^q \frac{1}{|B_{kr}|} \int_{B_{kr}} |X_i X_j u(x)| dx + ck^{2+Q/p} \left(\frac{1}{|B_{kr}|} \int_{B_{kr}} |Lu(x)|^p dx \right)^{1/p}$$

$$+ ck^{2+Q/p} \left(a_R^\sharp \right)^{1/\beta p} \sum_{i,j=1}^q \left(\frac{1}{|B_{kr}|} \int_{B_{kr}} |X_i X_j u(x)|^{p\alpha} dx \right)^{1/\alpha p}$$

$$\le \frac{c}{k} \sum_{i,j=1}^q M(X_i X_j u)(\overline{x}) + ck^{2+Q/p} (M(|Lu|^p)(\overline{x}))^{1/p}$$

$$+ ck^{2+Q/p} (a_R^\#)^{1/\beta p} \sum_{i,j=1}^q (M(|X_i X_j u|^{p\alpha})(\overline{x}))^{1/\alpha p} .$$

Note that in this estimate the constant matrix does not appear any longer. The constants c are independent of k, r and the estimate holds for any $k \ge 4\Lambda^3$ and $r > 0$. We can then take the supremum with respect to $r \in (0, \varepsilon_{m+2})$, getting

$$(X_i X_j u)_{\Omega_{m+2}, \Omega_{m+3}}^\# (\overline{x}) \le \frac{N}{k} \sum_{i,j=1}^q M(X_i X_j u)(\overline{x})$$

$$+ Nk^{2+Q/p} \left\{ (M(|Lu|^p)(\overline{x}))^{1/p} + \left(a_{m+2,R}^\sharp \right)^{1/\beta p} (M(|X_i X_j u|^{p\alpha})(\overline{x}))^{1/\alpha p} \right\} .$$

22

■

We are now in position to give the

Proof of Theorem 2. Assume that B_R and $B_{\gamma R}$ are as in the statement of Theorem 3. Fix $p \in (1, \infty)$ and choose $\alpha, \beta, p_1 \in (1, \infty)$ such that $\alpha p_1 < p$ and $\alpha^{-1} + \beta^{-1} = 1$. Apply Theorem 16 to these α, β, p_1 and the ball $B_{\gamma R}$ (but with $u \in C_0^\infty (B_R)$) writing, for $x \in B_{\gamma R}$:

$$(X_i X_j u)^{\#}_{\Omega_{m+2}, \Omega_{m+3}} (x) \le \frac{c}{k} \sum_{i,j=1}^{q} M(X_i X_j u)(x) + ck^{2+Q/p_1} \left(M(|Lu|^{p_1})(x) \right)^{1/p_1}$$

$$+ ck^{2+Q/p_1} \left(a^{\sharp}_{m+2,\gamma R} \right)^{1/\beta p_1} \sum_{i,j=1}^{q} \left(M(|X_i X_j u|^{p_1 \alpha})(x) \right)^{1/\alpha p_1}.$$

Then, taking $L^p (B_{\gamma R})$ norms of both sides we get

$$\left\| (X_i X_j u)^{\#}_{\Omega_{m+2}, \Omega_{m+3}} \right\|_{L^p(B_{\gamma R})} \le \frac{c}{k} \sum_{i,j=1}^{q} \| M(X_i X_j u) \|_{L^p(B_{\gamma R})}$$

$$+ ck^{2+Q/p_1} \left(\int_{B_{\gamma R}} \left(M(|Lu|^{p_1})(x) \right)^{p/p_1} dx \right)^{1/p}$$

$$+ ck^{2+Q/p_1} \left(a^{\sharp}_{m+2,\gamma R} \right)^{1/\beta p_1} \sum_{i,j=1}^{q} \left(\int_{B_{\gamma R}} \left(M(|X_i X_j u|^{p_1 \alpha})(x) \right)^{p/\alpha p_1} dx \right)^{1/p}.$$

$$(29)$$

Note that, since $u \in C_0^\infty(B_R)$,

$$\int_{B_R} X_i X_j u\,(x)\, dx = 0.$$

This follows from the structure of the vector fields X_i in Carnot groups, since

$$X_i f = \sum_{j=1}^{n} b_{ij}(x)\, \partial_{x_j} f = \sum_{j=1}^{n} \partial_{x_j} \left(b_{ij}(x)\, f \right).$$

Hence we can apply Theorem 3 writing

$$\sum_{i,j=1}^{q} \| X_i X_j u \|_{L^p(B_R)} \le c \sum_{i,j=1}^{q} \left\| (X_i X_j u)^{\#}_{\Omega_{m+1}, \Omega_{m+2}} \right\|_{L^p(B_{\gamma R})}$$

applying the p, p/p_1 and $p/\alpha p_1$-maximal inequality (7) on the right hand side of (29) (recall that u is compactly supported in B_R):

$$\le \frac{c}{k} \sum_{i,j=1}^{q} \| X_i X_j u \|_{L^p(B_R)}$$

$$+ ck^{2+Q/p_1} \left\{ \| Lu \|_{L^p(B_R)} + \left(a^{\sharp}_{m+2,\gamma R} \right)^{1/\beta p_1} \sum_{i,j=1}^{q} \| X_i X_j u \|_{L^p(B_R)} \right\}.$$

23

Since this inequality holds for any $k \geq 4\Lambda^3$, we can now choose k so that $c/k < 1/2$, getting

$$\sum_{i,j=1}^{q} \|X_i X_j u\|_{L^p(B_R)} \leq c \|Lu\|_{L^p(B_R)} + c \left(a_{m+2,\gamma R}^{\sharp}\right)^{1/\beta p_1} \sum_{i,j=1}^{q} \|X_i X_j u\|_{L^p(B_R)}.$$

Finally, exploiting the VMO_{loc} assumption on the coefficients a_{ij} we can choose R small enough to have $c \left(a_{m+2,\gamma R}^{\sharp}\right)^{1/\beta p_1} < 1/2$, so that

$$\sum_{i,j=1}^{q} \|X_i X_j u\|_{L^p(B_R)} \leq c \|Lu\|_{L^p(B_R)}$$

and we are done. ∎

References

[1] S. Agmon, A. Douglis, L. Nirenberg: *Estimates near the boundary of solutions of elliptic partial differential equations under general boundary conditions. Part I,* Comm. Pure Applied Math., vol. 12, (1959), 623-727; *Part II,* Comm. Pure Applied Math. vol. 17, Issue 1, (1964), 35–92.

[2] A. Bonfiglioli, E. Lanconelli, F. Uguzzoni: *Stratified Lie groups and potential theory for their sub-Laplacians.* Springer Monographs in Mathematics. Springer, Berlin, 2007.

[3] M. Bramanti, L. Brandolini. *L^p-estimates for uniformly hypoelliptic operators with discontinuous coefficients on homogeneous groups.* Rend. Sem. Mat. dell'Univ. e del Politec. di Torino, Vol. 58, 4 (2000), 389–433.

[4] M. Bramanti, L. Brandolini: *L^p-estimates for nonvariational hypoelliptic operators with VMO coefficients.* Trans. Amer. Math. Soc. 352 (2000), no. 2, 781–822.

[5] M. Bramanti, M. C. Cerutti: *$W_p^{1,2}$-solvability for the Cauchy-Dirichlet problem for parabolic equations with VMO coefficients.* Comm. Partial Differential Equations 18 (1993), no. 9-10, 1735–1763.

[6] M. Bramanti, M. S. Fanciullo: *The local sharp maximal function and BMO in locally homogeneous spaces.* (2015) Preprint. http://arxiv.org/abs/1511.02384

[7] M. Bramanti, M. Zhu. *Local real analysis in locally homogeneous spaces.* Manuscripta Mathematica. 138, 477–528 (2012).

[8] A. P. Calderón, A. Zygmund: *On the existence of certain singular integrals.* Acta Math. 88, (1952). 85–139.

[9] F. Chiarenza, M. Frasca, P. Longo: $W^{1,2}$-solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients. Trans. Amer. Math. Soc. 336 (1993), no. 2, 841–853.

[10] R. R. Coifman, R. Rochberg, G. Weiss: Factorization theorems for Hardy spaces in several variables. Ann. of Math. (2) 103 (1976), no. 3, 611–635.

[11] R. Coifman; G. Weiss. Analyse harmonique non-commutative sur certains espaces homogènes. Lecture Notes in Mathematics 242, Springer-Verlag, Berlin-Heidelberg- New York 1971.

[12] G. B. Folland. Subelliptic estimates and function spaces on nilpotent Lie groups. Arkiv för Math. 13, (1975), 161-207.

[13] D. Jerison: The Poincaré inequality for vector fields satisfying Hörmander's condition. Duke Math. J. 53 (1986), no. 2, 503–523.

[14] J. J. Kohn: Pseudo-differential operators and hypoellipticity. Partial differential equations (Proc. Sympos. Pure Math., Vol. XXIII, Univ. California, Berkeley, Calif., 1971), pp. 61–69. Amer. Math. Soc., Providence, R.I., 1973.

[15] N. V. Krylov: Parabolic and Elliptic Equations with VMO coefficients. Comm. in P.D.E.s, 32 (2007), 453-475.

[16] L. Hörmander: Hypoelliptic second order differential equations. Acta Math. 119 1967 147–171.

[17] A. Nagel, E. M. Stein, S. Wainger: Balls and metrics defined by vector fields. I. Basic properties. Acta Math. 155 (1985), no. 1-2, 103–147.

[18] E. M. Stein: Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals. Princeton Univ. Press. Princeton, New Jersey 1993.

Dipartimento di Matematica
Politecnico di Milano
Via Bonardi 9, 20133 Milano, Italy
marco.bramanti@polimi.it

Instituto de Matematica Aplicada del Litoral (CONICET-UNL)
Departamento de Matematica (FIQ-UNL)
Santa Fe, Argentina
mtoschi@santafe-conicet.gov.ar