SIAM J. FINANCIAL MATH.
Vol. 6, pp. 900–918

On the Level-Slope-Curvature Effect in Yield Curves and Eventual Total Positivity*

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Abstract. Principal components analysis has become widely used in a variety of fields. In finance and, more specifically, in the theory of interest rate derivative modeling, its use has been pioneered by Litterman and Scheinkman [*J. Fixed Income*, 1 (1991), pp. 54–61]. Their key finding was that a few components explain most of the variance of treasury zero-coupon rates and that the first three eigenvectors represent level, slope, and curvature (LSC) changes on the curve. This result has been, since then, observed in various markets. Over the years, there have been several attempts at modeling correlation matrices displaying the observed effects as well as trying to understand what properties of those matrices are responsible for them. Using recent results of the theory of total positiveness [O. Kushel, *Matrices with Totally Positive Powers and Their Generalizations*, 2014], we characterize these matrices and, as an application, we shed light on the critique to the methodology raised by Lekkos [*J. Derivatives*, 8 (2000), pp. 72–83].

Key words. interest rate models, principal components analysis, totally positive matrices

AMS subject classifications. 91B70, 91B24, 91B25, 91G30, 91G70

DOI. 10.1137/140998354

1. Introduction.

Ever since Litterman and Scheinkman observed that the eigenstructure of the correlation matrix of treasury rates had some particular characteristics [15], the same analysis has been replicated by various people in different markets.

Specifically, the first three eigenvalues of the correlation matrices of time changes of interest rates or, in the same spirit, returns of commodity futures are simple, and their corresponding eigenvectors represent level, slope, and curvature (LSC) moves on the curves. Moreover, these three eigenvalues explain most of the variance that the whole curve carries. This result is one of the most widely used facts in the modeling of interest rates either for derivative pricing and hedging or for risk management. For example, thanks to the Heath–Jarrow–Morton approach [9, 10], which dates from around the same time as Litterman and Scheinkman's work, one could use these three vectors to build the dynamics of yield curves or commodity futures curves.

The fact that LSC changes can be replicated for different markets motivated a fair amount of work during the past decade. For example, in [7], the authors proved that in the case in which correlation between maturities s and t decays exponentially in $|s-t|$ ($\rho^{|s-t|}$, $0<\rho<1$)

*Received by the editors December 2, 2014; accepted for publication (in revised form) July 2, 2015; published electronically September 10, 2015.
http://www.siam.org/journals/sifin/6/99835.html

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the eigenvectors approach cosine functions as ρ approaches 1. The result ignited some interest in finding what properties are shared by all correlations coming from yield curves [16, 19].

Lord and Pelsser [16] made a step forward in this direction, and they gave for the first time a precise definition of LSC that involves only the changes of signs of the eigenvectors of the correlation matrices instead of looking at their shapes. In the same work they gave sufficient conditions for matrices to satisfy and using the theory of total positivity (see for example [18]) they found sufficient conditions that guarantee the LSC property for a given matrix. In their definition, a matrix satisfies the LSC property if the first three eigenvectors have zero, one, and two sign changes, respectively. A sufficient condition that guarantees LSC turns out to be a generalization of total positivity, and they show through various examples (see also [19]) that the reverse is not true. In other words, one can find matrices satisfying the LSC property that violate total positiveness. The crucial ingredient in order to prove these kinds of results is the theorem of Perron and Frobenius (see, for example, [18]). Interestingly, as is proved in [17] and [13] (see also Lemma 2.1 in [8]), a certain converse of the theorem of Perron and Frobenius is true. Nevertheless, the converse uncovers a condition that is stronger that the definition of LSC given by Lord and Pelsser, but, as we will argue later in this paper, is a condition that agrees more with the intuition behind LSC analysis. Because of that, we give a new definition of level-slope-curvature (Definition 3.19) that involves the whole behavior of the eigenvectors as opposed to relying solely on their zero-crossings.

Aside from showing that the eigenstructure of yield curve correlations can be expressed in terms of the total positiveness of the matrices, it is our interest to bring attention to Lekkos's critique of the methodology. Armed with the mentioned result, we can prove that pretty much any matrix that comes from a *curve market* (yield cures, commodity futures) will satisfy the LSC property. In [14], Lekkos showed this with an example and by looking at empirical data. In this work we present a more general result that includes Lekkos's examples. His critique, as far as we can see, has been almost completely ignored by practitioners and in the literature.

It is important to point out that it is not only in the derivatives world that the LSC effect has been widely analyzed and used to build models and measure risks. In finance and economics the related literature is quite extensive. To cite just a few recent examples, in [11], the authors quantify to what extent the variation in economic activity and inflation in the United States influences the market prices of level, slope, and curvature risks in Treasury markets, and in [1] the notion of LSC is generalized to account for various yield curves (those of Canada, the United Kingdom, Japan, Germany, Australia, New Zealand, and Switzerland).

The paper is organized as follows. In section 2 we briefly show how the LSC effect is used in the context of term structure modeling. In section 3 we present the needed results from the theory of total positiveness and define what we mean by LSC (which is not just based on the number of zero-crossings). In addition we present the characterization of the matrices satisfying the property. Section 4 applies the theory to the parametric model studied in [7], whereas section 5 reframes Lekkos' critique in terms of the concepts introduced in section 3. Finally, section 6 concludes with a few remarks.

2. LSC and term structure modeling. The phenomenon called LSC has attracted a lot of attention for the past couple of decades. Very loosely defined, LSC refers to the fact that, when we take a yield curve (or commodity forward curve) and we diagonalize the increments (or returns), the first three eigenvectors account for almost all the variance, and their shapes resemble shifts in level, slope, and curvature, respectively. There is a clear intuition behind these type of shifts: a shock in the short rate will most likely produce a shift on the curve (*level*); however, it is clear that the move will not be perfectly parallel, and a decreasing or increasing adjustment with respect to the first shock effect will be needed. This second effect is called *slope*, and it corresponds to the second eigenvector. The same reasoning (with respect to the second eigenvector) carries through with the next eigenvector, called *curvature*.

This description has made its way into the derivative modeling field, risk management, as well as macro-finance models. In the world of derivative modeling the combination of such a strong result and the interest rate term structure framework pioneered by Heath, Jarrow, and A. Morton [9, 10] has proved to be quite successful. Up until the point when the Heath–Jarrow–Morton (HJM) framework was proposed, term structure models contained either one or two factors, but in the case in which lower correlations were needed, it was not clear how to accommodate further sources of risk. As we have said before, Litterman and Scheinkman's results can be replicated in commodity markets. The first example of one such study was in the case of copper futures, and it was done by Cortazar and Schwartz [5].

We show now how HJM "+" LSC work in this case (more details can be found in [7]). Suppose that we want to model the curve of a certain commodity, and we denote by $F(t, T_k)$ the futures price at time t of the underlying commodity expiring at time T_k, $k = 1, \dots, m$. Here, t represents the running time, with the first futures contract expiring at a later time ($t < T_1 < \cdots < T_m$). The dynamics of the whole curve, in the risk-neutral world, can be written as

$$\frac{dF(t, T_k)}{F(t, T_k)} = \sum_{i=1}^{m} \sigma_i(t, T_k) dW^i(t), \quad 1 \le k \le m, \tag{2.1}$$

where $W^1, \dots, W^m$ are m independent Brownian motions under the equivalent martingale measure and $\sigma_i(t, T_k)$ are volatility functions satisfying the technical conditions specified in [10]. In particular, the volatility functions need to satisfy:

$$\int_0^{T_k} \sigma_i^2(t, T_k) dt < 0$$

to guarantee the integrability with respect to the Brownian motion.

For the applications that we have in mind the volatility functions will be determined empirically. In order to fix them, we analyze the eigenstructure of the correlation matrix corresponding to the historical returns of the futures contracts. This methodology will ultimately allow us to capture the variance of the curve with the minimum number of factors (which leads to a less computationally intensive model). With this objective in mind, we impose the following factor-analytic structure on the covariance in (2.1):

$$\sigma_i(t, T_k) = \sigma_k \sqrt{\lambda_i} V_i(T_k - t).$$

Here σ_k is the same across the m factors (and it represents the implied volatility for F_k), and $\lambda_1, \dots, \lambda_m, V_1, \dots, V_m$ are the eigenvalues and eigenvectors of the correlation matrix of the returns of the $F(t,T_k)$'s. The advantage of this approach is that, in most cases, one finds that a lot of the λ's are small, so that by discarding those, we achieve dimensionality reduction.

We can then rewrite (2.1) as

$$\frac{dF(t,T_k)}{F(t,T_k)} = \sigma_k \sum_{i=1}^{m} \sqrt{\lambda_i} V_i(T_k - t) dW^i(t), \tag{2.2}$$

which leaves F's log-normally distributed over any time step Δt, with volatility

$$\sqrt{\int_t^{t+\Delta t} \sigma_k^2 \, ds} = \sigma_k \sqrt{\Delta t}. \tag{2.3}$$

All the usual caveats apply when calibrating (2.3) to the market. In the case of a flat term structure of volatilities of the kth contract, we need only one implied volatility per contract to calibrate the model. For nonflat, including seasonal, term structures of volatilities we need to either observe the term structure for each individual contract or come up with reasonable assumptions about the volatility surface.

As we can see, in order to specify the model in (2.2) we need to fix m and, also, the scalars λ_i and the vectors V_i for $1 \le i \le m$. It is here where the spectral structure (LSC) is used. In other words, given that, say, three factors explain most of the variance, we could fix $m = 3$ and take the first three eigenvalues and eigenvectors of the corresponding correlation matrix as the λ_i and V_i, $1 \le i \le m$.

3. LSC and totally positive matrices.

To the best of our knowledge, only Lord and Pelsser gave a proper definition of what LSC means. Up to that point, definitions were loose. Their definition states that a correlation matrix has *level* if the first eigenvector can be chosen to have only positive components, has *slope* if the second eigenvector crosses zero once, and has *curvature* when the third eigenvector crosses zero twice.

More specifically, given a vector $x \in \mathbb{R}^N$, they define the number of sign changes as follows:

1. $S^-(x)$ = number of sign changes in $x_1, \dots, x_N$ with zero terms discarded.
2. $S^+(x)$ = maximum number of sign changes in $x_1, \dots, x_N$ with zero terms arbitrarily assigned the values $+1$ and -1.

The definition of the LSC effect that Lord and Pelsser provided is the following.

Definition 3.1. *$A \in \mathbb{R}^{N\times N}$ is said to have the LSC ($A \in LP_3$) property if its first three eigenvectors x_1, x_2, x_3 satisfy:*

1. $S^-(x_1) = 0$.
2. $S^-(x_2) = 1$.
3. $S^-(x_3) = 2$.

In this definition the fact that the first three eigenvalues are simple is understood.

Analogously, we will say that such a matrix has the slope property ($A \in LP_2$) if the first two conditions hold in Definition 3.1.

Our first goal is to motivate an alternative definition of *slope* and *curvature* that makes a lot more sense in the context of our application, the modeling of yield curves (or commodity forward curves). Before that, we review the *level* definition and a characterization in terms of a property of the matrix.

3.1. Eventual positiveness, level. This section is meant to be a remainder of the Perron–Frobenius theorem. We will make heavy use of this theorem to understand the first eigenvector of yield-curve correlation matrices, and, moreover, it will serve us well to study the behavior of the subsequent eigenvectors (which we will do in the next section).

Definition 3.2. *A matrix $A \in \mathbb{R}^{N\times N}$ is positive ($A > 0$) if all elements of the matrix are greater than* 0*.*

Definition 3.3. *A matrix $A \in \mathbb{R}^{N\times N}$ is said to be* eventually positive *if there exists a positive integer k_0 such that $A^k > 0$ for all $k \geq k_0$.*

Definition 3.4. *A matrix $A \in \mathbb{R}^{N\times N}$ satisfies the strong Perron–Frobenius property if it has a unique, positive, dominant in absolute value eigenvalue λ_1 and the corresponding eigenvector can be chosen to have positive entries.*

A crucial result relating positiveness and the nature of the spectral structure of matrices is the celebrated Perron–Forbenius theorem (see, for example, [17]), given next.

Theorem 3.5. *For a symmetric matrix $A \in \mathbb{R}^{N\times N}$ the following properties are equivalent:*

1. *A satisfies the strong Perron–Frobenius property.*
2. *A is eventually positive.*

So, Perron–Frobenius (and its converse) takes care of the *level* case. Eventual positiveness is equivalent to having a positive first eigenvector (plus the fact that the corresponding eigenvalue is positive and simple). In order for us to be able to say something about the subsequent eigenvectors we need to introduce the notion of compound matrices.

3.2. Slope, curvature, and higher orders. As we have just said, to get a deeper understanding of the eigenstructure we need to study further some properties of the corresponding matrices. We will now introduce new matrices (compound matrices) that are related to the minors of the original ones. For example, the 2nd compound matrix of A will be denoted by $A^{(2)}$, and it will consist of the minors of A. The reason for this construct is that the eigenvectors and eigenvalues of the resulting compound matrices are intimately related to those of A (Kronecker's theorem). This fact together with the Perron–Frobenius property will be enough to understand the rest of the spectral structure of A.

Definition 3.6. *Let $x_1, \ldots, x_j$ $(2 \leq j \leq N)$ be any vectors in $\mathbb{R}^N$ defined by their coordinates: $x_i = (x_i^1, \ldots, x_i^N)$, $i = 1, \ldots, j$. Then the vector $x_1 \wedge \cdots \wedge x_j \in \mathbb{R}^{\binom{N}{j}}$ with coordinates of the form*

$$(x_1 \wedge \cdots \wedge x_j)^{\mathbf{l}} := \det \begin{pmatrix} x_1^{i_1} & \cdots & x_j^{i_1} \\ \cdots & \cdots & \cdots \\ x_1^{i_j} & \cdots & x_j^{i_j} \end{pmatrix},$$

where $\mathbf{l} = (i_1, \ldots, i_j) \subseteq [N]$ in the lexicographic ordering is called an exterior product of $x_1, \ldots, x_j$.

Definition 3.7. *Given $N \in \mathbb{N}$, we define $I_{p,N}$ as the set of all p order tuples from $(1, \ldots, N)$. In other words: $I_{p,N} = \{\mathbf{i} = (i_1, \ldots, i_p) | 1 \leq i_1 < \cdots < i_p \leq N\}$.*

Definition 3.8. *The pth compound matrix of A ($A^{(p)}$) is the square matrix of dimension $\binom{N}{p}$ containing all the p-minors (the determinants of all the matrices obtained by choosing $\mathbf{i} \in I_{p,N}$ in lexicographic order).*

Example 3.9. Consider the following matrix in $\mathbb{R}^{3\times3}$:

$$M_\rho = \begin{pmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{pmatrix}.$$

If we denote by $\det(M([i,j][k,l])$ the determinant of the 2×2 matrix formed with rows i and j and columns k and l, then the 2nd compound $M_\rho^{(2)}$is defined as

$$M_\rho^{(2)} = \begin{pmatrix} \det(M([1,2][1,2]) & \det(M([1,2][1,3]) & \det(M([1,2][2,3]) \\ \det(M([1,3][1,2]) & \det(M([1,3][1,3]) & \det(M([1,3][2,3]) \\ \det(M([2,3][1,2]) & \det(M([2,3][1,3]) & \det(M([2,3][2,3]) \end{pmatrix}$$
$$= \begin{pmatrix} 1-\rho^2 & \rho-\rho^3 & 0 \\ \rho-\rho^3 & 1-\rho^4 & \rho-\rho^3 \\ 0 & \rho-\rho^3 & 1-\rho^2 \end{pmatrix}.$$

We can now ask ourselves the question that opened this section: would the eigenvectors and eigenvalues of the compound matrices $M_\rho^{(2)}$ bear any relationship with the eigenvectors and eigenvalues of the original matrix M_ρ? The answer to this question is at the heart of the following theorem due to Kronecker (see, for example [13]).

Theorem 3.10 (Kronecker). *Let $\{\lambda_i\}_{i=1}^N$ be the set of all eigenvalues of an $N\times N$ matrix A, repeated according to multiplicity. Then all the possible products of the form $\lambda_{i_1},\dots,\lambda_{i_j}$, where $1\le i_1<\cdots<i_j\le N$, forms the set of all the possible eigenvalues of the jth compound matrix $A^{(j)}$, repeated according to multiplicity. If $x_{i_1},\dots,x_{i_j}$ are linearly independent eigenvectors of A, corresponding to eigenvalues $\lambda_{i_1},\dots,\lambda_{i_j}$, respectively, then their exterior product $x_{i_1}\wedge\cdots\wedge x_{i_j}$ is an eigenvector of $A^{(j)}$, corresponding to eigenvalue $\lambda_{i_1},\dots,\lambda_{i_j}$.*

As we have anticipated, Kronecker's theorem relates the eigenvalues and eigenvectors of a matrix to those of its compounds. To make use of it we will introduce the notion of total positiveness [18, 6], which means positiveness for the compound matrices.

Definition 3.11. *A matrix $A\in\mathbb{R}^{N\times N}$ is totally positive of order k (STP$_k$) if $A^{(p)}$ is positive for all $p\le k$.*

Definition 3.12. *A matrix $A\in\mathbb{R}^{N\times N}$ is eventually totally positive of order k (ESTP$_k$) if $A^{(p)}$ is eventually positive for all $p\le k$.*

Definition 3.13. *$A\in\mathbb{R}^{N\times N}$ is said to have the Gantmacher–Krein property of order k (GK$_k$) if it has at least k positive simple eigenvalues satisfying $\lambda_1>\lambda_2>\cdots>\lambda_k>|\lambda_{k+1}|\ge\cdots\ge|\lambda_N|$ and if the eigenvectors corresponding to the top eigenvalues $\lambda_1,\dots,\lambda_k$ $(x_1,\dots,x_k)$ satisfy that for any $1\le j\le k$,*

$$S^+\left(\sum_i^j c_i x_i\right) \le j-1 \qquad (3.1)$$

for any $c_1,\dots,c_j$, where at least one of them is not zero, where $S^+(\sum_i^j c_i x_i)$ is the maximum number of sign changes in the vector $\sum_i^j c_i x_i$ where zero coordinates are arbitrarily assigned values ±1.

Remark 3.14. For $k = 1$, (3.1) states that the first eigenvector is strictly positive or strictly negative. For $k > 1$, (3.1) implies that the eigenvectors cross zero the right number of times. However, the condition is stronger than number of crossings, as we will see in the following example.

Example 3.15. Let us consider the matrix

$$M_2 = \begin{pmatrix} 1 & .9 & 0.8 & 0.7 \\ 0.9 & 1 & 0.8 & 0.6 \\ 0.8 & 0.8 & 1 & 0.9 \\ 0.7 & 0.6 & 0.9 & 1 \end{pmatrix},$$

which appears as Example 6 in [19]. Its eigenvalues are

$$\lambda_1 = 3.35, \quad \lambda_2 = 0.485, \quad \lambda_3 = 0.122, \quad \lambda_4 = 0.0387,$$

and the corresponding eigenvectors are

$$v_1 = [0.5076, 0.4929, 0.5221, 0.4762]', \quad v_2 = [-0.3845, -0.5599, 0.2857, 0.6761]',$$
$$v_3 = [0.7032, -0.4351, -0.5026, 0.2519]', \quad v_4 = [-0.3163, 0.5042, -0.6270, 0.5026]'.$$

As we can see, M_2 satisfies the LSC property in the sense of Lord and Pelsser (level, slope, and curvature: first eigenvector does not cross 0, second eigenvector crosses 0 once, and third eigenvector crosses 0 twice). However, M_2 violate the GK_2 property if we take, for example, $c_1 = c_2 = 1$, since in this case $c_1v_1 + c_2v_2 = [0.1231, -0.067, 0.8078, 1.1523]$ that changes signs twice. Note that $S^-(c_1v_1 + c_2v_2) = 2$.

Since the case GK_1 is totally characterized by eventually positive matrices, the open question is whether there is a similar characterization for GK_k for any k. The answer is given in the next theorem, which is a generalization of the Theorem 7 in [13]. There are two reasons why we want to generalize the result in Theorem 7 in [13]. On one hand, as it was stated in the introduction, this result will provide an answer to the question posed by Lord and Pelsser; i.e., it will characterize the family of matrices satisfying LSC (with a new definition, which we will call strong LSC), and, on the other hand, it will strengthen the case made by Lekkos [14].

For the following lemma, which we use in the proof of Theorem 3.17, we refer the reader to [3, Lemma 5.1].

Lemma 3.16 (Ando's lemma). *Let $x_1, \dots, x_j$ be real vectors in $\mathbb{R}^N$, $j < N$. In order that*

$$S^+\left(\sum_i^j c_i x_i\right) \le j - 1$$

whenever $c_i \in \mathbb{R}$, where at least one of them is not zero, it is necessary and sufficient that $x_1 \wedge \cdots \wedge x_j$ be strictly positive or strictly negative.

We now state and prove our main result as follows.

Theorem 3.17. *If $\Sigma \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ is symmetric, then the following statements are equivalent:*

1. $\Sigma \in ESTP_k$.
2. $\Sigma \in GK_k$.

Proof. If $\Sigma \in ESTP_k$, then all of the $\Sigma^{(j)}$ are eventually positive for $j = 1, \ldots, k$. Now, since Σ is eventually positive, it has the strong Perron–Frobenius property; i.e., there is a simple, positive eigenvalue λ_1 which is bigger than the rest of them. The same is true for $\Sigma^{(2)}$; it has a simple, positive eigenvalue strictly bigger than the remaining ones. Now, according to Kronecker's theorem, the eigenvalues of $\Sigma^{(2)}$ are exactly given by the products $\lambda_{i_1}\lambda_{i_2}$, $1 \le i_1 < i_2 \le N$, where $\lambda_{i_1}, \lambda_{i_2}$ are eigenvalues of Σ. Therefore the highest eigenvalue of the compound matrix must be $\lambda_1\lambda_2$. From here we get that $\lambda_2 > 0$. Iteratively, we can use the same argument to find $\lambda_1, \ldots, \lambda_k$ positive simple eigenvalues higher than the remaining $N - k$ of them. This proves that the first k eigenvalues are simple and positive. Let us now take j satisfying $1 \le j \le k$. The first eigenvector of $\Sigma^{(j)}$ (with eigenvalue $\lambda_1, \ldots, \lambda_j$) is given by $x_1 \wedge \cdots \wedge x_j$. Since $\Sigma \in ESTP_k$, then $x_1 \wedge \cdots \wedge x_j$ can be taken to be positive. By Ando's lemma we conclude that for any $c_1, \ldots, c_j$ so that $\sum_{i=1}^{j} c_i^2 \neq 0$, (3.1) is satisfied.

Now, let us see the reverse. Assume that $\Sigma \in GK_k$. Then $\lambda_1 > \lambda_2 > \cdots > \lambda_k > |\lambda_{k+1}| \ge |\lambda_{k+2}| \ge \cdots \ge 0$, and the corresponding $x_1, \ldots, x_k$ satisfy (3.1) for any $1 \le j \le k$ and any $c_1, \ldots, c_j$ so that $\sum_{i=1}^{j} c_i^2 \neq 0$.

If we now fix $j \in 1, \ldots, k$ and we take a look at $\Sigma^{(j)}$, we see that, due to Kronecker's theorem, the first eigenvalue of $\Sigma^{(j)}$ is $\lambda_1, \ldots, \lambda_j$. This eigenvalue is simple, positive, and strictly dominant with eigenvector $x_1 \wedge \cdots \wedge x_j$. The result then follows, applying Ando's lemma again. ■

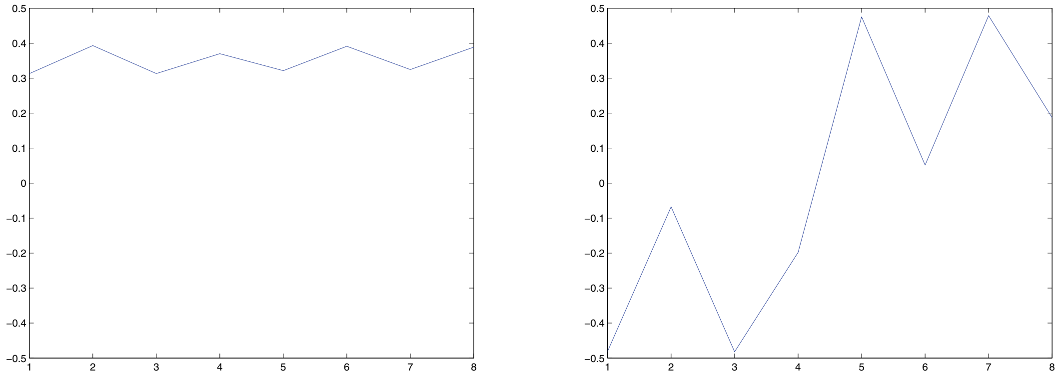
3.3. An example from Lord and Pelsser. As Lord and Pelsser point out, total positivity does not provide a characterization of LSC. We have already mentioned this fact in the previous section and showed what information is missing in Example 6 from [19]. For the sake of completeness we will now do likewise for a matrix that appears in [16]. The matrix M_1 is not $ESTP_2$; its second eigenvector does, however, cross zero only once (moreover, every one of its eigenvectors crosses zero $j - 1$ times, where j is the order of the eigenvector):

$$M_1 = \begin{pmatrix} 1 & 0.8396 & 0.8297 & 0.8204 \\ 0.8396 & 1 & 0.9695 & 0.901 \\ 0.8297 & 0.9695 & 1 & 0.9785 \\ 0.8204 & 0.901 & 0.9785 & 1 \end{pmatrix}.$$

Let us look at the first two eigenvectors: $v1 = [0.4736, 0.5057, 0.5152, 0.5045]'$ and $v_2 = [-0.8735, 0.1649, 0.3315, 0.3162]'$. As we said above, v_1 and v_2 cross zero the "right" number of times. However, it is not hard to see that there exist c_1 and c_2 that make $\sum_{i=1}^{2} c_i v_i$ violate the needed condition for M_1 to be in GK_2.

3.4. Why zero-crossings are not enough. Let us consider the following matrix:

$$N = \begin{pmatrix} 1.0000 & 0.8024 & 0.9206 & 0.8359 & 0.2981 & 0.7288 & 0.2955 & 0.6339 \\ 0.8024 & 1.0000 & 0.8220 & 0.9099 & 0.7247 & 0.9411 & 0.7574 & 0.9360 \\ 0.9206 & 0.8220 & 1.0000 & 0.8368 & 0.3026 & 0.7197 & 0.2958 & 0.6216 \\ 0.8359 & 0.9099 & 0.8368 & 1.0000 & 0.6005 & 0.8739 & 0.6025 & 0.8322 \\ 0.2981 & 0.7247 & 0.3026 & 0.6005 & 1.0000 & 0.8118 & 0.9765 & 0.9059 \\ 0.7288 & 0.9411 & 0.7197 & 0.8739 & 0.8118 & 1.0000 & 0.8208 & 0.9570 \\ 0.2955 & 0.7574 & 0.2958 & 0.6025 & 0.9765 & 0.8208 & 1.0000 & 0.9185 \\ 0.6339 & 0.9360 & 0.6216 & 0.8322 & 0.9059 & 0.9570 & 0.9185 & 1.0000 \end{pmatrix}.$$



First eigenvector. *Second eigenvector.*

Figure 1. *First two eigenvectors of the matrix N.*

Its first two eigenvectors are shown in Figure 1. We see that the first one is almost constant, and the *level* effect is clear. According to the definition of Lord and Pelsser, given that the second eigenvector crosses zero only once, the matrix N has the *slope* effect ($N \in LP_2$). Nevertheless, even if the second eigenvector crosses zero only once, it is not clear what the meaning of *slope* effect is here. As we said before, the *slope* should be an adjustment of the shock that is reflected in the first eigenvector since that shock is in general decreasing or increasing in the curve. This is not the case for the example since the effect is more *wavering* than increasing or decreasing. The theory of total positivity tells us what the right definition should be in order to fully characterize the corresponding correlation matrices. The following theorem provides more intuition about the relationship that first and second eigenvectors have to fulfill in order for the matrix to be in GK_2 (for the proof, see the appendix).

Theorem 3.18. *Suppose that that the matrix has first positive eigenvector x_1. Then the second eigenvector x_2 satisfies that $S^+(c_1x_1 + c_2x_2) \le 1$ whenever $c_1 \neq 0$ or $c_2 \neq 0$ if and only if the vector $v_1 = (v_1^i)_{i=1}^N$ with $v_1^i = x_2^i/x_1^i$ has decreasing or increasing coordinates.*

Theorem 3.17 and the example from the previous section motivate us to give the following alternative definition of the LSC effect.

Definition 3.19. *$A \in \mathbb{R}^{N\times N}$ is said to have the strong LSC property if it has the Gantmacher–Krein property of order 3 ($A \in GK_3$). Analogously we say that the matrix A has the strong slope effect if A has the Gantmacher–Krein property of order 2 ($A \in GK_2$).*

Let us now compare both definitions. A matrix is said to have the LSC property (LP_3) if its first three eigenvectors cross zero the right number of times. Intuitively, the definition of strong LSC requires not only that the vectors satisfy the conditions for the matrix to be in LP_3, but also that no vector on the generated subspaces violate the condition either.

In order to say whether the matrix N has the strong slope effect we would have to check whether $N \in \mathrm{GK}_2$. To do this we compute $N^{(2)}$ and check whether it is eventually totally positive or, which is the same, whether it satisfies the strong Perron–Frobenius property. As we suspect, the answer for the matrix N is negative (the top eigenvector of $N^{(2)}$ is shown in Figure 2).

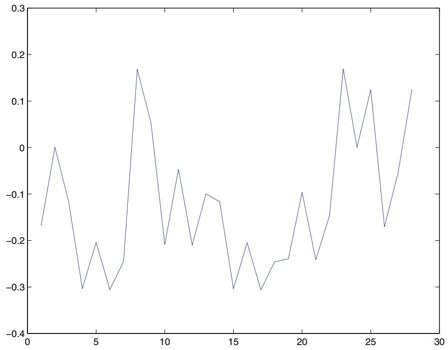


Figure 2. *First eigenvector of* $N^{(2)}$.

4. Exponentially decaying correlations.

In [7], Forzani and Tolmasky suggested that a typical correlation matrix coming from yield curves can be approximated by matrices having exponentially decaying (as a function $|i-j|$) correlations. We will now check that these matrices satisfy the strong LSC condition. For example, if we consider $0<\rho<1$, in $\mathbb{R}^{7\times 7}$ this would give us

$$M_\rho = \begin{pmatrix} 1 & \rho & \rho^2 & \rho^3 & \rho^4 & \rho^5 & \rho^6 \\ \rho & 1 & \rho & \rho^2 & \rho^3 & \rho^4 & \rho^5 \\ \rho^2 & \rho & 1 & \rho & \rho^2 & \rho^3 & \rho^4 \\ \rho^3 & \rho^2 & \rho & 1 & \rho & \rho^2 & \rho^3 \\ \rho^4 & \rho^3 & \rho^2 & \rho & 1 & \rho & \rho^2 \\ \rho^5 & \rho^4 & \rho^3 & \rho^2 & \rho & 1 & \rho \\ \rho^6 & \rho^5 & \rho^4 & \rho^3 & \rho^2 & \rho & 1 \end{pmatrix}.$$

In order to prove that M_ρ satisfies the strong LSC condition we should prove that the first two corresponding compound matrices are eventually positive ("level" is obvious since the matrix is positive). We will make use of the following lemma (a proof can be found in [4, Theorem 3.2.1]).

Lemma 4.1. *A nonnegative, symmetric matrix A is irreducible if and only if its associated graph is connected.*

Also, we recall the statement of the Perron–Frobenius theorem for irreducible matrices.

Theorem 4.2 (Perron–Frobenius for irreducible matrices). *An irreducible, nonnegative matrix A has the strong Perron–Frobenius property.*

Let us now look at the 2nd-compound matrix $M_\rho^{(2)}$, which is formed by choosing pairs $\mathbf{i}=(i_1,i_2)$, $\mathbf{j}=(j_1,j_2)\in I_{2,5}$. Its entries are defined as

$$m_{i,j}^{(2)} = \det\begin{pmatrix} \rho^{|i_1-j_1|} & \rho^{|i_1-j_2|} \\ \rho^{|i_2-j_1|} & \rho^{|i_2-j_2|} \end{pmatrix}.$$

We have to distinguish between the following cases:

1. $i_1 < i_2 \le j_1 < j_2$.
2. $j_1 < j_2 \le i_1 < i_2$.
3. $i_1 \le j_1 \le 2_1 \le j_2$.
4. $i_1 \le j_1 \le j_2 \le i_2$.
5. $j_1 \le i_1 \le i_2 \le j_2$.
6. $j_1 \le i_1 \le j_2 \le i_2$.

It is not difficult to check that in the first two cases the determinant is zero, whereas in the rest of the cases is strictly positive. In particular, $M_\rho^{(2)}$ turns out to be nonegative. Now, in order for us to be able to prove that $M_\rho^{(2)}$ satisfies the strong Perron–Frobenius property we are going to make use of Lemma 4.1. To see that the the graph associated with $M_\rho^{(2)}$ is connected it is enough to see that the elements on its superdiagonal are strictly positive (since this implies that, on the graph, each node i is linked to node $i+1$, $i \le n-1$, where n is the number of nodes). Let us assume that M_ρ is in $\mathbb{R}^{N\times N}$, and let us take $\mathbf{i} = (i_1, i_2)$ and $\mathbf{j} = (j_1, j_2)$. The elements just above the diagonal correspond to the elements that come from determinants of 2×2 minors in which $\mathbf{j}$ is the element in $I_{2,N}$ which immediately follows $\mathbf{i}$. Therefore, there are two possible cases:

1. $i_2 \le N-1$: In this case $\mathbf{i} = (i_1, i_2)$ and $\mathbf{j} = (i_1, i_2+1)$, and we are then looking at

$$\det \begin{pmatrix} 1 & \rho^{i_2+1-i_1} \\ \rho^{i_2-i_1} & \rho \end{pmatrix} = \rho - \rho^{2(i_2-i_1)+1} > 0$$

since $i_1 > i_1$.

2. $i_1 \le N-2$: Then $i_2 = N$, $\mathbf{i} = (i_1, i_2)$, and $\mathbf{j} = (i_1+1, i_1+2)$, and the minor results in:

$$\det = \begin{pmatrix} \rho & \rho^2 \\ \rho^{N-(i_1+1)} & \rho^{N-(i_1+2)} \end{pmatrix} = \rho^{N-i_1-1} - \rho^{N-i_1+1} > 0$$

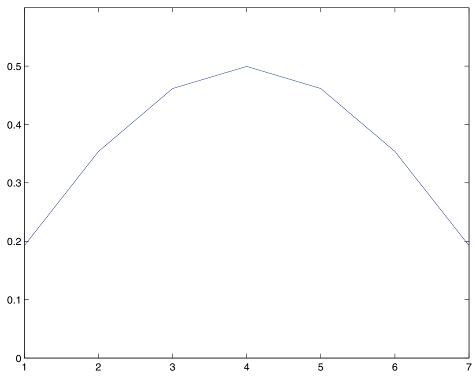
in virtue of $N = i_2 > i_1 + 1$.

This proves that the superdiagonal is formed of positive elements, and therefore the associated graph is connected, and by Lemma 4.1 the matrix $M_\rho^{(2)}$ is irreducible. Since we have seen already that it is nonnegative, we can conclude that it satisfies the strong Perron–Frobenius property. As a consequence, by Theorem 3.5, $M_rho^{(2)}$ is eventually positive.

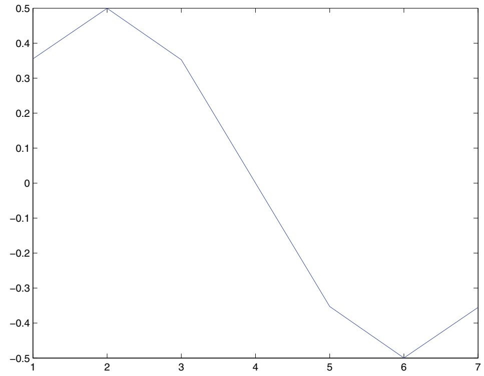
The case of $M_\rho^{(3)}$ is tedious, but it can be proved in a similar fashion. This proves that $M_\rho \in GK_3$.

Remark 4.3. In [7] the authors proved a more specific characterization of the eigenstructure of the matrices M_ρ in the limit when ρ approaches 1. The present proof takes care of all the cases in which $0 < \rho < 1$, but it does not tell us what the eigenvectors look like exactly; all the information we get is that they satisfy (3.1).

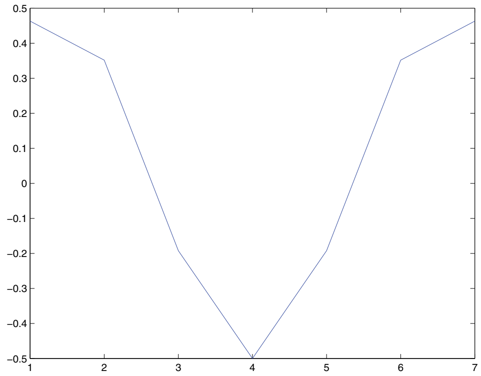
Remark 4.4. As was pointed out by one of the anonymous referees, the eigenvectors of the matrix M_ρ coincide (in reverse order of importance) with the eigenvectors of its inverse



First eigenvector.



Second eigenvector.



Third eigenvector.

Figure 3. *First three eigenvectors of the matrix M_ρ.*

(see Figure 3):

$$
(4.1) \qquad M_\rho^{-1} = \frac{1}{1-\rho^2} \begin{pmatrix} 1 & -\rho & 0 & 0 & 0 & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & 0 & 0 & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & -\rho & 0 & 0 & 0 \\ 0 & 0 & -\rho & 1+\rho^2 & -\rho & 0 & 0 \\ 0 & 0 & 0 & -\rho & 1+\rho^2 & -\rho & 0 \\ 0 & 0 & 0 & 0 & -\rho & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & 0 & 0 & -\rho & 1 \end{pmatrix}.
$$

The advantage of looking at M_ρ^{-1} instead of M_ρ is that the form of the eigenvectors can be obtained by solving a system of difference equations. It is shown in [2] (see also [20]) that these eigenvectors are sine functions. In this way it is not difficult to see that the eigenvectors cross zero the right number of times. Nevertheless, since the frequencies are only known to be solutions to an equation (and not known explicitly) it is not immediate to see that the Gantmacher–Krein condition is satisfied.

Remark 4.5. It is interesting to note that if we assume that the covariance matrix of the spot rates is of the form M_ρ, then the corresponding forwards do not necessarily satisfy the strong LSC property. Indeed, let us fix $N = 4$, $\rho = .9$ and suppose that the variances of spot rates are all equal to 1. The corresponding correlation matrix for the forward rates is then

$$N = \begin{pmatrix} 1 & 0.68 & 0.42 & 0.26 \\ 0.68 & 1 & 0.44 & 0.27 \\ 0.42 & 0.44 & 1 & 0.26 \\ 0.26 & 0.27 & 0.26 & 1 \end{pmatrix}.$$

By looking at the signs of the first eigenvector, it is easy to check that the 3rd compound ($N^{(3)}$) is not in ESTP_3. And, at the same time, it is not difficult to see that the top three eigenvectors do not satisfy the condition (3.1).

5. Lekkos' critique. As Lekkos points out, "*Factor analysis concludes that three underlying economic factors affecting the level, steepness and curvature of the term structure are sufficient to describe the dynamic evolution of interest rates.*" Over the past 25 years researchers and practitioners have found the same result in different markets. A very natural question to ask is, then, what are the common properties of the correlation (or covariance) matrices obtained for this result to be true across markets? The answer to this question provided by Lekkos is that the effect is just an artifact created by the fact that longer spot rates contain the information of shorter ones in themselves. To show this, he proposes to analyze a, somewhat extreme, example: start by assuming that forward rates are independent, compute the corresponding spot rates, and check what principal components analysis shows in this case. Surprisingly, LSC emerges once again. In this section we show, using the theory developed so far, why this is so. It turns out that even in the case in which forwards are independent, the correlation matrices corresponding to spot rates belong to GK_3. Therefore, the moves in spot rates can be well explained with three factors, whereas the forwards, which encode the same information, are independent random variables (and, therefore, admit no dimensionality reduction).

We now look a bit closer at Lekkos' argument. Spot (or zero-coupon) rates and forward rates encode exactly the same information. More specifically, spot rates are just weighted averages of forward rates. For the sake of completeness we will now recall the definition of a forward rate. If we denote by $R(T)$ the spot rate for time T, we define $f(T_1, T_2)$, the forward rate for the period starting at time T_1 and ending at time T_2, as the rate satisfying

$$e^{R(T_1)T_1} e^{f(T_1,T_2)(T_2-T_1)} = e^{R(T_2)T_2}.$$

For example, if we take $T_1 = 1$ and $T_2 = 2$,

$$R(2) = \frac{1}{2}R(1) + \frac{1}{2}f(1,2).$$

If we assume that $R(1)$ is the shortest observed spot rate, then $R(1) = f(0,1)$ and then $R(2)$ yields the average of the first two forward rates. Analogously,

$$R(3) = \frac{1}{3}f(0,1) + \frac{1}{3}f(1,2) + \frac{1}{3}f(2,3).$$

If we consider a curve consisting of only five tenors (say, 1-year, 2-year, 3-year, 4-year, and 5-year rates) the relationship between spot and forward rates can be written as

$$\begin{pmatrix} R(1) \\ R(2) \\ R(3) \\ R(4) \\ R(5) \end{pmatrix} = W \begin{pmatrix} f(0,1) \\ f(1,2) \\ f(2,3) \\ f(3,4) \\ f(4,5) \end{pmatrix},$$

where the matrix W that transforms forward rates into spot rates has the form

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{pmatrix}. \tag{5.1}$$

If we now call Σ_f the covariance of the forward rates (assumed to be the identity in this example), we get that the covariance of the zero-coupon rates, Σ_R, equals $W\Sigma_f W^T$:

$$\Sigma_R = WW^T = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1/2 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/3 & 1/3 & 1/4 & 1/5 \\ 1/4 & 1/4 & 1/4 & 1/4 & 1/5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{pmatrix}.$$

Our goal is to prove that the compound matrices $\Sigma_R^{(2)}$ and $\Sigma_R^{(3)}$ are eventually (strictly) positive. Or, equivalently, that they satisfy the strong Perron–Frobenius condition. For this, we will make use of the following two results (Propositions 1.3 and 4.1 in [18]).

Lemma 5.1. *Assume that A is an $n \times m$ (strictly) totally positive matrix. Let B denote the matrix obtained from A by reversing the order of both its rows and columns; i.e., if $A = (a_{ij})$, $i = 1, \dots, n, j = 1, \dots, m$, then $B = (b_{ij})$, $i = 1, \dots, n, j = 1, \dots, m$, where $b_{ij} = a_{n+1-i,m+1-j}$, $i = 1, \dots, n, j = 1, \dots, m$. Then the matrix B is (strictly) totally positive.*

Lemma 5.2. *Let $b_1, \dots, b_n$ be n distinct numbers. Set $a_{ij} = b_{min(i,j)}$, $i, j = 1, \dots, n$. Then $A = (a_{ij})$, $i, j = 1, \dots, n$, is a totally positive matrix if and only if $0 \le b_1 < \cdots < b_n$.*

By Lemma 5.1 the matrix Σ_R will turn out to be totally positive if we can prove that the matrix

$$\begin{pmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \\ 1/5 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/5 & 1/4 & 1/3 & 1/3 & 1/3 \\ 1/5 & 1/4 & 1/3 & 1/2 & 1/2 \\ 1/5 & 1/4 & 1/3 & 1/2 & 1 \end{pmatrix}$$

is itself totally positive, which follows from Lemma 5.2.

The corresponding correlation matrix for Σ_R is of the form

$$\begin{pmatrix} 1 & \sqrt{1/2} & \sqrt{1/3} & \sqrt{1/4} & \sqrt{1/5} \\ \sqrt{1/2} & 1 & \sqrt{2/3} & \sqrt{1/2} & \sqrt{2/5} \\ \sqrt{1/3} & \sqrt{2/3} & 1 & \sqrt{3/4} & \sqrt{3/5} \\ \sqrt{1/4} & \sqrt{1/2} & \sqrt{3/4} & 1 & \sqrt{4/5} \\ \sqrt{1/5} & \sqrt{2/5} & \sqrt{3/5} & \sqrt{4/5} & 1 \end{pmatrix},$$

which total positivity follows from Green's theorem (see Theorem 4.2 in [18]).

Theorem 5.3. *If $c_i, d_j,\ i,j = 1,\dots,N$, are all either strictly positive or strictly negative, then the matrix $A \in \mathbb{R}^{N\times N}$ consisting of the elements $a_{i,j} = c_{\min(i,j)} d_{\max(i,j)}$ is totally positive if and only if $0 \le \frac{c_1}{d_1} \le \cdots \le \frac{c_N}{d_N}$.*

5.1. Perturbations of rank 1.

The covariance matrices of forward rates do not show the same structure as the ones of the zero-coupon rates. However, all the examples considered in [14] do display a *level* effect. Motivated by this fact, we are interested in studying what happens if we set the covariance matrix of the forward rates to be a rank-1 perturbation of the identity matrix. Moreover, we will take this perturbation to be of the form of a level effect. Will the corresponding covariance (or correlation) matrices for the zero-coupon rates satisfy the LSC property in this case?

To see this let us take $\Sigma_R = W(I + hee^T)W^T = WW^T + hee^T$, where W is of the same form as in (5.1) and e is the exact level vector (a constant vector). Then the covariance Σ_R becomes

$$\begin{pmatrix} h+1 & h+1/2 & h+1/3 & h+1/4 & h+1/5 \\ h+1/2 & h+1/2 & h+1/3 & h+1/4 & h+1/5 \\ h+1/3 & h+1/3 & h+1/3 & h+1/4 & h+1/5 \\ h+1/4 & h+1/4 & h+1/4 & h+1/4 & h+1/5 \\ h+1/5 & h+1/5 & h+1/5 & h+1/5 & h+1/5 \end{pmatrix},$$

the correlation is

$$\begin{pmatrix} 1 & \sqrt{\frac{h+1/2}{h+1}} & \sqrt{\frac{h+1/3}{h+1}} & \sqrt{\frac{h+1/4}{h+1}} & \sqrt{\frac{h+1/5}{h+1}} \\ \sqrt{\frac{h+1/2}{h+1}} & 1 & \sqrt{\frac{h+1/3}{h+1/2}} & \sqrt{\frac{h+1/4}{h+1/2}} & \sqrt{\frac{h+1/5}{h+1/2}} \\ \sqrt{\frac{h+1/3}{h+1}} & \sqrt{\frac{h+1/3}{h+1/2}} & 1 & \sqrt{\frac{h+/4}{h+1/3}} & \sqrt{\frac{h+1/5}{h+1/3}} \\ \sqrt{\frac{h+1/4}{h+1}} & \sqrt{\frac{h+1/4}{h+1/2}} & \sqrt{\frac{h+1/4}{h+1/3}} & 1 & \sqrt{\frac{h+1/5}{h+1/4}} \\ \sqrt{\frac{h+1/5}{h+1}} & \sqrt{\frac{h+1/5}{h+1/2}} & \sqrt{\frac{h+1/5}{h+1/3}} & \sqrt{\frac{h+1/5}{h+1/4}} & 1 \end{pmatrix},$$

and both matrices can be shown to be totally positive by following the same arguments as in the previous section.

5.2. Lekkos' matrices and the class $\Xi(0,1,2,3)$.

Salinelli and Sgarra [19] defined the class of correlation matrices $\Xi(0,1,2,3)$. Their purpose was to capture the properties that make a matrix satisfy the LSC property. A matrix $\mathbf{R} = (\rho_{i,j})$ is in $\Xi(0,1,2,3)$ if it satisfies the following properties (see also [16]):

($P0$) positivity, i.e., $\rho_{i,j} > 0$ for all $i, j = 1, \ldots, N$.
($P1$) strict decreasingness of superdiagonal row elements:

$$\forall \text{ fixed } i \geq 1, \quad \rho_{i,j} \text{ decreases with respect to } j \geq i.$$

($P2$) strict increasingness of superdiagonal column elements:

$$\forall \text{ fixed } j \geq 1, \quad \rho_{i,j} \text{ increases with respect to } i \leq j.$$

($P3$) strict increasingness of secondary superdiagonal elements:

$$\forall \text{ fixed } p > 0, \quad \rho_{i,i+p} \text{ increases with respect to } i.$$

In their work, Salinelli and Sgarra prove some results relating LSC and the conditions (P0)–(P3). We will now check whether or not some of the matrices we have been looking at so far belong to $\Xi(0,1,2,3)$.

Consider, for example, the covariance of seven forward rates $\Sigma_f = I$ and the corresponding Σ_R's. The correlation matrix of the spot rates turns out to be

$$C_1 = \begin{pmatrix} 1.0000 & 0.7071 & 0.5774 & 0.5000 & 0.4472 & 0.4082 & 0.3780 \\ 0.7071 & 1.0000 & 0.8165 & 0.7071 & 0.6325 & 0.5774 & 0.5345 \\ 0.5774 & 0.8165 & 1.0000 & 0.8660 & 0.7746 & 0.7071 & 0.6547 \\ 0.5000 & 0.7071 & 0.8660 & 1.0000 & 0.8944 & 0.8165 & 0.7559 \\ 0.4472 & 0.6325 & 0.7746 & 0.8944 & 1.0000 & 0.9129 & 0.8452 \\ 0.4082 & 0.5774 & 0.7071 & 0.8165 & 0.9129 & 1.0000 & 0.9258 \\ 0.3780 & 0.5345 & 0.6547 & 0.7559 & 0.8452 & 0.9258 & 1.0000 \end{pmatrix}.$$

We have already proved that these types of matrices are eventually strictly totally positive of order 7, and therefore they satisfy the condition GK_7. As we can see, these types of matrices satisfy conditions (P0)–(P3), and then they are in $\Xi(0,1,2,3)$. With C as the correlation matrix, the first eigenvector "explains" 74% of the variance.

If the forwards are not exactly independent but display a level effect, we can, as in section 5.1, pose $\Sigma_R = W(I + huu')W$, where u is the "level" vector, $u = (1,1,1,1,1,1,1)'$, and h can be taken to be so that the matrix $(I + huu')$ is the covariance of the forward rates. For example, if we take $h = 1$, the first eigenvector now explains 81% of the variance. By doing this, the correlation corresponding to the spot rates becomes

$$C_2 = \begin{pmatrix} 1.0000 & 0.7500 & 0.6455 & 0.5863 & 0.5477 & 0.5204 & 0.5000 \\ 0.7500 & 1.0000 & 0.8607 & 0.7817 & 0.7303 & 0.6939 & 0.6667 \\ 0.6455 & 0.8607 & 1.0000 & 0.9083 & 0.8485 & 0.8062 & 0.7746 \\ 0.5863 & 0.7817 & 0.9083 & 1.0000 & 0.9342 & 0.8876 & 0.8528 \\ 0.5477 & 0.7303 & 0.8485 & 0.9342 & 1.0000 & 0.9501 & 0.9129 \\ 0.5204 & 0.6939 & 0.8062 & 0.8876 & 0.9501 & 1.0000 & 0.9608 \\ 0.5000 & 0.6667 & 0.7746 & 0.8528 & 0.9129 & 0.9608 & 1.0000 \end{pmatrix}.$$

Once more, C_2 is in $ESTP_7$, and C_2 also belongs to the set $\Xi(0,1,2,3)$.

In comparison, the matrices studied in [7] and presented in section 4 satisfy (P0), (P1), and (P2), but they do not satisfy the condition (P3). Therefore, these types of matrices do not belong to $\Xi(0,1,2,3)$, but, as we saw, they do satisfy the strong LSC property. For example, if we fix $\rho = .75$, the matrix we find is

$$M_{.75} = \begin{pmatrix} 1.0000 & 0.7500 & 0.5625 & 0.4219 & 0.3164 & 0.2373 & 0.1780 \\ 0.7500 & 1.0000 & 0.7500 & 0.5625 & 0.4219 & 0.3164 & 0.2373 \\ 0.5625 & 0.7500 & 1.0000 & 0.7500 & 0.5625 & 0.4219 & 0.3164 \\ 0.4219 & 0.5625 & 0.7500 & 1.0000 & 0.7500 & 0.5625 & 0.4219 \\ 0.3164 & 0.4219 & 0.5625 & 0.7500 & 1.0000 & 0.7500 & 0.5625 \\ 0.2373 & 0.3164 & 0.4219 & 0.5625 & 0.7500 & 1.0000 & 0.7500 \\ 0.1780 & 0.2373 & 0.3164 & 0.4219 & 0.5625 & 0.7500 & 1.0000 \end{pmatrix}.$$

6. Fact or artifact? Conclusion. With the right definition in relation to the behavior of the eigenvectors, we have provided a characterization of the matrices that have the LSC property. It turns out that looking at the number of zero-crossings of the eigenvectors is not enough; rather the condition on the number of zero-crossings has to be satisfied by all the vectors in certain subspaces. To show this we have relied heavily on some recent results in linear algebra, particularly the work of Kushel [13]. Understanding the structure of these matrices allows us to see why the spectral structure of covariance (or correlation) matrices is so similar in different interest rate markets. So, even if it is definitely true that yield curves display the LSC effect, to a large extent the effect is just bound to be true by construction. In other words, independently from any empirical study, we know that these type of structures will be present for any "curve" (yield curve and also commodity forward curve). This fact was pointed out already in [14]. Therefore, in order for us to identify particularities of a given (curve) market, we should consider first filtering out the effect created by the matrices W_N. Once we do that, we should test how many eigenvalues are relevant in the correlation matrix of the corresponding forwards. Also, we should check whether the eigenvectors keep the properties in these cases. Our suspicion, based on the results reported in [14] (also in [12]), is that the structures will turn out to be way less homogeneous across markets.

Appendix. Proof of Theorem 3.18. Suppose that that the matrix has first positive eigenvector x_1. Now, the second one x_2 satisfies that $S^+(x_1 + c_2 x_2) \leq 1$ when $c_2 \neq 0$ if and only if the vector $v = (v^i)_{i=1}^N$ with $v_i = x_2^i / x_1^i$ has decreasing or increasing coordinates.

Proof. We proceed in two parts.

Step 1. Proof that if the vector $v = (v^i)_{i=1}^N$ with $v_i = x_2^i / x_1^i$ has decreasing or increasing coordinates, then $S^+(x_1 + c_2 x_2) \leq 1$ when $c_2 \neq 0$. Suppose that v_i has decreasing coordinates; then for $c > 0$

$$1 + cv^{i+1} < 1 + cv^i,$$

and it follows that $1 + cv$ changes signs not more than once. That implies $S^+(x_1 + c_2 x_2) \leq 1$ when $c_2 \neq 0$.

Step 2. Proof that if $S^+(x_1 + c_2 x_2) \leq 1$ when $c_2 \neq 0$, then the vector $v = (v^i)_{i=1}^N$ with $v_i = x_2^i / x_1^i$ has decreasing or increasing coordinates. Suppose that $1 + cv$ changes signs not more than once, and suppose that v does not have decreasing coordinates; then there exists i such that

$$v^i > v^{i+1},$$
$$v^{i+2} > v^{i+1}.$$

We consider two cases:

1. $v^{i+1} > v^{i+2} > v^i$. Now, since v changes signs only once, we have the following cases (we have more, but they are equivalent):
   - $v^i > 0$ and $v^{i+2} > 0$ implies $v^{i+1} > 0$. In this case we take any $c \in (-\frac{1}{v^{i+1}}, -\frac{1}{v^{i+2}})$, and we get a contradiction.
   - $v^i > 0$ and $v^{i+2} < 0$ and $v^{i+1} < 0$. In this case we take any $c \in (-\frac{1}{v^{i+1}}, -\frac{1}{v^{i+2}})$, and we get a contradiction.
   - $v^i > 0$ and $v^{i+2} < 0$ and $v^{i+1} > 0$. In this case we take any $c \in (-\frac{1}{v^{i+1}}, -\frac{1}{v^i})$, and we get a contradiction.
2. $v^{i+1} < v^i < v^{i+2}$. Analogous, changing i by $i + 2$ and $i + 2$ by i. ■

Acknowledgments. We would like to thank Leslie Hogben for pointing out the reference to Lemma 2.1 in [8], and to one of the referees for the suggestion that led to Remark 4.4.

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