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Characterizations of the boundedness of generalized fractional maximal functions and related operators in Orlicz spaces

Estefanía Dalmasso* and **Gladis Pradolini****

Instituto de Matemática Aplicada del Litoral, UNL, CONICET, FIQ, Predio Dr. Alberto Cassano del CCT-CONICET-Santa Fe, Colectora Ruta Nacional No. 168, (3000), Santa Fe, Argentina

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Given $0 < \alpha < n$ and a Young function η, we consider the generalized fractional maximal operator $M_{\alpha,\eta}$ defined by

$$M_{\alpha,\eta} f(x) = \sup_{B \ni x} |B|^{\alpha/n} ||f||_{\eta,B},$$

where the supremum is taken over every ball B contained in $\mathbb{R}^n$. In this article, we give necessary and sufficient Dini type conditions on the functions $\mathcal{A}$, $\mathcal{B}$ and η such that $M_{\alpha,\eta}$ is bounded from the Orlicz space $L^{\mathcal{A}}(\mathbb{R}^n)$ into the Orlicz space $L^{\mathcal{B}}(\mathbb{R}^n)$. We also present a version of this result for open subsets of $\mathbb{R}^n$ with finite measure. Both results generalize those contained in [6] and [14] when $\eta(t) = t$, respectively. As a consequence, we obtain a characterization of the functions involved in the boundedness of the higher order commutators of the fractional integral operator with BMO symbols. Moreover, we give sufficient conditions that guarantee the continuity in Orlicz spaces of a large class of fractional integral operators of convolution type with less regular kernels and their commutators, which are controlled by $M_{\alpha,\eta}$.

1 Introduction and main results

Norm inequalities for several classical operators of harmonic analysis have been widely studied in the context of Orlicz spaces. It is well known that many of such operators fail to have continuity properties when they act between certain Lebesgue spaces and, in some situations, the Orlicz spaces appear as adequate substitutes.

For example, for the Hardy–Littlewood maximal operator M, boundedness results between general Orlicz spaces were given in [2], [5], [12], [14], [16] and [27]. Since M controls, in some sense, the singular integral operators, the mentioned results allow us to derive continuity properties for them. According to this, the Hilbert and Riesz transforms were also studied in this setting (see, for instance, [6], [14] and [18]).

When dealing with other type of singular integrals or their commutators, the maximal operators that govern their behavior are defined in terms of a Young function η, M_η. The continuity properties in Orlicz spaces for such maximal operators were first studied in [17] in the Euclidean context. In this article the author considers the operator M^k, the k-th iteration of the Hardy–Littlewood maximal operator M, which is known to be pointwise equivalent to M_η with $\eta(t) = t(1 + \log^+ t)^{k-1}$. When η is a general Young function, the boundedness of M_η in Orlicz spaces was analized in [15] in the framework of spaces of homogeneous type.

Concerning fractional type operators, the authors in [6] and [14] gave necessary and sufficient conditions on certain functions $\mathcal{A}$ and $\mathcal{B}$ for the maximal fractional operator M_α, $0 \leq \alpha < n$, and the fractional integral operator I_α to be bounded between the associated Orlicz spaces $L^{\mathcal{A}}$ and $L^{\mathcal{B}}$ (see below for the definition of these spaces).

When dealing with generalizations of I_α and their commutators, the maximal functions that control them, in some sense, are the fractional maximal operators $M_{\alpha,\eta}$ associated with a Young function η, where η is sometimes related with the regularity properties of the kernel of the operator, or η is a $L \log L$ type function when dealing

* Corresponding author: e-mail: edalmasso@santafe-conicet.gov.ar, Phone: +54 451 1370 ext. 1575, Fax: +54 451 1370 ext. 4004
** e-mail: gpradolini@santafe-conicet.gov.ar, Phone: +54 451 1370 ext. 4114 Fax: +54 451 1370 ext. 4004

with commutators, or both. In relation with the correspondence between regularity and maximal operators, the more regularity on the kernel, the better the maximal operator.

Let us define the operator $M_{\alpha,\eta}$. Given an open subset Ω of $\mathbb{R}^n$, for $0 \le \alpha < n$ and $f \in L^1_{loc}(\Omega)$, $M_{\alpha,\eta}$ is defined by

$$M_{\alpha,\eta} f(x) = \sup_{B \ni x} |B \cap \Omega|^{\alpha/n} ||f||_{\eta,B},$$

where the supremum is taken over every Euclidean ball $B = B(x_0, R)$ with $x_0 \in \Omega$ and $R > 0$, and $|| \cdot ||_{\eta,B}$ denotes the Luxemburg-type average given by

$$||f||_{\eta,B} = \inf \left\{ \lambda > 0 : \frac{1}{|B \cap \Omega|} \int_B \eta \left(\frac{|f(x)|}{\lambda} \right) \le 1 \right\}.$$

When $\alpha = 0$, we simply write $M_{0,\eta} = M_\eta$. It is well known that this type of maximal functions control a large class of operators such as generalized singular integrals and their commutators.

Our main aim is to characterize the functions involved in the boundedness on Orlicz spaces of the fractional maximal operator $M_{\alpha,\eta}$. Concretely, we give necessary and sufficient conditions on the functions $\mathcal{A}$, $\mathcal{B}$ and η such that $M_{\alpha,\eta} : L^{\mathcal{A}}(\Omega) \hookrightarrow L^{\mathcal{B}}(\Omega)$ for $\Omega = \mathbb{R}^n$ and open subsets Ω of $\mathbb{R}^n$ with finite measure. Our results generalize those related with M_α contained in [6] and [14] and they have a correspondence with the result in [15] for the case $\alpha = 0$ and $\Omega = \mathbb{R}^n$. As a consequence, we obtain similar continuity properties for other operators of convolution type with kernels satisfying certain size condition and having different degrees of regularity, as well as their commutators, which are controlled, in some sense, by $M_{\alpha,\eta}$.

Throughout this article, we consider the functions that define the Orlicz spaces to be of the form

$$\mathcal{A}(t) = \int_0^t a(s)\, ds \quad \text{and} \quad \mathcal{B}(t) = \int_0^t b(s)\, ds,$$

where a and b are left-continuous functions defined on $[0, \infty)$ with $a(0) = b(0) = 0$, such that a is positive on $(0, \infty)$ and nondecreasing, and b is nonnegative (see the next section for the definition of Orlicz spaces). The function η associated with the maximal operator we are dealing with is a submultiplicative Young function, that is, $\eta : [0, \infty) \to [0, \infty)$ is convex, increasing with $\eta(0) = 0$ and $\lim_{t\to\infty} \eta(t) = +\infty$, and it satisfies $\eta(ts) \le \eta(t)\eta(s)$ for every $t, s > 0$. We will also assume, without loss of generality, that η is normalized, that is, $\eta(1) = 1$. Under these conditions, we obtain the following main results. The first one gives the characterization when $\Omega = \mathbb{R}^n$ and the second one when Ω has finite measure.

Theorem 1.1 *Let $0 < \alpha < n$, and let $\mathcal{A}$, $\mathcal{B}$ and η be defined as above. Let ξ be a Young function such that $\xi^{-1}(t) = t^{-\frac{\alpha}{n}} \eta^{-1}(t)$. Then, the following statements are equivalent.*

(i) *There exist positive constants C_1 and C_2 such that*

$$\int_0^{C_1 t \mathcal{A}(t)^{-\alpha/n}} \frac{b(\lambda)}{\lambda} \xi' \left(\frac{C_1 t \mathcal{A}(t)^{-\alpha/n}}{\lambda} \right) d\lambda \le C_2 \frac{\mathcal{A}(t)^{1+\alpha/n}}{t},$$

for every $t > 0$.

(ii) $M_{\alpha,\eta} : L^{\mathcal{A}}(\mathbb{R}^n) \hookrightarrow L^{\mathcal{B}}(\mathbb{R}^n)$*, that is, there exists a positive constant K such that the inequality*

$$\left|\left| M_{\alpha,\eta} f \right|\right|_{L^{\mathcal{B}}(\mathbb{R}^n)} \le K ||f||_{L^{\mathcal{A}}(\mathbb{R}^n)}$$

holds for every $f \in L^{\mathcal{A}}(\mathbb{R}^n)$.

Remark 1.2 When $\eta(t) = t$, the theorem above improves the corresponding result proved in [6], where the author gives necessary and sufficient conditions on the functions $\mathcal{A}$ and $\mathcal{B}$ that are more difficult to handle than the Dini type condition given in (i). Moreover, as our result shows, the function b needs not to be increasing, which allows us to include more spaces then the ones considered in [6].

When $\alpha = 0$, the characterization of the Dini-type condition for M_η was obtained in [15] in the setting of spaces of homogeneous type, and it is in correspondence with the result for $0 < \alpha < n$, by noticing that, in this case, $\xi = \eta$.

Remark 1.3 If we consider $\mathcal{A}(t) = t^p$ and $\mathcal{B}(t) = t^q$ with $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$ in Theorem 1.1, we have that $M_{\alpha,\eta} : L^p(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ if and only if

$$\int_0^{C_1 t^{1-\frac{\alpha p}{n}}} q\lambda^{q-1}\xi'\left(\frac{C_1 t^{1-\frac{\alpha p}{n}}}{\lambda}\right)\frac{d\lambda}{\lambda} \le C_2 t^{p(1+\frac{\alpha}{n})-1} = C_2 t^{p-\left(1-\frac{\alpha p}{n}\right)},$$

for some positive constants C_1 and C_2, and each $t > 0$, where $\xi^{-1}(t) = t^{-\alpha/n}\eta^{-1}(t)$. It is easy to see from the relation between p and q that this inequality can be rewritten as

$$\int_1^\infty \frac{\xi(s)}{s^q}\frac{ds}{s} \le C,$$

which means that ξ belongs to the class B_q introduced in [24]. It is easy to check from the formula $\xi^{-1}(t) = t^{-\alpha/n}\eta^{-1}(t)$ that $\xi \in B_q$ if and only if $\eta^{q/p} \in B_q$. In the case $\alpha = 0$, we have that $q = p$ and Theorem 1.1 is the fractional version of the result given in [24, Theorem 1.7].

Remark 1.4 From the previous remark, when considering $\mathcal{A}(t) = t^p$ and $\mathcal{B}(t) = t^q$ with $1 \le p < n/\alpha$ and $1/q = 1/p - \alpha/n$, the functions η that verify the Dini condition are those such that $\eta^{q/p} \in B_q$. Examples of these functions are given by $\eta(t) = t^r$ for every $1 \le r < p$, or $\eta(t) = t^r(1 + \log^+ t)^\delta$ and $\eta(t) = t^r(1 + \log(1 + \log^+ t))^\delta$ for every $\delta > 0$ and $1 \le r < p$. Other examples are functions with negative exponents on the logarithm, like $\eta(t) = t^r(1 + \log^+ t)^{-\delta}$ for every $1 \le r < p$ and $\delta \ge 0$. For $\mathcal{A}$ and $\mathcal{B}$ of $L \log L$ type, it is easy to check that $\eta(t) = t^r$ with $1 \le r < n/\alpha$ also verifies the Dini condition if

$$\mathcal{A}(t) = t^p(1 + \log^+ t)^\epsilon, \quad \mathcal{B}(t) = t^{\frac{np}{n-\alpha p}}(1 + \log^+ t)^\gamma$$

where $r < p < n/\alpha$ and $\gamma \ge n\epsilon/(n - \alpha p)$ for $\epsilon, \gamma \ge 0$.

Even though our original purpose was to give boundedness properties on Orlicz spaces defined over $\mathbb{R}^n$, the use of the ideas of [14] in the proof of the theorem above also allowed us to obtain the following characterization over finite measure domains, generalizing the result for M_α given in that article.

Theorem 1.5 *Let Ω be an open subset of $\mathbb{R}^n$ with $|\Omega| < \infty$. Let $0 < \alpha < n$, $\mathcal{A}$, $\mathcal{B}$, η and ξ be as in Theorem 1.1 such that the functions $t \mapsto t^{1-\alpha/n}a(t)^{-\alpha/n}$ and $t \mapsto \eta^{-1}(\mathcal{A}(t))/t$ are increasing and tend to infinity when $t \to \infty$, and $\mathcal{B}$ is of positive lower-type. The following statements are equivalent.*

(i) *There exist positive constants C_1 and C_2 such that*

$$\int_1^{C_1 t^{1-\alpha/n}a(t)^{-\alpha/n}} \frac{b(\lambda)}{\lambda}\,\xi'\left(\frac{C_1 t^{1-\alpha/n}a(t)^{-\alpha/n}}{\lambda}\right)d\lambda \le C_2 t^{\alpha/n}a(t)^{1+\alpha/n},$$

for every $t \ge 1$.

(ii) $M_{\alpha,\eta} : L^{\mathcal{A}}(\Omega) \hookrightarrow L^{\mathcal{B}}(\Omega)$.

Remark 1.6 Although Theorem 1.5 is proved for the non-centered fractional maximal operator, a similar result can be obtained for its centered version, let us say $M^c_{\alpha,\eta}$, as it will be obvious from the proof of the theorem above and the fact that $M^c_{\alpha,\eta}f(x) \le M_{\alpha,\eta}f(x)$, even though they are not equivalent.

Then, in the particular case of the fractional maximal operator M_α, that is $\eta(t) = t$, we have proved that, under the corresponding hypotheses on $\mathcal{A}$ and $\mathcal{B}$, M_α and M^c_α are bounded between the associated Orlicz spaces if and only if there exist two positive constants C_1 and C_2 such that

$$\int_1^{C_1 t^{1-\alpha/n}a(t)^{-\alpha/n}} \frac{b(\lambda)}{\lambda^{\frac{n}{n-\alpha}}}\,d\lambda \le C_2 a(t)^{\frac{n}{n-\alpha}} \tag{1.1}$$

for every $t \ge 1$. This result was proved in [14] for M^c_α and, thus, our result is a generalization of the corresponding result given in that article. Notice that, if $\eta(t) = t$, $\eta^{-1}(\mathcal{A}(t))/t = \mathcal{A}(t)/t$ and from the fact that $a(t/2)/2 \le \mathcal{A}(t)/t \le a(t)$, the hypotheses of Theorem 1.5 are equivalent to the corresponding ones of [14].

From Theorem 1.1 we can obtain the next characterization of the boundedness of $M_{\alpha,\eta}$ over $\mathbb{R}^n$ in terms of the continuity properties of the classical fractional maximal operator with weights, that is, nonnegative and locally integrable functions over $\mathbb{R}^n$. The theorem below gives the analogous result in the fractional case for M_η in [15]. When $\mathcal{A}(t) = \mathcal{B}(t) = t^p$, $1 < p < \infty$, the corresponding result was proved in [24] for $\alpha = 0$. The interest in this kind of characterization appears when it is required a dual version of Fefferman–Stein's inequality for nonlinear operators. Inequalities in the spirit of (1.2) were also studied in [26] for $0 \le \alpha < n$ and η certain power function.

Theorem 1.7 *Let $0 < \alpha < n$ and let $\mathcal{A}$, $\mathcal{B}$, η and ξ be as in Theorem 1.1. The following statements are equivalent.*

(i) $M_{\alpha,\eta} : L^{\mathcal{A}}(\mathbb{R}^n) \hookrightarrow L^{\mathcal{B}}(\mathbb{R}^n)$.

(ii) *There exists a positive constant C such that*

$$\left\| \frac{M_\alpha f}{M_{\tilde{\eta}} u} \right\|_{L^{\mathcal{B}}(\mathbb{R}^n)} \le C \left\| \frac{f}{u} \right\|_{L^{\mathcal{A}}(\mathbb{R}^n)}, \tag{1.2}$$

for every $f \ge 0$ and for every weight u, where $\tilde{\eta}$ is the complementary Young function of η.

It is well known that the operator $M_{\alpha,\eta}$ controls a large class of fractional type operators (see, for instance, [3], [4], [7], [10], [11] and [13]). As a consequence of Theorem 1.1, in §4 we will derive boundedness results for this class of operators. The classical example is the k-th order commutator of I_α with symbol $\mathfrak{b} \in BMO$, that is,

$$I^k_{\alpha,\mathfrak{b}} f(x) = \int_{\mathbb{R}^n} (\mathfrak{b}(x) - \mathfrak{b}(y))^k \frac{f(y)}{|x-y|^{n-\alpha}}\, dy,$$

where $k \in \mathbb{N} \cup \{0\}$ and $\mathfrak{b}$ satisfies

$$||\mathfrak{b}||_{BMO} := \sup_B \frac{1}{|B|} \int_B \left| \mathfrak{b}(x) - \frac{1}{|B|}\int_B \mathfrak{b} \right| dx < \infty.$$

Clearly, $I^0_{\alpha,\mathfrak{b}} = I_\alpha$. For this operator we prove the following characterization.

Theorem 1.8 *Let α, $\mathcal{A}$ and $\mathcal{B}$ be as in Theorem 1.1, $\mathfrak{b} \in BMO$ and $k \in \mathbb{N} \cup \{0\}$. Suppose that $\mathcal{B} \in \Delta_2$ and has positive lower-type $q > n/(n-\alpha)$, and for $k > 0$, that also $\widetilde{\mathcal{B}}$ satisfy the Δ_2-condition. Then, the following statements are equivalent.*

(i) *There exist positive constants C_1 and C_2 such that*

$$\int_0^{C_1 t^{1-\alpha/n} a(t)^{-\frac{\alpha}{n}}} \frac{b(\lambda)}{\lambda^{\frac{n}{n-\alpha}}} \left(1 + \log^+\left(\frac{C_1 t^{1-\alpha/n} a(t)^{-\frac{\alpha}{n}}}{\lambda}\right)\right)^{\frac{kn}{n-\alpha}} d\lambda \le C_2 a(t)^{\frac{n}{n-\alpha}}$$

holds for every $t > 0$.

(ii) *There exists a positive constant $K = K(||\mathfrak{b}||_{BMO})$ such that the inequality*

$$\left\| I^k_{\alpha,\mathfrak{b}} f \right\|_{L^{\mathcal{B}}(\mathbb{R}^n)} \le K ||f||_{L^{\mathcal{A}}(\mathbb{R}^n)}$$

holds for every $f \in L^\infty_c(\mathbb{R}^n)$.

Remark 1.9 The Δ_2-condition on $\mathcal{B}$ and $\widetilde{\mathcal{B}}$, and the positive lower-type on $\mathcal{B}$ are not really necessary for the implication (ii) ⇒ (i) in the theorem above.

Remark 1.10 Note that the Dini condition (i) is the same as condition (i) of Theorem 1.1 with $\xi(t) = \eta(t)^{\frac{n}{n-\alpha}}$, where $\eta(t) = t(1 + \log^+ t)^k$ is the Young function that defines the fractional maximal operator $M_{\alpha,\eta}$ that controls $I^k_{\alpha,\mathfrak{b}}$ in some sense (see Theorem 4.8).

2 Preliminaries

Before we proceed with the proofs of the main results, we shall introduce some preliminary definitions and properties concerning Orlicz spaces.

Let Ω be an open subset of $\mathbb{R}^n$. Given an increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$, we define the Orlicz space $L^\Phi(\Omega)$ as the set of all measurable functions for which there exists a positive number λ such that

$$\int_\Omega \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1.$$

This definition induces the Luxemburg norm for this space, given by

$$||f||_{L^\Phi(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\},$$

and $\left(L^\Phi(\Omega), \|\cdot||_{L^\Phi(\Omega)}\right)$ is a Banach space. For more information and properties of these spaces see, for instance, [25]. Clearly, when $\Phi(t) = t^p$ with $1 \le p < \infty$, we recover the norm $||f||_{L^p(\Omega)}$.

If Φ is an N-function, that is, Φ is a Young function that satisfies

$$\lim_{t\to 0^+} \frac{\Phi(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to +\infty} \frac{\Phi(t)}{t} = +\infty,$$

its complementary N-function $\widetilde{\Phi}$ can be defined by means of the inequalities

$$t \le \Phi^{-1}(t)\widetilde{\Phi}^{-1}(t) \le 2t, \ \forall\, t > 0,$$

and the following generalization of Hölder's inequality

$$\int_\Omega |fg| \le 2||f||_\Phi ||g||_{\widetilde{\Phi}} \tag{2.1}$$

holds (see [23]). It is easy to see that the above inequality allows to prove that

$$||f||_{L^\Phi(\Omega)} \approx \sup_{||g||_{L^{\widetilde{\Phi}}(\Omega)} \le 1} \int_\Omega f(x)g(x)\, dx, \tag{2.2}$$

whenever the left-hand side is finite (see [25]). This gives an important tool in order to obtain boundedness results in §4.

There is also a version of Hölder's inequality for the Luxemburg-type averages

$$\frac{1}{|B|}\int_B |fg| \le 2||f||_{\Phi,B}||g||_{\widetilde{\Phi},B}, \tag{2.3}$$

where $||\cdot||_{\Phi,B}$ were defined in the Introduction. Although $\Phi(t) = t$ is not an N-function, inequalities (2.1) and (2.3) still hold, where $||\cdot||_{\widetilde{\Phi}}$ and $||\cdot||_{\widetilde{\Phi},B}$ must be understood as $||\cdot||_{L^\infty(\mathbb{R}^n)}$ and $||\cdot||_{L^\infty(B)}$ respectively.

Moreover, there is a further generalization of the inequality above. If Φ, Ψ and Θ are nonnegative, nondecreasing and left-continuous functions satisfying the relation $\Phi^{-1}(t)\Psi^{-1}(t) \le \Theta^{-1}(t)$, the following generalized Hölder's inequality, proved in [23], holds:

$$||fg||_{\Theta,B} \le 2||f||_{\Phi,B}||g||_{\Psi,B}. \tag{2.4}$$

We must also consider certain subclasses of Young functions. We say that a Young function Φ is in the class Δ_2, or satisfies the Δ_2-condition if $\Phi(2t) \le C\Phi(t)$ for certain positive constant C and every $t \ge 0$. This is equivalent to say that Φ has finite upper-type, that is, there exist constants $C > 0$ and $0 < q < \infty$ such that $\Phi(st) \le Cs^q\Phi(t)$ for every $s \ge 1$ and every $t \ge 0$. Similarly, we can define the positive lower-type: there exist constants $C > 0$ and $0 < q < \infty$ such that $\Phi(st) \le Cs^q\Phi(t)$ for every $0 \le s \le 1$ and every $t \ge 0$.

3 Proofs of the main results

In this section we will give the proofs of Theorems 1.1, 1.5 and 1.7. We shall postpone the proof of Theorem 1.8 until the next section.

An auxiliary result that we will be using in the proof of Theorems 1.1 and 1.5 is the following modular inequality for the generalized maximal operator M_ξ, ξ a submultiplicative Young function, that can be easily deduced from the proof of [15, Theorem 2.3]. When $M_\xi = M$, the analogous result was already proved in [22].

Theorem 3.1 *Let ψ, ϕ be two nonnegative functions and let $\Psi(t) = \int_0^t \psi(s)\,ds$ and $\Phi(t) = \int_0^t \phi(s)\,ds$. Let Ω be an open subset of $\mathbb{R}^n$ and let $D_0 = t_0 = 0$ if $|\Omega| = +\infty$ and $D_0 = \Phi(1)|\Omega|$ and $t_0 = 1$ if $|\Omega| < \infty$. If ξ is a submultiplicative Young function such that for every $t > t_0$*

$$\int_{t_0}^{D_1 t} \frac{\phi(\lambda)}{\lambda}\,\xi'\left(\frac{D_1 t}{\lambda}\right) d\lambda \le D_2 \psi(D_2 t), \tag{3.1}$$

for some constants $D_1 > 1$ and $D_2 > 0$, there exists a positive constant D such that

$$\int_\Omega \Phi(M_\xi f(x))\,dx \le D_0 + D \int_\Omega \Psi(D|f(x)|)\,dx.$$

Proof of Theorem 1.1. We will first show that (i) implies (ii). By homogeneity, it is enough to consider a function $f \in L^{\mathcal{A}}(\mathbb{R}^n)$ with $||f||_{L^{\mathcal{A}}(\mathbb{R}^n)} = 1$. For such a function, we need to find a positive constant C, independent of f, such that

$$\int_{\mathbb{R}^n} \mathcal{B}\left(\frac{M_{\alpha,\eta} f(x)}{C}\right) dx \le 1.$$

Let us write $|f(x)| = g(x)h(x)$ where

$$g(x) = |f(x)|\mathcal{A}(|f(x)|)^{-\alpha/n}\chi_{\{f\neq 0\}}(x) \quad \text{and} \quad h(x) = \mathcal{A}(|f(x)|)^{\alpha/n}\chi_{\{f\neq 0\}}(x).$$

Since $\eta^{-1}(t) = t^{\alpha/n}\xi^{-1}(t)$, from Hölder's inequality (2.4) we have that

$$M_{\alpha,\eta} f(x) \le 2M_\xi(g)(x)||h||_{L^{n/\alpha}(\mathbb{R}^n)} \le 2M_\xi(g)(x), \quad x \in \mathbb{R}^n, \tag{3.2}$$

since, from the hypothesis on f, $||h||_{L^{\mathcal{A}}(\mathbb{R}^n)} \le 1$. Then,

$$\int_{\mathbb{R}^n} \mathcal{B}\left(\frac{M_{\alpha,\eta} f(x)}{C}\right) dx \le \int_{\mathbb{R}^n} \mathcal{B}\left(\frac{M_\xi g(x)}{C/2}\right) dx = \int_{\mathbb{R}^n} \mathcal{B}\left(M_\xi(2g/C)(x)\right) dx. \tag{3.3}$$

Let us now consider the function

$$c_\xi(t) = \int_0^t \frac{b(\lambda)}{\lambda}\xi'\left(\frac{t}{\lambda}\right) d\lambda, \tag{3.4}$$

which is well-defined on $[0, \infty)$. In fact, by condition (i), $c_\xi(t_1) < \infty$ for some t_1, and thus, $c_\xi(t) < \infty$ for every $0 \le t \le t_1$ since ξ' is increasing. For $t \ge t_1$, since η is submultiplicative, it is easy to see that η verifies

$$\frac{\eta(t)}{t} \le \eta'(t) \le \frac{\eta(2t)}{t}. \tag{3.5}$$

From the definition of ξ, one can show that ξ is also submultiplicative and satisfies (3.5). This yields

$$\xi'(st) \le \frac{\xi(2st)}{st} \le \frac{\xi(2s)\xi(t)}{st} = 2\frac{\xi(2s)}{2s}\frac{\xi(t)}{t} \le 2\xi'(2s)\xi'(t).$$

From this estimate and the fact that $c_\xi(t_1) < \infty$

$$\begin{aligned} c_\xi(t) &= \int_0^{t_1} \frac{b(\lambda)}{\lambda}\xi'\left(\frac{t}{\lambda}\right) d\lambda + \int_{t_1}^t \frac{b(\lambda)}{\lambda}\xi'\left(\frac{t}{\lambda}\right) d\lambda \\ &\le \xi'\left(\frac{2t}{t_1}\right) c_\xi(t_1) + \int_{t_1}^t \frac{b(\lambda)}{\lambda}\xi'\left(\frac{t}{\lambda}\right) d\lambda < \infty, \end{aligned}$$

where the last term is finite since the integrand is continuous.

Then, by taking $\mathcal{C}_\xi(t) = \int_0^t c_\xi(s)\,ds$, condition (3.1) of Theorem 3.1 trivially holds with $\phi = b$ and $\psi = c_\xi$, and any constants $D_1 > 1$ and $D_2 \ge D_1$. Thus, from that theorem, there exists $D > 0$ such that

$$\int_{\mathbb{R}^n} \mathcal{B}\left(M_\xi(2g/C)(x)\right) dx \le D \int_{\mathbb{R}^n} \mathcal{C}_\xi(2Dg(x)/C)\,dx. \tag{3.6}$$

Let $C = \max\{2D/C_1, 2D^2C_2\}$, where C_1 and C_2 are the constants of (i). From (3.3), (3.6) and using the fact that $\mathcal{C}_\xi(t) \le tc_\xi(t)$, it follows that

$$\begin{aligned}\int_{\mathbb{R}^n} \mathcal{B}\left(\frac{M_{\alpha,\eta} f(x)}{C}\right) dx &\le D \int_{\mathbb{R}^n} \mathcal{C}_\xi(2Dg(x)/C)\, dx \\ &\le D \int_{\mathbb{R}^n} \frac{2Dg(x)}{C} c_\xi \left(\frac{2Dg(x)}{C}\right) dx \\ &\le \frac{2D^2}{C} \int_{\mathbb{R}^n} g(x) c_\xi \left(C_1 g(x)\right) dx.\end{aligned}$$

Note that, from condition (i) and (3.4),

$$c_\xi(C_1 t\mathcal{A}(t)^{-\alpha/n}) = \int_0^{C_1 t\mathcal{A}(t)^{-\alpha/n}} \frac{b(\lambda)}{\lambda} \xi' \left(\frac{C_1 t\mathcal{A}(t)^{-\alpha/n}}{\lambda}\right) d\lambda \le C_2 \frac{\mathcal{A}(t)^{1+\alpha/n}}{t}, \quad \forall t > 0.$$

Then, by applying this inequality to $t = |f(x)| > 0$,

$$\begin{aligned} g(x)c_\xi\left(C_1 g(x)\right) &= |f(x)|\mathcal{A}(|f(x)|)^{-\alpha/n} c_\xi\left(C_1|f(x)|\mathcal{A}(|f(x)|)^{-\alpha/n}\right) \\ &\le |f(x)|\mathcal{A}(|f(x)|)^{-\alpha/n} \frac{\mathcal{A}(|f(x)|)^{1+\alpha/n}}{|f(x)|} = C_2\mathcal{A}(|f(x)|). \end{aligned} \tag{3.7}$$

Finally, from this estimate and the definition of the constant C, we get

$$\int_{\mathbb{R}^n} \mathcal{B}\left(\frac{M_{\alpha,\eta} f(x)}{C}\right) dx \le \frac{2D^2C_2}{C} \int_{\mathbb{R}^n} \mathcal{A}(|f(x)|)dx \le \int_{\mathbb{R}^n} \mathcal{A}(|f(x)|)\, dx \le 1.$$

In order to prove the converse, let us now consider $\delta > 0$ and $f_\delta = \chi_{B(0,\delta)}$. Then, $||f_\delta||_{L^{\mathcal{A}}(\mathbb{R}^n)} = 1/\mathcal{A}^{-1}(\omega_n^{-1}\delta^{-n})$, where $\omega_n = |B(0,1)|$. We will estimate the measure of the set $\{x \in \mathbb{R}^n : M_{\alpha,\eta} f_\delta(x) > s\}$ for certain values of $s > 0$.

By considering $0 < s < \omega_n^{\alpha/n}\delta^\alpha/\xi^{-1}(2^{n+1})$, from the definition of ξ^{-1} we get

$$\begin{aligned} |\{x \in \mathbb{R}^n : M_{\alpha,\eta} f_\delta(x) > s\}| &\ge \left|\left\{x \in \mathbb{R}^n : \delta < |x| \text{ and } |B(x, 2|x|)|^{\alpha/n} ||f_\delta||_{\eta, B(x,2|x|)} > s\right\}\right| \\ &= \left|\left\{x \in \mathbb{R}^n : \delta < |x| \text{ and } \frac{\omega_n^{\alpha/n}(2|x|)^\alpha}{\eta^{-1}((2|x|/\delta)^n)} > s\right\}\right| \\ &= \left|\left\{x \in \mathbb{R}^n : \delta < |x| \text{ and } \frac{\omega_n^{\alpha/n}\delta^\alpha}{\xi^{-1}((2|x|/\delta)^n)} > s\right\}\right| \\ &= \left|\left\{x \in \mathbb{R}^n : \delta < |x| < \frac{\delta}{2}\xi\left(\frac{\omega_n^{\alpha/n}\delta^\alpha}{s}\right)^{\frac{1}{n}}\right\}\right| \\ &= \frac{\omega_n\delta^n}{2^n}\xi\left(\frac{\omega_n^{\alpha/n}\delta^\alpha}{s}\right) - \omega_n\delta^n \end{aligned}$$

where we have used that, for each $\delta > 0$, the set $\left\{x \in \mathbb{R}^n : \delta < |x| < \frac{\delta}{2}\xi\left(\omega_n^{\alpha/n}\delta^\alpha/s\right)^{1/n}\right\}$ is non-empty since ξ^{-1} is increasing and $s < \omega_n^{\alpha/n}\delta^\alpha/\xi^{-1}(2^n)$. By using again that $s < \omega_n^{\alpha/n}\delta^\alpha/\xi^{-1}(2^{n+1})$ and the property $\xi'(t/2)/2 \le \xi(t)/t$, we obtain

$$|\{x \in \mathbb{R}^n : M_{\alpha,\eta} f_\delta(x) > s\}| \ge \frac{\omega_n\delta^n}{2^{n+1}}\xi\left(\frac{\omega_n^{\alpha/n}\delta^\alpha}{s}\right) \ge \frac{\omega_n^{1+\frac{\alpha}{n}}\delta^{n+\alpha}}{2^{n+2}s}\xi'\left(\frac{\omega_n^{\alpha/n}\delta^\alpha}{2s}\right). \tag{3.8}$$

Now, from (ii) and the previous estimate we get

$$\begin{aligned}1 &\geq \int_{\mathbb{R}^n} \mathcal{B}\left(\frac{M_{\alpha,\eta} f_\delta(x)}{K||f_\delta||_{L^{\mathcal{A}}(\mathbb{R}^n)}}\right) dx \\ &= \int_0^\infty b(\lambda)|\{x \in \mathbb{R}^n : M_{\alpha,\eta} f_\delta(x) > \lambda K||f_\delta||_{L^{\mathcal{A}}(\mathbb{R}^n)}\}|\, d\lambda \\ &\geq \frac{\omega_n^{1+\frac{\alpha}{n}} \delta^{n+\alpha}}{2^{n+2} K||f_\delta||_{L^{\mathcal{A}}(\mathbb{R}^n)}} \int_0^{\frac{\omega_n^{\alpha/n}\delta^\alpha}{K\xi^{-1}(2^{n+1})||f_\delta||_{L^{\mathcal{A}}(\mathbb{R}^n)}}} \frac{b(\lambda)}{\lambda} \xi'\left(\frac{\omega_n^{\alpha/n}\delta^\alpha}{2\lambda K||f_\delta||_{L^{\mathcal{A}}(\mathbb{R}^n)}}\right) d\lambda \\ &= \frac{\omega_n^{1+\frac{\alpha}{n}} \delta^{n+\alpha} \mathcal{A}^{-1}(\omega_n^{-1}\delta^{-n})}{2^{n+2} K} \int_0^{\frac{\omega_n^{\alpha/n}\delta^\alpha \mathcal{A}^{-1}(\omega_n^{-1}\delta^{-n})}{K\xi^{-1}(2^{n+1})}} \frac{b(\lambda)}{\lambda} \xi'\left(\frac{\omega_n^{\alpha/n}\delta^\alpha \mathcal{A}^{-1}(\omega_n^{-1}\delta^{-n})}{2K\lambda}\right) d\lambda \\ &\geq \frac{\omega_n^{1+\frac{\alpha}{n}} \delta^{n+\alpha} \mathcal{A}^{-1}(\omega_n^{-1}\delta^{-n})}{2^{n+2} K} \int_0^{C_1\omega_n^{\alpha/n}\delta^\alpha \mathcal{A}^{-1}(\omega_n^{-1}\delta^{-n})} \frac{b(\lambda)}{\lambda} \xi'\left(\frac{C_1\omega_n^{\alpha/n}\delta^\alpha \mathcal{A}^{-1}(\omega_n^{-1}\delta^{-n})}{\lambda}\right) d\lambda.\end{aligned} \tag{3.9}$$

where $C_1 = 1/\big(K\xi^{-1}\big(2^{n+1}\big)\big)$. By taking $t = \mathcal{A}^{-1}\big(\omega_n^{-1}\delta^{-n}\big) > 0$ since $\mathcal{A}^{-1}$ is positive on $(0, \infty)$, we get that

$$\int_0^{C_1 t\mathcal{A}(t)^{-\frac{\alpha}{n}}} \frac{b(\lambda)}{\lambda} \xi'\left(\frac{C_1 t\mathcal{A}(t)^{-\frac{\alpha}{n}}}{\lambda}\right) d\lambda \leq C_2 \frac{\mathcal{A}(t)^{1+\frac{\alpha}{n}}}{t},$$

for every $t > 0$ and for some positive constants C_1 and C_2, that is, condition (i) holds. □

Proof of Theorem 1.5. Let us first prove (i) ⇒ (ii). Fix $f \in L^{\mathcal{A}}(\Omega)$ with $||f||_{L^{\mathcal{A}}(\Omega)} = 1$. Then, it is enough to show that there exists a positive constant C, independent of f, for which

$$\int_\Omega \mathcal{B}\left(\frac{M_{\alpha,\eta} f(x)}{C}\right) dx \leq 1.$$

We split $|f(x)| = f_1 + f_2$ where $f_1(x) = |f(x)|\chi_{\{|f|>1\}}(x)$ and $f_2(x) = |f(x)|\chi_{\{|f|\leq 1\}}(x)$. Thus, we have $M_{\alpha,\eta} f \leq M_{\alpha,\eta} f_1 + M_{\alpha,\eta} f_2$. Let us estimate each term separately.

For the case of f_2, we write as before $f_2 = g.h$ where $g(x) = |f(x)|\chi_{\{|f|\leq 1\}}(x)$ and $h(x) = \chi_{\{|f|\leq 1\}}(x)$. From the relation $\eta^{-1}(t) = t^{\alpha/n}\xi^{-1}(t)$, we can apply the generalized Hölder's inequality (2.4) to obtain

$$M_{\alpha,\eta} f_2(x) \leq 2M_\xi g(x)||h||_{L^{n/\alpha}(\Omega)} \leq 2|\Omega|^{\alpha/n} = 2K_1.$$

On the other hand, if we define the functions $g_1(x) = |f(x)|\mathcal{A}(|f(x)|)^{-\alpha/n}\chi_{\{|f|>1\}}(x)$ and $h_1(x) = \mathcal{A}(|f(x)|)^{\alpha/n}\chi_{\{|f|>1\}}(x)$, we have that $f_1 = g_1.h_1$ and we can use again Hölder's inequality (2.4) to get

$$M_{\alpha,\eta} f_1(x) \leq 2M_\xi(g_1)(x)||h_1||_{L^{n/\alpha}(\Omega)} \leq 2M_\xi(g_1)(x),$$

since $||h_1||_{L^{n/\alpha}(\Omega)} \leq 1$. Therefore,

$$\begin{aligned}\int_\Omega \mathcal{B}\left(\frac{M_{\alpha,\eta} f(x)}{C}\right) dx &\leq \int_\Omega \mathcal{B}\left(\frac{2M_\xi g_1(x) + 2K_1}{C}\right) dx \\ &\leq \int_\Omega \mathcal{B}\left(4C^{-1}\max\left\{M_\xi g_1(x), K_1\right\}\right) dx \\ &\leq \int_\Omega \mathcal{B}\left(M_\xi\big(4C^{-1}g_1\big)(x)\right) dx + |\Omega|\mathcal{B}\big(4C^{-1}K_1\big).\end{aligned} \tag{3.10}$$

Let us now consider the function

$$c_\xi(t) = \begin{cases} 0, & 0 \leq t < 1, \\ \int_1^t \frac{b(\lambda)}{\lambda} \xi'\left(\frac{t}{\lambda}\right) d\lambda, & t \geq 1, \end{cases} \tag{3.11}$$

which is clearly well-defined on $[0,\infty)$. From this definition, it follows immediately that condition (3.1) of Theorem 3.1 holds with $\phi = b$ and $\psi = c_\xi$, for any constants $D_1 > 1$ and $D_2 \geq D_1$. Thus, by taking $\mathcal{C}_\xi(t) = \int_0^t c_\xi(s)\,ds$, we have that

$$\int_\Omega \mathcal{B}\left(M_\xi\left(4C^{-1}g_1\right)(x)\right)dx \leq |\Omega|\mathcal{B}(1) + D\int_\Omega \mathcal{C}_\xi\left(4C^{-1}Dg_1(x)\right)dx. \tag{3.12}$$

Let $K_2 = |\Omega|\left(\mathcal{B}(1) + \mathcal{B}\left(4C^{-1}K_1\right)\right)$ and choose $C \geq 4D/C_1$ where C_1 is the constant appearing in (i). Then, from (3.10), (3.12) and using that $\mathcal{C}_\xi(t) \leq tc_\xi(t)$, we deduce that

$$\begin{aligned}\int_\Omega \mathcal{B}\left(\frac{M_{\alpha,\eta}f(x)}{C}\right)dx &\leq K_2 + D\int_\Omega \mathcal{C}_\xi(4C^{-1}Dg_1(x))\,dx\\ &\leq K_2 + D\int_\Omega 4C^{-1}Dg_1(x)c_\xi\left(4C^{-1}Dg_1(x)\right)dx\\ &\leq K_2 + C_1\int_\Omega g_1(x)c_\xi\left(C_1g_1(x)\right)dx.\end{aligned}$$

As in (3.7), it is easy to see from (i) and (3.11) that $g_1(x)c_\xi\left(C_1g_1(x)\right) \leq C_2\mathcal{A}(|f(x)|)$. Therefore, we obtain

$$\int_\Omega \mathcal{B}\left(\frac{M_{\alpha,\eta}f(x)}{C}\right)dx \leq K_2 + C_1C_2\int_\Omega \mathcal{A}(|f(x)|)\,dx \leq K_2 + C_1C_2.$$

Finally, if we know that $K_2 + C_1C_2 \leq 1$, the sufficiency of the Dini type condition (i) is proved. On the contrary, if $K_2 + C_1C_2 > 1$, since $\mathcal{B}$ has positive lower-type, there exist positive constants c, q such that $\mathcal{B}(st) \leq cs^q\mathcal{B}(t)$ for every $0 \leq s \leq 1$ and every $t \geq 0$. Then, it follows that

$$\int_\Omega \mathcal{B}\left(\frac{M_{\alpha,\eta}f(x)}{C[c(K_2 + C_1C_2)]^{1/q}}\right)dx \leq c\left(\frac{1}{[c(K_2 + C_1C_2)]^{1/q}}\right)^q\int_\Omega \mathcal{B}\left(\frac{M_{\alpha,\eta}f(x)}{C}\right)dx \leq 1.$$

Conversely, let us assume, without loss of generality, that there exists $x_0 \in \Omega$ such that the ball $B(x_0, 1)$ is contained in Ω. Let us consider $f_\delta = \chi_{B(x_0,\delta)}$ for $0 < \delta \leq 2^{-1/n}$ and, as in the Euclidean case, we will measure the level sets of $M_{\alpha,\eta}(f_\delta)$, $E_s = \{x \in \Omega : M_{\alpha,\eta}f_\delta(x) > s\}$.

Let $\omega_n^{\alpha/n}\delta^\alpha/\xi^{-1}((2/\delta)^n) < s < \omega_n^{\alpha/n}\delta^\alpha/\xi^{-1}\left(2^{n+1}\right)$. Then, by following the same arguments as in the proof of Theorem 1.1, we obtain

$$\begin{aligned}|E_s| &\geq \left|\left\{x \in \Omega : \delta < |x - x_0| < 1/4 \text{ and } |B(x, 2|x - x_0|)|^{\alpha/n}||f_\delta||_{\eta,B(x,2|x-x_0|)} > s\right\}\right|\\ &\geq \left|\left\{x \in \Omega : \delta < |x - x_0| < \frac{\delta}{2}\xi\left(\frac{\omega_n^{\alpha/n}\delta^\alpha}{s}\right)^{\frac{1}{n}}\right\}\right|.\end{aligned}$$

For the values of s considered, it is easy to see that

$$\delta < \frac{\delta}{2}\xi\left(\frac{\omega_n^{\alpha/n}\delta^\alpha}{s}\right)^{1/n} < 1.$$

This guarantees that the annulus above is non-empty and is entirely contained in Ω. Thus,

$$|\{x \in \Omega : M_{\alpha,\eta}f_\delta(x) > s\}| \geq \frac{\omega_n^{1+\frac{\alpha}{n}}\delta^{n+\alpha}}{2^{n+2}s}\xi'\left(\frac{\omega_n^{\alpha/n}\delta^\alpha}{2s}\right),$$

by applying similar arguments to those used in the case $\Omega = \mathbb{R}^n$.

By following the ideas in (3.9), assuming now that (ii) holds, we have

$$\begin{aligned}
1 &\geq \int_\Omega \mathcal{B}\left(\frac{M_{\alpha,\eta} f_\delta(x)}{K\|f_\delta\|_{L^{\mathcal{A}}(\Omega)}}\right) dx \\
&= \int_0^\infty b(\lambda)\left|\left\{x \in \Omega : M_{\alpha,\eta} f_\delta(x) > \lambda K \|f_\delta\|_{L^{\mathcal{A}}(\Omega)}\right\}\right| d\lambda \\
&\geq \frac{\omega_n^{1+\frac{\alpha}{n}}\delta^{n+\alpha}}{2^{n+2}K\|f_\delta\|_{L^{\mathcal{A}}(\mathbb{R}^n)}} \int_{\frac{\omega_n^{\alpha/n}\delta^\alpha}{K\xi^{-1}((2/\delta)^n)\|f_\delta\|_{L^{\mathcal{A}}(\mathbb{R}^n)}}}^{\frac{\omega_n^{\alpha/n}\delta^\alpha}{K\xi^{-1}(2^{n+1})\|f_\delta\|_{L^{\mathcal{A}}(\Omega)}}} \frac{b(\lambda)}{\lambda}\xi'\left(\frac{\omega_n^{\alpha/n}\delta^\alpha}{2\lambda K\|f_\delta\|_{L^{\mathcal{A}}(\Omega)}}\right) d\lambda \\
&\geq \frac{\omega_n^{1+\frac{\alpha}{n}}\delta^{n+\alpha}\mathcal{A}^{-1}(\omega_n^{-1}\delta^{-n})}{2^{n+2}K} \int_{\frac{2^\alpha\omega_n^{\alpha/n}\mathcal{A}^{-1}(\omega_n^{-1}\delta^{-n})}{K\eta^{-1}(2^n\delta^{-n})}}^{C_1\omega_n^{\alpha/n}\delta^\alpha\mathcal{A}^{-1}(\omega_n^{-1}\delta^{-n})} \frac{b(\lambda)}{\lambda}\xi'\left(\frac{C_1\omega_n^{\alpha/n}\delta^\alpha\mathcal{A}^{-1}(\omega_n^{-1}\delta^{-n})}{\lambda}\right) d\lambda,
\end{aligned}$$

where C_1 is the constant defined in the proof of Theorem 1.1. Then, if we take $t = \mathcal{A}^{-1}(\omega_n^{-1}\delta^{-n})$, we obtain the following condition

$$\int_{\frac{2^\alpha\omega_n^{\alpha/n}t}{K\eta^{-1}(2^n\omega_n\mathcal{A}(t))}}^{C_1 t\mathcal{A}(t)^{-\alpha/n}} \frac{b(\lambda)}{\lambda}\xi'\left(\frac{C_1 t\mathcal{A}(t)^{-\alpha/n}}{\lambda}\right) d\lambda \leq C_2\frac{\mathcal{A}(t)^{1+\alpha/n}}{t},$$

for each $t \geq t_1 = \mathcal{A}^{-1}(2\omega_n^{-1})$.

Let us now define

$$h(t) = \frac{2^\alpha\omega_n^{\alpha/n}t}{K\eta^{-1}(2^n\omega_n\mathcal{A}(t))},$$

which is smaller than $\frac{2^\alpha\omega_n^{\alpha/n}}{K\eta^{-1}(2^n\omega_n)}\frac{t}{\eta^{-1}(\mathcal{A}(t))}$ since η is submultiplicative. From the hypothesis on $\eta^{-1}(\mathcal{A}(t))/t$, we know that $h(t) \to 0$ when $t \to +\infty$. On the other hand, we also have that $t^{1-\alpha/n}a(t)^{-\alpha/n} \to \infty$ when $t \to +\infty$. Consequently, we can assert that there exists $t_0 \geq t_1$ such that, if $t \geq t_0$, then $h(t) \leq 1$ and $C_1 t^{1-\alpha/n}a(t)^{-\alpha/n} > 1$. For those values of t, by using that $\mathcal{A}(t) \leq ta(t)$, we get

$$\begin{aligned}
&\int_1^{C_1 t^{1-\alpha/n}a(t)^{-\alpha/n}} \frac{b(\lambda)}{\lambda}\xi'\left(\frac{C_1 t^{1-\alpha/n}a(t)^{-\alpha/n}}{\lambda}\right) d\lambda \\
&\quad\leq \int_1^{C_1 t\mathcal{A}(t)^{-\alpha/n}} \frac{b(\lambda)}{\lambda}\,\xi'\left(\frac{C_1 t\mathcal{A}(t)^{-\alpha/n}}{\lambda}\right) d\lambda \\
&\quad\leq C_2\frac{\mathcal{A}(t)^{1+\alpha/n}}{t} \leq C_2 t^{\alpha/n}a(t)^{1+\alpha/n}.
\end{aligned}$$

If $t_0 \leq 1$, the Dini type condition holds for every $t \geq 1$. If $t_0 > 1$, we claim that the above inequality can be extended to every $1 \leq t < t_0$. In fact, since $t^{1-\alpha/n}a(t)^{-\alpha/n}$ and $a(t)$ are increasing, and the above condition holds for $t = t_0$, if $1 \leq t < t_0$ it follows that

$$\begin{aligned}
&\int_1^{C_1 t^{1-\alpha/n}a(t)^{-\alpha/n}} \frac{b(\lambda)}{\lambda}\xi'\left(\frac{C_1 t^{1-\alpha/n}a(t)^{-\alpha/n}}{\lambda}\right) d\lambda \\
&\quad\leq \int_1^{C_1 t_0^{1-\alpha/n}a(t_0)^{-\alpha/n}} \frac{b(\lambda)}{\lambda}\xi'\left(\frac{C_1 t_0^{1-\alpha/n}a(t_0)^{-\alpha/n}}{\lambda}\right) d\lambda \\
&\quad\leq C_2 t_0^{\alpha/n}a(t_0)^{1+\alpha/n} = K_0 \\
&\quad\leq \frac{K_0}{a(1)^{1+\alpha/n}} t^{\alpha/n}a(t)^{1+\alpha/n}.
\end{aligned}$$

By taking $\widetilde{C_2} = \max\{C_2, K_0 a(1)^{-1-\alpha/n}\}$, we obtain

$$\int_1^{C_1 t^{1-\alpha/n} a(t)^{-\alpha/n}} \frac{b(\lambda)}{\lambda} \xi' \left(\frac{C_1 t^{1-\alpha/n} a(t)^{-\alpha/n}}{\lambda} \right) d\lambda \le \widetilde{C_2} t^{\alpha/n} a(t)^{1+\alpha/n}$$

for every $t \ge 1$.

Proof of Theorem 1.7. It is immediate from the pointwise inequality

$$M_\alpha f(x) \le 2 M_{\alpha,\eta}(f/u)(x) M_{\widetilde{\eta}} u(x), \; x \in \mathbb{R}^n,$$

(see (2.3)) that (i) $\Rightarrow$ (ii).

By virtue of Theorem 1.1, the converse follows if we show that (ii) implies the Dini type condition (i) of that theorem. Thus, we will estimate the measure of the sets $\{x \in \mathbb{R}^n : M_\alpha f(x) > s M_{\widetilde{\eta}} u(x)\}$, for certain values of $s > 0$.

Let us consider, for $\delta > 0$, $f = u = \chi_{B(0,\delta)}$. Thus, $||f/u||_{L^{\mathcal{A}}(\mathbb{R}^n)} = 1/\mathcal{A}^{-1}(\omega_n^{-1}\delta^{-n})$, where $\omega_n = |B(0,1)|$. For $|x| > 2\delta$, let us suppose, for the moment, that the following estimate holds:

$$M_{\widetilde{\eta}} u(x) = M_{\widetilde{\eta}} \chi_{B(0,\delta)}(x) \le 1/\widetilde{\eta}^{-1}((|x|/4\delta)^n). \tag{3.13}$$

Thus, by considering $0 < s < 6^{\alpha-n}\omega^{\alpha/n}\delta^\alpha/\xi^{-1}(1/(2^n-1))$ and using that

$$\frac{1}{\widetilde{\eta}^{-1}(t)} \le \frac{\eta^{-1}(t)}{t} = \xi^{-1}(t) t^{\alpha/n-1}, \; t > 0,$$

by following similar arguments as in the proof of Theorem 1.1, we can get

$$|\{x \in \mathbb{R}^n : M_\alpha f(x) > s M_{\widetilde{\eta}} u(x)\}| \ge \frac{2^{n-1} 6^{\alpha-n} \omega^{1+\alpha/n} \delta^{n+\alpha}}{s} \xi' \left(\frac{6^{\alpha-n}\omega^{\alpha/n}\delta^\alpha}{2s} \right).$$

Up to a constant, the inequality above is of the form of (3.8) obtained in Theorem 1.1. Therefore, it is clear that we can continue in the same way in order to have the desired Dini type condition.

Let us now prove (3.13). From the definition of $M_{\widetilde{\eta}}$,

$$M_{\widetilde{\eta}} \chi_{B(0,\delta)}(x) = \sup_{B \ni x : B \cap B(0,\delta) \neq \emptyset} \frac{1}{\widetilde{\eta}^{-1}\left(\frac{|B|}{|B \cap B(0,\delta)|}\right)} \le \sup_{B \ni x : B \cap B(0,\delta) \neq \emptyset} \frac{1}{\widetilde{\eta}^{-1}\left(\frac{|B|}{\omega_n \delta^n}\right)}.$$

If $B = B(y, R)$ is such that $B \cap B(0,\delta) \neq \emptyset$ and we consider $|x| > 2\delta$, it is clear that $2R > |x| - \delta > |x|/2$. Then, $R \ge |x|/4$ and

$$M_{\widetilde{\eta}} \chi_{B(0,\delta)}(x) \le \sup_{B \ni x : B \cap B(0,\delta) \neq \emptyset} \frac{1}{\widetilde{\eta}^{-1}((R/\delta)^n)} \le \frac{1}{\widetilde{\eta}^{-1}((|x|/4\delta)^n)}.$$ □

4 Applications of Theorem 1.1 to fractional type operators

We will now introduce a class of fractional integral operators of convolution type that can be controlled by the fractional maximal operator we are dealing with, which includes the classical fractional integral operator. This control will allow us to derive continuity properties on Orlicz spaces for these operators.

In [4] the authors consider fractional operators of convolution type of the form

$$T_\alpha f(x) = \int_{\mathbb{R}^n} K_\alpha(x-y) f(y)\, dy, \quad 0 \le \alpha < n, \tag{4.1}$$

where the kernel K_α satisfies a size type condition and a certain fractional Hörmander condition associated with a given Young function. We say that K_α satisfies the size condition S_α if there exists a positive constant C such that

$$\int_{|x| \sim s} |K_\alpha(x)|\, dx \le C s^\alpha,$$

where $|x| \sim s$ denotes the set $\{x \in \mathbb{R}^n : s < |x| \le 2s\}$.

The Hörmander type condition is defined as follows. For a given Young function Φ we say that $K_\alpha \in H_{\alpha,\Phi}$ if there exist $c \geq 1$ and $C > 0$ such that for every $y \in \mathbb{R}^n$ and $R > c|y|$

$$\sum_{m=1}^{\infty}(2^m R)^{n-\alpha}\|K_\alpha(\cdot - y) - K_\alpha(\cdot)\|_{\Phi,|x|\sim 2^m R} \leq C.$$

If $\Phi(t) = t^r$ with $1 \leq r < \infty$, the class $H_{\alpha,\Phi}$ will be denoted by $H_{\alpha,r}$. We can generalize the Hörmander type condition to L^∞ by considering the corresponding norms restricted to the sets $|x| \sim 2^m R$ and the class, in that case, will be denoted by $H_{\alpha,\infty}$.

Another class that we should take into account is the following. We say that $K_\alpha \in H^*_{\alpha,\infty}$ if there exist constants $C > 0$ and $c > 1$ such that

$$|K_\alpha(x-y) - K_\alpha(x)| \leq C\frac{|y|}{|x|^{n+1-\alpha}}, \quad |x| > c|y|.$$

All the mentioned classes satisfy the inclusions

$$H^*_{\alpha,\infty} \subset H_{\alpha,\infty} \subset H_{\alpha,r} \subset H_{\alpha,s} \subset H_{\alpha,1}$$

for every $1 \leq s \leq r \leq \infty$. Moreover, if Φ is a Young function satisfying either $\Phi(t) \leq Ct^r$ or $Ct^s \leq \Phi(t)$ for large values of t, then either $H_{\alpha,r} \subset H_{\alpha,\Phi}$ or $H_{\alpha,\Phi} \subset H_{\alpha,s}$ holds, respectively.

The interest in considering the classes $H_{\alpha,\Phi}$ for a general Young function Φ is justified in [21] in the case $\alpha = 0$. In that article, the authors show the existence of an operator whose kernel K belongs to every H_r, $1 \leq r < \infty$, but $K \notin H_\infty$. This implies that the mentioned operator can be controlled, in the norm of the L^p spaces, by any maximal operator $M_{r'}$, where $1/r + 1/r' = 1$ and $1 < r < \infty$, but it cannot be asserted that the same inequality holds with M, which is smaller than $M_{r'}$. However, that kernel K does belong to a generalized Hörmander class H_{Φ_ϵ} for every $\epsilon > 0$, where $\Phi_\epsilon(t) \approx e^{t^{1/\epsilon}}$. This says that $H_\infty \subsetneq H_{\Phi_\epsilon} \subset \cap_{1\leq r<\infty} H_r$, which yields that the control by means of $M_{r'}$ can be improved by replacing it with the maximal operator $M_{\widetilde{\Phi_\epsilon}} \approx M_{L(\log L)^\epsilon} \lesssim C_\epsilon M_{r'}$ for each $1 < r < \infty$. In other words, the Hörmander conditions associated with Young functions in the kernels of convolution type operators can give us better boundedness results for them since they can be controlled by smaller maximal operators.

As we said before, a classical example of an operator T_α of the form (4.1) is the fractional integral operator I_α, $0 < \alpha < n$, whose kernel $K_\alpha(x) = |x|^{\alpha-n}$ satisfies the condition S_α and the Hörmander type condition $H^*_{\alpha,\infty}$, as it can be easily checked.

Fractional integrals with less regular kernels than I_α are also known. For example, in [19], the author studied fractional integrals given by a multiplier. Specifically, if $m : \mathbb{R}^n \to \mathbb{R}$ is a given function, the multiplier operator T_m is defined, by means of the Fourier transform, as $\widehat{T_m f}(\zeta) = m(\zeta)\widehat{f}(\zeta)$ for f in the Schwartz class. Under some conditions on the derivatives of m, the operator T_m can be written as the limit of simpler convolution operators T_m^N. The associated kernels K_α^N are in the class $S_\alpha \cap H_{\alpha,r}$ with constant independent of N, for certain values of $r > 1$ determined by the properties on the function m.

Another example for this kind of fractional operators are fractional integrals with rough kernels, that is, convolution type operators with kernel $K_\alpha(x) = \Omega(x)|x|^{\alpha-n}$ where Ω is a function defined on the unit sphere S^{n-1} of $\mathbb{R}^n$ with integral zero (see, for example, [8] and [9]). By extending Ω to $\mathbb{R}^n \setminus \{0\}$ radially, the extension $\bar{\Omega}$ is an homogeneous function of degree 0. In [4, Proposition 4.2], the authors showed that the kernel of T_α satisfies $K_\alpha \in S_\alpha \cap H_{\alpha,\Phi}$, for certain Young function Φ, provided that $\Omega \in L^\Phi(S^{n-1})$ with

$$\int_0^1 \omega_\Phi(t)\,\frac{dt}{t} < \infty,$$

where ω_Φ is the L^Φ-modulus of continuity defined by

$$\omega_\Phi(t) = \sup_{|y|\leq t} ||\bar{\Omega}(\cdot + y) - \bar{\Omega}(\cdot)||_{\Phi,S^{n-1}} < \infty, \quad \forall t \geq 0.$$

The following result, proved in [4], shows the control of these fractional integral operators by a generalized fractional maximal operator associated with a certain N-function.

Theorem 4.1 ([4]) *Let $0 < \alpha < n$ and let Φ be an N-function. Let T_α be the fractional operator with kernel $K_\alpha \in S_\alpha \cap H_{\alpha,\Phi}$. Then, for any $0 < \delta < 1$, there exists a positive constant C_δ such that*

$$M_\delta^\sharp(T_\alpha f)(x) \le C_\delta M_{\alpha,\widetilde{\Phi}} f(x), \tag{4.2}$$

for every measurable function f, where

$$M^\sharp(g)(x) := \sup_{B\ni x} \inf_{a\in\mathbb{R}} \int_B |g(y) - a|\, dy,$$

and $M_\delta^\sharp g := M^\sharp\big(|g|^\delta\big)^{1/\delta}$.

Remark 4.2 If we consider T_α to be a fractional integral given by a multiplier, (4.2) holds with $M_{\alpha,r'} = (M_\alpha)_{r'}$ provided that the kernel belongs to the class $H_{\alpha,r}$. When T_α has a rough kernel with $\Omega \in L^\Phi\big(S^{n-1}\big)$, $\int_{S^{n-1}} \Omega = 0$ and $\int_0^1 \omega_\Phi(t)\frac{dt}{t} < \infty$, we obtain (4.2) with the corresponding N-function Φ.

When T_α is a fractional integral operator whose kernel satisfies the Hörmander condition $H_{\alpha,\infty}$ or the stronger condition $H^*_{\alpha,\infty}$, it was proved in [4] that the control is given by means of the fractional maximal operator M_α, as the next results shows. When $T_\alpha = I_\alpha$, the fractional integral operator, the mentioned estimate was already proved in [1].

Theorem 4.3 ([4]) *Let $0 < \alpha < n$ and let T_α be a fractional operator with kernel $K_\alpha \in S_\alpha \cap H_{\alpha,\infty}$ or $K_\alpha \in S_\alpha \cap H^*_{\alpha,\infty}$. Then, for any $0 < \delta < 1$, there exists a positive constant C_δ such that*

$$M_\delta^\sharp(T_\alpha f)(x) \le C_\delta M_\alpha f(x), \tag{4.3}$$

for every measurable function f.

Theorems 4.1 and 4.3 allow us to give sufficient conditions such that the operator T_α is bounded from $L^{\mathcal{A}}(\mathbb{R}^n)$ into $L^{\mathcal{B}}(\mathbb{R}^n)$.

Theorem 4.4 *For $0 < \alpha < n$ and an N-function η, let T_α be a fractional operator defined in* (4.1) *with kernel $K_\alpha \in S_\alpha \cap H_{\alpha,\tilde{\eta}}$. Let $\mathcal{A}$, $\mathcal{B}$ and ξ be as in Theorem 1.1 with $\mathcal{B} \in \Delta_2$, and suppose that (i) of Theorem 1.1 holds. Then*

$$||T_\alpha f||_{L^{\mathcal{B}}(\mathbb{R}^n)} \le C||f||_{L^{\mathcal{A}}(\mathbb{R}^n)},$$

for every $f \in L_c^\infty(\mathbb{R}^n)$, whenever the left-hand side is finite.

If $K_\alpha \in S_\alpha \cap H_{\alpha,\infty}$ or $K_\alpha \in S_\alpha \cap H^*_{\alpha,\infty}$, and we suppose that (1.1) *holds, the result is also true.*

In order to prove the theorem above, we shall use the following results, that will be also useful in all of the boundedness results of this section. The first one shows how can the previous control theorems be applied; the second one establishes a characterization of the boundedness of the Hardy–Littlewood maximal operator M on Orlicz spaces by means of a Δ_2-condition. This characterization was given in [12], although in [5] the authors showed that the same result holds by means of a Dini type condition which is equivalent to the condition given in [12]. Similar results were also obtained in [27].

Theorem 4.5 ([20]) *There exists a positive constant C such that for every locally integrable function g and every measurable function f satisfying $|\{x : |f(x)| > \lambda\}| < \infty$ for each $\lambda > 0$, the following inequality*

$$\int_{\mathbb{R}^n} |fg|\, dx \le C \int_{\mathbb{R}^n} M^\sharp f\, Mg\, dx$$

holds.

Theorem 4.6 ([12]) *Let Φ be an N-function. The following statements are equivalent.*

(i) *The Hardy–Littlewood maximal operator is bounded on $L^\Phi(\mathbb{R}^n)$,*
(ii) *$\widetilde{\Phi}$, the complementary function of Φ, satisfies the Δ_2-condition.*

We are now in position to prove Theorem 4.4.

Proof of Theorem 4.4. Fix $f \in L_c^\infty(\mathbb{R}^n)$ with $||T_\alpha f||_{L^{\mathcal{B}}(\mathbb{R}^n)} < \infty$. For $0 < \delta < 1$, let $\mathcal{B}_\delta(t) = \mathcal{B}(t^{1/\delta})$. Then, we can write

$$||T_\alpha f||_{L^{\mathcal{B}}(\mathbb{R}^n)} = \left\| |T_\alpha f|^\delta \right\|_{L^{\mathcal{B}_\delta}(\mathbb{R}^n)}^{1/\delta} = \sup_{||g||_{L^{\widetilde{\mathcal{B}_\delta}}(\mathbb{R}^n)} \le 1} \left(\int_{\mathbb{R}^n} |T_\alpha f|^\delta |g| \, dx \right)^{1/\delta}.$$

We wish to apply Theorem 4.5 with $|T_\alpha f|^\delta$ and $|g|$. In order to do so, we need to prove that $|\{x \in \mathbb{R}^n : |T_\alpha f| > \lambda\}| < \infty$ for every $\lambda > 0$. But this can be easily obtained from the Δ_2-condition on $\mathcal{B}$ and the hypothesis $||T_\alpha f||_{L^{\mathcal{B}}(\mathbb{R}^n)} < \infty$. In fact, the property on $\mathcal{B}$ says that $\mathcal{B}$ has finite upper-type q, for some $q > 0$, and then

$$\begin{aligned} |\{x \in \mathbb{R}^n : |T_\alpha f(x)| > \lambda\}| &\le \int_{\{x\in\mathbb{R}^n : |T_\alpha f(x)|>\lambda\}} \frac{|T_\alpha f(x)|}{\lambda} \, dx \\ &\le \frac{1}{\mathcal{B}(1)} \int_{\{x\in\mathbb{R}^n : |T_\alpha f(x)|>\lambda\}} \mathcal{B}\left(\frac{|T_\alpha f(x)|}{||T_\alpha f||_{L^{\mathcal{B}}(\mathbb{R}^n)}} \frac{||T_\alpha f||_{L^{\mathcal{B}}(\mathbb{R}^n)}}{\lambda} \right) dx \\ &\le \frac{1}{\mathcal{B}(1)} \min\left\{ 1, \frac{||T_\alpha f||^q_{L^{\mathcal{B}}(\mathbb{R}^n)}}{\lambda^q} \right\} \int_{\mathbb{R}^n} \mathcal{B}\left(\frac{|T_\alpha f(x)|}{||T_\alpha f||_{L^{\mathcal{B}}(\mathbb{R}^n)}} \right) dx \\ &\le \frac{||T_\alpha f||^q_{L^{\mathcal{B}}(\mathbb{R}^n)}}{\mathcal{B}(1)\lambda^q} < \infty. \end{aligned} \tag{4.4}$$

Thus, by Theorem 4.5, applying Hölder's inequality (2.1) and (4.2) we obtain that

$$\begin{aligned} ||T_\alpha f||_{L^{\mathcal{B}}(\mathbb{R}^n)} &\le C \sup_{||g||_{L^{\widetilde{\mathcal{B}_\delta}}(\mathbb{R}^n)} \le 1} \left(\int_{\mathbb{R}^n} M^\sharp\left(|T_\alpha f|^\delta\right) Mg \, dx \right)^{1/\delta} \\ &\le C \sup_{||g||_{L^{\widetilde{\mathcal{B}_\delta}}(\mathbb{R}^n)} \le 1} \left\| M^\sharp\left(|T_\alpha f|^\delta\right) \right\|^{1/\delta}_{L^{\mathcal{B}_\delta}(\mathbb{R}^n)} \|Mg\|^{1/\delta}_{L^{\widetilde{\mathcal{B}_\delta}}(\mathbb{R}^n)} \\ &= C \sup_{||g||_{L^{\widetilde{\mathcal{B}_\delta}}(\mathbb{R}^n)} \le 1} \left\| M^\sharp_\delta\left(|T_\alpha f|\right) \right\|_{L^{\mathcal{B}}(\mathbb{R}^n)} \|Mg\|^{1/\delta}_{L^{\widetilde{\mathcal{B}_\delta}}(\mathbb{R}^n)} \\ &\le C \sup_{||g||_{L^{\widetilde{\mathcal{B}_\delta}}(\mathbb{R}^n)} \le 1} \|M_{\alpha,\eta} f\|_{L^{\mathcal{B}}(\mathbb{R}^n)} \|Mg\|^{1/\delta}_{L^{\widetilde{\mathcal{B}_\delta}}(\mathbb{R}^n)} \end{aligned}$$

Since $\mathcal{B} \in \Delta_2$, also $\mathcal{B}_\delta \in \Delta_2$, and from Theorem 4.6 $||Mg||_{L^{\widetilde{\mathcal{B}_\delta}}(\mathbb{R}^n)} \le C\|g\|_{L^{\widetilde{\mathcal{B}_\delta}}(\mathbb{R}^n)}$. Thus, by using Theorem 1.1 we obtain

$$||T_\alpha f||_{L^{\mathcal{B}}(\mathbb{R}^n)} \le C \sup_{||g||_{L^{\widetilde{\mathcal{B}_\delta}}(\mathbb{R}^n)} \le 1} \|M_{\alpha,\eta} f\|_{L^{\mathcal{B}}(\mathbb{R}^n)} \|g\|^{1/\delta}_{L^{\widetilde{\mathcal{B}_\delta}}(\mathbb{R}^n)} \le C\|f\|_{L^{\mathcal{A}}(\mathbb{R}^n)}.$$

If $K_\alpha \in S_\alpha \cap H_{\alpha,\infty}$ or $K_\alpha \in S_\alpha \cap H^*_{\alpha,\infty}$, we can repeat the proof above by applying instead Theorem 4.3 which gives us the control of T_α by M_α, and we use the corresponding Dini condition (1.1). □

Remark 4.7 Notice that, as a particular case, the theorem above gives us the corresponding result for the operator I_α. Actually, if $\mathcal{B}$ has finite upper-type $q > n/(n-\alpha)$, $||I_\alpha f||_{L^{\mathcal{B}}}$ is finite, and, since its kernel is nonnegative, we can extend the result for every function in $L^{\mathcal{A}}(\mathbb{R}^n)$ by an approximation argument.

We will now introduce the higher order commutators of T_α, in order to state similar boundedness results for them. In the case of the commutators of I_α, we were able to obtain a characterization of their continuity properties on Orlicz spaces by means of a Dini type condition, as we showed in Theorem 1.8.

Given $\mathfrak{b} \in BMO$, the k-th order commutator of T_α for $k \in \mathbb{N} \cup \{0\}$ and $0 < \alpha < n$ is defined by

$$T^k_{\alpha,\mathfrak{b}} f(x) = \int_{\mathbb{R}^n} (\mathfrak{b}(x) - \mathfrak{b}(y))^k K_\alpha(x-y) f(y) \, dy.$$

When $k = 0$, $T^k_{\alpha,\mathfrak{b}} = T_\alpha$. We will now suppose that the kernel K_α satisfies the S_α condition and a Hörmander condition related to the order k, the condition $H_{\alpha,\Phi,k}$. This means that there exist $c \geq 1$ and $C > 0$ such that for every $y \in \mathbb{R}^n$ and $R > c|y|$

$$\sum_{m=1}^{\infty}(2^m R)^{n-\alpha} m^k \|K_\alpha(\cdot - y) - K_\alpha(\cdot)\|_{\Phi,|x|\sim 2^m R} \leq C.$$

We say that $K_\alpha \in H_{\alpha,\infty,k}$ if the norm in the condition above is taken over L^∞.

In [4], the authors also proved an estimate in the spirit of Theorem 4.1 for the commutators $T^k_{\alpha,\mathfrak{b}}$, which is given in the following result. In the case of the commutators of I_α, it was proved in [10] and [7] for $k = 1$ and in [3] for any $k \in \mathbb{N}$ in the more general framework of spaces of homogeneous type.

Theorem 4.8 ([4]) *Let $k \in \mathbb{N}$ and let Φ and Ψ be two N-functions such that $\widetilde{\Phi}^{-1}(t)\Psi^{-1}(t) \leq \varphi_k^{-1}(t)$, where $\varphi_k(t) = t(1 + \log^+ t)^k$. If $0 < \alpha < n$ and T_α is a fractional integral operator with kernel $K \in S_\alpha \cap H_{\alpha,\Psi,k}$, for any $\mathfrak{b} \in BMO$ and $0 < \delta < \epsilon < 1$, there exists $C = C_{\delta,\epsilon}$ such that for every $f \in L^\infty_c(\mathbb{R}^n)$*

$$M^\sharp_\delta\left(T^k_{\alpha,\mathfrak{b}} f\right)(x) \leq C \sum_{j=0}^{k-1} ||\mathfrak{b}||^{k-j}_{BMO} M_\epsilon(T^j_{\alpha,\mathfrak{b}} f)(x) + C||\mathfrak{b}||^k_{BMO} M_{\alpha,\widetilde{\Phi}} f(x).$$

If the kernel verifies $K \in S_\alpha \cap H_{\alpha,\infty,k}$ or $K \in S_\alpha \cap H^*_{\alpha,\infty}$, the above inequality takes the form

$$M^\sharp_\delta\left(T^k_{\alpha,\mathfrak{b}} f\right)(x) \leq C \sum_{j=0}^{k-1} ||\mathfrak{b}||^{k-j}_{BMO} M_\epsilon \left(T^j_{\alpha,\mathfrak{b}} f\right)(x) + C||\mathfrak{b}||^k_{BMO} M_{\alpha,\varphi_k} f(x).$$

The theorem above allow us to derive the following boundedness result for $T^k_{\alpha,\mathfrak{b}}$.

Theorem 4.9 *Let $0 < \alpha < n$, $k \in \mathbb{N} \cup \{0\}$ and N-functions η and Φ such that $\eta^{-1}(t)\Phi^{-1}(t) \leq \varphi_k^{-1}(t)$, being $\varphi_k(t) = t(1 + \log^+ t)^k$. Let T_α be the fractional operator defined in (4.1) with kernel $K \in S_\alpha \cap H_{\alpha,\Phi,k}$, and assume that T_α is bounded from $L^{p_0}(\mathbb{R}^n)$ into $L^{p_0}(\mathbb{R}^n)$, for some $p_0, q_0 > 1$. Let $\mathcal{A}$, $\mathcal{B}$ and ξ be as in Theorem 1.1 such that $\mathcal{B} \in \Delta_2$ and also $\widetilde{\mathcal{B}} \in \Delta_2$ for $k > 0$, and suppose that (i) of Theorem 1.1 holds. Then, for any $\mathfrak{b} \in BMO$*

$$\left\|T^k_{\alpha,\mathfrak{b}} f\right\|_{L^{\mathcal{B}}(\mathbb{R}^n)} \leq C(||\mathfrak{b}||_{BMO})||f||_{L^{\mathcal{A}}(\mathbb{R}^n)},$$

for every $f \in L^\infty_c(\mathbb{R}^n)$, whenever $||T_\alpha f||_{L^{\mathcal{B}}(\mathbb{R}^n)} < \infty$.

If $K_\alpha \in S_\alpha \cap H_{\alpha,\infty}$ or $K_\alpha \in S_\alpha \cap H^*_{\alpha,\infty}$, and we suppose that (i) of Theorem 1.1 holds with $\xi(t) = (\varphi_k(t))^{\frac{n}{n-\alpha}}$, the result is also true.

Remark 4.10 If $k = 0$ and we take $\Phi \approx \widetilde{\eta}$ in the theorem above, we recover Theorem 4.4.

Remark 4.11 The functions of the form $\mathcal{B}(t) = t^\beta(1 + \log^+ t)^\gamma$ with $\beta > 1$ and $\gamma \geq 0$ are in the hypotheses of the previous theorem, that is, both $\mathcal{B}$ and $\widetilde{\mathcal{B}}(t) \approx t^{\frac{\beta}{\beta-1}}/(1 + \log^+ t)^{\frac{\gamma}{\beta-1}}$ satisfy the Δ_2-condition.

Proof of Theorem 4.9. We will prove it by an induction argument. If $k = 0$, the result holds by virtue of Theorem 4.4 and Remark 4.10. Suppose now that the result holds for every $0 \leq j \leq k - 1$. We will prove it for k.

Let $||\mathfrak{b}||_{BMO} = 1$ and $f \in L^\infty_c(\mathbb{R}^n)$ such that $||T_\alpha f||_{L^{\mathcal{B}}(\mathbb{R}^n)} < \infty$, where the kernel of T_α satisfies $K \in S_\alpha \cap H_{\alpha,\Phi,k}$. We will first consider the case $\mathfrak{b} \in L^\infty(\mathbb{R}^n)$.

Let us prove that $\left\|T^k_{\alpha,\mathfrak{b}} f\right\|_{L^{\mathcal{B}}(\mathbb{R}^n)} < \infty$. By using the following formula

$$T^k_{\alpha,\mathfrak{b}} f(x) = \sum_{m=0}^{k} C_{m,k}\mathfrak{b}(x)^{k-m} T_\alpha(\mathfrak{b}^m f)(x), \tag{4.5}$$

and the fact that $\mathfrak{b} \in L^\infty(\mathbb{R}^n)$, we have

$$||T^k_{\alpha,\mathfrak{b}} f||_{L^{\mathcal{B}}(\mathbb{R}^n)} = \left\| \sum_{m=0}^{k} C_{m,k} \mathfrak{b}^{k-m} T_\alpha(\mathfrak{b}^m f) \right\|_{L^{\mathcal{B}}(\mathbb{R}^n)} \tag{4.6}$$

$$\le \sum_{m=0}^{k} C_{m,k} ||\mathfrak{b}||^{k-m}_{L^\infty(\mathbb{R}^n)} ||T_\alpha(\mathfrak{b}^m f)||_{L^{\mathcal{B}}(\mathbb{R}^n)}$$

$$\le \sum_{m=0}^{k} C_{m,k} ||\mathfrak{b}||^{k}_{L^\infty(\mathbb{R}^n)} ||T_\alpha f||_{L^{\mathcal{B}}(\mathbb{R}^n)} \lesssim ||\mathfrak{b}||^{k}_{L^\infty(\mathbb{R}^n)} ||T_\alpha f||_{L^{\mathcal{B}}(\mathbb{R}^n)} < \infty.$$

Then, we can apply Lemma 2.2 in order to get

$$\left\| T^k_{\alpha,\mathfrak{b}} f \right\|_{L^{\mathcal{B}}(\mathbb{R}^n)} \lesssim \sup_{||g||_{L^{\widetilde{\mathcal{B}_\delta}}(\mathbb{R}^n)} \le 1} \left(\int_{\mathbb{R}^n} \left| T^k_{\alpha,\mathfrak{b}} f \right|^\delta |g(x)|\, dx \right)^{1/\delta}. \tag{4.7}$$

On the other hand, proceeding as in (4.4), we obtain that for each $\lambda > 0$ the measure of the set $\{x \in \mathbb{R}^n : \left|T^k_{\alpha,\mathfrak{b}} f(x)\right| > \lambda\}$ is finite. Therefore, by Lemma 2.2, and using Theorems 4.5 and 4.6, we have

$$\left\| T^k_{\alpha,\mathfrak{b}} f \right\|_{L^{\mathcal{B}}(\mathbb{R}^n)} \le C \left\| M^\sharp_\delta \left(|T^k_{\alpha,\mathfrak{b}} f|\right) \right\|_{L^{\mathcal{B}}(\mathbb{R}^n)} \le C \sum_{j=0}^{k-1} \left\| M_\epsilon \left(T^j_{\alpha,\mathfrak{b}} f \right) \right\|_{L^{\mathcal{B}}(\mathbb{R}^n)} + \|M_{\alpha,\eta} f\|_{L^{\mathcal{B}}(\mathbb{R}^n)},$$

where in the last inequality we applied Theorem 4.8. From the hypotheses on $\mathcal{A}$, $\mathcal{B}$ and η, the last term is bounded by $||f||_{L^{\mathcal{A}}(\mathbb{R}^n)}$. The other terms are bounded by $\left\| T^j_{\alpha,\mathfrak{b}} f \right\|_{L^{\mathcal{B}}(\mathbb{R}^n)}$ for each $0 \le j \le k-1$ respectively, since $\widetilde{\mathcal{B}_\epsilon} \in \Delta_2$ for every $0 < \epsilon < 1$ and thanks to Theorem 4.6.

Now, notice that for every $0 \le j \le k-1$, $\varphi_j(t) \le \varphi_k(t)$, so

$$\eta^{-1}(t)\Phi^{-1}(t) \le \varphi_k^{-1}(t) \le \varphi_j^{-1}(t)$$

and $K \in S_\alpha \cap H_{\alpha,\Phi,k} \subset S_\alpha \cap H_{\alpha,\Phi,j}$. Then, for each $0 \le j \le k-1$, from the inductive hypothesis,

$$\left\| T^j_{\alpha,\mathfrak{b}} f \right\|_{L^{\mathcal{B}}(\mathbb{R}^n)} \le C||f||_{L^{\mathcal{A}}(\mathbb{R}^n)}. \tag{4.8}$$

Thus, $\left\| T^k_{\alpha,\mathfrak{b}} f \right\|_{L^{\mathcal{B}}(\mathbb{R}^n)} \le C||f||_{L^{\mathcal{A}}(\mathbb{R}^n)}$ for every $\mathfrak{b} \in L^\infty(\mathbb{R}^n)$, with constant C independent of $||\mathfrak{b}||_{L^\infty(\mathbb{R}^n)}$.

It is enough to show that we can extend the result for every $\mathfrak{b} \in BMO$. Define, for each $N \in \mathbb{N}$, the functions $\mathfrak{b}_N$ as

$$\mathfrak{b}_N(x) = \begin{cases} \mathfrak{b} & \text{if } -N \le \mathfrak{b}(x) < N, \\ N & \text{if } \mathfrak{b}(x) > N, \\ -N & \text{if } \mathfrak{b}(x) < -N. \end{cases}$$

It is easy to see that for each N, $|\mathfrak{b}_N(x) - \mathfrak{b}_N(y)| \le |\mathfrak{b}(x) - \mathfrak{b}(y)|$, which yields $||\mathfrak{b}_N||_{BMO} \le 2||\mathfrak{b}||_{BMO} = 2$. Moreover, since $\mathfrak{b}_N \in L^\infty(\mathbb{R}^n)$ and $f \in L^\infty_c(\mathbb{R}^n)$, we have that $(\mathfrak{b}_N)^m f \to \mathfrak{b}^m f$ when $N \to \infty$ on $L^{p_0}(\mathbb{R}^n)$ and, from the boundedness properties on T_α, $T_\alpha((\mathfrak{b}_N)^m f) \to T_\alpha(\mathfrak{b}^m f)$ when $N \to \infty$ on $L^{q_0}(\mathbb{R}^n)$. Thus, there is a subsequence such that both limits exist almost everywhere. By applying again (4.5) we obtain that $T^j_{\alpha,\mathfrak{b}_{N_l}} f \to T^j_{\alpha,\mathfrak{b}} f$ when $l \to \infty$ almost everywhere. Therefore, from the continuity of $\mathcal{B}$, Fatou's Lemma and the fact that $\left\| T^j_{\alpha,\mathfrak{b}_{N_l}} f \right\|_{L^{\mathcal{B}}(\mathbb{R}^n)} \le C||f||_{L^{\mathcal{A}}(\mathbb{R}^n)}$ we get

$$\int_{\mathbb{R}^n} \mathcal{B}\left(\frac{|T^j_{\alpha,\mathfrak{b}} f(x)|}{2||f||_{L^{\mathcal{A}}(\mathbb{R}^n)}} \right) dx = \int_{\mathbb{R}^n} \lim_{l\to\infty} \mathcal{B}\left(\frac{|T^j_{\alpha,\mathfrak{b}_{N_l}} f(x)|}{2||f||_{L^{\mathcal{A}}(\mathbb{R}^n)}} \right) dx$$

$$\le \liminf_{l\to\infty} \int_{\mathbb{R}^n} \mathcal{B}\left(\frac{|T^j_{\alpha,\mathfrak{b}_{N_l}} f(x)|}{2||f||_{L^{\mathcal{A}}(\mathbb{R}^n)}} \right) dx \le 1,$$

which means that $\left\|T^{j}_{\alpha,\mathfrak{b}} f\right\|_{L^{\mathcal{B}}(\mathbb{R}^n)} \leq 2\|f\|_{L^{\mathcal{A}}(\mathbb{R}^n)} < \infty$. Finally, by the homogeneity of the norm we have that $\left\|T^{j}_{\alpha,\mathfrak{b}} f\right\|_{L^{\mathcal{B}}(\mathbb{R}^n)} \leq 2\|\mathfrak{b}\|^{j}_{BMO}\|f\|_{L^{\mathcal{A}}(\mathbb{R}^n)} < \infty$ for every $0 \leq j \leq k-1$ as desired.

If $K \in S_\alpha \cap H_{\alpha,\infty,k}$ or $K \in S_\alpha \cap H^*_{\alpha,\infty}$, the only difference we will find in the previous proof is that the maximal operator that controls T_α is the fractional maximal operator M_{α,φ_k}, as Theorem 4.8 shows. It is easy to see that the function $\xi(t) = (\varphi_k(t))^{\frac{n}{n-\alpha}}$ satisfies the relation $\xi^{-1}(t) \approx t^{-\alpha/n}\varphi_k^{-1}(t)$, so the Dini type condition (i) of Theorem 1.1 that involves this function and $\mathcal{A}$ and $\mathcal{B}$ is sufficient for the boundedness of M_{α,φ_k}. The rest of the proof is the same.

We give finally the proof of the characterization of the boundedness of higher order commutators of the fractional integral operator in Orlicz spaces.

Proof of Theorem 1.8. Since the kernel of I_α belongs to $S_\alpha \cap H^*_{\alpha,\infty}$, the proof of Theorem 4.9 shows that condition (i) is sufficient.

To see that the Dini type condition *(i)* is necessary, we will make a lower estimate of the k-th order commutator of I_α, for the particular symbol $\mathfrak{b}(x) = \log|x| \in BMO$ and $f_\delta = \chi_{B(0,\delta)}$ for $\delta > 0$.

Let us suppose that $|x| > 2\delta$. Then, for each $y \in B(0,\delta)$,

$$\mathfrak{b}(x) - \mathfrak{b}(y) \geq \log(|x|/\delta) = \log 2 + \log(|x|/2\delta) \geq C(1 + \log^+(|x|/2\delta)).$$

Thus, for those values of x,

$$\begin{aligned} I^k_{\alpha,\mathfrak{b}} f_\delta(x) &= \int_{B(0,\delta)} \frac{(\mathfrak{b}(x) - \mathfrak{b}(y))^k}{|x-y|^{n-\alpha}}\, dy \\ &\geq C^k(1 + \log^+(|x|/2\delta))^k \frac{\omega_n \delta^n}{(3|x|/2)^{n-\alpha}} \\ &= C(\alpha, n, k)\, \omega_n^{\alpha/n}\delta^\alpha \frac{(1 + \log^+(|x|/2\delta))^k}{(|x|/2\delta)^{n-\alpha}}. \end{aligned}$$

By defining $\eta(t) = t^{1/(n-\alpha)}(1 + \log^+ t)^{k/(n-\alpha)}$, it is easy to see that there exists a constant $D > 1$ such that

$$D^{-1}\eta^{-1}(t) \leq t^{n-\alpha}(1 + \log^+ t)^{-k} \leq D\eta^{-1}(t),$$

so $I^k_{\alpha,\mathfrak{b}} f_\delta(x) \geq C(\alpha, n, k)\omega_n^{\alpha/n}\delta^\alpha / D\eta^{-1}(|x|/2\delta) = \tilde{C}\omega_n^{\alpha/n}\delta^\alpha/\eta^{-1}(|x|/2\delta)$ for $|x| > 2\delta$.

Then, by considering $0 < \lambda < \tilde{C}\omega_n^{\alpha/n}\delta^\alpha/\eta^{-1}(2^{1/n})$, we have that

$$\begin{aligned} \left|\{x \in \mathbb{R}^n : |I^k_{\alpha,\mathfrak{b}} f_\delta(x)| > \lambda\}\right| &\geq \left|\left\{x \in \mathbb{R}^n : |x| > 2\delta \text{ and } \tilde{C}\omega_n^{\alpha/n}\delta^\alpha/\eta^{-1}(|x|/2\delta) > \lambda\right\}\right| \\ &\geq \left|\left\{x \in \mathbb{R}^n : 2\delta < |x| < 2\delta\eta\left(\frac{\tilde{C}\omega_n^{\alpha/n}\delta^\alpha}{\lambda}\right)\right\}\right| \\ &= \omega_n 2^n\delta^n\eta\left(\frac{\tilde{C}\omega_n^{\alpha/n}\delta^\alpha}{\lambda}\right)^n - 2^n\delta^n \geq \omega_n 2^{n-1}\delta^n\eta\left(\frac{\tilde{C}\omega_n^{\alpha/n}\delta^\alpha}{\lambda}\right)^n \\ &= \omega_n 2^{n-1}\delta^n\xi\left(\frac{\tilde{C}\omega_n^{\alpha/n}\delta^\alpha}{\lambda}\right), \end{aligned}$$

being $\xi(t) = t^{\frac{n}{n-\alpha}}(1 + \log^+ t)^{\frac{kn}{n-\alpha}}$, which is a submultiplicative Young function. Hence, from the above estimate we obtain

$$\left|\{x \in \mathbb{R}^n : \left|I^k_{\alpha,\mathfrak{b}} f_\delta(x)\right| > \lambda\}\right| \geq \frac{\tilde{C}2^{n-2}\omega_n^{1+\alpha/n}\delta^{n+\alpha}}{\lambda}\xi'\left(\frac{\tilde{C}\omega_n^{\alpha/n}\delta^\alpha}{2\lambda}\right).$$

By following the ideas in (3.9), we obtain the desired Dini type condition. □

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