

Refinable spaces and local approximation estimates for hierarchical splines

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We study the local approximation properties in hierarchical spline spaces through multiscale quasi-interpolation operators. This construction suggests the analysis of a subspace of the classical hierarchical spline space (Vuong *et al.* (2011) A hierarchical approach to adaptive local refinement in isogeometric analysis. *Comput. Methods Appl. Mech. Eng.*, **200**, 3554–3567) which still satisfies the essential properties of the full space. The B-spline basis of such a subspace can be constructed using parent–children relations only, making it well adapted to local refinement algorithms.

Keywords: adaptivity in isogeometric analysis; hierarchical splines; quasi-interpolation; local refinement.

1. Introduction

Local adaptivity in numerical methods for partial differential equations makes it possible to solve real problems leading to a suitable approximation of the desired solution without exceeding the limits of available software. When considering isogeometric methods (Hughes *et al.*, 2005; Cottrell *et al.*, 2009), from a theoretical point of view, the design of efficient and robust strategies for local refinement constitutes a challenging problem because the tensor product structure of B-splines (de Boor, 2001; Schumaker, 2007) is broken.

Hierarchical B-splines (HB-splines) based on the construction presented in Kraft (1997, 1998) and Vuong *et al.* (2011) are a promising approach, because their construction and properties are closely related to the ones of hierarchical finite elements. Moreover, truncated hierarchical B-splines (THB-splines) have been introduced in Giannelli *et al.* (2012), where their use as a framework for isogeometric analysis that provides local refinement possibilities has been analysed; see also Giannelli *et al.* (2014). Local approximation estimates for hierarchical spline spaces have been studied in Speleers & Manni (2016) using quasi-interpolants described in terms of the truncated hierarchical basis, while the use of

THB-splines in conjunction with residual-based error indicators has been proposed in [Buffa & Giannelli \(2016\)](#) under a few assumptions on the meshes. It is important to remark that truncation is indeed a possible strategy to recover partition of unity and convex hull property. On the other hand, the procedure of truncation requires a specific construction that entails complicated basis function supports (that may be nonconvex and/or not connected) and their use may produce a non-negligible overhead with an adaptive strategy.

In the present paper, we take the point of view of classical HB-splines and we study their structure with a special attention to all those properties that may be needed or may facilitate their use with an adaptive isogeometric method.

When considering an underlying sequence of nested tensor-product spline spaces and the corresponding B-spline bases, a particular way of selecting B-spline basis functions from each different level in order to build a hierarchical basis \mathcal{H} has been established in [Kraft \(1998\)](#) (see also [Vuong *et al.*, 2011](#)). This hierarchical basis \mathcal{H} enjoys some important properties:

- \mathcal{H} is a set of linearly independent B-spline functions.
- It is possible to identify uniquely the basis \mathcal{H} from the knowledge of a hierarchical mesh.
- All functions in the coarsest underlying tensor-product spline space belong to the hierarchical space $\text{span } \mathcal{H}$.
- Under certain assumptions about the hierarchy of subdomains associated with \mathcal{H} , it is possible to define a multiscale quasi-interpolant operator in $\text{span } \mathcal{H}$ obtaining optimal orders of local approximation.
- Any enlargement of the hierarchy of the subdomains associated with \mathcal{H} gives rise to a *refined* basis $\tilde{\mathcal{H}}^*$ in the sense that $\text{span } \mathcal{H} \subset \text{span } \tilde{\mathcal{H}}^*$.

On the other hand, unlike tensor-product B-spline bases, the functions in the hierarchical basis \mathcal{H} do not constitute a partition of unity. If we consider the corresponding coefficients $\{a_\beta\}_{\beta \in \mathcal{H}} \subset \mathbb{R}$ in order to form such a partition, i.e.,

$$\sum_{\beta \in \mathcal{H}} a_\beta \beta \equiv 1, \quad (1.1)$$

it is known that the coefficients a_β are non-negative, but in fact, some of them can be equal to zero.

In this article we analyse the local approximation properties of hierarchical splines spaces through the construction of a multiscale quasi-interpolant operator. [Kraft \(1998\)](#) has introduced such an operator for the case of bivariate spline spaces on infinite uniform knot vectors and has studied its pointwise approximation properties. We extend his results to the case of open knot vectors with possible multiple internal knots in d -dimensional domains, for $d \geq 1$, and we also provide local approximation estimates in L^q -norms, for $1 \leq q \leq \infty$.

Furthermore, we propose a new hierarchical spline space, through a construction of a set of basis functions named $\tilde{\mathcal{H}}$ that satisfies all important properties just listed above. In particular, $\tilde{\mathcal{H}} \subset \mathcal{H}$, and therefore, in general, the *new* hierarchical space $\text{span } \tilde{\mathcal{H}}$ may be smaller than $\text{span } \mathcal{H}$. However, the new basis is easier to manage and to update when performing local adaptive refinement, and in this case, the coefficients for (1.1) are strictly positive. It is worth mentioning that, although our approach follows some ideas from [Grinspun *et al.* \(2002\)](#) and [Krysl *et al.* \(2003\)](#), the function space obtained in these articles for

the case of high order splines may not be uniquely determined from the knowledge of the hierarchical mesh and moreover the generating set may be linearly dependent.

This paper is organized as follows. In Section 2 we introduce the notation and the assumptions for the underlying tensor-product spline spaces to be considered, and in Section 3 we briefly introduce the standard HB-spline basis and prove some results which will be useful later. In Section 4 we construct a multiscale quasi-interpolant operator and study the local approximation properties in hierarchical spline spaces. In Section 5 we define a simplified HB-spline basis and prove some of its basic properties. Finally, we conclude the article with some final remarks in Section 6.

2. Spline spaces and B-spline bases

2.1 Univariate B-spline bases

Let $\mathcal{E}_{p,n} := \{\xi_j\}_{j=1}^{n+p+1}$ be a p -open knot vector, i.e., a *sequence* such that

$$0 = \xi_1 = \dots = \xi_{p+1} < \xi_{p+2} \leq \dots \leq \xi_n < \xi_{n+1} = \dots = \xi_{n+p+1} = 1,$$

where the two positive integers p and n denote a given polynomial degree, and the corresponding number of B-splines defined over the subdivision $\mathcal{E}_{p,n}$, respectively. Here, $n \geq p + 1$. We also introduce the *set* $Z_{p,n} := \{\zeta_j\}_{j=1}^{\tilde{n}}$ of breakpoints (i.e., knots without repetitions) and denote by m_j the multiplicity of the breakpoint ζ_j , such that

$$\mathcal{E}_{p,n} = \underbrace{\{\zeta_1, \dots, \zeta_1\}}_{m_1 \text{ times}}, \underbrace{\{\zeta_2, \dots, \zeta_2\}}_{m_2 \text{ times}}, \dots, \underbrace{\{\zeta_{\tilde{n}}, \dots, \zeta_{\tilde{n}}\}}_{m_{\tilde{n}} \text{ times}},$$

with $\sum_{i=1}^{\tilde{n}} m_i = n + p + 1$. Note that the two extreme knots are repeated $p + 1$ times, i.e., $m_1 = m_{\tilde{n}} = p + 1$. We assume that an internal knot can be repeated at most $p + 1$ times, i.e., $m_j \leq p + 1$, for $j = 2, \dots, \tilde{n} - 1$.

Let $\mathcal{B}(\mathcal{E}_{p,n}) := \{b_1, b_2, \dots, b_n\}$ be the B-spline basis (cf. de Boor, 2001; Schumaker, 2007) associated with the knot vector $\mathcal{E}_{p,n}$. The *local knot vector* of b_j is given by

$$\mathcal{E}_{b_j} := \{\xi_j, \dots, \xi_{j+p+1}\},$$

which is a subsequence of $p + 2$ consecutive knots of $\mathcal{E}_{p,n}$. We remark that

$$\text{supp } b_j = [\xi_j, \xi_{j+p+1}].$$

Let $\mathcal{I}(\mathcal{E}_{p,n})$ be the mesh defined by

$$\mathcal{I}(\mathcal{E}_{p,n}) := \{[\zeta_j, \zeta_{j+1}] \mid j = 1, \dots, \tilde{n} - 1\}.$$

For each $I = [\zeta_j, \zeta_{j+1}] \in \mathcal{I}(\mathcal{E}_{p,n})$ there exists a unique $k = \sum_{i=1}^j m_i$ such that $I = [\xi_k, \xi_{k+1}]$ and $\xi_k \neq \xi_{k+1}$. The union of the supports of the B-splines acting on I identifies the *support extension* \tilde{I} , namely

$$\tilde{I} := [\xi_{k-p}, \xi_{k+p+1}].$$

Let $\mathcal{E}_0 := \mathcal{E}_{p,n}$ be a given p -open knot vector. We consider a sequence $\{\mathcal{E}_\ell\}_{\ell \in \mathbb{N}}$ of successive refinements of \mathcal{E}_0 , i.e.,

$$\mathcal{E}_0 \subset \mathcal{E}_1 \subset \dots, \quad (2.1)$$

where \mathcal{E}_ℓ is a p -open knot vector, and $\mathcal{E}_\ell \subset \mathcal{E}_{\ell+1}$ means that \mathcal{E}_ℓ is a subsequence of $\mathcal{E}_{\ell+1}$, for $\ell \in \mathbb{N}_0$. In other words, condition (2.1) says that if ξ is a knot in \mathcal{E}_ℓ with multiplicity m , then ξ is also a knot in $\mathcal{E}_{\ell+1}$ with multiplicity at least m , for $\ell \in \mathbb{N}_0$.

Let $\mathcal{B}_\ell := \mathcal{B}(\mathcal{E}_\ell)$ be the B-spline basis, for $\ell \in \mathbb{N}_0$.

DEFINITION 2.1 Let $\beta_\ell \in \mathcal{B}_\ell$ be given and let \mathcal{E}_{β_ℓ} be the corresponding local knot vector. Let $\mathcal{E}_{\beta_\ell}^{(\ell+1)} \subset \mathcal{E}_{\ell+1}$ be the knot vector obtained from \mathcal{E}_{β_ℓ} after inserting the knots of $\mathcal{E}_{\ell+1}$ which are in the interior of $\text{supp } \beta_\ell$. We say that $\beta_{\ell+1} \in \mathcal{B}_{\ell+1}$ is a child of $\beta_\ell \in \mathcal{B}_\ell$ if the local knot vector of $\beta_{\ell+1}$, which is denoted by $\mathcal{E}_{\beta_{\ell+1}}$, is a subsequence of $\mathcal{E}_{\beta_\ell}^{(\ell+1)}$. In other words, the children of β_ℓ are the B-splines in $\mathcal{B}_{\ell+1}$ whose local knot vector consists of $p+2$ consecutive knots of $\mathcal{E}_{\beta_\ell}^{(\ell+1)}$. We let

$$\mathcal{C}(\beta_\ell) := \{\beta_{\ell+1} \in \mathcal{B}_{\ell+1} \mid \beta_{\ell+1} \text{ is a child of } \beta_\ell\}.$$

Conversely, if $\beta_{\ell+1} \in \mathcal{B}_{\ell+1}$ is given, we define the set of parents of $\beta_{\ell+1}$ by

$$\mathcal{P}(\beta_{\ell+1}) := \{\beta_\ell \in \mathcal{B}_\ell \mid \beta_{\ell+1} \text{ is a child of } \beta_\ell\}.$$

It is easy to check that the last definition means that $\beta_{\ell+1} \in \mathcal{B}_{\ell+1}$ is a child of $\beta_\ell \in \mathcal{B}_\ell$ if and only if

- (i) $\min \mathcal{E}_{\beta_\ell} \leq \min \mathcal{E}_{\beta_{\ell+1}} \leq \max \mathcal{E}_{\beta_{\ell+1}} \leq \max \mathcal{E}_{\beta_\ell}$.
- (ii) If $\xi \in \mathcal{E}_{\beta_{\ell+1}}$ matches any of the end points of \mathcal{E}_{β_ℓ} then the multiplicity of ξ in $\mathcal{E}_{\beta_{\ell+1}}$ is less or equal to the multiplicity of ξ in \mathcal{E}_{β_ℓ} .

In particular, notice that if $\beta_{\ell+1}$ is a child of β_ℓ then $\text{supp } \beta_{\ell+1} \subset \text{supp } \beta_\ell$.

Since $\mathcal{E}_\ell \subset \mathcal{E}_{\ell+1}$, using the so-called knot insertion formula, all B-splines of level ℓ can be written as a linear combination of B-splines of level $\ell+1$. More precisely, if $\beta_\ell \in \mathcal{B}_\ell$ then

$$\beta_\ell = \sum_{\beta_{\ell+1} \in \mathcal{C}(\beta_\ell)} c_{\beta_{\ell+1}}(\beta_\ell) \beta_{\ell+1}, \quad (2.2)$$

where the coefficients $c_{\beta_{\ell+1}}(\beta_\ell)$ are positive, and $\mathcal{C}(\beta_\ell) \subset \mathcal{B}_{\ell+1}$ is the set of children of β_ℓ (cf. Fig. 1).

2.2 Tensor-product B-spline bases

Let $d \geq 1$. In order to define a tensor-product d -variate spline function space on $\Omega := [0, 1]^d \subset \mathbb{R}^d$, we consider $\mathbf{p} := (p_1, p_2, \dots, p_d)$ the vector of polynomial degrees with respect to each coordinate direction and $\mathbf{n} := (n_1, n_2, \dots, n_d)$, where $n_i \geq p_i + 1$. For $i = 1, 2, \dots, d$, let $\mathcal{E}_{p_i, n_i} := \{\xi_j^{(i)}\}_{j=1}^{n_i+p_i+1}$ be a p_i -open knot vector, i.e.,

$$0 = \xi_1^{(i)} = \dots = \xi_{p_i+1}^{(i)} < \xi_{p_i+2}^{(i)} \leq \dots \leq \xi_{n_i}^{(i)} < \xi_{n_i+1}^{(i)} = \dots = \xi_{n_i+p_i+1}^{(i)} = 1,$$

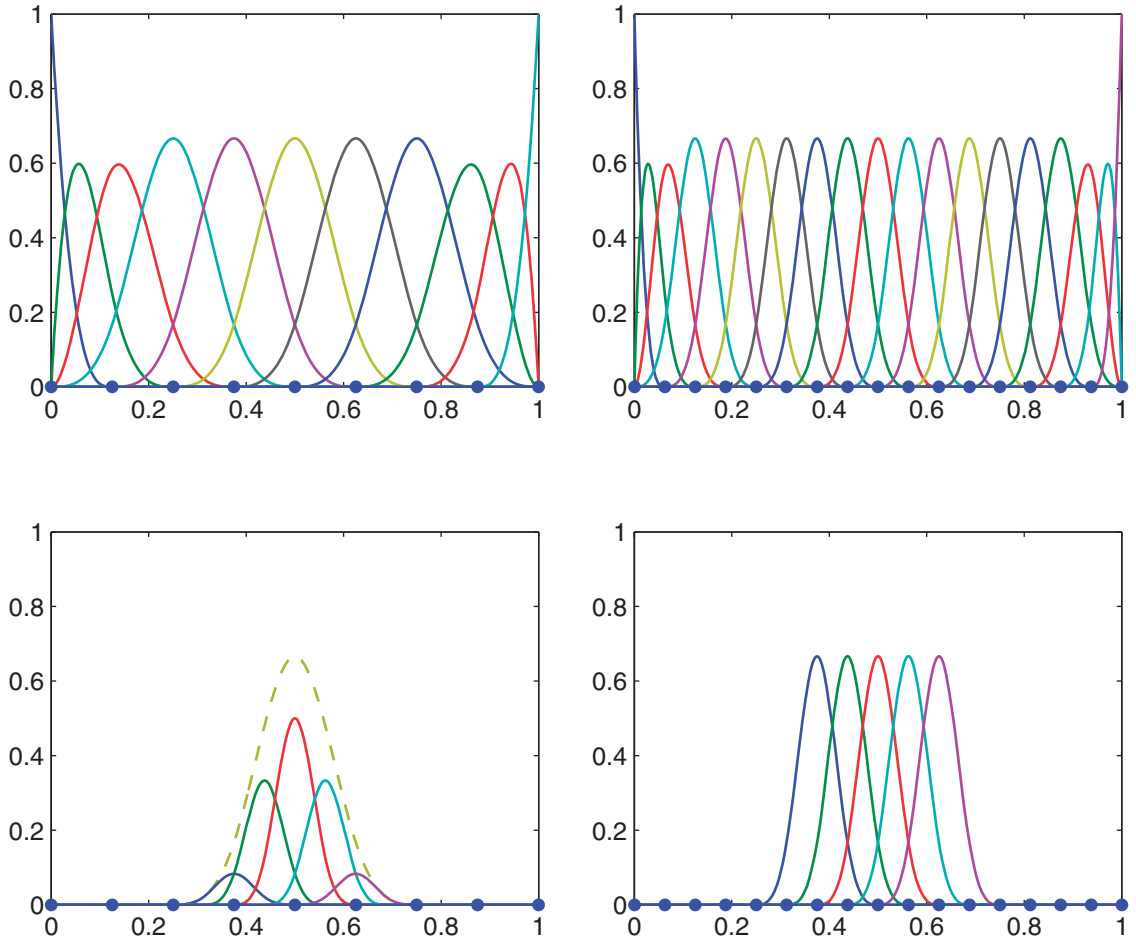


FIG. 1. As an example, we consider cubic splines (i.e., $p = 3$) of maximum smoothness. The B-spline basis functions in \mathcal{B}_ℓ are shown at the top on the left. Assuming that the level $\ell + 1$ is obtained from the level ℓ just by dyadic refinement, the B-spline basis functions in $\mathcal{B}_{\ell+1}$ are shown at the top on the right. If $\beta_\ell \in \mathcal{B}_\ell$ is the dashed B-spline, the refinement relation (2.2) provides its decomposition in terms of its children (at the bottom on the left) which are the B-splines represented at the bottom on the right.

where the two extreme knots are repeated $p_i + 1$ times and any internal knot can be repeated at most $p_i + 1$ times. We denote by $\mathcal{S}_{\mathbf{p},\mathbf{n}}$ the tensor-product spline space spanned by the B-spline basis $\mathcal{B}_{\mathbf{p},\mathbf{n}}$ defined as the tensor-product of the univariate B-spline bases $\mathcal{B}(\mathcal{E}_{p_1,n_1}), \dots, \mathcal{B}(\mathcal{E}_{p_d,n_d})$. Let $\mathcal{Q}_{\mathbf{p},\mathbf{n}}$ be tensor-product mesh consisting of the elements $Q = I_1 \times \dots \times I_d$, where I_i is an element (closed interval) of the i th univariate mesh $\mathcal{I}(\mathcal{E}_{p_i,n_i})$, for $i = 1, \dots, d$.

Now, we consider a given sequence $\{\mathcal{S}_\ell\}_{\ell \in \mathbb{N}_0}$ of tensor-product d -variate spline spaces such that

$$\mathcal{S}_0 \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathcal{S}_3 \subset \dots, \quad (2.3)$$

with the corresponding tensor-product B-spline bases denoted by

$$\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \dots,$$

respectively. More precisely, if $\mathbf{p} = (p_1, p_2, \dots, p_d)$ is given, for $\ell \in \mathbb{N}_0$, $\mathcal{S}_\ell := \mathcal{S}_{\mathbf{p}, \mathbf{n}_\ell}$ is the tensor-product spline space and $\mathcal{B}_\ell := \mathcal{B}_{\mathbf{p}, \mathbf{n}_\ell}$ is the corresponding B-spline basis, for some $\mathbf{n}_\ell = (n_1^\ell, n_2^\ell, \dots, n_d^\ell)$. In order to guarantee (2.3), we assume that if ξ is a knot in $\mathcal{E}_{p_i, n_i^\ell}$ with multiplicity m then ξ is also a knot in $\mathcal{E}_{p_i, n_i^{\ell+1}}$ with multiplicity at least m , for $i = 1, \dots, d$ and $\ell \in \mathbb{N}_0$. Furthermore, we denote by $\mathcal{Q}_\ell := \mathcal{Q}_{\mathbf{p}, \mathbf{n}_\ell}$ the tensor-product mesh and we say that $Q \in \mathcal{Q}_\ell$ is a *cell of level ℓ* . We state some well-known properties of the B-spline basis functions that will be useful in this presentation (de Boor, 2001; Schumaker, 2007):

- *Local linear independence.* For any nonempty open set $O \subset \Omega$, the functions in \mathcal{B}_ℓ that do not vanish identically on O are linearly independent on O .
- *Positive partition of unity.* The B-spline basis functions of level ℓ form a partition of the unity on Ω , i.e.,

$$\sum_{\beta \in \mathcal{B}_\ell} \beta \equiv 1, \quad \text{on } \Omega. \quad (2.4)$$

- *Two-scale relation between consecutive levels.* The B-splines of level ℓ can be written as a linear combination of B-splines of level $\ell + 1$. More precisely,

$$\beta_\ell = \sum_{\beta_{\ell+1} \in \mathcal{C}(\beta_\ell)} c_{\beta_{\ell+1}}(\beta_\ell) \beta_{\ell+1}, \quad \forall \beta_\ell \in \mathcal{B}_\ell, \quad (2.5)$$

where the coefficients $c_{\beta_{\ell+1}}(\beta_\ell)$ are positive and can be computed using the corresponding coefficients in the univariate two-scale relation (2.2) and Kronecker products. Here, $\mathcal{C}(\beta_\ell)$ is the set of children of β_ℓ , and we say that $\beta_{\ell+1} \in \mathcal{B}_{\ell+1}$ is a child of β_ℓ if the i th univariate B-spline which defines $\beta_{\ell+1}$ is a child of the i th univariate B-spline defining β_ℓ , for each coordinate direction $i = 1, \dots, d$.

REMARK 2.2 Notice that if we define $c_{\beta_{\ell+1}}(\beta_\ell) := 0$ when $\beta_{\ell+1}$ is not a child of β_ℓ then equation (2.5) can be written as

$$\beta_\ell = \sum_{\beta_{\ell+1} \in \mathcal{B}_{\ell+1}} c_{\beta_{\ell+1}}(\beta_\ell) \beta_{\ell+1}, \quad \forall \beta_\ell \in \mathcal{B}_\ell. \quad (2.6)$$

In particular, we remark that

$$\mathcal{C}(\beta_\ell) = \{\beta_{\ell+1} \in \mathcal{B}_{\ell+1} \mid c_{\beta_{\ell+1}}(\beta_\ell) > 0\} \subset \{\beta_{\ell+1} \in \mathcal{B}_{\ell+1} \mid \text{supp } \beta_{\ell+1} \subset \text{supp } \beta_\ell\}. \quad (2.7)$$

Finally, we also consider the set of parents of $\beta_{\ell+1} \in \mathcal{B}_{\ell+1}$ given by

$$\mathcal{P}(\beta_{\ell+1}) := \{\beta_\ell \in \mathcal{B}_\ell \mid \beta_{\ell+1} \text{ is a child of } \beta_\ell\}.$$

3. HB-spline basis

DEFINITION 3.1 If $n \in \mathbb{N}$, we say that $\underline{\Omega}_n := \{\Omega_0, \Omega_1, \dots, \Omega_n\}$ is a *hierarchy of subdomains of Ω of depth n* if

- (i) Ω_ℓ is the union of cells of level $\ell - 1$, for $\ell = 1, 2, \dots, n$.
- (ii) $\Omega = \Omega_0 \supset \Omega_1 \supset \dots \supset \Omega_{n-1} \supset \Omega_n = \emptyset$.

We now define the HB-spline basis $\mathcal{H} = \mathcal{H}(\underline{\Omega}_n)$ in the following recursive way:

$$\begin{cases} \mathcal{H}_0 := \mathcal{B}_0, \\ \mathcal{H}_{\ell+1} := \{\beta \in \mathcal{H}_\ell \mid \text{supp } \beta \not\subset \Omega_{\ell+1}\} \cup \{\beta \in \mathcal{B}_{\ell+1} \mid \text{supp } \beta \subset \Omega_{\ell+1}\}, & \ell = 0, \dots, n-2. \\ \mathcal{H} := \mathcal{H}_{n-1}. \end{cases} \quad (3.1)$$

If $\mathcal{D}_\ell := \mathcal{H}_\ell \setminus \mathcal{H}_{\ell+1}$ is the set of *deactivated* B-splines of level ℓ then

$$\mathcal{D}_\ell = \{\beta \in \mathcal{H}_\ell \mid \text{supp } \beta \subset \Omega_{\ell+1}\} = \{\beta \in \mathcal{B}_\ell \mid \text{supp } \beta \subset \Omega_{\ell+1}\}.$$

Notice that in order to get $\mathcal{H}_{\ell+1}$ from \mathcal{H}_ℓ we replace the set \mathcal{D}_ℓ by

$$\{\beta \in \mathcal{B}_{\ell+1} \mid \text{supp } \beta \subset \Omega_{\ell+1}\}.$$

Moreover, it is easy to check that

$$\mathcal{H} = \bigcup_{\ell=0}^{n-1} \{\beta \in \mathcal{B}_\ell \mid \text{supp } \beta \subset \Omega_\ell \wedge \text{supp } \beta \not\subset \Omega_{\ell+1}\}. \quad (3.2)$$

We say that β is *active* if $\beta \in \mathcal{H}$. The corresponding underlying mesh $\mathcal{Q} = \mathcal{Q}(\underline{\Omega}_n)$ is given by

$$\mathcal{Q} := \bigcup_{\ell=0}^{n-1} \{Q \in \mathcal{Q}_\ell \mid Q \subset \Omega_\ell \wedge Q \not\subset \Omega_{\ell+1}\}, \quad (3.3)$$

and we say that Q is an *active cell* if $Q \in \mathcal{Q}$, or that Q is an *active cell of level ℓ* if $Q \in \mathcal{Q} \cap \mathcal{Q}_\ell$.

Our definition of the HB-spline basis \mathcal{H} is slightly different from the one given in Kraft (1998) and Vuong *et al.* (2011), because they consider the domains Ω_ℓ , the cells Q and the function supports as open sets. That definition does not allow to rebuild the hierarchical space uniquely from the only knowledge of the hierarchical mesh (i.e., the active cells of each level); see, for example, Kraft (1998, Fig. 2.5) or Kraft (1997, Fig. 2).

We know (cf. Vuong *et al.*, 2011) that functions in \mathcal{H} constitutes a linearly independent set and that

$$\mathcal{S}_0 = \text{span } \mathcal{B}_0 \subset \text{span } \mathcal{H}.$$

Unlike the B-spline bases \mathcal{B}_ℓ for tensor-product spline spaces, the HB-spline basis \mathcal{H} does not constitute a partition of unity. However, from Vuong *et al.* (2011, Lemma 5), we know that a partition of unity can be obtained by using a proper weighting. More precisely, we can prove the following result.

LEMMA 3.2 (Partition of unity in \mathcal{H}). Let \mathcal{H} be the HB-spline basis associated with the hierarchy of subdomains of depth n , $\underline{\Omega}_n := \{\Omega_0, \Omega_1, \dots, \Omega_n\}$. Let $a_{\beta_0} := 1$ for all $\beta_0 \in \mathcal{B}_0$ and

$$a_{\beta_{\ell+1}} := \sum_{\substack{\beta_\ell \in \mathcal{B}_\ell \\ \text{supp } \beta_\ell \subset \Omega_{\ell+1}}} a_{\beta_\ell} c_{\beta_{\ell+1}}(\beta_\ell), \quad \forall \beta_{\ell+1} \in \mathcal{B}_{\ell+1}, \text{supp } \beta_{\ell+1} \subset \Omega_{\ell+1}, \quad (3.4)$$

for $\ell = 0, 1, \dots, n-2$. Then,

$$\sum_{\beta \in \mathcal{H}} a_\beta \beta \equiv 1, \quad \text{on } \Omega.$$

REMARK 3.3 In view of the linear independence of functions in \mathcal{H} , we have that the set $\{a_\beta\}_{\beta \in \mathcal{H}}$ is uniquely determined. On the other hand, we remark that the definition (3.4) depends on the hierarchy of subdomains $\underline{\Omega}_n$, and that the coefficients a_β are defined not only for $\beta \in \mathcal{H}$, but also for $\beta \in \bigcup_{\ell=0}^{n-1} \{\beta_\ell \in \mathcal{B}_\ell \mid \text{supp } \beta_\ell \subset \Omega_\ell\}$.

Proof. By (2.4) we have that $\sum_{\beta \in \mathcal{H}_0} a_\beta \beta \equiv 1$, on Ω . Assume now that for a fixed ℓ ($0 \leq \ell \leq n-2$), $\sum_{\beta \in \mathcal{H}_\ell} a_\beta \beta \equiv 1$, on Ω . Thus, using (3.4) and (2.6) we have that

$$\begin{aligned} \sum_{\beta \in \mathcal{H}_{\ell+1}} a_\beta \beta &= \sum_{\substack{\beta \in \mathcal{H}_\ell \\ \text{supp } \beta \not\subset \Omega_{\ell+1}}} a_\beta \beta + \sum_{\substack{\beta_{\ell+1} \in \mathcal{B}_{\ell+1} \\ \text{supp } \beta_{\ell+1} \subset \Omega_{\ell+1}}} a_{\beta_{\ell+1}} \beta_{\ell+1} \\ &= \sum_{\substack{\beta \in \mathcal{H}_\ell \\ \text{supp } \beta \not\subset \Omega_{\ell+1}}} a_\beta \beta + \sum_{\substack{\beta_\ell \in \mathcal{B}_\ell \\ \text{supp } \beta_\ell \subset \Omega_{\ell+1}}} a_{\beta_\ell} \beta_\ell \\ &= \sum_{\beta \in \mathcal{H}_\ell} a_\beta \beta. \end{aligned}$$

Thus, $\sum_{\beta \in \mathcal{H}_{\ell+1}} a_\beta \beta \equiv 1$, on Ω .

Finally, the proof is complete regarding that $\mathcal{H} = \mathcal{H}_{n-1}$. \square

Notice that in the last lemma $a_\beta \geq 0$, for all $\beta \in \mathcal{H}$. The next result characterizes the functions β whose weight a_β is equal to zero.

THEOREM 3.4 Let $\beta_{\ell+1} \in \mathcal{B}_{\ell+1}$ be such that $\text{supp } \beta_{\ell+1} \subset \Omega_{\ell+1}$, for some $\ell = 0, 1, \dots, n-2$. Then, the following statements are equivalent:

- (i) $a_{\beta_{\ell+1}} = 0$.
- (ii) $\forall \beta_\ell \in \mathcal{P}(\beta_{\ell+1}), (a_{\beta_\ell} > 0 \wedge \text{supp } \beta_\ell \subset \Omega_\ell \implies \text{supp } \beta_\ell \not\subset \Omega_{\ell+1})$.

Proof. From (3.4), we have that $a_{\beta_{\ell+1}} = 0$ if and only if for all $\beta_\ell \in \mathcal{B}_\ell$, with $\text{supp } \beta_\ell \subset \Omega_{\ell+1}$,

$$a_{\beta_\ell} = 0 \quad \text{or} \quad c_{\beta_{\ell+1}}(\beta_\ell) = 0,$$

or, equivalently, if and only if, all parents β_ℓ of $\beta_{\ell+1}$ with $a_{\beta_\ell} > 0$ satisfy $\text{supp } \beta_\ell \not\subset \Omega_{\ell+1}$. \square

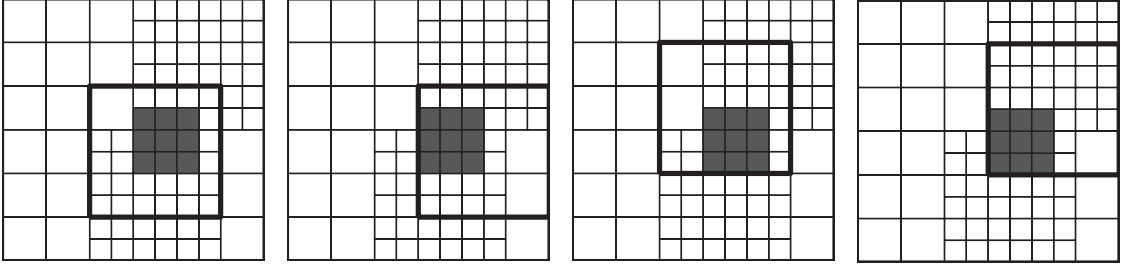


FIG. 2. As an illustration of Theorem 3.4 we consider a two-level mesh for biquadratic splines (i.e., $\mathbf{p} = (2, 2)$) of maximum smoothness. If β denotes the highlighted B-spline of Level 1, since the four parents of β are active B-splines of level 0 (i.e., their supports are not included in Ω_1), we have that $a_\beta = 0$.

Notice that there is no function $\beta_0 \in \mathcal{B}_0$ satisfying $a_{\beta_0} = 0$. Now, the last theorem implies that $a_{\beta_1} = 0$ for $\beta_1 \in \mathcal{B}_1$ with $\text{supp } \beta_1 \subset \Omega_1$ if and only if all its parents are active (cf. Fig. 2). Roughly speaking, Ω_1 is too narrow around $\text{supp } \beta_1$. Notice that, as soon as one of the β_1 's parents is deactivated, a_{β_1} will become positive in the new configuration.

We conclude this section with the following result, which states that each deactivated B-spline of level ℓ can be written as a linear combination of functions in the hierarchical basis of the subsequent levels, i.e., $\ell + 1, \dots, n - 1$.

LEMMA 3.5 Let \mathcal{H} be the HB-spline basis associated with the hierarchy of subdomains of depth n , $\underline{\Omega}_n := \{\Omega_0, \Omega_1, \dots, \Omega_n\}$. Then,

$$\mathcal{D}_\ell \subset \text{span} \left(\mathcal{H} \cap \bigcup_{k=\ell+1}^{n-1} \mathcal{B}_k \right), \quad (3.5)$$

for $\ell = 0, 1, \dots, n - 2$.

Proof. Notice that (3.5) holds for $\ell = n - 2$ due to (2.6). Let us assume that (3.5) holds for some ℓ and prove that it holds for $\ell - 1$. Let $\beta_{\ell-1} \in \mathcal{D}_{\ell-1}$. Since $\beta_{\ell-1} \in \mathcal{B}_{\ell-1}$ and $\text{supp } \beta_{\ell-1} \subset \Omega_\ell$, we have that

$$\beta_{\ell-1} = \sum_{\substack{\beta_\ell \in \mathcal{B}_\ell \\ \text{supp } \beta_\ell \subset \Omega_\ell}} c_{\beta_\ell}(\beta_{\ell-1}) \beta_\ell = \sum_{\substack{\beta_\ell \in \mathcal{B}_\ell \\ \text{supp } \beta_\ell \subset \Omega_{\ell+1}}} c_{\beta_\ell}(\beta_{\ell-1}) \beta_\ell + \sum_{\beta_\ell \in \mathcal{H} \cap \mathcal{B}_\ell} c_{\beta_\ell}(\beta_{\ell-1}) \beta_\ell.$$

Thus, $\beta_{\ell-1} \in \text{span} \left(\mathcal{H} \cap \bigcup_{k=\ell}^{n-1} \mathcal{B}_k \right)$, which concludes the proof. \square

4. Approximation properties and quasi-interpolation

Let $\underline{\Omega}_n := \{\Omega_0, \Omega_1, \dots, \Omega_n\}$ be a hierarchy of subdomains of Ω of depth n and let \mathcal{Q} be the hierarchical mesh given by (3.3). Let \mathcal{H} be the hierarchical basis defined in (3.2).

A multiscale quasi-interpolant operator has been introduced by Kraft (1998), where pointwise approximation estimates were established, for the case $\Omega = \mathbb{R}^2$, where there is no boundary. There, the (infinite) knot vector in each direction considered for building the initial tensor-product space \mathcal{S}_0 was the set of

the integer numbers \mathbb{Z} , and then dyadic refinement is performed to obtain the subsequent levels. This multiscale operator has been extended in [Speleers & Manni \(2016, Corollary 3\)](#) to a general multivariate hierarchical spline setting allowing nonuniform knot vectors. In this section, we extend the Kraft construction to the case of open knot vectors with possible multiple internal knots. Moreover, we present a multiscale quasi-interpolant operator, which provides suitable local approximation orders in L^q -norm, for any $1 \leq q \leq \infty$. We remark that the results presented in this section can be considered as an alternative way to those in [Speleers & Manni \(2016\)](#) to analyse the local approximation properties in span \mathcal{H} , but our approach will allow in [Section 5.1](#) to obtain such properties also for the simplified hierarchical space defined in the next section.

For each coordinate direction $i = 1, \dots, d$, we assume that the sequence of knot vectors $\{\mathcal{E}_{p_i, n_i^\ell}\}_{\ell \in \mathbb{N}_0}$ satisfies as follows:

- The sequence of meshes $\{\mathcal{S}(\mathcal{E}_{p_i, n_i^\ell})\}_{\ell \in \mathbb{N}_0}$ is locally quasi-uniform with parameter $\theta_i > 0$.¹ We let $\theta := \max_{i=1, \dots, d} \theta_i$.
- The meshsize in each direction is at least halved when moving from a level to the next one, i.e.,

$$h_{\ell+1, i} \leq \frac{1}{2} h_{\ell, i}, \quad (4.1)$$

for all $\ell \in \mathbb{N}_0$, where $h_{\ell, i}$ is the maximum length of the intervals in $\mathcal{S}(\mathcal{E}_{p_i, n_i^\ell})$.

REMARK 4.1 Notice that if the initial knot vectors for each direction, i.e., $\mathcal{E}_{p_1, n_1^0}, \mathcal{E}_{p_2, n_2^0}, \dots, \mathcal{E}_{p_d, n_d^0}$, are chosen arbitrarily and we perform dyadic refinement for obtaining the subsequent levels, we will obtain a sequence of meshes which satisfies the two conditions just stated.

In order to define the multiscale quasi-interpolant operator in the hierarchical space, we need to introduce first some *local* quasi-interpolant operators P_ℓ , for $\ell = 0, 1, \dots, n-1$, satisfying certain suitable properties. These last operators can be defined using the ideas from [Lee et al. \(2001\)](#), where each operator is defined using a underlying local approximation method.

4.1 A local approximation method

We recall that $\mathbf{p} := (p_1, p_2, \dots, p_d)$ and denote by $\mathbb{P}_{\mathbf{p}}$ the set of tensor-product polynomials with degree at most p_i in the coordinate direction x_i , for $i = 1, 2, \dots, d$. Let $N := \dim \mathbb{P}_{\mathbf{p}} = \prod_{i=1}^d (p_i + 1)$.

Let $\ell \in \mathbb{N}_0$ be arbitrary and fixed. For $Q \in \mathcal{Q}_\ell$ given, we consider the basis $\mathcal{B}_Q := \{\beta_1^Q, \dots, \beta_N^Q\}$ of $\mathbb{P}_{\mathbf{p}}$, consisting of the B-spline basis functions in \mathcal{B}_ℓ which are nonzero on Q .

Let $\Pi_Q : L^1(Q) \rightarrow \mathbb{P}_{\mathbf{p}}$ be the L^2 -projection operator defined by

$$\int_Q (f - \Pi_Q f) g = 0, \quad \forall g \in \mathbb{P}_{\mathbf{p}}. \quad (4.2)$$

¹ More precisely, for each coordinate direction $i = 1, 2, \dots, d$, there exists a constant $\theta_i > 0$ such that for all $\ell \in \mathbb{N}_0$, if I_1 and I_2 are elements in $\mathcal{S}(\mathcal{E}_{p_i, n_i^\ell})$ sharing a breakpoint in Z_{p_i, n_i^ℓ} then $\theta_i^{-1} \leq \frac{|I_1|}{|I_2|} \leq \theta_i$.

Notice that

$$\Pi_Q f = \sum_{i=1}^N \lambda_i^Q(f) \beta_i^Q, \quad \forall f \in L^1(Q), \quad (4.3)$$

where $\lambda^Q(f) := (\lambda_1^Q(f), \dots, \lambda_N^Q(f))^T$ is the solution of the linear system

$$M_Q \mathbf{x} = F_Q,$$

where

$$M_Q = \left(\int_Q \beta_j^Q \beta_i^Q \right)_{i,j=1,\dots,N} \in \mathbb{R}^{N \times N}, \quad \text{and} \quad F_Q = \left(\int_Q f \beta_i^Q \right)_{i=1,\dots,N} \in \mathbb{R}^{N \times 1}.$$

Since Π_Q preserves polynomials in $\mathbb{P}_{\mathbf{p}}$, we have that $\{\lambda_i^Q : L^1(Q) \rightarrow \mathbb{R} \mid i = 1, \dots, N\}$ is a dual basis for \mathcal{B}_Q in the sense that

$$\lambda_i^Q(\beta_j^Q) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}, \quad i, j = 1, \dots, N. \quad (4.4)$$

As a consequence of the L^∞ -local stability of the B-spline basis we have the following result (cf. Buffa *et al.*, 2015).

LEMMA 4.2 Let q be such that $1 \leq q \leq \infty$. Let $Q \in \mathcal{Q}_\ell$ and let $\Pi_Q : L^1(Q) \rightarrow \mathbb{P}_{\mathbf{p}}$ be the L^2 -projection operator defined by (4.2). Then, there exists a constant $C_{\mathbf{p},\theta} > 0$ which depends only on \mathbf{p} and θ such that

$$\|\lambda^Q(f)\|_\infty \leq C_{\mathbf{p},\theta} |Q|^{-\frac{1}{q}} \|f\|_{L^q(Q)}, \quad \forall f \in L^q(Q),$$

where $\lambda^Q(f) = (\lambda_1^Q(f), \dots, \lambda_N^Q(f))^T$ are the coefficients of $\Pi_Q(f)$ with respect to the local basis \mathcal{B}_Q as given (4.3).

4.2 A locally supported dual basis

For $\ell = 0, 1, \dots, n-1$, we define ω_ℓ as the union of the elements of level ℓ whose support extension is contained in Ω_ℓ , i.e.,

$$\omega_\ell := \bigcup_{\substack{Q \in \mathcal{Q}_\ell \\ \tilde{Q} \subset \Omega_\ell}} Q,$$

where \tilde{Q} denotes the support extension of Q , given by $\tilde{Q} = \bigcup_{\substack{\beta \in \mathcal{B}_\ell \\ \text{supp } \beta \supset Q}} \text{supp } \beta$. In other words, ω_ℓ consists of

the elements of level ℓ , where the full tensor-product space of level ℓ can be exactly represented in the hierarchical space.

Let

$$\mathcal{B}_{\ell, \omega_\ell} := \{\beta \in \mathcal{B}_\ell \mid \exists Q \in \mathcal{Q}_\ell : Q \subset \text{supp } \beta \cap \omega_\ell\}, \quad (4.5)$$

and

$$\mathcal{S}_{\omega_\ell} := \text{span } \mathcal{B}_{\ell, \omega_\ell}.$$

We remark that $\mathcal{B}_{\ell, \omega_\ell} \subset \{\beta \in \mathcal{B}_\ell \mid \text{supp } \beta \subset \Omega_\ell\}$, but in general, $\mathcal{B}_{\ell, \omega_\ell} \subsetneq \{\beta \in \mathcal{B}_\ell \mid \text{supp } \beta \subset \Omega_\ell\}$ (see Fig. 4). The goal of this paragraph is to define a dual basis for the multivariate B-spline basis $\mathcal{B}_{\ell, \omega_\ell}$, i.e., a set of linear functionals

$$\{\lambda_\beta : L^1(\Omega) \rightarrow \mathbb{R} \mid \beta \in \mathcal{B}_{\ell, \omega_\ell}\},$$

such that $\lambda_{\beta_i}(\beta_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$, for all $\beta_i, \beta_j \in \mathcal{B}_{\ell, \omega_\ell}$. We will use the technique presented in Lee *et al.* (2001) together with the local L^2 -projection defined in (4.2). Roughly speaking, we define the functional λ_β as a local projection onto some $Q_\beta \in \mathcal{Q}_\ell$ such that $Q_\beta \subset \text{supp } \beta \cap \omega_\ell$. More precisely, for each $\beta \in \mathcal{B}_{\ell, \omega_\ell}$, we choose $Q_\beta \in \mathcal{Q}_\ell$ such that $Q_\beta \subset \text{supp } \beta \cap \omega_\ell$ and let

$$\lambda_\beta := \lambda_{i_0}^{Q_\beta},$$

where $i_0 = i_0(\beta, Q_\beta)$ with $1 \leq i_0 \leq N$ is such that $\beta_{i_0}^{Q_\beta} \equiv \beta$ on Q_β .

As an immediate consequence of (4.4) and Lemma 4.2 we have the following result.

PROPOSITION 4.3 Let $\{\lambda_\beta : L^1(\Omega) \rightarrow \mathbb{R} \mid \beta \in \mathcal{B}_{\ell, \omega_\ell}\}$ be the set of linear functionals just defined above. Then, the following properties hold:

- (i) *Local support:* If Q_β denotes the element in \mathcal{Q}_ℓ chosen for the definition of λ_β then λ_β is supported in Q_β , i.e.,

$$\forall f \in L^q(\Omega), \quad f|_{Q_\beta} \equiv 0 \quad \implies \quad \lambda_\beta(f) = 0.$$

- (ii) *Dual basis:* For $\beta_i, \beta_j \in \mathcal{B}_{\ell, \omega_\ell}$, $\lambda_{\beta_i}(\beta_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$

- (iii) *L^q -Stability:* If $\beta \in \mathcal{B}_{\ell, \omega_\ell}$ and $f \in L^q(Q_\beta)$,

$$|\lambda_\beta(f)| \leq C_{\mathbf{p}, \beta} |Q_\beta|^{-\frac{1}{q}} \|f\|_{L^q(Q_\beta)}. \quad (4.6)$$

4.3 Localized quasi-interpolant operators in the tensor-product spaces

Now, we are in position of defining a quasi-interpolant operator for each level ℓ , using the dual bases introduced in the previous paragraph. For $\ell = 0, 1, \dots, n-1$, let $P_\ell : L^q(\Omega) \rightarrow \text{span } \mathcal{B}_{\ell, \omega_\ell} \subset \mathcal{S}_\ell$ be given by

$$P_\ell f := \sum_{\beta \in \mathcal{B}_{\ell, \omega_\ell}} \lambda_\beta(f) \beta, \quad \forall f \in L^q(\Omega). \quad (4.7)$$

The next result summarizes the main properties of P_ℓ .

THEOREM 4.4 For $\ell = 0, 1, \dots, n-1$, let P_ℓ be the operator given by (4.7). Then, the following properties hold:

- (i) P_ℓ preserves splines in $\mathcal{S}_{\omega_\ell}$, i.e., $P_\ell s = s$, for all $s \in \mathcal{S}_{\omega_\ell}$.
- (ii) P_ℓ is supported in ω_ℓ , i.e.,

$$\forall f \in L^q(\Omega), \quad f|_{\omega_\ell} \equiv 0 \quad \implies \quad P_\ell f \equiv 0. \quad (4.8)$$

- (1) For all $s \in \mathcal{S}_\ell$,

$$P_\ell s \equiv s, \quad \text{on } \omega_\ell.$$

- (2) *Stability*: The quasi-interpolant operator P_ℓ satisfies

$$\|P_\ell f\|_{L^q(\Omega_\ell)} \leq C_S \|f\|_{L^q(\omega_\ell)}, \quad \forall f \in L^q(\omega_\ell), \quad (4.9)$$

where the constant $C_S > 0$ only depends on \mathbf{p} and θ .

- (3) *Approximation*: Let $\mathbf{s} := (s_1, s_2, \dots, s_d)$ be such that $s_i \leq p_i + 1$, for $i = 1, 2, \dots, d$. For $f \in L^q_s(\Omega) := \{g \in L^1_{\text{loc}}(\Omega) \mid D_{x_i}^{r_i} g \in L^q(\Omega), 0 \leq r_i \leq s_i, i = 1, \dots, d\}$,

$$\|f - P_\ell f\|_{L^q(\omega_\ell)} \leq C_A \sum_{i=1}^d h_{\ell, i}^{s_i} \|D_{x_i}^{s_i} f\|_{L^q(\Omega_\ell)}, \quad (4.10)$$

where the constant $C_A > 0$ depends on $d, \mathbf{s}, \mathbf{p}$ and θ .

Proof. (i) This is an immediate consequence of Proposition 4.3 (ii).

(ii) This follows from (4.6) and (4.7).

(iii) This is a consequence of (i) and (ii).

(iv) Let $Q \in \mathcal{Q}_\ell$ such that $Q \subset \Omega_\ell$. Taking into account (4.6) and (2.4) we have that

$$|P_\ell f| \leq \max_{\substack{\beta \in \mathcal{B}_\ell \\ \text{supp } \beta \supset Q}} |\lambda_\beta(f)| \leq C |Q|^{-\frac{1}{q}} \|f\|_{L^q(\tilde{Q} \cap \omega_\ell)}, \quad \text{on } Q,$$

for all $f \in L^q(\omega_\ell)$, where $\tilde{Q} = \bigcup_{\substack{\beta \in \mathcal{B}_\ell \\ \text{supp } \beta \supset Q}} \text{supp } \beta$. Here, the constant $C > 0$ depends on \mathbf{p} and θ . Then,

$$\|P_\ell f\|_{L^q(Q)} \leq C \|f\|_{L^q(\tilde{Q} \cap \omega_\ell)}. \quad (4.11)$$

Now, (4.9) follows from the last equation.

(v) Let $Q \in \mathcal{Q}_\ell$ such that $Q \subset \omega_\ell$. By results on multidimensional Taylor expansions, there exists $p_{\tilde{Q}} \in \mathbb{P}_{\mathbf{p}}$ such that

$$\|f - p_{\tilde{Q}}\|_{L^q(\tilde{Q})} \leq C_T \sum_{i=1}^d h_{\ell,i}^{s_i} \|D_{x_i}^{s_i} f\|_{L^q(\tilde{Q})}, \quad (4.12)$$

where the constant $C_T > 0$ only depends on $d, \mathbf{s}, \mathbf{p}$ and θ . Taking into account (4.11), (iii) and (4.12), we have that

$$\begin{aligned} \|f - P_\ell f\|_{L^q(Q)} &\leq \|f - p_{\tilde{Q}}\|_{L^q(Q)} + \|p_{\tilde{Q}} - P_\ell f\|_{L^q(Q)} \\ &= \|f - p_{\tilde{Q}}\|_{L^q(Q)} + \|P_\ell(p_{\tilde{Q}} - f)\|_{L^q(Q)} \\ &\leq (1 + C) \|f - p_{\tilde{Q}}\|_{L^q(\tilde{Q})} \\ &\leq (1 + C) C_T \sum_{i=1}^d h_{\ell,i}^{s_i} \|D_{x_i}^{s_i} f\|_{L^q(\tilde{Q})}. \end{aligned}$$

Now, (4.10) follows. \square

4.4 A multiscale quasi-interpolant operator

Let P_ℓ be the operator given by (4.7), for each $\ell = 0, 1, \dots, n-1$. We define $\Pi : L^q(\Omega) \rightarrow \text{span } \mathcal{H}$ by

$$\begin{cases} \Pi_0 := P_0, \\ \Pi_{\ell+1} := \Pi_\ell + P_{\ell+1}(\text{id} - \Pi_\ell), & \ell = 0, \dots, n-2. \\ \Pi := \Pi_{n-1}. \end{cases} \quad (4.13)$$

REMARK 4.5 Notice that Theorem 4.4 (i) implies that $P_0 s = s$, for all $s \in \mathcal{S}_0$. Thus, as an immediate consequence of the definition of Π given in (4.13), we have that

$$\Pi s = s, \quad \forall s \in \mathcal{S}_0,$$

i.e., Π preserves splines in the initial level and in particular tensor-product polynomials in $\mathbb{P}_{\mathbf{p}}$.

REMARK 4.6 Taking into account the definitions of Π and P_ℓ given by (4.13) and (4.7), respectively, we have that

$$\Pi : L^q(\Omega) \rightarrow \sum_{\ell=0}^{n-1} \mathcal{S}_{\omega_\ell} := \left\{ \sum_{\ell=0}^{n-1} s_\ell \mid s_\ell \in \mathcal{S}_{\omega_\ell}, \quad \ell = 0, 1, \dots, n-1 \right\}.$$

When restricting to a fixed ω_ℓ , the following result allows to write the multiscale quasi-interpolant Π in terms of the operators $P_\ell, P_{\ell+1}, \dots, P_{n-1}$.

THEOREM 4.7 If

$$\omega_{n-1} \subset \omega_{n-2} \subset \cdots \subset \omega_2 \subset \omega_1 \subset \omega_0,$$

then

$$\Pi f = P_\ell f + \sum_{k=\ell+1}^{n-1} P_k(f - P_{k-1}f), \quad \text{on } \omega_\ell \quad (\ell = 0, 1, \dots, n-1),$$

for $f \in L^q(\Omega)$.

Before proving Theorem 4.7 we state the following elementary result.

LEMMA 4.8 For $f \in L^q(\Omega)$,

$$\Pi_\ell f = P_\ell f, \quad \text{on } \omega_\ell \quad (\ell = 0, 1, \dots, n-1).$$

Proof. Let $1 \leq \ell \leq n-1$ and $f \in L^q(\Omega)$. Since $\Pi_{\ell-1}f \in \mathcal{S}_{\ell-1} \subset \mathcal{S}_\ell$, by Theorem 4.4 (iii), we have that $P_\ell \Pi_{\ell-1}f = \Pi_{\ell-1}f$, on ω_ℓ . Now, the definition of Π_ℓ yields $\Pi_\ell f = \Pi_{\ell-1}f + P_\ell(f - \Pi_{\ell-1}f) = P_\ell f$, on ω_ℓ . \square

Proof of Theorem 4.7. From the definition of Π given by (4.13) we have that

$$\Pi = \Pi_\ell + \sum_{k=\ell+1}^{n-1} P_k(\text{id} - \Pi_{k-1}).$$

Now, since $\omega_k \subset \omega_{k-1}$, using that P_k is supported in ω_k (cf. (4.8)) and Lemma 4.8 we have that

$$\sum_{k=\ell+1}^{n-1} P_k(\text{id} - \Pi_{k-1}) = \sum_{k=\ell+1}^{n-1} P_k(\text{id} - P_{k-1}),$$

which concludes the proof. \square

Now, we state and prove the main result of this section.

THEOREM 4.9 (Quasi-interpolation in hierarchical spline spaces) Assume that

$$\omega_{n-1} \subset \omega_{n-2} \subset \cdots \subset \omega_2 \subset \omega_1 \subset \omega_0.$$

Let $\mathbf{s} := (s_1, s_2, \dots, s_d)$ be such that $1 \leq s_i \leq p_i + 1$, for $i = 1, 2, \dots, d$. If $\Pi : L^q(\Omega) \rightarrow \text{span } \mathcal{H}$ is the multiscale quasi-interpolant given by (4.13) then

$$\|f - \Pi f\|_{L^q(\omega_\ell)} \leq C_A(1 + 2C_S) \sum_{i=1}^d h_{\ell,i}^{s_i} \|D_{x_i}^{s_i} f\|_{L^q(\Omega_\ell)}, \quad \ell = 0, 1, \dots, n-1,$$

for $f \in L^s_q(\Omega)$.

Proof. Let $f \in L^s_q(\Omega)$ and let ℓ be such that $0 \leq \ell \leq n-1$. Then, using Theorem 4.7, (4.9), that $\omega_k \subset \omega_{k-1}$, (4.10) and (4.1), we have that

$$\begin{aligned} \|f - \Pi f\|_{L^q(\omega_\ell)} &\leq \|f - P_\ell f\|_{L^q(\omega_\ell)} + \sum_{k=\ell+1}^{n-1} \|P_k(f - P_{k-1}f)\|_{L^q(\omega_\ell)} \\ &= \|f - P_\ell f\|_{L^q(\omega_\ell)} + \sum_{k=\ell+1}^{n-1} \|P_k(f - P_{k-1}f)\|_{L^q(\Omega_k)} \\ &\leq \|f - P_\ell f\|_{L^q(\omega_\ell)} + C_S \sum_{k=\ell+1}^{n-1} \|f - P_{k-1}f\|_{L^q(\omega_k)} \\ &\leq \|f - P_\ell f\|_{L^q(\omega_\ell)} + C_S \sum_{k=\ell+1}^{n-1} \|f - P_{k-1}f\|_{L^q(\omega_{k-1})} \\ &\leq C_A \left(\sum_{i=1}^d h_{\ell,i}^{s_i} \|D_{x_i}^{s_i} f\|_{L^q(\Omega_\ell)} + C_S \sum_{k=\ell+1}^{n-1} \sum_{i=1}^d h_{k-1,i}^{s_i} \|D_{x_i}^{s_i} f\|_{L^q(\Omega_{k-1})} \right) \\ &\leq C_A \sum_{i=1}^d \left(1 + C_S \sum_{k=\ell+1}^{n-1} \frac{1}{2^{(k-1-\ell)s_i}} \right) h_{\ell,i}^{s_i} \|D_{x_i}^{s_i} f\|_{L^q(\Omega_\ell)} \\ &\leq C_A \sum_{i=1}^d \left(1 + C_S \sum_{k=0}^{\infty} \frac{1}{2^{ks_i}} \right) h_{\ell,i}^{s_i} \|D_{x_i}^{s_i} f\|_{L^q(\Omega_\ell)} \\ &= C_A \sum_{i=1}^d \left(1 + \frac{C_S}{1 - 2^{-s_i}} \right) h_{\ell,i}^{s_i} \|D_{x_i}^{s_i} f\|_{L^q(\Omega_\ell)}. \end{aligned}$$

□

REMARK 4.10 If the hierarchy of subdomains $\underline{\Omega}_n := \{\Omega_0, \Omega_1, \dots, \Omega_n\}$ satisfies

$$\Omega_\ell \subset \omega_{\ell-1}, \quad \ell = 1, \dots, n, \quad (4.14)$$

we say that the mesh \mathcal{Q} is *strictly admissible* (of Class 2, cf. Buffa & Giannelli, 2016). In particular, if the mesh is strictly admissible, in view of Giannelli *et al.* (2014, Proposition 20), we have that the functions

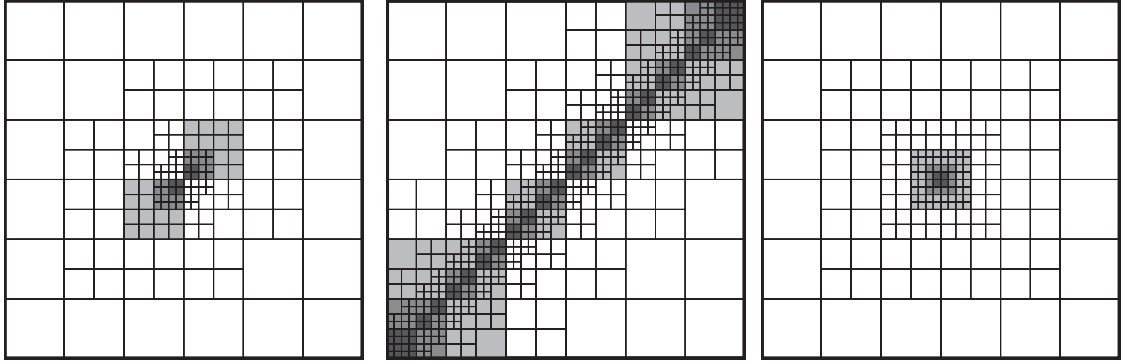


FIG. 3. Some examples of four-level meshes for splines of maximum smoothness; we consider biquadratics ($\mathbf{p} = (2, 2)$) on the left and in the middle, and bicubics ($\mathbf{p} = (3, 3)$) on the right. In all cases, the meshes are not strictly admissible, but they satisfy (4.15). The domains ω_1, ω_2 and ω_3 are highlighted in grey from the lightest to the darkest.

in the truncated basis (Giannelli *et al.*, 2012), which take nonzero values on any active cell belong to at most two different levels.

Notice that if a mesh is strictly admissible then satisfies

$$\omega_{n-1} \subset \omega_{n-2} \subset \cdots \subset \omega_2 \subset \omega_1 \subset \omega_0. \quad (4.15)$$

On the other hand, in Fig. 3 we show some nonstrictly admissible meshes which satisfy (4.15).

We conclude this section by applying Theorem 4.9 to the case of strictly admissible meshes. More precisely, we obtain optimal rates of convergence in each level of the hierarchical mesh when considering the asymptotic behaviour (cf. Speleers & Manni, 2016, Example 2).

COROLLARY 4.11 Assume that each level is obtained by dyadic refinement of the elements of the previous one (see Remark 4.1). If the mesh is strictly admissible (cf. (4.14)) then

$$\|f - \Pi f\|_{L^q(\Omega_\ell)} \leq C \sum_{i=1}^d h_{\ell,i}^{s_i} \|D_{x_i}^{s_i} f\|_{L^q(\Omega_{\ell-1})} \quad (\ell = 1, \dots, n-1),$$

for all $f \in L^s_q(\Omega)$, where the constant $C > 0$ depends only on d, \mathbf{s} and \mathbf{p} .

5. A new and easier hierarchical spline space

Assume that we have already computed the set \mathcal{H}_ℓ for given ℓ (cf. (3.1)). Now, in order to compute $\mathcal{H}_{\ell+1}$ we need to select the new B-splines to be added, i.e., $\{\beta \in \mathcal{B}_{\ell+1} \mid \text{supp } \beta \subset \Omega_{\ell+1}\}$. Once we know the B-spline basis functions to be deactivated $\mathcal{D}_\ell = \{\beta \in \mathcal{B}_\ell \mid \text{supp } \beta \subset \Omega_{\ell+1}\}$, notice that it is not enough replacing the functions in \mathcal{D}_ℓ by their children, because in general,

$$\bigcup_{\beta \in \mathcal{D}_\ell} \mathcal{C}(\beta) \subsetneq \{\beta \in \mathcal{B}_{\ell+1} \mid \text{supp } \beta \subset \Omega_{\ell+1}\}.$$

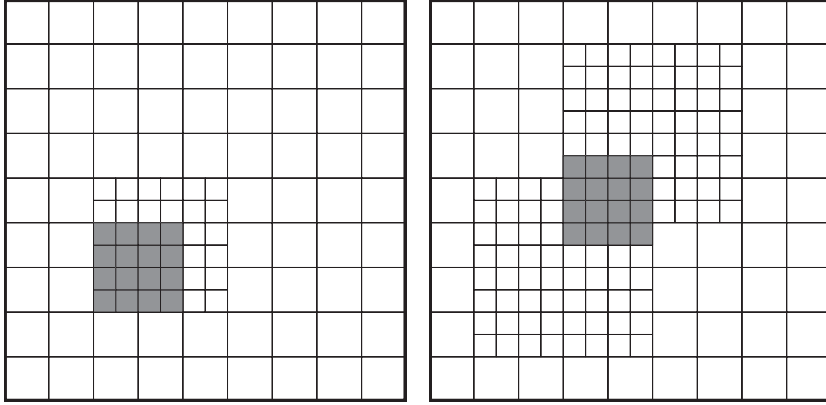


FIG. 4. Some examples of two-level meshes for bicubic splines (i.e., $\mathbf{p} = (3, 3)$) of maximum smoothness. In both cases, the highlighted B-splines of Level 1 have support included in Ω_1 , but they are not children of any deactivated B-spline of level 0.

In Fig. 4 we show some examples of this situation. This observation suggests a simplified way of selecting B-splines at different levels which consists in adding solely the children of the deactivated functions. This idea has been already used in the context of finite elements in Krysl *et al.* (2003) (see also Grinspun *et al.* (2002) and Jiang & Dolbow (2015), where adaptive refinement methods are implemented for hierarchical splines). Doing so, we obtain a new hierarchical space whose basis that we call $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}(\underline{\Omega}_n)$ is defined as follows:

$$\begin{cases} \tilde{\mathcal{H}}_0 := \mathcal{B}_0, \\ \tilde{\mathcal{H}}_{\ell+1} := \{\beta \in \tilde{\mathcal{H}}_\ell \mid \text{supp } \beta \not\subset \Omega_{\ell+1}\} \cup \bigcup_{\substack{\beta \in \tilde{\mathcal{H}}_\ell \\ \text{supp } \beta \subset \Omega_{\ell+1}}} \mathcal{C}(\beta), & \ell = 0, \dots, n-2. \\ \tilde{\mathcal{H}} := \tilde{\mathcal{H}}_{n-1}. \end{cases} \quad (5.1)$$

In this case, if $\tilde{\mathcal{D}}_\ell := \tilde{\mathcal{H}}_\ell \setminus \tilde{\mathcal{H}}_{\ell+1}$, we have that

$$\tilde{\mathcal{D}}_\ell = \{\beta \in \tilde{\mathcal{H}}_\ell \mid \text{supp } \beta \subset \Omega_{\ell+1}\} \subset \{\beta \in \mathcal{B}_\ell \mid \text{supp } \beta \subset \Omega_{\ell+1}\}, \quad (5.2)$$

but now, we can get $\tilde{\mathcal{H}}_{\ell+1}$ from $\tilde{\mathcal{H}}_\ell$ by replacing the B-splines in $\tilde{\mathcal{D}}_\ell$ by their children.

Thus, it seems that building the basis $\tilde{\mathcal{H}}$ is easier than the basis \mathcal{H} . In particular, there is no need of traversing the mesh in order to identify the B-splines to add in each recursive step of (5.1). However, as an immediate consequence of the following lemma we have that

$$\tilde{\mathcal{H}} \subset \mathcal{H},$$

and therefore, in general, $\text{span } \tilde{\mathcal{H}}$ can be smaller than $\text{span } \mathcal{H}$.

LEMMA 5.1

$$\tilde{\mathcal{H}}_\ell \subset \mathcal{H}_\ell, \quad \ell = 0, 1, \dots, n-1. \quad (5.3)$$

Proof. Notice that (5.3) holds for $\ell = 0$ due to $\tilde{\mathcal{H}}_0 = \mathcal{H}_0 = \mathcal{B}_0$. Now, using mathematical induction and taking into account (5.2) and (2.7) the proof can be completed. \square

Thus, since $\tilde{\mathcal{H}} \subset \mathcal{H}$, when considering the basis $\tilde{\mathcal{H}}$ instead \mathcal{H} for discretizations in isogeometric methods, it will be important to understand which functions we are discarding from the basis and the properties of the space $\text{span } \tilde{\mathcal{H}}$. Regarding the set of coefficients $\{a_\beta\}_{\beta \in \mathcal{H}}$ for the partition of unity in \mathcal{H} (cf. Lemma 3.2) and using Theorem 3.4, we can establish the following characterization for functions in $\tilde{\mathcal{H}}$.

THEOREM 5.2 For $\ell = 0, 1, \dots, n-1$,

$$\tilde{\mathcal{H}}_\ell = \{\beta \in \mathcal{H}_\ell \mid a_\beta > 0\}. \quad (5.4)$$

In particular,

$$\tilde{\mathcal{H}} = \{\beta \in \mathcal{H} \mid a_\beta > 0\}.$$

Proof. Since that $a_\beta = 1$, for all $\beta \in \mathcal{B}_0$, we have that (5.4) holds for $\ell = 0$. Assume now that (5.4) holds for some ℓ and prove that it holds for $\ell + 1$.

Let $\beta \in \tilde{\mathcal{H}}_{\ell+1}$. If $\beta \in \tilde{\mathcal{H}}_{\ell+1} \cap \tilde{\mathcal{H}}_\ell$, using the induction hypothesis we have that $a_\beta > 0$. On the other hand, if $\beta \in \tilde{\mathcal{H}}_{\ell+1} \setminus \tilde{\mathcal{H}}_\ell$, there exists $\beta_\ell \in \tilde{\mathcal{H}}_\ell \cap \mathcal{P}(\beta)$ such that $\text{supp } \beta_\ell \subset \Omega_{\ell+1}$ and Theorem 3.4 yields $a_\beta > 0$. Thus, by Lemma 5.1 we have that $\tilde{\mathcal{H}}_{\ell+1} \subset \{\beta \in \mathcal{H}_{\ell+1} \mid a_\beta > 0\}$.

Now, let $\beta \in \mathcal{H}_{\ell+1}$ satisfying $a_\beta > 0$. If $\beta \in \mathcal{H}_{\ell+1} \cap \mathcal{H}_\ell$, using the induction hypothesis we have that $\beta \in \tilde{\mathcal{H}}_{\ell+1}$. On the other hand, if $\beta \in \mathcal{H}_{\ell+1} \setminus \mathcal{H}_\ell$, we have that $\beta \in \mathcal{B}_{\ell+1}$ and $\text{supp } \beta \subset \Omega_{\ell+1}$. Thus, using Theorem 3.4 we have that there exists $\beta_\ell \in \mathcal{P}(\beta)$ such that $a_{\beta_\ell} > 0$ and $\text{supp } \beta_\ell \subset \Omega_{\ell+1}$. The induction hypothesis now implies that $\beta_\ell \in \tilde{\mathcal{H}}_\ell$ and therefore $\beta \in \tilde{\mathcal{H}}_{\ell+1}$. In consequence, $\{\beta \in \mathcal{H}_{\ell+1} \mid a_\beta > 0\} \subset \tilde{\mathcal{H}}_{\ell+1}$, which concludes the proof. \square

Notice that the functions in $\tilde{\mathcal{H}}$ are linearly independent because $\tilde{\mathcal{H}} \subset \mathcal{H}$. On the other hand, from (5.1) it follows that

$$\text{span } \tilde{\mathcal{H}}_\ell \subset \text{span } \tilde{\mathcal{H}}_{\ell+1}, \quad \ell = 0, 1, \dots, n-2, \quad (5.5)$$

and therefore, taking into account that $\tilde{\mathcal{H}}_0 = \mathcal{B}_0$ and $\tilde{\mathcal{H}}_{n-1} = \tilde{\mathcal{H}}$,

$$\mathcal{S}_0 = \text{span } \mathcal{B}_0 \subset \text{span } \tilde{\mathcal{H}}.$$

REMARK 5.3 Since $\text{span } \mathcal{B}_0 \subset \text{span } \tilde{\mathcal{H}}$, we have that tensor-product polynomials in \mathbb{P}_p belong to $\text{span } \tilde{\mathcal{H}}$.

In Section 5.1 we study the local approximation properties of the space $\text{span } \tilde{\mathcal{H}}$, through multiscale quasi-interpolant operators.

Finally, taking into account Theorems 5.2 and 3.4, we can prove the analogous result of Lemma 3.5 when considering the basis $\tilde{\mathcal{H}}$.

LEMMA 5.4 Let $\tilde{\mathcal{H}}$ be the HB-spline basis defined by (5.1) associated with the hierarchy of subdomains of depth n , $\underline{\Omega}_n := \{\Omega_0, \Omega_1, \dots, \Omega_n\}$. Then,

$$\tilde{\mathcal{D}}_\ell \subset \text{span} \left(\tilde{\mathcal{H}} \cap \bigcup_{k=\ell+1}^{n-1} \mathcal{B}_k \right), \quad (5.6)$$

for $\ell = 0, 1, \dots, n-2$.

Proof. Notice that (5.6) holds for $\ell = n-2$ due to (2.5). Let us assume that (5.6) holds for some ℓ and prove that it holds for $\ell-1$. Let $\beta_{\ell-1} \in \tilde{\mathcal{D}}_{\ell-1}$. Since $\beta_{\ell-1} \in \mathcal{B}_{\ell-1}$ and $\text{supp } \beta_{\ell-1} \subset \Omega_\ell$, we have that

$$\beta_{\ell-1} = \sum_{\beta_\ell \in \mathcal{C}(\beta_{\ell-1})} c_{\beta_\ell}(\beta_{\ell-1}) \beta_\ell = \sum_{\beta_\ell \in \tilde{\mathcal{D}}_\ell} c_{\beta_\ell}(\beta_{\ell-1}) \beta_\ell + \sum_{\beta_\ell \in \mathcal{C}(\beta_{\ell-1}) \setminus \tilde{\mathcal{D}}_\ell} c_{\beta_\ell}(\beta_{\ell-1}) \beta_\ell.$$

Thus, $\beta_{\ell-1} \in \text{span} \left(\tilde{\mathcal{H}} \cap \bigcup_{k=\ell}^{n-1} \mathcal{B}_k \right)$, which concludes the proof. \square

5.1 Quasi-interpolation and local approximation properties in $\text{span } \tilde{\mathcal{H}}$

In this section we assume that each level is obtained by dyadic refinement of the elements of the previous one (see Remark 4.1). The following related auxiliary technical result will be useful. The proof is presented in Appendix A.

LEMMA 5.5 Let $\mathcal{B}_{\ell, \omega_\ell}$ be the set of B-splines defined in (4.5), for $\ell = 0, 1, \dots, n-1$. Then,

$$\mathcal{B}_{\ell+1, \omega_{\ell+1}} \subset \bigcup_{\substack{\beta \in \mathcal{B}_\ell \\ \text{supp } \beta \subset \Omega_{\ell+1}}} \mathcal{C}(\beta), \quad \ell = 0, 1, \dots, n-2.$$

This lemma allows us to prove the following proposition, which, together with the results presented in the previous section, shows that the space $\text{span } \mathcal{H}$ is *rich enough* and in particular contains all the local spaces $\text{span } \mathcal{B}_{\ell, \omega_\ell}$.

PROPOSITION 5.6 Assume that

$$\omega_{n-1} \subset \omega_{n-2} \subset \dots \subset \omega_2 \subset \omega_1 \subset \omega_0. \quad (5.7)$$

Then,

$$\mathcal{B}_{\ell, \omega_\ell} \subset \tilde{\mathcal{H}}_\ell, \quad \ell = 0, 1, \dots, n-1.$$

Proof. Taking into account the definition of \mathcal{H}_ℓ given in (3.1) and the characterization of $\tilde{\mathcal{H}}_\ell$ in (5.4), we have that $\{\beta \in \mathcal{B}_\ell \mid \text{supp } \beta \subset \Omega_\ell \wedge a_\beta > 0\} \subset \tilde{\mathcal{H}}_\ell$, for $\ell = 0, 1, \dots, n-1$. On the other hand, for $\beta \in \mathcal{B}_{\ell, \omega_\ell}$, we have that $\text{supp } \beta \subset \Omega_\ell$ and thus a_β is well-defined (cf. (3.4)). Thus, it will be enough to prove that

$$\mathcal{B}_{\ell, \omega_\ell} \subset \{\beta \in \mathcal{B}_\ell \mid a_\beta > 0\}, \quad \ell = 0, 1, \dots, n-1. \quad (5.8)$$

Notice that (5.8) holds for $\ell = 0$ due to $a_{\beta_0} = 1 > 0$, for all $\beta_0 \in \mathcal{B}_0 = \mathcal{B}_{0,\omega_0}$. Now, assume that (5.8) holds for some ℓ . Let $\beta_{\ell+1} \in \mathcal{B}_{\ell+1,\omega_{\ell+1}}$. In view of Lemma 5.5, there exists $\beta_\ell \in \mathcal{P}(\beta_{\ell+1})$ such that $\text{supp } \beta_\ell \subset \Omega_{\ell+1}$. Moreover, taking into account (5.7), the definition of $\mathcal{B}_{\ell+1,\omega_{\ell+1}}$ implies that there exists $Q_{\ell+1} \in \mathcal{Q}_{\ell+1}$ such that

$$Q_{\ell+1} \subset \text{supp } \beta_{\ell+1} \cap \omega_{\ell+1} \subset \text{supp } \beta_\ell \cap \omega_\ell,$$

which in turn yields $\beta_\ell \in \mathcal{B}_{\ell,\omega_\ell}$. Finally, taking into account the induction hypothesis, we have that $a_{\beta_\ell} > 0$ and now using Theorem 3.4 we conclude that $a_{\beta_{\ell+1}} > 0$. \square

In view of Remark 4.6, the immediate consequence of Proposition 5.6 is that the multiscale quasi-interpolant operator defined in Section 4 does construct interpolating functions belonging to $\tilde{\mathcal{H}}$ and not only to \mathcal{H} . Thus, the approximation estimates from Theorem 4.9 apply verbatim to $\tilde{\mathcal{H}}$. We express this fact in the following simple corollary.

COROLLARY 5.7 Assume that

$$\omega_{n-1} \subset \omega_{n-2} \subset \cdots \subset \omega_2 \subset \omega_1 \subset \omega_0.$$

Let $1 \leq q \leq \infty$ and $\Pi : L^q(\Omega) \rightarrow \text{span } \mathcal{H}$ be the multiscale quasi-interpolant operator defined in (4.13). Then,

$$\Pi : L^q(\Omega) \rightarrow \text{span } \tilde{\mathcal{H}}.$$

5.2 Refinement of hierarchical spline spaces

When thinking of hierarchical splines within a refinement and an adaptation process, it is very important to have a precise link between the *enlargement* of the hierarchy of subdomains $\underline{\Omega}_n = \{\Omega_0, \Omega_1, \dots, \Omega_n\}$ and the refinement of the corresponding hierarchical space. This issue has been addressed for *classical* hierarchical splines in Giannelli *et al.* (2014) and here we address it for $\tilde{\mathcal{H}}$.

DEFINITION 5.8 Let $\underline{\Omega}_n := \{\Omega_0, \Omega_1, \dots, \Omega_n\}$ and $\underline{\Omega}_{n+1}^* := \{\Omega_0^*, \Omega_1^*, \dots, \Omega_n^*, \Omega_{n+1}^*\}$ be hierarchies of subdomains of Ω of depth (at most) n and $n+1$, respectively. We say that $\underline{\Omega}_{n+1}^*$ is an *enlargement* of $\underline{\Omega}_n$ if

$$\Omega_\ell \subset \Omega_\ell^*, \quad \ell = 1, 2, \dots, n.$$

Let $\underline{\Omega}_{n+1}^*$ be an enlargement of $\underline{\Omega}_n$. Now, the corresponding HB-spline basis \mathcal{H}^* and *refined* mesh \mathcal{Q}^* are given by

$$\mathcal{H}^* := \bigcup_{\ell=0}^n \{\beta \in \mathcal{B}_\ell \mid \text{supp } \beta \subset \Omega_\ell^* \wedge \text{supp } \beta \not\subset \Omega_{\ell+1}^*\},$$

and

$$\mathcal{Q}^* := \bigcup_{\ell=0}^n \{Q \in \mathcal{Q}_\ell \mid Q \subset \Omega_\ell^* \wedge Q \not\subset \Omega_{\ell+1}^*\}.$$

In Giannelli *et al.* (2014) it has been proved that

$$\text{span } \mathcal{H} \subset \text{span } \mathcal{H}^*. \quad (5.9)$$

Let $\{a_\beta^*\}_{\beta \in \mathcal{H}^*}$ denote the sequence of coefficients (with respect to the hierarchy $\underline{\Omega}_{n+1}^*$) given by Lemma 3.2. Thus, we have that

$$\sum_{\beta \in \mathcal{H}^*} a_\beta^* \beta \equiv 1, \quad \text{on } \Omega,$$

and thus, we can consider

$$\tilde{\mathcal{H}}^* := \{\beta \in \mathcal{H}^* \mid a_\beta^* > 0\}.$$

The following theorem establishes the analogous of (5.9) when considering the basis $\tilde{\mathcal{H}}$.

THEOREM 5.9

$$\text{span } \tilde{\mathcal{H}} \subset \text{span } \tilde{\mathcal{H}}^*.$$

In order to prove this result we need the following auxiliary lemma.

LEMMA 5.10 If $\beta \in \mathcal{B}_\ell$ and $\text{supp } \beta \subset \Omega_\ell$, for $\ell = 0, 1, \dots, n-1$, then

$$a_\beta^* \geq a_\beta.$$

Proof. The assertion holds for $\ell = 0$ due to $a_\beta^* = a_\beta = 1$ for all $\beta \in \mathcal{B}_0$. Now, assume that for some ℓ we have that

$$a_{\beta_\ell}^* \geq a_{\beta_\ell}, \quad \text{for } \beta_\ell \in \mathcal{B}_\ell, \text{ such that } \text{supp } \beta_\ell \subset \Omega_\ell. \quad (5.10)$$

Let $\beta_{\ell+1} \in \mathcal{B}_{\ell+1}$ such that $\text{supp } \beta_{\ell+1} \subset \Omega_{\ell+1}$. Since $\Omega_{\ell+1} \subset \Omega_{\ell+1}^*$, using (5.10) and the definitions of $a_{\beta_{\ell+1}}$ and $a_{\beta_{\ell+1}}^*$, we have that

$$a_{\beta_{\ell+1}}^* = \sum_{\substack{\beta_\ell \in \mathcal{B}_\ell \\ \text{supp } \beta_\ell \subset \Omega_{\ell+1}^*}} a_{\beta_\ell}^* c_{\beta_{\ell+1}}(\beta_\ell) \geq \sum_{\substack{\beta_\ell \in \mathcal{B}_\ell \\ \text{supp } \beta_\ell \subset \Omega_{\ell+1}}} a_{\beta_\ell}^* c_{\beta_{\ell+1}}(\beta_\ell) \geq \sum_{\substack{\beta_\ell \in \mathcal{B}_\ell \\ \text{supp } \beta_\ell \subset \Omega_{\ell+1}}} a_{\beta_\ell} c_{\beta_{\ell+1}}(\beta_\ell) = a_{\beta_{\ell+1}}.$$

□

Proof of Theorem 5.9. Let $\beta \in \tilde{\mathcal{H}}$ and let ℓ be such that $\beta \in \mathcal{B}_\ell$. Since $\text{supp } \beta \subset \Omega_\ell \subset \Omega_\ell^*$, we have that $\beta \in \mathcal{H}_\ell^*$. On the other hand, Lemma 5.10 implies that $a_\beta^* \geq a_\beta > 0$, and (5.4) yields $\beta \in \tilde{\mathcal{H}}_\ell^*$. Finally, taking into account (5.5) we have that $\beta \in \text{span } \tilde{\mathcal{H}}^*$. \square

6. Concluding remarks

In this paper, after studying the approximation properties of hierarchical splines as defined in Kraft (1998), we propose an alternative hierarchical spline space, following the approach in Grinspun *et al.* (2002), through a construction of a set of basis functions named $\tilde{\mathcal{H}}$, that enjoys several properties and may be considered as a valuable alternative to truncated hierarchical splines (Giannelli *et al.*, 2012). We can summarize and comment upon our results as follows:

- The basis that we construct simplifies the implementation and data structures needed to carry hierarchical splines because the refinement can be performed through the parent–children relations between B-splines. Unlike the classical hierarchical space, where algorithms traversing the mesh are needed to identify the new active B-splines, we just add children of already active B-splines. Moreover, we believe that our construction can be suitably used in conjunction with function-based error indicators, i.e., error indicators that mark functions (and not elements) to be refined: in our framework, a marked function would be simply replaced by some of its children. These aspects will be studied in a forthcoming paper.
- The weighted basis $\{a_\beta \beta \mid \beta \in \tilde{\mathcal{H}}\}$ constitutes a convex partition of unity and has the advantage of preserving simple basis function supports (only hypercubes) and also, in principle, asks for the use of simple spline evaluation formulae.
- We have extended the multiscale quasi-interpolant proposed in Kraft (1998) to the cases of general and open knot vectors, and we have provided local approximation estimates in L^q -norms, for $1 \leq q \leq \infty$. This interpolant is built in a multiscale fashion, it is not a projector in general and is not based on dual functionals, unlike the one presented in Speleers & Manni (2016) based on THB-splines. On the other hand, it verifies optimal local approximation estimates.

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Appendix A

Now we present the proof of Lemma 5.5. Here, if A and B are two knot vectors (i.e., sequences), we say that $A \subset B$ if A is a subsequence of B . On the other hand, given two arbitrary sequences A and B , we denote by $A \cap B$ the largest subsequence of A and B .

Proof of Lemma 5.5. For clarity of presentation, we consider first the univariate case $d = 1$. Let $p = p_1$ be the polynomial degree and let ℓ be fixed satisfying $0 \leq \ell \leq n - 2$. Let \mathcal{E}_ℓ and $\mathcal{E}_{\ell+1}$ be the open knot vectors associated with the spline spaces \mathcal{S}_ℓ and $\mathcal{S}_{\ell+1}$, respectively. Let

$$\mathcal{E}_{\ell+1} := \{\xi_1, \xi_2, \dots, \xi_{\#\mathcal{B}_{\ell+1+p+1}}\}.$$

Let $\beta_{\ell+1} \in \mathcal{B}_{\ell+1, \omega_{\ell+1}}$. Then, there exists $Q \in \mathcal{Q}_{\ell+1}$ such that $Q \subset \text{supp } \beta_{\ell+1}$ and $Q \subset \omega_{\ell+1}$. Thus, there exists k such that $Q = [\xi_k, \xi_{k+1}]$. By the definition of $\omega_{\ell+1}$, we have that

$$\tilde{Q} = [\xi_{k-p}, \xi_{k+p+1}] \subset \Omega_{\ell+1}.$$

Notice that

$$\mathcal{E}_{\tilde{Q}} := \{\xi_{k-p}, \dots, \xi_{k+p+1}\}$$

consists of $2p + 2$ consecutive knots in $\mathcal{E}_{\ell+1}$. Since $\mathcal{E}_{\ell+1}$ is obtained from \mathcal{E}_{ℓ} by dyadic refinement, we have that $\#(\mathcal{E}_{\tilde{Q}} \cap \mathcal{E}_{\ell}) \geq p + 1$. If $\#(\mathcal{E}_{\tilde{Q}} \cap \mathcal{E}_{\ell}) = p + 1$, it is easy to check that we can add one knot $\hat{\xi} \in \mathcal{E}_{\ell}$ such that $\#((\mathcal{E}_{\tilde{Q}} \cup \{\hat{\xi}\}) \cap \mathcal{E}_{\ell}) = p + 2$ and $\{\xi \in (\mathcal{E}_{\tilde{Q}} \cup \{\hat{\xi}\}) \cap \mathcal{E}_{\ell}\} \subset \Omega_{\ell+1}$. Therefore, there exists $\mathcal{E}_{\tilde{Q}}^{\ell} \subset \mathcal{E}_{\ell}$ such that

$$\#\mathcal{E}_{\tilde{Q}}^{\ell} \geq p + 2, \quad (\mathcal{E}_{\beta_{\ell+1}} \cap \mathcal{E}_{\ell}) \subset (\mathcal{E}_{\tilde{Q}} \cap \mathcal{E}_{\ell}) \subset \mathcal{E}_{\tilde{Q}}^{\ell}, \quad \{\xi \in \mathcal{E}_{\tilde{Q}}^{\ell}\} \subset \Omega_{\ell+1}. \quad (\text{A.1})$$

Let $r := \#(\mathcal{E}_{\beta_{\ell+1}} \cap \mathcal{E}_{\ell})$ and notice that $r \leq p + 1$. We consider two cases:

- (i) $\min \mathcal{E}_{\beta_{\ell+1}}$ or $\max \mathcal{E}_{\beta_{\ell+1}}$ matches a knot in \mathcal{E}_{ℓ} : By (A.1), there exists $\beta_{\ell} \in \mathcal{B}_{\ell}$ such that $\mathcal{E}_{\beta_{\ell+1}} \cap \mathcal{E}_{\ell} \subset \mathcal{E}_{\beta_{\ell}} \subset \mathcal{E}_{\tilde{Q}}^{\ell}$. Thus, $\beta_{\ell+1} \in \mathcal{C}(\beta_{\ell})$ and $\text{supp } \beta_{\ell} \subset \Omega_{\ell+1}$.
- (ii) Neither $\min \mathcal{E}_{\beta_{\ell+1}}$ nor $\max \mathcal{E}_{\beta_{\ell+1}}$ matches knots in \mathcal{E}_{ℓ} : Since $\#\mathcal{E}_{\beta_{\ell+1}} = p + 2$, in this case $r \leq p$. Thus, there exists $\beta_{\ell} \in \mathcal{B}_{\ell}$ such that $\mathcal{E}_{\beta_{\ell+1}} \cap \mathcal{E}_{\ell} \subset \mathcal{E}_{\beta_{\ell}} \subset \mathcal{E}_{\tilde{Q}}^{\ell}$. Again, $\beta_{\ell+1} \in \mathcal{C}(\beta_{\ell})$ and $\text{supp } \beta_{\ell} \subset \Omega_{\ell+1}$.

Finally, for the multivariate case $d > 1$, we can apply this argument in each coordinate direction. \square