Algebra Univers. 71 (2014) 71–100
DOI 10.1007/s00012-014-0266-3
Published online January 22, 2014

Algebra Universalis

Monadic MV-algebras I: a study of subvarieties

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Abstract. In this paper, we study and classify some important subvarieties of the variety of monadic MV-algebras. We introduce the notion of width of a monadic MV-algebra and we prove that the equational class of monadic MV-algebras of finite width k is generated by the monadic MV-algebra $[\mathbf{0},\mathbf{1}]^k$. We describe completely the lattice of subvarieties of the subvariety $\mathcal{V}([\mathbf{0},\mathbf{1}]^k)$ generated by $[\mathbf{0},\mathbf{1}]^k$. We prove that the subvariety generated by a subdirectly irreducible monadic MV-algebra of finite width depends on the order and rank of $\forall\mathbf{A}$, the partition associated to $\mathbf{A}$ of the set of coatoms of the boolean subalgebra $\mathbf{B}(\mathbf{A})$ of its complemented elements, and the width of the algebra. We also give an equational basis for each proper subvariety in $\mathcal{V}([\mathbf{0},\mathbf{1}]^k)$. Finally, we give some results about subvarieties of infinite width.

1. Introduction

To give an algebraic proof of the completeness of the Łukasiewicz infinite-valued sentential calculus, Chang introduced MV-algebras in [2]. In [11], Komori gave a complete description of the lattice of all subvarieties of MV-algebras and showed that each proper subvariety is finitely axiomatizable. Moreover, he proved that each proper subvariety of MV-algebras is generated by a finite set of totally ordered MV-algebras (MV-chains) of finite rank. After that, in [9], Di Nola and Lettieri gave equational bases for all MV-varieties.

Monadic MV-algebras, MMV-algebras for short, were introduced and studied by Rutledge in [13] as an algebraic model for the monadic predicate calculus of Łukasiewicz infinite-valued logic, in which only a single individual variable occurs. He gave MMV-algebras the name of monadic Chang algebras. Rutledge followed Halmos' study of monadic boolean algebras and represented each subdirectly irreducible MMV-algebra as a subalgebra of a functional MMV-algebra. From this representation, he proved the completeness of the monadic predicate calculus.

As usual, a functional MMV-algebra is defined as follows. Let us consider the MV-algebra $\mathbf{V}^X$ of all functions from a nonempty set X to an MV-algebra $\mathbf{V}$, where the operations $\oplus$, $\neg$, and 0 are defined pointwise. If for $p \in V^X$, there exist the supremum and the infimum of the set $\{p(y) : y \in X\}$, then we

Presented by C. Tsinakis.

Received December 10, 2012; accepted in final form August 20, 2013.
2010 *Mathematics Subject Classification*: Primary: 06D35; Secondary: 08B15, 03G25.
Key words and phrases: monadic MV-algebras, functional representation, subvarieties, equational bases.

Birkhäuser

define the constant functions $\exists_\vee(p)(x) = \sup\{p(y) : y \in X\}$ and $\forall_\wedge(p)(x) = \inf\{p(y) : y \in X\}$, for every $x \in X$. A *functional MMV-algebra* $\mathbf{A}'$ is an MMV-algebra whose MV-reduct is an MV-subalgebra of $\mathbf{V}^X$ and such that the existential and universal operators are the functions $\exists_\vee$ and $\forall_\wedge$, respectively. Observe that $\mathbf{A}'$ satisfies that

(1) if $p \in A'$, then the elements $\sup\{p(y) : y \in X\}$ and $\inf\{p(y) : y \in X\}$ exist in $\mathbf{V}$,
(2) if $p \in A'$, then the constant functions $\exists_\vee(p)$ and $\forall_\wedge(p)$ are in A'.

By a *functional representation* of an MMV-algebra $\mathbf{A}$, we mean simply a functional MMV-algebra $\mathbf{A}'$ such that $\mathbf{A}$ is isomorphic to $\mathbf{A}'$.

As MMV-algebras form the algebraic semantics of the monadic predicate infinite-valued calculus of Łukasiewicz, then the subvarieties of the variety $\mathcal{MMV}$ of MMV-algebras are in one-to-one correspondence with the intermediate logics.

In this paper, we study and classify some important subvarieties of MMV-algebras. From Rutledge's representation of an MMV-algebra, we introduce the notion of width of an MMV-algebra. We prove that if $\mathbf{A}$ is a subdirectly irreducible MMV-algebra whose width is less than or equal to a finite positive integer k, then $\mathbf{A}$ is isomorphic to a subalgebra of the functional MMV-algebra $(\forall\mathbf{A})^k$. We also prove that the equational class of all MMV-algebras of width k is generated by $[\mathbf{0},\mathbf{1}]^k$, and we give the identity (α^k) that characterizes it.

We describe completely the lattice of subvarieties of the subvariety of MMV-algebras $\mathcal{V}([\mathbf{0},\mathbf{1}]^k)$ generated by $[\mathbf{0},\mathbf{1}]^k$. One of the most important results in this paper is that the subvariety generated by a subdirectly irreducible MMV-algebra $\mathbf{A} \in \mathcal{V}([\mathbf{0},\mathbf{1}]^k)$ depends on the order and rank of $\forall\mathbf{A}$, its width, and the partition associated to $\mathbf{A}$ of the set of coatoms of the boolean subalgebra $\mathbf{B}(\mathbf{A})$ of its complemented elements. We also give the identities that characterize each proper subvariety in $\mathcal{V}([\mathbf{0},\mathbf{1}]^k)$.

Finally in this paper, we give some results about subvarieties of infinite width, but the problem of classification and axiomatization of these subvarieties in general is still open. We prove that the variety generated by a functional MMV-algebra $[\mathbf{0},\mathbf{1}]^X$, where X is infinite, is the variety generated by the set $\{[\mathbf{0},\mathbf{1}]^k : k \text{ positive integer}\}$. As a consequence, we give a finite set of generators for some simple subvarieties.

This work is the first of three. These papers can be considered as a unity and they are part of the Ph.D. Thesis [5]. In the second paper, we study the class of $\{\to,\forall,1\}$-subreducts of MMV-algebras. We introduce the equations that characterize this class and we prove that it is a variety. An algebra in this variety is called a monadic Łukasiewicz implication algebra. The main goal of this work is that the width of a monadic Łukasiewicz implication algebra $\mathbf{A}$ and the order of the Łukasiewicz implication algebra $\forall\mathbf{A}$ determine the subvariety that the algebra generates, and this result determines completely the lattice of subvarieties of the variety [7]. The last of the papers studies the class of $\{\odot, \to, \forall, 1\}$-subreducts of monadic MV-algebras. In this case, we also prove that this class is an equational class and we introduce a set of equations that describe it. An algebra in this variety is called a monadic Wajsberg hoop. One of the most important results in this last paper is that the subvariety that generates a subdirectly irreducible monadic Wajsberg hoop $\mathbf{A}$ depends on its width and the subvariety of Wajsberg hoop that $\forall\mathbf{A}$ generates. We also study and classify the subvarieties of cancellative monadic Wajsberg hoops.

This paper is structured as follows. In Section 2, we give the basic definitions and results about MV-algebras and MMV-algebras that we need in this paper. In Section 3, we characterize the directly indecomposable members of $\mathcal{MMV}$. In Section 4, we give the notion of width of an MMV-algebra and we prove that if $\mathbf{A}$ is a subdirectly irreducible MMV-algebra whose width is less than or equal to k, then $\mathbf{A}$ is isomorphic to a subalgebra of $(\forall\mathbf{A})^k$. We also prove that the equational class of MMV-algebras of width k is generated by the MMV-algebra $[\mathbf{0},\mathbf{1}]^k$, and we give the identity that characterizes it. In Section 5, we study the subvarieties generated by an algebra $\mathbf{A}$ of finite width and such that $\forall\mathbf{A}$ has finite rank. In Section 5.1, we begin by studying the subvarieties generated by simple algebras of width k. We clarify the inclusion property between them and we give the identities that characterize them. In Section 5.2, we prove that the subvariety generated by a non-simple subdirectly irreducible MMV-algebra $\mathbf{A}$ of finite width depends on the rank of $\forall\mathbf{A}$, the partition associated to $\mathbf{A}$ of the coatoms of the boolean subalgebra $\mathbf{B}(\mathbf{A})$ of the complemented elements of $\mathbf{A}$, and its width. We also give the identities that characterize each subvariety of this type. In Section 6, we describe the lattice of subvarieties of the variety of MMV-algebras generated by $[\mathbf{0},\mathbf{1}]^k$ and we give an equational basis for each proper subvariety in it. Finally, in Section 7, we study some subvarieties generated by functional MMV-algebras of infinite width.

2. Preliminaries

In this section, we include the basic definitions and results on MV-algebras and monadic MV-algebras that we need in the rest of the paper. We start by recalling the definition of MV-algebras. These algebras were introduced by C. C. Chang in [2] as algebraic models for Łukasiewicz infinitely-valued logic. We refer the reader to [4].

An *MV-algebra* is an algebra $\mathbf{A} = \langle A; \oplus, \neg, 0\rangle$ of type $(2,1,0)$ satisfying the following identities:

(MV1) $x \oplus (y \oplus z) \approx (x \oplus y) \oplus z$,
(MV2) $x \oplus y \approx y \oplus x$,
(MV3) $x \oplus 0 \approx x$,
(MV4) $\neg\neg x \approx x$,
(MV5) $x \oplus \neg 0 \approx \neg 0$,
(MV6) $\neg(\neg x \oplus y) \oplus y \approx \neg(\neg y \oplus x) \oplus x$.

We denote by $\mathcal{MV}$ the equational class of all MV-algebras. If K is a set of MV-algebras, we denote by $\mathcal{V}_{\mathcal{MV}}(K)$ the subvariety of $\mathcal{MV}$ generated by K. Where there is no risk of confusion, we just write $\mathcal{V}(K)$. In the particular case that K is equal to a single algebra $\mathbf{A}$, then we write simply $\mathcal{V}_{\mathcal{MV}}(\mathbf{A})$.

On each MV-algebra $\mathbf{A}$, we define the constant 1 and the operations $\odot$ and $\rightarrow$ as follows: $1 := \neg 0$, $x \odot y := \neg(\neg x \oplus \neg y)$, and $x \rightarrow y := \neg x \oplus y$. We write $x \leq y$ if and only if $\neg x \oplus y = 1$. It follows that $\leq$ is a partial order, called the *natural order* of $\mathbf{A}$. An MV-algebra whose natural order is total is called an MV-chain. On each MV-algebra, the natural order determines a lattice structure. Specifically, $x \vee y = (x \odot \neg y) \oplus y$ and $x \wedge y = x \odot (\neg x \oplus y)$. MV-algebras are non-idempotent generalizations of boolean algebras. Indeed, boolean algebras are just the MV-algebras obeying the additional identity $x \oplus x \approx x$. Let $\mathbf{A}$ be an MV-algebra and $B(\mathbf{A}) = \{a \in A : a \oplus a = a\}$ be the set of all idempotent elements of $\mathbf{A}$. Then $\mathbf{B}(\mathbf{A}) = \langle B(\mathbf{A}); \oplus, \neg, 0\rangle$ is a subalgebra of $\mathbf{A}$, which is also a boolean algebra. Indeed, it is the greatest boolean subalgebra of $\mathbf{A}$.

The real interval $[0,1]$, enriched with the operations $a \oplus b = \min\{1, a+b\}$ and $\neg a = 1 - a$, is an MV-algebra that we denote by $[\mathbf{0},\mathbf{1}]$. Chang proved in [3] that this algebra generates the variety $\mathcal{MV}$. Let $\mathbb{N}$ be the set of all the positive integers. For every $n \in \mathbb{N}$, we denote by $\mathbf{S}_n = \langle S_n = \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\}; \oplus, \neg, 0\rangle$ the finite MV-subalgebra of $[\mathbf{0},\mathbf{1}]$ with $n+1$ elements.

Mundici [12] defined a functor Γ between MV-algebras and (abelian) ℓ-groups with strong unit u, and proved that Γ is a categorical equivalence. For every abelian ℓ-group $\mathbf{G}$, the functor Γ equips the unit interval $[0,u]$ with the operations $x \oplus y = u \wedge (x+y)$, $\neg x = u - x$ and $1 = u$. The resulting structure $\Gamma(\mathbf{G}, u) = \langle [0,u]; \oplus, \neg, 0\rangle$ is an MV-algebra. Set $\mathbf{S}_{n,\omega} = \Gamma(\mathbb{Z} \times \mathbb{Z}, \langle n, 0\rangle)$, where $\mathbb{Z}$ is the totally ordered additive group of integers and $\mathbb{Z} \times \mathbb{Z}$ is the lexicographic product of $\mathbb{Z}$ by itself. Note that $\mathbf{S}_n$ is isomorphic to $\Gamma(\mathbb{Z}, n)$, and we write $\mathbf{S}_n \cong \Gamma(\mathbb{Z}, n)$.

A subset F of an MV-algebra $\mathbf{A}$ is a *filter* if it is closed under $\odot$, and $a \leq b$, $a \in F$ imply $b \in F$. Let $\mathrm{Fg}(X)$ denote the filter generated by $X \subseteq A$. It is easy to check that $\mathrm{Fg}(X) = \{b \in A : a_1 \odot a_2 \odot \cdots \odot a_n \leq b,\ \ a_1, a_2, \ldots, a_n \in X\}$. A filter F is called *prime* if and only if $F \neq A$ and whenever $a \vee b \in F$, then either $a \in F$ or $b \in F$. A filter F is called *maximal* if and only if it is proper and no proper filter of $\mathbf{A}$ strictly contains F. Every maximal filter is prime, but not conversely. Also, F is prime if and only if $\mathbf{A}/F$ is totally ordered. The intersection of all maximal filters, the *radical* of $\mathbf{A}$, is denoted by $\mathrm{Rad}(\mathbf{A})$.

For every $a \in A$ and $n \in \mathbb{N}$, we write a^n instead of $a \odot \cdots \odot a$ (n times). For each $a \in A$ such that $a \neq 1$, we say that $\mathrm{ord}(a) = n$ if n is the least positive integer such that $a^n = 0$. If no such integer exists, we write $\mathrm{ord}(a) = \omega$. We write $\mathrm{ord}(\mathbf{A}) = m$ if $m = \sup\{n \in \mathbb{N} : \text{there is } a \in A - \{1\} \text{ with } \mathrm{ord}(a) = n\}$, and following [11], we define $\mathrm{rank}(\mathbf{A}) = \mathrm{ord}(\mathbf{A}/\operatorname{Rad}(\mathbf{A}))$. It is known that if $\mathbf{A}$ is an MV-chain, then $\mathbf{A}/\operatorname{Rad}(\mathbf{A})$ is a simple MV-algebra. So $\mathbf{A}/\operatorname{Rad}(\mathbf{A})$ is isomorphic to a subalgebra of $[\mathbf{0},\mathbf{1}]$. Moreover, for each non-trivial MV-algebra

$\mathbf{A}$, we have that $\operatorname{rank}(\mathbf{A}) \leq n$ if and only if $\mathbf{A}$ satisfies

$$((n+1)x^n)^2 \approx 2x^{n+1}, \tag{ρ_n}$$

if and only if $\mathbf{A} \in \mathcal{V}(\{\mathbf{S}_{1,\omega}, \ldots, \mathbf{S}_{n,\omega}\})$, if and only if $\mathbf{A}/\operatorname{Rad}(\mathbf{A}) \in \mathcal{V}(\{\mathbf{S}_1, \ldots, \mathbf{S}_n\})$ [9].

In [11], Komori gave a complete description of the lattice of subvarieties of $\mathcal{MV}$ and showed that each proper subvariety of MV-algebras is generated by a finite set of MV-chains of finite rank. Indeed, he proved that a class $\mathcal{V}$ of MV-algebras is a proper variety if and only if there are two finite sets I and J of positive integers such that $I \cup J$ is nonempty and $\mathcal{V} = \mathcal{V}\left(\{\mathbf{S}_i\}_{i \in I} \cup \{\mathbf{S}_{j,\omega}\}_{j \in J}\right)$. Furthermore, in [9], Di Nola and Lettieri gave equational bases for all MV-varieties. They proved that if $\mathcal{V}$ is a proper subvariety of $\mathcal{MV}$, then for any MV-algebra $\mathbf{A}$, we have that $\mathbf{A} \in \mathcal{V}$ if and only if $\mathbf{A}$ satisfies the identities

$$((n+1)x^n)^2 \approx 2x^{n+1} \tag{ρ_n}$$

where $n = \max\{I \cup J\}$,

$$\left(px^{p-1}\right)^{n+1} \approx (n+1)x^p \tag{γ_{np}}$$

for every positive integer $1 < p < n$ such that p is not a divisor of any $i \in I \cup J$, and

$$(n+1)x^q \approx (n+2)x^q$$

for every $q \in \bigcup_{r \in I} \left(D(r) \setminus \bigcup_{s \in J} D(s)\right)$, where $D(r)$ and $D(s)$ are the sets of positive divisors of r and s, respectively.

An algebra $\mathbf{A} = \langle A; \oplus, \neg, \exists, 0\rangle$ of type $(2,1,1,0)$ is called a *monadic MV-algebra* (an MMV-algebra for short) if $\langle A; \oplus, \neg, 0\rangle$ is an MV-algebra and $\exists$ satisfies the following identities:

(MMV1) $x \leq \exists x$,	(MMV4) $\exists(\exists x \oplus \exists y) \approx \exists x \oplus \exists y$,
(MMV2) $\exists(x \vee y) \approx \exists x \vee \exists y$,	(MMV5) $\exists(x \odot x) \approx \exists x \odot \exists x$,
(MMV3) $\exists\neg\exists x \approx \neg\exists x$,	(MMV6) $\exists(x \oplus x) \approx \exists x \oplus \exists x$.

In an MMV-algebra $\mathbf{A}$, we define $\forall \colon A \to A$ by $\forall a = \neg\exists\neg a$, for every $a \in A$. Clearly, $\exists a = \neg\forall\neg a$. In the following lemma, we state that $\forall$ satisfies identities dual to (MMV1)–(MMV6).

Lemma 2.1. *In every MMV-algebra $\mathbf{A}$, the following equations are satisfied.*

(MMV7) $\forall x \leq x$,	(MMV10) $\forall(\forall x \odot \forall y) \approx \forall x \odot \forall y$,
(MMV8) $\forall(x \wedge y) \approx \forall x \wedge \forall y$,	(MMV11) $\forall(x \odot x) \approx \forall x \odot \forall x$,
(MMV9) $\forall\neg\forall x \approx \neg\forall x$,	(MMV12) $\forall(x \oplus x) \approx \forall x \oplus \forall x$.

For our purposes, it is more convenient to consider the operator $\forall$ instead of $\exists$. So from now on, we consider an algebra $\mathbf{A} = \langle A; \oplus, \neg, \forall, 0\rangle$ as an MMV-algebra if $\forall$ satisfies the identities of Lemma 2.1. We often write $\langle A; \forall\rangle$ for short.

The variety of monadic MV-algebras is denoted by $\mathcal{MMV}$. The next lemma collects some basic properties of MMV-algebras.

Lemma 2.2. [13, 6] *Let* $\mathbf{A} \in \mathcal{MMV}$. *For every* $a, b \in A$, *the following properties hold:*

(MMV13) $\forall 0 = 0$	(MMV16) $\forall(\neg a \oplus b) \leq \neg\forall a \oplus \forall b$,
(MMV14) $\forall\forall a = \forall a$,	(MMV17) $\forall(\forall a \vee \forall b) = \forall a \vee \forall b$,
(MMV15) $\forall(\forall a \oplus \forall b) = \forall a \oplus \forall b$,	(MMV18) $\forall(a \vee \forall b) = \forall a \vee \forall b$.

Consider the set $\forall A = \{a \in A : a = \forall a\} = \{a \in A : a = \exists a\}$. It is an immediate consequence of (MMV9), (MMV13), (MMV14), and (MMV15) that $\forall\mathbf{A}$ is a subalgebra of $\mathbf{A}$.

In every MMV-algebra $\mathbf{A}$, congruences are determined by monadic filters. A subset $F \subseteq A$ is said to be a *monadic filter* of $\mathbf{A}$ if F is a filter of $\mathbf{A}$ and $\forall a \in F$ whenever $a \in F$. For any set $X \subseteq A$, let $\mathrm{FMg}(X)$ denote the monadic filter generated by X. It is easy to check that $\mathrm{FMg}(X) = \{b \in A : \forall a_1 \odot \forall a_2 \odot \cdots \odot \forall a_n \leq b,\ \ a_1, a_2, \ldots, a_n \in X\}$. Note that $\mathrm{FMg}(X) = \mathrm{Fg}(\forall X)$. If F is a monadic filter of $\mathbf{A}$, then the relation θ_F defined on A by $a\theta_F b$ if and only if $(a \to b) \odot (a \to b) \in F$ is a congruence. Moreover, the correspondence $F \mapsto \theta_F$ is an isomorphism between the lattice of monadic filters and the lattice of congruences of an MMV-algebra. On the other hand, there exists an isomorphism between the lattice of monadic filters of $\mathbf{A}$ and the lattice of filters of $\forall\mathbf{A}$ given by the correspondence $F \mapsto F \cap \forall A$ [13]. From this, it is not difficult to see that any MMV-algebra $\mathbf{A}$ is isomorphic to a subdirect product of MMV-algebras $\mathbf{A}_i$ such that $\forall\mathbf{A}_i$ is totally ordered.

The following result will also be necessary.

Theorem 2.3. [13] *Let* $\mathbf{A}$ *be an MMV-algebra such that* $\forall\mathbf{A}$ *is totally ordered. For each* $a \in A$ *with* $a \neq 1$, *there is a prime filter* P_a *of* $\mathbf{A}$ *such that*

(1) $a \notin P_a$,
(2) $P_a \cap \forall A = \{1\}$, *and*
(3) *if* $r < 1$, *then* $a \vee \forall r \notin P_a$.

From this theorem, Rutledge proved the following characterization, which will be needed.

Proposition 2.4. [13] *If* $\mathbf{A}$ *is an MMV-algebra such that* $\forall\mathbf{A}$ *is totally ordered, then the MV-reduct of* $\mathbf{A}$ *is isomorphic to a subdirect product of totally ordered MV-algebras* $\mathbf{B}_i$, *for* $i \in I$, *where the canonical projections* $\pi_i \colon \mathbf{A} \to \mathbf{B}_i$ *satisfy that* $\forall\mathbf{A} \cong \pi_i(\forall\mathbf{A}) \subseteq \mathbf{B}_i$.

If $\mathbf{A}$ is a *finite* subdirectly irreducible MMV-algebra, then $\mathbf{A}$ is isomorphic to $(\forall\mathbf{A})^k$ for some positive integer k, where $\oplus$, $\neg$, and 0 are defined pointwise and $\forall_\wedge \colon (\forall A)^k \to (\forall A)^k$ is defined by

$$\forall_\wedge(\langle a_1, a_2, \ldots, a_n\rangle) = \langle a_1 \wedge a_2 \wedge \cdots \wedge a_n, \ldots, a_1 \wedge a_2 \wedge \cdots \wedge a_n\rangle.$$

Let us notice that $\exists_{\vee}(\langle a_1, a_2, \ldots, a_n\rangle) = \langle a_1 \vee a_2 \vee \cdots \vee a_n, \ldots, a_1 \vee a_2 \vee \cdots \vee a_n\rangle$. Moreover, $\forall\mathbf{A}$ is isomorphic to the diagonal subalgebra of the product [8].

For each integer $n \geq 1$, let $\mathcal{K}_n$ be the class of MMV-algebras that satisfy the identity

$$x^n \approx x^{n+1}. \tag{δ_n}$$

Then $\mathcal{K}_1$ is the variety of monadic boolean algebras, and it is clear that if $n \leq l$, then $\mathcal{K}_n \subseteq \mathcal{K}_l$. If $\mathbf{A}$ is a finite subdirectly irreducible MMV-algebra in $\mathcal{K}_n$, then $\mathbf{A} \cong \mathbf{S}_m^k$ for some m such that $1 \leq m \leq n$ and for some $k \in \mathbb{N}$ (see [8]). In addition, the variety $\mathcal{K}_n = \mathcal{V}(\{\mathbf{S}_m^k : k \in \mathbb{N}, 1 \leq m \leq n\})$ and the variety $\mathcal{MMV} = \mathcal{V}(\{\mathbf{S}_n^k : n, k \in \mathbb{N}\})$ [6].

Let $\mathbf{A}$ be an MMV-algebra. We define the monadic radical of $\mathbf{A}$, denoted by $\mathrm{RadMon}(\mathbf{A})$, as the intersection of all maximal monadic filters of $\mathbf{A}$. It is easy to see that $\mathrm{Rad}(\forall\mathbf{A}) = \mathrm{RadMon}(\mathbf{A}) \cap \forall A$. In particular, $\mathrm{Rad}(\forall\mathbf{A}) = \{1\}$ if and only if $\mathrm{RadMon}(\mathbf{A}) = \{1\}$.

Let us recall that in every MV-algebra $\mathbf{A}$, $x \in \mathrm{Rad}(\mathbf{A})$ if and only if $2x^n = 1$ for every positive integer n. Then if $\mathbf{A}$ is an MMV-algebra, $\mathrm{Rad}(\mathbf{A})$ is a monadic filter (see identity (MMV20) in Lemma 4.2). It is not difficult to see that $\mathrm{Rad}(\mathbf{A}) = \mathrm{RadMon}(\mathbf{A})$.

Let us consider the set $B(\mathbf{A})$ of boolean elements of an MMV-algebra $\mathbf{A}$. We know that $\mathbf{B}(\mathbf{A})$ is an MV-subalgebra of the MV-reduct of $\mathbf{A}$. Furthermore, if $a \in B(\mathbf{A})$, then $a = a \oplus a$. So $\forall a = \forall(a \oplus a) = \forall a \oplus \forall a$. Then $\forall a \in B(\mathbf{A})$. Thus, $\mathbf{B}(\mathbf{A})$ is an MMV-subalgebra of $\mathbf{A}$. If $\mathbf{A}$ is a subdirectly irreducible MMV-algebra, then $\forall\mathbf{A}$ is a chain and $\mathbf{B}(\mathbf{A})$ is a simple monadic boolean algebra. Let us recall that a monadic boolean algebra $\mathbf{B}$ is simple if and only if $\mathbf{B}$ is subdirectly irreducible if and only if $\forall\colon B \to B$ is defined by

$$\forall a = \begin{cases} 0 & \text{if } a < 1, \\ 1 & \text{if } a = 1. \end{cases}$$

3. Direct products

In this section, we characterize the directly indecomposable members of the variety $\mathcal{MMV}$. We prove that an MMV-algebra $\mathbf{A}$ is directly indecomposable if and only if the monadic boolean algebra $\mathbf{B}(\mathbf{A})$ is simple.

Let us recall that in an MV-algebra $\mathbf{A}$, we have that $[b) = \{a \in A : b \leq a\}$ is a filter of $\mathbf{A}$ if and only if $[b) = \mathrm{Fg}(b)$ if and only if $b \in B(\mathbf{A})$. From this, we easily have the following.

Lemma 3.1. *Let* $\mathbf{A}$ *be an MMV-algebra. The following are equivalent:*

(1) $b \in B(\mathbf{A})$ *and* $\forall b = b$,
(2) $[b) \in \mathcal{F}_M(\mathbf{A})$, *where* $\mathcal{F}_M(\mathbf{A})$ *is the set of monadic filters of* $\mathbf{A}$,
(3) $[b) = \mathrm{FMg}(b)$.

It is straightforward to see the following result.

Lemma 3.2. *Let* $\mathbf{A}$ *be an MMV-algebra and* $b \in B(\mathbf{A}) - \{1\}$*. Let* $\neg_b \colon A \to A$ *and* $\forall_b \colon A \to A$ *be defined by* $\neg_b x := \neg x \vee b$ *and* $\forall_b(x) := \forall x \vee b$*, respectively. Then* $[\mathbf{b}) = \langle [b); \oplus, \neg_b, \forall_b, 0\rangle$ *is an MMV-algebra.*

Corollary 3.3. *For every MMV-algebra* $\mathbf{A}$ *and* $b \in B(\mathbf{A}) - \{1\}$ *such that we have* $\forall b = b$*, let us define the function* $h_b \colon A \to A$ *by* $h_b(x) := x \vee b$*. Then* $[\mathbf{b}) = \langle [b); \oplus, \neg_b, \forall, b\rangle$ *is an MMV-algebra, and* h_b *is a homomorphism from* $\mathbf{A}$ *onto* $[\mathbf{b})$ *with* $\mathrm{Ker}(h_b) = [\neg b)$*.*

Corollary 3.4. *For every MMV-algebra* $\mathbf{A}$ *and* $b \in B(\mathbf{A}) - \{1\}$ *such that* $\forall b = b$*, we have:*

(a) *the MMV-algebras* $[\mathbf{b})$ *and* $\mathbf{A}/[\neg b)$ *are isomorphic,*
(b) $[\mathbf{b})$ *is a subalgebra of* $\mathbf{A}$ *if and only if* $b = 0$*,*
(c) $B([\mathbf{b})) = [b) \cap B(\mathbf{A})$ *and in addition, if* $[b)$ *is a chain, then* b *is a coatom of the boolean algebra* $\mathbf{B}(\mathbf{A})$*.*

Lemma 3.5. *Let* $\mathbf{P} = \prod_{i\in I} \mathbf{A}_i$ *be the direct product of* $\{\mathbf{A}_i\}_{i\in I}$*, a nonempty family of MMV-algebras. Then there is a set* $\{b_i : i \in I\} \subseteq B(\mathbf{P}) \cap \forall(P)$ *satisfying the following conditions:*

(a) $\bigwedge_{i\in I} b_i = 0$*,*
(b) *if* $i \neq j$*, then* $b_i \vee b_j = 1$*,*
(c) $\mathbf{A}_i$ *is isomorphic to* $[\mathbf{b}_i)$ *for each* i*.*

Proof. For each $i \in I$, let $b_i \colon I \to \bigcup_{i\in I} A_i$ be defined by $b_i(i) = 0$ and $b_i(j) = 1$ for $i \neq j$. Then $b_i \in B(\mathbf{P})$ and $\forall b_i = b_i$, and we have (a) and (b). Let $\pi_i \colon \mathbf{P} \to \mathbf{A}_i$ be the canonical projection, and let $h_{b_i} \colon \mathbf{P} \to [\mathbf{b}_i)$ be defined as previously. Then $\mathrm{Ker}(h_{b_i}) = [\neg b_i) = \{f \in P : f(i) = 1\} = \mathrm{Ker}(\pi_i)$. We conclude that $[\mathbf{b}_i)$ is isomorphic to $\mathbf{A}_i$ for each $i \in I$. □

Lemma 3.6. *Let* $\mathbf{A}$ *be an MMV-algebra. If for* $k \geq 2$ *there are boolean elements* $b_1, \ldots, b_k$ *such that*

(a) $\forall b_i = b_i$ *for each* i*,*
(b) *if* $i \neq j$*, then* $b_i \vee b_j = 1$*, and*
(c) $b_1 \wedge \cdots \wedge b_k = 0$*,*

then $\mathbf{A}$ *is isomorphic to* $[\mathbf{b}_1) \times \cdots \times [\mathbf{b}_k)$*.*

Proof. Let $h \colon \mathbf{A} \to [\mathbf{b}_1) \times \cdots \times [\mathbf{b}_k)$ be defined by $h(a) = \langle a \vee b_1, \ldots, a \vee b_k\rangle$. From (c), we have that $\bigcap_{i=1}^{k} [\neg b_i) = \{1\}$. Then h is a monomorphism. Let $\langle a_1, \ldots, a_k\rangle \in [b_1) \times \cdots \times [b_k)$. Then from (b), $h(a_1 \wedge \cdots \wedge a_k) = \langle a_1, \ldots, a_k\rangle$. So h is also surjective. Thus, h is an isomorphism. □

As a consequence of the above lemmas, we have the following.

Theorem 3.7. *An MMV-algebra* $\mathbf{A}$ *is directly indecomposable if and only if the boolean monadic algebra* $\mathbf{B}(\mathbf{A})$ *is simple.*

Corollary 3.8. *If* $\mathbf{A}$ *is an MMV-algebra and* $b \in B(\mathbf{A})$ *is a coatom of* $\mathbf{B}(\mathbf{A})$ *such that* $\forall b = b$*, then the MMV-algebra* $[\mathbf{b})$ *is directly indecomposable.*

4. MMV-algebras of finite width

In this section, we introduce the notion of width of an MMV-algebra and we prove that if $\mathbf{A}$ is a subdirectly irreducible MMV-algebra of width less than or equal to k, then $\mathbf{A}$ is isomorphic to a subalgebra of $(\forall\mathbf{A})^k$. We also prove that the equational class of all MMV-algebras of width k is generated by the MMV-algebra $[\mathbf{0},\mathbf{1}]^k$.

The following result is due to Rutledge.

Theorem 4.1. [13] *Let $\mathbf{A}$ be an MMV-algebra such that $\forall\mathbf{A}$ is totally ordered. Then $\mathbf{A}$ is isomorphic to a functional MMV-algebra whose elements are functions from a set I to an* MV-chain $\mathbf{V}$*.*

The set I of Theorem 4.1 is the set of all prime filters $\{P_a : a \in A - \{1\}\}$ given in Theorem 2.3. The MV-chain $\mathbf{V}$ has a quite convoluted construction. For our purposes, it is enough to note that there exists an MV-monomorphism from $\mathbf{A}/P_a$ to $\mathbf{V}$ for each $P_a \in I$. We refer the reader to the monograph [13] for details on the construction of $\mathbf{V}$.

It is not difficult to see that $[\mathbf{0},\mathbf{1}]^{\mathbb{N}} = \langle [0,1]^{\mathbb{N}}; \oplus, \neg, \forall_\wedge, 0\rangle$ is an MMV-algebra. Furthermore, $[\mathbf{0},\mathbf{1}]^{\mathbb{N}}$ generates the variety $\mathcal{MMV}$. Indeed, for each positive integers n and k, we have that $\mathbf{S}_n^k$ is a subalgebra of $[\mathbf{0},\mathbf{1}]^{\mathbb{N}}$. Then $\mathcal{V}(\mathbf{S}_n^k) \subseteq \mathcal{V}\left([\mathbf{0},\mathbf{1}]^{\mathbb{N}}\right)$. We also know that $\mathcal{MMV} = \mathcal{V}\left(\{\mathbf{S}_n^k : n,k \in \mathbb{N}\}\right)$ (see [6]). Thus, $\mathcal{MMV} = \mathcal{V}\left(\{\mathbf{S}_n^k : n,k \in \mathbb{N}\}\right) \subseteq \mathcal{V}([\mathbf{0},\mathbf{1}]^{\mathbb{N}}) \subseteq \mathcal{MMV}$.

Since $[\mathbf{0},\mathbf{1}]^{\mathbb{N}}$ generates the variety $\mathcal{MMV}$, we have the following lemma.

Lemma 4.2. *The following identities are satisfied in every MMV-algebra, for each positive integer n.*

(MMV19) $\forall(nx) \approx n(\forall x)$, (MMV21) $\exists(x^n) \approx (\exists x)^n$,
(MMV20) $\forall(x^n) \approx (\forall x)^n$, (MMV22) $\exists(nx) \approx n(\exists x)$.

It is known that if $\mathbf{A}$ is a subdirectly irreducible MMV-chain, then $\forall\mathbf{A} = \mathbf{A}$ [8]. So the next lemma follows.

Lemma 4.3. *Let $\mathbf{A}$ be a subdirectly irreducible MMV-algebra. Then $\mathbf{A}$ is a chain if and only if $\mathbf{A}$ satisfies*

$$\forall x \approx x. \tag{α^1}$$

For each integer $k \geq 2$, let us consider the identity

$$\forall\Big(\bigvee_{i=1}^{k+1} \bigwedge X_i^-\Big) \to \bigvee_{j=1}^{k+1} \forall x_j \approx 1, \tag{α^k}$$

where $X = \{x_1, x_2, \ldots, x_k, x_{k+1}\}$, $X_i^- = X - \{x_i\}$, and $\bigwedge X_i^-$ is the infimum of all the elements of X_i^-.

Let us observe that (α^k) can be written as

$$\bigwedge_{\{1\leq i<j\leq k+1\}} \forall(x_i \vee x_j) \to \bigvee_{j=1}^{k+1} \forall x_j \approx 1. \qquad (\alpha^k)$$

Let us consider the set I of Theorem 4.1 and the set $\bar{I}$ of all minimal prime filters of $\mathbf{A}$. For each $P_i \in \bar{I}$, we have that $P_i \subseteq P_a$ for some $P_a \in I$. Then $P_i \cap \forall A = \{1\}$ for each $P_i \in \bar{I}$. From this, and by an argument similar to the one in the proof of Proposition 2.4, we have that the MV-reduct of $\mathbf{A}$ is a subdirectly product of MV-algebras $\mathbf{A}/P_i$ totally ordered, where the projections $\pi_i\colon \mathbf{A} \to \mathbf{A}/P_i$ satisfy that $\forall\mathbf{A} \cong \pi_i(\forall\mathbf{A}) \subseteq \mathbf{A}/P_i$. We say that this representation of $\mathbf{A}$ is *minimal* because the intersection of all the filters except one is always different from $\{1\}$, and the intersection of all the filters is equal to $\{1\}$. Then, the MV-algebra $\mathbf{A}$ is a subdirect product of $\mathbf{A}/P_i$, for $P_i \in \bar{I}$.

Proposition 4.4. *If $\mathbf{A}$ is an MMV-algebra such that (α^k) holds in $\mathbf{A}$ and $\forall\mathbf{A}$ is totally ordered, then the set $\bar{I}$ has at most k elements.*

Proof. Suppose that the cardinal of $\bar{I}$ is greater than k. Let us consider $k+1$ elements y_j, for $j \in \{1,\dots,k+1\}$, such that for all j, $y_j \in \bigcap_{i\neq j} P_i$, $y_j \notin P_j$, and $y_j < 1$. From the above paragraph, we have that $\mathbf{A}$ is isomorphic to a monadic functional subalgebra of $\mathbf{V}^{\bar{I}}$. Since $y_j \in P_i$ for all $P_i \in \bar{I}$, except for P_j, the representation of the element y_j in $V^{\bar{I}}$ has all its components equal to 1, except in the place j. That is,

$$y_j(i) = \begin{cases} 1 & \text{if } i \neq j, \\ v_j & \text{if } i = j, \end{cases}$$

where $v_j < 1$. Then $\forall_\wedge(y_j) = \langle v_j,\dots,v_j\rangle < \langle 1,\dots,1\rangle$. If $i \neq j$, then $y_j \vee y_i = \langle 1,\dots,1\rangle$. Let us denote by v the supremum $\bigvee_{j=1}^{k+1} v_j$. Since $\mathbf{V}$ is a chain, we have that $v < 1$. Then

$$\begin{aligned}\bigwedge_{\{1\leq i<j\leq k+1\}} \forall_\wedge(y_i \vee y_j) \to \bigvee_{j=1}^{k+1} \forall_\wedge(y_j) &= \langle 1,\dots,1\rangle \to \langle v,\dots,v\rangle \\ &= \langle v,\dots,v\rangle < \langle 1,\dots,1\rangle.\end{aligned}$$

This contradicts (α^k). □

Proposition 4.5. *If $\mathbf{A}$ is an MMV-subalgebra of a functional MMV-algebra $\langle V^n;\forall\rangle$ such that $\mathbf{V}$ is an MV-chain, n is a positive integer, and $\forall\mathbf{A}$ is a chain, then $\mathbf{A}$ is isomorphic to a subalgebra of $(\forall\mathbf{A})^n$.*

Proof. For each $i \in \{1,\dots,n\}$, let us consider the epimorphism $\pi_i{\restriction_A}\colon A \to V$. We are going to show that for each i, $\pi_i{\restriction_A}(\forall A) = \pi_i{\restriction_A}(A)$. Clearly, $\pi_i{\restriction_A}(\forall A) \subseteq \pi_i{\restriction_A}(A)$. Let us prove that for every $b \in A$, there exists $c \in \forall A$ such that $\pi_i(b) = \pi_i(c)$. The proof of this is an induction argument on n. The case $n = 1$ is trivial because $A = \forall A$. Let us suppose that it is true for $n = k$. Let $A \subseteq V^{k+1}$ and $a = \langle a_1, a_2, \ldots, a_k, a_{k+1} \rangle \in A$. Without loss of generality, we can assume that $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_{k+1}$ since $\mathbf{V}$ is a chain. If $i = 1$ or $i = k+1$, we have that $\pi_1(a) = a_1 = \pi_1(\forall a)$ and $\pi_{k+1}(a) = a_{k+1} = \pi_{k+1}(\exists a)$. In addition, $a \to \forall a = \langle 1, a_2 \to a_1, \ldots, a_{k+1} \to a_1 \rangle$ and $\exists a \to a = \langle a_{k+1} \to a_1, a_{k+1} \to a_2, \ldots, a_{k+1} \to a_k, 1 \rangle$. Thus,

$$(a \to \forall a) \vee (\exists a \to a) = \langle 1, (a_2 \to a_1) \vee (a_{k+1} \to a_2), \ldots, (a_k \to a_1) \vee (a_{k+1} \to a_k), 1 \rangle.$$

Let $\mathbf{B}$ be the subalgebra of $\mathbf{V}^{k+1}$ on the set $B = \{a \in V^{k+1} : a_1 = a_{k+1}\}$. Then $\mathbf{B} \cong \mathbf{V}^k$ and $(a \to \forall a) \vee (\exists a \to a) \in B$. Furthermore, $(a \to \forall a) \vee (\exists a \to a) \in A \cap B$.

Let $1 < i < k+1$. Then $\pi_i((a \to \forall a) \vee (\exists a \to a)) = (a_i \to a_1) \vee (a_{k+1} \to a_i)$. Since $\mathbf{V}$ is a chain, two cases arise.

Suppose first that $a_i \to a_1 \geq a_{k+1} \to a_i$. Then $\pi_i((a \to \forall a) \vee (\exists a \to a)) = a_i \to a_1$. So $((a \to \forall a) \vee (\exists a \to a)) \to \forall a = \langle e_j \rangle_{1 \leq j \leq k+1}$, where

$$e_j = \begin{cases} a_1 & \text{if } j = 1 \text{ or } j = k+1, \\ ((a_j \to a_1) \vee (a_{k+1} \to a_j)) \to a_1 & \text{if } j \notin \{1, i, k+1\}, \\ (a_i \to a_1) \to a_1 & \text{if } j = i. \end{cases}$$

Then the i-component of $((a \to \forall a) \vee (\exists a \to a)) \to \forall a$ is equal to $a_i \vee a_1 = a_i$. Also, $((a \to \forall a) \vee (\exists a \to a)) \to \forall a \in B \cap A$, and from the induction hypothesis on $\mathbf{A} \cap \mathbf{B} \cong \mathbf{A} \cap \mathbf{V}^k$, there is $c \in \forall(A \cap B) \subseteq \forall A$ such that $\pi_i(c) = a_i$.

The other case to consider is that in which $a_i \to a_1 \leq a_{k+1} \to a_i$. Then $\pi_i((a \to \forall a) \vee (\exists a \to a)) = a_{k+1} \to a_i$. So $((a \to \forall a) \vee (\exists a \to a)) \odot \exists a = \langle e_j \rangle_{1 \leq j \leq k+1}$, where

$$e_j = \begin{cases} a_{k+1} & \text{if } j = 1 \text{ or } j = k+1, \\ ((a_j \to a_1) \vee (a_{k+1} \to a_j)) \odot a_{k+1} & \text{if } j \notin \{1, i, k+1\}, \\ (a_{k+1} \to a_i) \odot a_{k+1} & \text{if } j = i. \end{cases}$$

Then $\pi_i(((a \to \forall a) \vee (\exists a \to a)) \odot \exists a) = a_{k+1} \wedge a_i = a_i$. Thus, we have that $((a \to \forall a) \vee (\exists a \to a)) \odot \exists a \in A \cap B$ and, by the induction hypothesis, we have that there is $d \in \forall(A \cap B) \subseteq \forall A$ such that $\pi_i(d) = a_i$. □

Corollary 4.6. *Let* $\mathbf{A}$ *be a subdirectly irreducible MMV-algebra satisfying* (α^k)*, then* $\mathbf{A}$ *is isomorphic to a subalgebra of* $(\forall \mathbf{A})^k$.

Definition 4.7. Let $\mathbf{A} \in \mathcal{MMV}$. We define the *width of* $\mathbf{A}$, denoted by $\mathrm{width}(\mathbf{A})$, as the least integer k such that (α^k) holds in $\mathbf{A}$. If k does not exist, then we say that the width of $\mathbf{A}$ is infinite and we write $\mathrm{width}(\mathbf{A}) = \omega$.

As a consequence of Corollary 4.6, we have the following result, which will be needed for the description of the subvarieties of MMV-algebras that satisfy (α^k).

Corollary 4.8. *If* $\mathbf{A}$ *is a subdirectly irreducible MMV-algebra that satisfies* (α^k)*, then the algebra of complemented elements* $\mathbf{B}(\mathbf{A})$ *is isomorphic to a subalgebra of the simple monadic boolean algebra* $\mathbf{2}^k$.

It is straightforward to prove the next result.

Lemma 4.9. *Let $X = \{1, \dots, k\}$ be a finite set and $\mathbf{A}$ be an MV-algebra (not necessarily an MV-chain). Let us consider the product $\mathbf{A}^X$ where $\oplus$, $\neg$, and 0 are defined pointwise and $\forall_\wedge \colon A^X \to A^X$ is defined by $\forall_\wedge(\langle a_1, \dots, a_n\rangle) = \langle a_1 \wedge \cdots \wedge a_n, \dots, a_1 \wedge \cdots \wedge a_n\rangle$. Then $\mathbf{A}^X = \langle A^X; \oplus, \neg, \forall_\wedge, 0\rangle$ is an MMV-algebra.*

Let us observe that the MMV-algebra $\mathbf{A}^X$ of the last lemma satisfies that $\forall_\wedge(\mathbf{A}^X)$ is isomorphic to $\mathbf{A}$. From now on, we denote $\mathbf{A}^X$ by $\mathbf{A}^k$ if X is the finite set $\{1, \dots, k\}$.

Proposition 4.10. *Let $\mathbf{A}$ and $\mathbf{B}$ be two MV-algebras such that $\mathbf{A} \in \mathcal{V}_{\mathcal{MV}}(\mathbf{B})$. Then, for each positive integer k, we have that $\mathbf{A}^k \in \mathcal{V}_{\mathcal{MMV}}(\mathbf{B}^k)$.*

Proof. Let $\mathbf{A} \in \mathcal{V}_{\mathcal{MV}}(\mathbf{B}) = \mathrm{H\,S\,P}_{\mathcal{MV}}(\mathbf{B})$. Then, there is $\mathbf{W} \in \mathrm{S\,P}(\mathbf{B})$ such that $\mathbf{A}$ is a homomorphic image of $\mathbf{W}$. Let $h \colon \mathbf{W} \to \mathbf{A}$ be the MV-epimorphism from $\mathbf{W}$ onto $\mathbf{A}$. This epimorphism induces naturally an MV-epimorphism $\bar{h} \colon \mathbf{W}^k \to \mathbf{A}^k$ defined by $\bar{h}(\langle w_1, \dots, w_k\rangle) = \langle h(w_1), \dots, h(w_k)\rangle$. From Lemma 4.9, we know that $\langle W^k; \forall_\wedge\rangle$ and $\langle A^k; \forall_\wedge\rangle$ are MMV-algebras. It is easy to see that $\bar{h}$ is a MMV-homomorphism.

On the other hand, $\mathbf{W}$ is a subalgebra of a direct product $\prod_{i\in I}\mathbf{B}$ of $\mathbf{B}$. It is straightforward to see that $\langle W^k; \forall_\wedge\rangle$ is an MMV-subalgebra of $\left(\prod_{i\in I}\mathbf{B}\right)^k$. Moreover, $\varphi \colon \left(\prod_{i\in I}\mathbf{B}\right)^k \to \prod_{i\in I}\mathbf{B}^k$ defined by $\varphi\langle (a_i^1)_{i\in I}, \dots, (a_i^k)_{i\in I}\rangle = (\langle a_i^1, \dots, a_i^k\rangle)_{i\in I}$ is an MMV-isomorphism. So $\mathbf{A}^k \in \mathrm{H\,S\,P}(\mathbf{B}^k)$, and this means that $\mathbf{A}^k \in \mathcal{V}_{\mathcal{MMV}}(\mathbf{B}^k)$. □

Consider the MMV-algebra $[\mathbf{0},\mathbf{1}]^k = \langle [0,1]^k; \oplus, \neg, \forall_\wedge, 0\rangle$. We will now see that the subvariety generated by $[\mathbf{0},\mathbf{1}]^k$ is the class of all MMV-algebras that satisfy (α^k).

Observe that $\oplus$, $\neg$, and $\forall_\wedge$ are continuous functions over $[0,1]^k$ with the product topology. Recall that this topology is induced by the metric $d(x,y) = \max_{1\le j\le k}\{|x_j - y_j|\}$. Then for each MMV-term $\tau(x_1, x_2, \dots, x_s)$, the function $\tau^{[0,1]^k} \colon \left([0,1]^k\right)^s \to [0,1]^k$ is continuous. It is straightforward to see that if $1 \le n_0 < n_1 < \cdots$ is an infinite sequence of positive integers, then the set $\bigcup\{S_{n_i}^k : i = 0, 1, \dots\}$ is dense in $[0,1]^k$. Then we have the following.

Lemma 4.11. *Let k be a fixed positive integer. If $1 \le n_0 < n_1 < \cdots$ is an infinite sequence of positive integers, then $\mathcal{V}(\{\mathbf{S}_{n_i}^k : i = 0, 1, \dots\}) = \mathcal{V}([\mathbf{0},\mathbf{1}]^k)$.*

Let us recall that if $\mathbf{A}$ is an infinite subalgebra of the MV-algebra $[\mathbf{0},\mathbf{1}]$, then A is a dense subchain of $[0,1]$ [4, Prop. 3.5.3]. From this and the continuity of the term functions over $[0,1]^k$, we have the following result.

Lemma 4.12. *If $\mathbf{A}$ is an infinite subalgebra of the MV-algebra $[\mathbf{0},\mathbf{1}]$, then $\mathcal{V}\left(\mathbf{A}^k\right) = \mathcal{V}\left([\mathbf{0},\mathbf{1}]^k\right)$.*

In the next theorem, we prove that (α^k) is the identity that characterizes the subvariety $\mathcal{V}([\mathbf{0},\mathbf{1}]^k)$ within $\mathcal{MMV}$.

Theorem 4.13. *The subvariety of $\mathcal{MMV}$ generated by the algebra $[\mathbf{0},\mathbf{1}]^k$ is characterized by the identity (α^k).*

Proof. Let us consider first the case $k = 1$. Clearly, $[\mathbf{0},\mathbf{1}]$ satisfies (α^1). Conversely, if $\mathbf{A}$ is a subdirectly irreducible algebra that satisfies (α^1), then from Lemma 4.3, we know that $\mathbf{A}$ is a chain. Then $\mathbf{A} \in \mathcal{V}([\mathbf{0},\mathbf{1}])$.

Let k be an integer such that $k \geq 2$, and let us see that (α^k) holds in $[\mathbf{0},\mathbf{1}]^k$. First observe that

$$\forall(\bigvee_{i=1}^{k+1} \bigwedge X_i^-) \to \bigvee_{j=1}^{k+1} \forall x_j \approx \bigvee_{j=1}^{k+1} \Big(\forall(\bigvee_{i=1}^{k+1} \bigwedge X_i^-) \to \forall x_j\Big).$$

Let $a_1, a_2, \ldots, a_k, a_{k+1} \in [0,1]^k$. Let $A = \{a_1, a_2, \ldots, a_k, a_{k+1}\}$, and for each $j \in \{1,\ldots,k+1\}$, let $A_j^- = A - \{a_j\}$. Since A has $k+1$ elements, we have that there is some j such that $\bigwedge A = \bigwedge A_j^-$. Then $\bigwedge A_j^- \leq a_j$. In addition, if $i \neq j$, then $\bigwedge A_i^- \leq a_j$. Thus, $\bigvee_{i=1}^{k+1} \bigwedge A_i^- \leq a_j$. Then $\forall \left(\bigvee_{i=1}^{k+1} \bigwedge A_i^-\right) \leq \forall a_j$ and from this, we have that $\forall \left(\bigvee_{i=1}^{k+1} \bigwedge A_i^-\right) \to \forall a_j = 1$. This implies that (α^k) is satisfied.

Let $\mathbf{A}$ be a subdirectly irreducible algebra that satisfies (α^k). From Corollary 4.6, we know that $\mathbf{A}$ is isomorphic to a subalgebra of $(\forall\mathbf{A})^k$. In addition, the MV-algebra $\forall\mathbf{A} \in \mathcal{V}_{\mathcal{MV}}([\mathbf{0},\mathbf{1}])$. Then from Proposition 4.10, we have that $(\forall\mathbf{A})^k \in \mathcal{V}_{\mathcal{MMV}}([\mathbf{0},\mathbf{1}]^k)$. Consequently, $\mathbf{A} \in \mathcal{V}_{\mathcal{MMV}}([\mathbf{0},\mathbf{1}]^k)$. □

Corollary 4.14. $\mathcal{V}([\mathbf{0},\mathbf{1}]^t) \subseteq \mathcal{V}([\mathbf{0},\mathbf{1}]^s)$ *if and only if* $t \leq s$.

Proof. Analogously to the proof of Theorem 4.13, we can see that if $t \leq s$, then $[\mathbf{0},\mathbf{1}]^t$ satisfies (α^s). Hence, if $t \leq s$, then $\mathcal{V}([\mathbf{0},\mathbf{1}]^t) \subseteq \mathcal{V}([\mathbf{0},\mathbf{1}]^s)$. Let us prove now that if $t > s$, then $[\mathbf{0},\mathbf{1}]^t$ does not satisfy (α^s). For that, let us consider the subalgebra $\mathbf{B}([\mathbf{0},\mathbf{1}]^t)$ of boolean elements of $[\mathbf{0},\mathbf{1}]^t$. Since $t > s$, we can consider a set $\{x_1,\ldots,x_{s+1}\}$ with $s+1$ coatoms of $\mathbf{B}([\mathbf{0},\mathbf{1}]^t)$. Then $\forall x_j = 0$ for each $j \in \{1,\ldots,s+1\}$. If $i \neq j$, then $x_i \vee x_j = 1$. Consequently,

$$\bigwedge_{1 \leq i < j \leq s+1} \forall(x_i \vee x_j) \to \bigvee_{j=1}^{s+1} \forall x_j = (1 \to 0) = 0.$$

So $[\mathbf{0},\mathbf{1}]^t$ does not satisfy (α^s). □

From Lemma 4.11 and since the variety $\mathcal{MMV}$ is generated by its finite members, we have the following result.

Lemma 4.15. *If $1 \leq k_1 < k_2 < \cdots$ is an infinite sequence of natural numbers, then $\mathcal{V}\left(\{[\mathbf{0},\mathbf{1}]^{k_i} : i = 1,2,\ldots\}\right) = \mathcal{MMV}$.*

Consequently, there is an ω-chain

$$\mathcal{V}([\mathbf{0},\mathbf{1}]) \subsetneq \mathcal{V}([\mathbf{0},\mathbf{1}]^2) \subsetneq \cdots \subsetneq \mathcal{V}([\mathbf{0},\mathbf{1}]^k) \subsetneq \cdots \subsetneq \mathcal{MMV}$$

in the lattice of subvarieties of $\mathcal{MMV}$.

Lemma 4.16. *If* $\mathbf{A}$ *is a subdirectly irreducible MMV-algebra with* $\mathbf{A} \cong (\forall\mathbf{A})^k$ *and* $\operatorname{rank}(\forall\mathbf{A}) = \omega$*, then* $\mathcal{V}(\mathbf{A}) = \mathcal{V}([\mathbf{0},\mathbf{1}]^k)$.

Proof. Analogously to the proof of Theorem 4.13, we can see that $\mathbf{A}$ satisfies (α^k). Then $\mathcal{V}(\mathbf{A}) \subseteq \mathcal{V}([\mathbf{0},\mathbf{1}]^k)$. On the other hand, we know that $\forall\mathbf{A}/\operatorname{Rad}(\forall\mathbf{A})$ is a simple MV-algebra. Then $\forall\mathbf{A}/\operatorname{Rad}(\forall\mathbf{A})$ is a subalgebra of $[\mathbf{0},\mathbf{1}]$ that is also infinite since $\operatorname{ord}(\forall\mathbf{A}/\operatorname{Rad}(\forall\mathbf{A})) = \omega$. So from Lemma 4.12 we conclude that $\mathcal{V}([\mathbf{0},\mathbf{1}]^k) = \mathcal{V}\left((\forall\mathbf{A}/\operatorname{Rad}(\forall\mathbf{A}))^k\right) \subseteq \mathcal{V}(\mathbf{A})$. Hence, $\mathcal{V}(\mathbf{A}) = \mathcal{V}([\mathbf{0},\mathbf{1}]^k)$. □

5. Subvarieties of finite width and rank

In this section, we describe for each positive integer k the subvarieties of width k. We divide this section into two. In Section 5.1, we study the subvarieties generated by simple algebras of width k. We clarify the inclusion relation between them and we give an equational basis for each one. In Section 5.2, we describe the subvarieties generated by non-simple algebras such that $\forall\mathbf{A}$ has finite rank n and width k. The aim of this section is to prove that the variety generated by a non-simple subdirectly irreducible MMV-algebra $\mathbf{A}$ of finite width k depends on the order and rank of $\forall\mathbf{A}$ and partition of $\{1,\ldots,k\}$ associated to the boolean algebra $\mathbf{B}(\mathbf{A})$ of its complemented elements.

5.1. Subvarieties generated by simple algebras. In the following, let k, s, n, and m be positive integer numbers.

Let us consider the following subvarieties of the variety $\mathcal{K}_n$ defined in Section 2. Let $\mathcal{K}_n^k$ be the subvariety generated by $\{\mathbf{S}_1^k, \mathbf{S}_2^k, \ldots, \mathbf{S}_n^k\}$. If $n = 1$, then $\mathcal{K}_1$ is the variety of monadic boolean algebras and it is a well-known fact that the lattice of subvarieties of $\mathcal{K}_1$ is an $\omega + 1$-chain

$$\mathcal{K}_1^1 \subsetneq \mathcal{K}_1^2 \subsetneq \cdots \subsetneq \mathcal{K}_1^k \subsetneq \cdots \subsetneq \mathcal{K}_1.$$

More generally, $\mathcal{K}_n^1 \subsetneq \mathcal{K}_n^2 \subsetneq \cdots \subsetneq \mathcal{K}_n^k \subsetneq \cdots \subsetneq \mathcal{K}_n$. Also, $\mathcal{K}_n^k \subseteq \mathcal{K}_m^s$ if and only if $n \leq m$ and $k \leq s$ [1].

Lemma 5.1. *An MMV-algebra* $\mathbf{A}$ *is in* $\mathcal{K}_n^k$ *if and only if* $\mathbf{A}$ *satisfies* (α^k) *and* (δ_n).

Proof. We know that if $m \leq n$, then $\mathbf{S}_m$ satisfies (δ_n). Then $\mathbf{S}_m^k$ satisfies (δ_n). In addition, $\mathbf{S}_m^k$ is a subalgebra of $[\mathbf{0},\mathbf{1}]^k$. So from Theorem 4.13 we have that $\mathbf{S}_m^k$ satisfies (α^k). Thus, if $\mathbf{A} \in \mathcal{K}_n^k = \mathcal{V}(\{\mathbf{S}_1^k, \mathbf{S}_2^k, \ldots, \mathbf{S}_n^k\})$, then $\mathbf{A}$ satisfies (α^k) and (δ_n).

Conversely, let $\mathbf{A}$ be a subdirectly irreducible algebra that satisfies (α^k) and (δ_n). Then $\forall\mathbf{A}$ is an MV-chain such that (δ_n) holds in $\forall\mathbf{A}$. This implies that $\forall\mathbf{A}$ is isomorphic to $\mathbf{S}_m$ for some $m \leq n$. From Corollary 4.6, we have that $\mathbf{A}$ is a subalgebra of $\mathbf{S}_m^k$. Hence, $\mathbf{A} \in \mathcal{K}_n^k = \mathcal{V}(\{\mathbf{S}_1^k, \mathbf{S}_2^k, \ldots, \mathbf{S}_n^k\})$. □

Let us see now subvarieties generated by a simple MMV-algebra.

Lemma 5.2. *If* $\mathbf{A} \in \mathcal{MMV}$ *is a subdirectly irreducible algebra such that* $\mathrm{ord}(\forall\mathbf{A}) = m$ *and* $\mathrm{width}(\mathbf{A}) = k$*, then* $\mathbf{A} \cong \mathbf{S}_m^k$*, and hence* $\mathcal{V}(\mathbf{A}) = \mathcal{V}(\mathbf{S}_m^k)$*.*

Proof. If $\mathbf{A}$ is subdirectly irreducible and $\mathrm{ord}(\forall\mathbf{A}) = m$, then $\forall\mathbf{A} \cong \mathbf{S}_m$. Since $\mathrm{width}(\mathbf{A}) = k$ and from Corollary 4.6, we have that $\mathbf{A}$ is a subalgebra of $\mathbf{S}_m^k$. Then $\mathbf{A} \cong \mathbf{S}_m^{k'}$ for some $k' \leq k$ [1]. Since $\mathrm{width}(\mathbf{A}) = k$, then $k = k'$. Then $\mathbf{A} \cong \mathbf{S}_m^k$, and consequently $\mathcal{V}(\mathbf{A}) = \mathcal{V}(\mathbf{S}_m^k)$. □

The subvariety generated by $\mathbf{S}_m^k$ is denoted by $\mathcal{MMV}_m^k$.

Corollary 5.3. *The subvariety* $\mathcal{MMV}_1^k = \mathcal{V}(\mathbf{S}_1^k)$ *is characterized by the identities* (δ_1) *and* (α^k)*, and the subvariety* $\mathcal{MMV}_2^k = \mathcal{V}(\mathbf{S}_2^k)$ *is characterized by* (δ_2) *and* (α^k)*.*

Proposition 5.4. *The subvariety* $\mathcal{MMV}_n^k = \mathcal{V}(\mathbf{S}_n^k)$*, for* $n \geq 3$*, is characterized by the identities:*

$$\forall\Big(\bigvee_{i=1}^{k+1} \bigwedge X_i^-\Big) \to \bigvee_{j=1}^{k+1} \forall x_j \approx 1, \tag{α^k}$$

$$x^n \approx x^{n+1}, \tag{δ_n}$$

$$(px^{p-1})^{n+1} \approx (n+1)x^p, \tag{γ_{np}}$$

for every integer $p = 2, \ldots, n-1$ *such that* p *is not a divisor of* n*.*

Proof. The subvariety of MV-algebras generated by $\mathbf{S}_n$ is characterized by equations (δ_n) and (γ_{np}). Then $\mathbf{S}_m^k$ satisfies (δ_n) and (γ_{np}) [9]. From Theorem 4.13 and taking into account that $\mathbf{S}_m^k$ is a subalgebra of $[\mathbf{0}, \mathbf{1}]^k$, we conclude that $\mathbf{S}_m^k$ also satisfies (α^k).

Let $\mathbf{A}$ be a subdirectly irreducible algebra that satisfies (α^k), (δ_n), and (γ_{np}). Then $\forall\mathbf{A}$ is an MV-chain that satisfies (δ_n) and (γ_{np}). So $\forall\mathbf{A}$ is isomorphic to $\mathbf{S}_m$ for some m such that m divides n. In addition, from Corollary 4.6, we have that $\mathbf{A}$ is isomorphic to a subalgebra of $\mathbf{S}_m^k$. This implies that $\mathbf{A} \in \mathcal{V}(\mathbf{S}_n^k)$. □

Since $\mathbf{S}_m^s$ is a subalgebra of $\mathbf{S}_n^k$ if and only if m divides n and $s \leq k$, we have the following relation between the varieties $\mathcal{MMV}_n^k$.

Lemma 5.5. *Let* n*,* m*,* k*, and* s *be positive integer numbers. Then we have* $\mathcal{MMV}_m^s \subseteq \mathcal{MMV}_n^k$ *if and only if* m *divides* n *and* $s \leq k$*.*

5.2. Subvarieties generated by non-simple algebras. Let us recall the following theorem, which is a consequence of Theorems 5.8 and 5.10 of the monograph [10].

Theorem 5.6. [10] *If* $\mathbf{V}$ *is an MV-chain of rank* n*, then* $\mathbf{V} \in \mathrm{I\,S\,P_U}_{\mathcal{MV}}(\mathbf{S}_{n,\omega})$*.*

Proposition 5.7. *If* $\mathbf{V}$ *is an MV-chain of rank* n*, then the MMV-algebra* $\mathbf{V}^k$ *belongs to* $\mathrm{I\,S\,P_U}(\mathbf{S}_{n,\omega}^k)$ *for each positive integer* k*.*

Proof. Let k be a positive integer and let $\mathbf{V}$ be an MV-chain of rank n. Then there is an MV-algebra $\mathbf{W} \in \mathrm{S}\,\mathrm{P}_{\mathrm{U}\mathcal{MV}}(\mathbf{S}_{n,\omega})$ such that $\mathbf{V}$ is isomorphic to $\mathbf{W}$, and there exist a set I, an ultrafilter U of the boolean algebra of subsets of I, and an injective MV-homomorphism $h\colon \mathbf{W} \to \prod_{i\in I} \mathbf{S}_{n,\omega}/U$. Let us observe that $\prod_{i\in I} \mathbf{S}_{n,\omega}/U$ is totally ordered since the property of being totally ordered is preserved under ultraproducts.

From Proposition 4.9, we know that $\langle \left(\prod_{i\in I} \mathbf{S}_{n,\omega}/U\right)^k; \forall_\wedge\rangle$ is an MMV-algebra where the operator $\forall_\wedge$ is defined by $\forall_\wedge \left\langle \langle a_i^1\rangle_{i\in I}/U, \dots, \langle a_i^k\rangle_{i\in I}/U\right\rangle = \langle c,\dots,c\rangle$ and $c = \bigwedge_{j=1}^k \langle a_i^j\rangle_{i\in I}/U$. In a natural way, the MV-monomorphism h induces an MMV-monomorphism $\bar{h}\colon \langle W^k; \forall_\wedge\rangle \to \langle \left(\prod_{i\in I} \mathbf{S}_{n,\omega}/U\right)^k; \forall_\wedge\rangle$ defined by $\bar{h}(\langle w_1,\dots,w_k\rangle) = \langle h(w_1),\dots,h(w_k)\rangle$.

Let us prove that $\langle \left(\prod_{i\in I} \mathbf{S}_{n,\omega}/U\right)^k; \forall_\wedge\rangle$ is isomorphic to $\langle \prod_{i\in I} \mathbf{S}_{n,\omega}^k/U; \forall\rangle$. Let

$$\phi\colon \Big(\prod_{i\in I} \mathbf{S}_{n,\omega}/U\Big)^k \to \prod_{i\in I} \mathbf{S}_{n,\omega}^k/U$$

be defined by $\phi\left(\left\langle \langle a_i^1\rangle_{i\in I}/U,\dots,\langle a_i^k\rangle_{i\in I}/U\right\rangle\right) = \langle a_i^1,\dots,a_i^k\rangle_{i\in I}/U$. It is clear that ϕ is an MV-epimorphism. Let us see now that ϕ is an injective MMV-homomorphism. Indeed,

$$\begin{aligned}
&\forall_\wedge\langle\langle a_i^1\rangle_{i\in I}/U,\dots,\langle a_i^k\rangle_{i\in I}/U\rangle = \langle \bigwedge_{j=1}^k (\langle a_i^j\rangle_{i\in I}/U),\dots, \bigwedge_{j=1}^k (\langle a_i^j\rangle_{i\in I}/U)\rangle \\
&= \langle\langle \bigwedge_{j=1}^k a_i^j\rangle_{i\in I}/U,\dots,\langle \bigwedge_{j=1}^k a_i^j\rangle_{i\in I}/U\rangle = \langle\langle c_i\rangle_{i\in I}/U,\dots,\langle c_i\rangle_{i\in I}/U\rangle,
\end{aligned}$$

where $c_i = \bigwedge_{j=1}^k a_i^j$. Then

$$\begin{aligned}
&\phi\langle\langle c_i\rangle_{i\in I}/U,\dots,\langle c_i\rangle_{i\in I}/U\rangle = \langle c_i,\dots,c_i\rangle_{i\in I}/U = \langle \forall_\wedge\langle a_i^1,\dots,a_i^k\rangle\rangle_{i\in I}/U \\
&= \forall\left(\langle a_i^1,\dots,a_i^k\rangle_{i\in I}/U\right) = \forall\left(\phi\left(\left\langle\langle a_i^1\rangle_{i\in I}/U,\dots,\langle a_i^k\rangle_{i\in I}/U\right\rangle\right)\right).
\end{aligned}$$

Let us suppose that $\langle a_i^1,\dots,a_i^k\rangle_{i\in I}/U = \langle 1,\dots,1\rangle_{i\in I}/U$. Then we have that $\{i\in I : \langle a_i^1,\dots,a_i^k\rangle = \langle 1,\dots,1\rangle\} \in U$. Thus, $\{i\in I : a_i^1 = \cdots = a_i^k = 1\} \in U$. Since for all j, $\{i\in I : a_i^1 = \cdots = a_i^k = 1\} \subseteq \{i\in I : a_i^j = 1\}$, we have that $\{i\in I : a_i^j = 1\} \in U$. Then $\langle a_i^j\rangle_{i\in I}/U = \langle 1\rangle_{i\in I}/U$, for all j. In consequence, ϕ is injective. Hence, $\mathbf{V}^k \in \mathrm{I}\,\mathrm{S}\,\mathrm{P}_{\mathrm{U}}(\mathbf{S}_{n,\omega}^k)$. □

As a consequence of Proposition 5.7 and Corollary 4.6, we have the following theorem.

Theorem 5.8. *If $\mathbf{A} \in \mathcal{MMV}$ is a subdirectly irreducible algebra such that* $\mathrm{width}(\mathbf{A}) = k$ *and $\forall\mathbf{A}$ is an MV-chain of rank n, then $\mathbf{A} \in \mathrm{I}\,\mathrm{S}\,\mathrm{P}_{\mathrm{U}}(\mathbf{S}_{n,\omega}^k)$. In particular, $\mathbf{A} \in \mathcal{V}(\mathbf{S}_{n,\omega}^k)$.*

Let us denote by $\mathcal{MMV}_{n,\omega}^k$ the subvariety of MMV-algebras that satisfy the identities (α^k), (ρ_n), and (γ_{np}), for each integer $p = 2,\dots,n-1$ that does not divide n. Let us prove that $\mathcal{MMV}_{n,\omega}^k$ is equal to the subvariety generated by $\mathbf{S}_{n,\omega}^k$.

Lemma 5.9. *Let n and k be positive integers; then $\mathcal{V}(\mathbf{S}^k_{n,\omega}) = \mathcal{MMV}^k_{n,\omega}$.*

Proof. The identities (ρ_n) and (γ_{np}) characterize the variety of MV-algebras generated by $\mathbf{S}_{n,\omega}$. Then $\mathbf{S}^k_{n,\omega}$ satisfies (ρ_n) and (γ_{np}) [9]. In addition, from Proposition 4.10, we know that $\mathbf{S}^k_{n,\omega} \in \mathcal{V}_{\mathcal{MMV}}([\mathbf{0},\mathbf{1}]^k)$. Thus, from Theorem 4.13, we have that $\mathbf{S}^k_{n,\omega}$ satisfies (α^k). It follows that $\mathcal{V}(\mathbf{S}^k_{n,\omega}) \subseteq \mathcal{MMV}^k_{n,\omega}$.

Let $\mathbf{A}$ be a subdirectly irreducible algebra that satisfies (ρ_n) and (γ_{np}), for each integer $1 < p < n$ such that p is not a divisor of n, and satisfies (α^k). From Theorem 5.8, we have that $\mathbf{A} \in \mathcal{V}(\mathbf{S}^k_{n,\omega})$. Then $\mathcal{MMV}^k_{n,\omega} \subseteq \mathcal{V}(\mathbf{S}^k_{n,\omega})$. □

Since $\mathbf{S}^s_{m,\omega}$ and $\mathbf{S}^s_m$ are subalgebras of $\mathbf{S}^k_{n,\omega}$ if and only if m divides n and $s \leq k$, we have the following inclusion relation.

Lemma 5.10. *Let n, m, k, and s be positive integers.*

(1) *$\mathcal{MMV}^s_m \subseteq \mathcal{MMV}^k_{n,\omega}$ if and only if m divides n and $s \leq k$.*
(2) *$\mathcal{MMV}^s_{m,\omega} \subseteq \mathcal{MMV}^k_{n,\omega}$ if and only if m divides n and $s \leq k$.*

Let $\mathbf{A}$ be a subdirectly irreducible MMV-algebra that satisfies (α^k) and such that $\forall\mathbf{A}$ is a non-simple MV-chain of rank n. From Corollary 4.6, we know that $\mathbf{A}$ is a subalgebra of the MMV-algebra $(\forall\mathbf{A})^k$. From now on, we identify $\mathbf{A}$ with this subalgebra. We also have that $\mathbf{A}/\operatorname{Rad}(\mathbf{A})$ is a subalgebra of $(\forall\mathbf{A})^k/\operatorname{Rad}((\forall\mathbf{A})^k) \cong (\forall\mathbf{A}/\operatorname{Rad}(\forall\mathbf{A}))^k$. So $\mathbf{A}/\operatorname{Rad}(\mathbf{A})$ is embeddable into $\mathbf{S}^k_n$. Consequently, $\mathbf{A}/\operatorname{Rad}(\mathbf{A})$ is isomorphic to $\mathbf{S}^s_n$ where $s \leq k$.

Let us note that $\mathbf{B}(\mathbf{S}^k_n)$ is isomorphic to $\mathbf{2}^k$. Indeed, $f \in B(\mathbf{S}^k_n)$ if and only if $f(i) \in \{0,1\}$ for all $i \in \{1,\ldots,k\}$. It is well known that there is a correspondence between the family of all subalgebras of $\mathbf{2}^k$ and the partitions of the set of coatoms of $\mathbf{2}^k$. In addition, the partitions of this set are in a natural correspondence with the partitions of the set $\{1,\ldots,k\}$. Then we have a one-to-one onto correspondence between the set of subalgebras of $\mathbf{2}^k$ and the set of all partitions of $\{1,\ldots,k\}$. Let $\mathbf{P} = \{P_1, P_2, \ldots, P_s\}$ be the partition of $\{1,\ldots,k\}$ determined by a subalgebra $\mathbf{B}_s \cong \mathbf{2}^s$ of $\mathbf{B}(\mathbf{S}^k_n)$. Then the elements f of the subalgebra $\mathbf{B}_s$ are $f \in S^k_n$ such that $f(r) \in \{0,1\}$ for all $r \in \{1,\ldots,k\}$, and such that $f(i) = f(j)$ if $i,j \in P_t$ for all $t \in \{1,\ldots,s\}$. Let us observe also that each coatom f^j, for $j \in \{1,\ldots,s\}$, of $\mathbf{B}_s$ is obtained by the meet of the coatoms of $\mathbf{B}(\mathbf{S}^k_n)$ corresponding to the element P_j in the partition $\mathbf{P}$. So f^j satisfies that $f^j(i) = 1$ if $i \notin P_j$, and $f^j(i) = 0$ if $i \in P_j$.

Lemma 5.11. *Let $\mathbf{A}$ be a subdirectly irreducible MMV-algebra that satisfies (α^k) and such that $\forall\mathbf{A}$ is a non-simple MV-chain of rank n. Then $\mathbf{A}/\operatorname{Rad}(\mathbf{A})$ is isomorphic to $\mathbf{S}^s_n$ if and only if $\mathbf{B}(\mathbf{A})$ is isomorphic to $\mathbf{2}^s$.*

Proof. Let us observe that $\mathbf{A}/\operatorname{Rad}(\mathbf{A})$ is isomorphic to $\mathbf{S}^s_n$ with $s \leq k$. Then $\mathbf{B}(\mathbf{A}/\operatorname{Rad}(\mathbf{A}))$ is isomorphic to $\mathbf{2}^s$. Since $\{1\} = B(\mathbf{A}) \cap \operatorname{Rad}(\mathbf{A})$, we have that $\kappa : \mathbf{A} \to \mathbf{A}/\operatorname{Rad}(\mathbf{A})$ is the natural epimorphism and $\kappa\restriction_{\mathbf{B}(\mathbf{A})} : \mathbf{B}(\mathbf{A}) \to \mathbf{B}(\mathbf{S}^s_n)$ is one to one. Moreover, $\kappa\restriction_{\mathbf{B}(\mathbf{A})}$ is onto. Indeed, let $b \in \mathbf{B}(\mathbf{S}^s_n)$, so there is $c \in \mathbf{A}$ such that $\kappa(c) = b$. From Corollary 4.6, we know that $\mathbf{A}$ is isomorphic to a subalgebra of $(\forall \mathbf{A})^k$. Since $\operatorname{rank}(\forall \mathbf{A}) = n$, we have that $2c^{n+1}$ is boolean. Then $\kappa(2c^{n+1}) = 2(\kappa(c))^{n+1} = 2b^{n+1} = b$. Thus, $\kappa\restriction_{\mathbf{B}(\mathbf{A})}$ is onto. □

Proposition 5.12. *If* $\mathbf{A}$ *is a subdirectly irreducible MMV-algebra that satisfies* (α^k), $\forall \mathbf{A}$ *is a non-simple MV-chain of rank* n, *and* $\mathbf{A}/\operatorname{Rad}(\mathbf{A})$ *is isomorphic to* $\mathbf{S}_n^k$, *then* $\mathcal{V}(\mathbf{A}) = \mathcal{V}(\mathbf{S}_{n,\omega}^k) = \mathcal{MMV}_{n,\omega}^k$.

Proof. If $\mathbf{A}/\operatorname{Rad}(\mathbf{A})$ is isomorphic to $\mathbf{S}_n^k$, then from Lemma 5.11, we know that $\mathbf{B}(\mathbf{A})$ is isomorphic to $\mathbf{2}^k$. Then $\mathbf{A}$ is isomorphic to $(\forall \mathbf{A})^k$. By Rutledge's representation, $\mathbf{A}$ is isomorphic as MMV-algebra to $\langle(\forall \mathbf{A})^k, \forall_\wedge\rangle$. So from Proposition 4.10, we conclude that $\mathcal{V}(\mathbf{A}) = \mathcal{V}((\forall \mathbf{A})^k) = \mathcal{V}(\mathbf{S}_{n,\omega}^k)$. □

Now we analyze the case in which $\mathbf{A}/\operatorname{Rad}(\mathbf{A})$ is isomorphic to a proper subalgebra of $\mathbf{S}_n^k$. For this, we introduce the identity

$$\bigwedge_{1\leq i<j\leq s+1} \forall(2x_i^{n+1} \vee 2x_j^{n+1}) \rightarrow \bigvee_{j=1}^{s+1} \forall(2x_j^{n+1}) \approx 1. \qquad (\beta_n^s)$$

Proposition 5.13. *Let* $\mathbf{A} \in \mathcal{MMV}$ *be a subdirectly irreducible algebra that satisfies that* $\operatorname{rank}(\forall \mathbf{A}) = n$ *and* $\operatorname{width}(\mathbf{A}) = k$. *Then the identity* (β_n^s) *holds in* $\mathbf{A}$ *if and only if* $\mathbf{B}(\mathbf{A})$ *is isomorphic to a subalgebra of* $\mathbf{2}^s$.

Proof. Let us suppose that (β_n^s) holds in $\mathbf{A}$ and $\mathbf{B}(\mathbf{A})$ is isomorphic to $\mathbf{2}^t$ with $s < t$. Let $\{a_1, \ldots, a_{s+1}\}$ be a subset of the set of coatoms of $\mathbf{B}(\mathbf{A})$ with $s+1$ pairwise different elements. Then $\bigwedge_{1\leq i<j\leq s+1} \forall(2a_i^{n+1} \vee 2a_j^{n+1}) = 1$ and $\bigvee_{j=1}^{s+1} \forall(2a_j^{n+1}) = 0$. This contradicts (β_n^s).

Let $\mathbf{A}$ be such that $\mathbf{B}(\mathbf{A})$ is isomorphic to a subalgebra of $\mathbf{2}^s$. Then the set of coatoms of $\mathbf{B}(\mathbf{A})$ has at most s elements. Suppose that (β_n^s) does not hold in $\mathbf{A}$. Then there exist $s+1$ elements $\{2a_i^{n+1} : i \in \{1, \ldots, s+1\}\}$ in $B(\mathbf{A})$ such that $\bigwedge_{1\leq i<j\leq s+1} \forall(2a_i^{n+1} \vee 2a_j^{n+1}) = 1$ and $\bigvee_{j=1}^{s+1} \forall(2a_j^{n+1}) = 0$. This means that $2a_i^{n+1} \vee 2a_j^{n+1} = 1$ for $i \neq j$, and $2a_i^{n+1} \neq 1$ for all i. So $\mathbf{B}(\mathbf{A})$ has at least $s+1$ coatoms, and this is a contradiction. □

Definition 5.14. Let $\mathbf{A} \in \mathcal{MMV}$ with $\operatorname{rank}(\forall \mathbf{A}) = n$ and $\operatorname{width}(\mathbf{A}) = k$. We define the *boolean width of* $\mathbf{A}$, denoted by $\operatorname{bwidth}(\mathbf{A})$, as the least integer s such that (β_n^s) holds in $\mathbf{A}$.

From Corollary 4.8, the boolean width of an MMV-algebra of finite width $\mathbf{A}$ exists and it is less than or equal to the width of $\mathbf{A}$. Moreover, from Proposition 5.13, we have that if the boolean width of $\mathbf{A}$ is equal to s, then $\mathbf{B}(\mathbf{A})$ is isomorphic to $\mathbf{2}^s$.

Let us denote by $(\forall \mathbf{A})^{k,1}$ the MMV-subalgebra of $(\forall \mathbf{A})^k$ generated by $\forall(\mathbf{A}^k) \cup \operatorname{Rad}(\mathbf{A}^k)$. Observe that $(\forall \mathbf{A})^{k,1}$ is the largest subalgebra of $(\forall \mathbf{A})^k$ that satisfies the identity (β_n^1). In particular, $\mathbf{S}_{n,\omega}^{k,1}$ is the MMV-subalgebra of $\mathbf{S}_{n,\omega}^k$ generated by the constant elements $\forall(\mathbf{S}_{n,\omega}^k)$ and $\operatorname{Rad}(\mathbf{S}_{n,\omega}^k)$.

Let us suppose now that $\operatorname{bwidth}(\mathbf{A}) = 1$. In the following, we prove that $\mathcal{V}(\mathbf{A}) = \mathcal{V}(\mathbf{S}_{n,\omega}^{k,1})$. For that purpose, we need some more results.

Lemma 5.15. *If* $\mathbf{A} \in \mathrm{I\,S\,P_U}(\mathbf{B})$ *and an identity of type* $\tau(x_1, \ldots, x_n) \approx 1$ *holds in* $\mathbf{A}$*, then there exists a subalgebra* $\mathbf{S}$ *of* $\mathbf{B}$ *such that* $\mathbf{A} \in \mathrm{I\,S\,P_U}(\mathbf{S})$ *and* $\tau \approx 1$ *holds in* $\mathbf{S}$*.*

Proof. Let $\mathbf{A} \in \mathrm{I\,S\,P_U}(\mathbf{B})$. This means that $\mathbf{A}$ is isomorphic to a subalgebra of an ultraproduct $\prod_{i\in I} \mathbf{B}/U$. We identify $\mathbf{A}$ with this subalgebra. Let $\mathbf{S}$ the subalgebra of $\mathbf{B}$ generated by all the elements $a(i)$, for each $\bar{a} \in A$ and $\{i\} \in U$. Then $\mathbf{S}$ satisfies $\tau \approx 1$. Let $\overline{y} \in \prod_{i\in I} \mathbf{S}/U$ be defined by the class of

$$y(i) = \begin{cases} a(i) & \text{if } \{i\} \in U, \\ 1 & \text{if } \{i\} \notin U. \end{cases}$$

As $\overline{y} = \overline{a}$, we have that $\mathbf{A}$ is isomorphic to a subalgebra of an ultraproduct $\prod_{i\in I} \mathbf{S}/U$ where $\mathbf{S}$ is a subalgebra of $\mathbf{B}$ that satisfies the identity $\tau \approx 1$. □

As a consequence of Lemma 5.15 and Theorem 5.8, if $\mathrm{bwidth}(\mathbf{A}) = 1$, then $\mathbf{A} \in \mathrm{I\,S\,P_U}(\mathbf{S})$ where $\mathbf{S}$ is a subalgebra of $\mathbf{S}^k_{n,\omega}$ such that $\mathrm{bwidth}(\mathbf{S}) = 1$.

Lemma 5.16. *If* $\mathbf{S}$ *is a subalgebra of* $\mathbf{S}^k_{n,\omega}$ *such that* $\mathrm{bwidth}(\mathbf{S}) = 1$*, then* $\mathbf{S}$ *is a subalgebra of* $\mathbf{S}^{k,1}_{n,\omega}$*.*

From the last lemma and Lemma 5.15, we have the following.

Corollary 5.17. *Let* $\mathbf{A}$ *be a subdirectly irreducible MMV-algebra such that* $\mathrm{rank}(\forall\mathbf{A}) = n$*,* $\mathrm{width}(\mathbf{A}) = k$ *and* $\mathrm{bwidth}(\mathbf{A}) = 1$*. Then* $\mathbf{A} \in \mathrm{I\,S\,P_U}(\mathbf{S}^{k,1}_{n,\omega})$*. In particular,* $\mathbf{A} \in \mathcal{V}(\mathbf{S}^{k,1}_{n,\omega})$*.*

Proposition 5.18. *Let* $\mathbf{A}$ *be a subdirectly irreducible MMV-algebra such that* $\mathrm{rank}(\forall\mathbf{A}) = n$*,* $\mathrm{width}(\mathbf{A}) = k$*, and* $\mathrm{bwidth}(\mathbf{A}) = 1$*. Then there exists a subalgebra* $\mathbf{B}$ *of* $\mathbf{A}$ *such that* $\mathbf{B}$ *is isomorphic to* $(\forall\mathbf{B})^{k,1}$ *and* $\mathrm{rank}(\forall\mathbf{B}) = n$*.*

Proof. If $\mathbf{A}$ is a subdirectly irreducible MMV-algebra such that $\mathrm{rank}(\forall\mathbf{A}) = n$, $\mathrm{width}(\mathbf{A}) = k$, and $\mathrm{bwidth}(\mathbf{A}) = 1$, then $\mathbf{A}$ is isomorphic to a subalgebra of $(\forall\mathbf{A})^{k,1}$. We proceed by induction on k. If $k = 1$, then $\mathbf{A}$ is a chain, and the proposition is obvious in this case. Let us suppose that if $\mathbf{D}$ is isomorphic to a subalgebra of $(\forall\mathbf{D})^{k,1}$, then there exists a subalgebra $\mathbf{E}$ of $\mathbf{D}$ such that $\mathbf{E}$ is isomorphic to $(\forall\mathbf{E})^{k,1}$. Let us consider $\mathbf{A}$, which is isomorphic to a subalgebra of $(\forall\mathbf{A})^{k+1,1}$ and the subalgebra $\mathbf{B}^*_{ij}$ of $(\forall\mathbf{A})^{k+1,1}$, where $B^*_{ij} = \{\chi \in (\forall\mathbf{A})^{k+1} : \chi(i) = \chi(j)\}$ for $i \neq j$. Note that $\mathrm{width}(\mathbf{B}^*_{ij}) = k$. Let us consider the subalgebra of $\mathbf{A}$ given by $\mathbf{B}_{ij} = \mathbf{A} \cap \mathbf{B}^*_{ij}$. Then $\forall(\mathbf{B}_{ij}) = \forall\mathbf{A}$ and $\mathbf{B}_{ij}$ is embedabble into $(\forall\mathbf{A})^{k,1}$. By the induction assumption, there exists a subalgebra $\mathbf{C}_{ij}$ of $\mathbf{B}_{ij}$ such that $\mathbf{C}_{ij}$ is isomorphic to $(\forall\mathbf{C}_{ij})^{k,1}$ and also $\mathrm{rank}(\mathbf{C}_{ij}) = n$ and $\forall(\mathbf{C}_{ij})$ is a subalgebra of $\forall\mathbf{A}$. Let us consider the subalgebra $\mathbf{C} = \bigcap_{i\neq j} \forall(\mathbf{C}_{ij})$ of $\forall\mathbf{A}$ with $\mathrm{rank}(\mathbf{C}) = n$. Let us prove that $\langle\mathbf{C}^{k+1,1}; \forall_\wedge\rangle$ is a subalgebra of $\mathbf{A}$. First note that $\mathbf{C}^{k+1,1}$ is a subalgebra of $(\forall\mathbf{A})^{k+1,1}$. Let $\chi \in C^{k+1,1}$. If $\chi(i) = \chi(j)$ for $i \neq j$, then $\chi \in C_{ij}$. Thus, $\chi \in A$. Let us suppose that $\chi(i) \neq \chi(j)$ for all $i \neq j$. Then we can assume that $\chi(1) \leq \chi(2) \leq \cdots \leq \chi(k+1)$. Consider

$$\chi_1 = \langle \chi(1), \chi(1), \chi(3), \ldots, \chi(k+1) \rangle \in C \cap B_{12} \subseteq A;$$
$$\chi_2 = \langle \chi(1), \chi(2), \chi(2), \ldots, \chi(k+1) \rangle \in C \cap B_{23} \subseteq A.$$

Then $\chi_1 \vee \chi_2 = \chi \in A$. □

Theorem 5.19. *Let* $\mathbf{A}$ *be a subdirectly irreducible MMV-algebra such that* $\operatorname{rank}(\forall\mathbf{A}) = n$*,* $\operatorname{width}(\mathbf{A}) = k$*, and* $\operatorname{bwidth}(\mathbf{A}) = 1$*. Then* $\mathcal{V}(\mathbf{A}) = \mathcal{V}(\mathbf{S}^{k,1}_{n,\omega})$*.*

Proof. From Proposition 5.18, there exists a subalgebra $\mathbf{B}$ of $\mathbf{A}$ such that $\mathbf{B}$ is isomorphic to $(\forall\mathbf{B})^{k,1}$. Observe that $\mathbf{B}$ satisfies (β^1_n). Then $\mathcal{V}(\mathbf{S}^{k,1}_{n,\omega}) = \mathcal{V}(\mathbf{B}) \subseteq \mathcal{V}(\mathbf{A})$. Finally, from Corollary 5.17, we have the desired result. □

Theorem 5.20. *If* $n = 1$ *or* $n = 2$*, then the identities* (β^1_n)*,* (α^k) *and* (ρ_n) *characterize the subvariety generated by* $\mathbf{S}^{k,1}_{n,\omega}$*. If* $n \geq 3$*, then* (β^1_n)*,* (α^k)*,* (ρ_n)*, and*

$$\left(px^{p-1}\right)^{n+1} \approx (n+1)x^p, \tag{γ_{np}}$$

for each natural number $1 < p < n$ *such that* p *does not divide* n*, characterize the subvariety generated by* $\mathbf{S}^{k,1}_{n,\omega}$*.*

Finally, let us suppose that $1 < \operatorname{bwidth}(\mathbf{A}) = s < k$. Then, as before, there is a partition $\mathbf{P} = \{P_1, \ldots, P_s\}$ of the set $\{1, \ldots, k\}$ associated to the subalgebra $\mathbf{B}(\mathbf{A}) \cong \mathbf{2}^s$. We will prove that the subvariety generated by $\mathbf{A}$ depends on the rank of $\forall\mathbf{A}$, its width, its boolean width, and the partition $\mathbf{P}$ associated to $\mathbf{B}(\mathbf{A})$.

Let b_i be the coatom determined by the element P_i of $\mathbf{P}$; denote the set $\{a \in A : b_i \leq a\}$ by $[b_i)$. From Lemma 3.2, we know that we can define a structure of MMV-algebra in $[b_i)$. As a consequence, if $\mathbf{A}$ is a subdirectly irreducible algebra such that $\operatorname{rank}(\forall\mathbf{A}) = n$, $\operatorname{width}(\mathbf{A}) = k$, $\operatorname{bwidth}(\mathbf{A}) = s$, and b_i is a coatom of $\mathbf{B}(\mathbf{A})$, then $[\mathbf{b}_i)$ is an MMV-algebra. We denote by p_i the cardinal of each $P_i \in \mathbf{P}$ associated to $\mathbf{B}(\mathbf{A})$. Then the MMV-algebra $\langle [b_i); \forall_{b_i} \rangle$ has width p_i, $\operatorname{rank}(\forall_{b_i}[\mathbf{b}_i)) = n$, and is indecomposable. Then $[\mathbf{b}_i)$ is isomorphic to a subalgebra of $(\forall\mathbf{A})^{p_i,1}$. Moreover, the identity

$$\forall\Big(\bigvee_{i=1}^{p_i+1} \bigwedge X_i^-\Big) \to \bigvee_{j=1}^{p_i+1} \forall x_j \approx 1, \tag{α^{p_i}}$$

where $X = \{x_1, \ldots, x_{p_i+1}\}$ and $X_i^- = X - \{x_i\}$, holds in $\langle [b_i); \forall_{b_i} \rangle$.

As MV-algebras, $\mathbf{A}$ is isomorphic to a subalgebra of $(\forall\mathbf{A})^{p_1,1} \times \cdots \times (\forall\mathbf{A})^{p_s,1}$. In addition, $\langle \mathbf{A}; \forall \rangle$ is isomorphic to a subalgebra of $\langle (\forall\mathbf{A})^{k,\mathbf{P}}; \forall_\wedge \rangle$ where we denote by $(\forall\mathbf{A})^{k,\mathbf{P}}$ the algebra $(\forall\mathbf{A})^{p_1,1} \times \cdots \times (\forall\mathbf{A})^{p_s,1}$ for short.

Lemma 5.21. *Let* $\mathbf{A}$ *be a subdirectly irreducible algebra such that* $\operatorname{rank}(\forall\mathbf{A}) = n$*,* $\operatorname{width}(\mathbf{A}) = k$*, and* $\operatorname{bwidth}(\mathbf{A}) = s$ *with* $s > 1$*. Let* $\mathbf{P} = \{P_1, \ldots, P_s\}$ *be the partition of* $\{1, \ldots, k\}$ *associated to* $\mathbf{B}(\mathbf{A})$*, where the cardinal of each subset*

$P_i \in \mathbf{P}$ is denoted by p_i. Let $(f_n^{k,\mathbf{P}})$ be the identity

$$\Big(\Big(\bigwedge_{i=1}^{s} 2x_i^{n+1} \leftrightarrow 0\Big) \wedge \Big(\bigwedge_{1\le i<j\le s} ((2x_i^{n+1} \vee 2x_j^{n+1}) \leftrightarrow 1)\Big)$$
$$\wedge \Big(\bigwedge_{i=1}^{s} (\forall 2x_i^{n+1} \leftrightarrow 0)\Big) \wedge \Big(\bigwedge_{i=1}^{s} (\exists 2x_i^{n+1} \leftrightarrow 1)\Big)\Big)$$
$$\to \Big(\bigvee_{\sigma\in\mathbb{P}(\{1,\ldots,s\})} (\bigwedge_{i=1}^{s} \alpha^{p_{\sigma(i)}}_{\forall_{2x^{n+1}_{\sigma(i)}}}(z_1^{\sigma(i)},\ldots,z^{\sigma(i)}_{p_{\sigma(i)}+1}))\Big) \approx 1,$$

where the operation $\leftrightarrow$ is defined by $x \leftrightarrow y = (x \to y) \wedge (y \to x)$, $\mathbb{P}(\{1,\ldots,s\}$ is the set of all permutations of the set $\{1,\ldots,s\}$, and

$$\alpha^{p_{\sigma(i)}}_{\forall_{2x^{n+1}_{\sigma(i)}}}(z_1^{\sigma(i)},\ldots,z^{\sigma(i)}_{p_{\sigma(i)}+1})$$

is an abbreviation of

$$\forall_{2x^{n+1}_{\sigma(i)}}\Big(\bigvee_{r=1}^{p_{\sigma(i)}+1} \bigwedge Z_r^-\Big) \to \bigvee_{j=1}^{p_{\sigma(i)}+1} \forall_{2x^{n+1}_{\sigma(i)}}(z_j^{\sigma(i)}) =$$
$$\Big(\forall(\bigvee_{r=1}^{p_{\sigma(i)}+1} \bigwedge Z_r^-) \vee 2x^{n+1}_{\sigma(i)}\Big) \to \bigvee_{j=1}^{p_{\sigma(i)}+1} (\forall z_j^{\sigma(i)} \vee 2x^{n+1}_{\sigma(i)}),$$

where $Z = \{z_1^{\sigma(i)},\ldots,z^{\sigma(i)}_{p_{\sigma(i)}+1}\}$ and $Z_r^- = Z - \{z_r^{\sigma(i)}\}$. Then $(f_n^{k,\mathbf{P}})$ holds in $\mathbf{A}$.

Proof. Let $a_1,\ldots,a_s$ be elements of A. Note that

$$\Big((\bigwedge_{i=1}^{s} 2a_i^{n+1}) \leftrightarrow 0\Big) \wedge \Big(\bigwedge_{i\neq j}((2a_i^{n+1} \vee 2a_j^{n+1}) \leftrightarrow 1)\Big)$$
$$\wedge\Big(\bigwedge_{i=1}^{s} (\forall 2a_i^{n+1} \leftrightarrow 0)\Big) \wedge \Big(\bigwedge_{i=1}^{s} (\exists 2a_i^{n+1} \leftrightarrow 1)\Big) = 1$$

if and only if the set $\{2a_1^{n+1},\ldots,2a_s^{n+1}\}$ is exactly the set of coatoms of $\mathbf{B}(\mathbf{A})$. In addition, if the last set is not the set of coatoms then

$$\Big((\bigwedge_{i=1}^{s} 2a_i^{n+1}) \leftrightarrow 0\Big) \wedge \Big(\bigwedge_{i\neq j}((2a_i^{n+1} \vee 2a_j^{n+1}) \leftrightarrow 1)\Big)$$
$$\wedge\Big(\bigwedge_{i=1}^{s} (\forall 2a_i^{n+1} \leftrightarrow 0)\Big) \wedge \Big(\bigwedge_{i=1}^{s} (\exists 2a_i^{n+1} \leftrightarrow 1)\Big) = 0.$$

We know there is a permutation σ' of $\{1,\ldots,s\}$ such that $\{2a^{n+1}_{\sigma'(i)} : 1 \le i \le s\}$ is the set of coatoms associated to $\mathbf{P}$. Moreover, $\langle [2a^{n+1}_{\sigma'(i)}); \forall_{2a^{n+1}_{\sigma'(i)}} \rangle$ satisfies that for all i,

$$\alpha^{p_{\sigma'(i)}}_{\forall_{2a^{n+1}_{\sigma'(i)}}}(z_1^{\sigma'(i)},\dots,z^{\sigma'(i)}_{p_{\sigma'(i)}+1}) = \forall_{2a^{n+1}_{\sigma'(i)}}\Big(\bigvee_{r=1}^{p_{\sigma'(i)}+1}\bigwedge Z_r^-\Big) \to \bigvee_{j=1}^{p_{\sigma'(i)}+1}\forall_{2a^{n+1}_{\sigma'(i)}} z_j^{\sigma'(i)}$$

$$= \Big(\forall\big(\bigvee_{r=1}^{p_{\sigma'(i)}+1}\bigwedge Z_r^-\big)\vee 2a^{n+1}_{\sigma'(i)}\Big) \to \bigvee_{j=1}^{p_{\sigma'(i)}+1}\Big(\forall z_j^{\sigma'(i)}\vee 2a^{n+1}_{\sigma'(i)}\Big) = 1.$$

Then we have that $\bigwedge_{i=1}^s \alpha^{p_{\sigma'(i)}}_{\forall_{2a^{n+1}_{\sigma'(i)}}}(z_1^{\sigma'(i)},\dots,z^{\sigma'(i)}_{p_{\sigma'(i)}+1}) = 1$ and, in consequence,

$$\bigvee_{\sigma\in\mathbb{P}(\{1,\dots,s\})}\Big(\bigwedge_{i=1}^{s}\alpha^{p_{\sigma(i)}}_{\forall_{2a^{n+1}_{\sigma(i)}}}(z_1^{\sigma(i)},\dots,z^{\sigma(i)}_{p_{\sigma(i)}+1})\Big) = 1.$$

Hence, the identity $(f_n^{k,\mathbf{P}})$ holds in $\mathbf{A}$. $\square$

Let $\mathbf{P} = \{P_1,\dots,P_s\}$ be a given partition of the set $\{1,\dots,k\}$. Let us consider the MMV-subalgebra

$$\mathbf{S}^{k,\mathbf{P}}_{n,\omega} = \{a\in \mathbf{S}^k_{n,\omega} : a(i)/\operatorname{Rad}(\mathbf{S}_{n,\omega}) = a(j)/\operatorname{Rad}(\mathbf{S}_{n,\omega}) \text{ if } i,j\in P_t \text{ for some } t\}.$$

Observe that $\forall_\wedge(\mathbf{S}^k_{n,\omega}) = \forall_\wedge(\mathbf{S}^{k,\mathbf{P}}_{n,\omega})$. Indeed, $\forall_\wedge(\mathbf{S}^{k,\mathbf{P}}_{n,\omega}) \subseteq \forall_\wedge(\mathbf{S}^k_{n,\omega})$ and if $b\in\forall_\wedge(\mathbf{S}^k_{n,\omega})$, we know that b is constant. So it is clear that $b\in\forall_\wedge(\mathbf{S}^{k,\mathbf{P}}_{n,\omega})$.

For each $i\in\{1,\dots,s\}$, let $p_i = |P_i|$ be the cardinal of $P_i\in\mathbf{P}$ and b_i the coatom of $\mathbf{S}^{k,\mathbf{P}}_{n,\omega}$ determined by $P_i\in\mathbf{P}$. Defining in the set $A_i = [b_i)\cap S^{k,\mathbf{P}}_{n,\omega} = \{a\in S^{k,\mathbf{P}}_{n,\omega} : b_i\le a\}$ the operator $\neg_{b_i}x := \neg x\vee b_i$, we know that $\mathbf{A}_i = \langle A_i;\oplus,\neg_{b_i},b_i\rangle$ is an MV-algebra. Then the MV-reduct of $\mathbf{S}^{k,\mathbf{P}}_{n,\omega}$ is isomorphic to the MV-algebra $\mathbf{A}_1\times\cdots\times\mathbf{A}_s$. Indeed, the operator $\psi\colon \mathbf{S}^{k,\mathbf{P}}_{n,\omega}\to\mathbf{A}_1\times\cdots\times\mathbf{A}_s$ defined by $\psi(a) = \langle a\vee b_1,\dots,a\vee b_s\rangle$ is an MV-isomorphism. In addition, $\mathbf{A}_i$ is isomorphic to $\mathbf{S}^{p_i,1}_{n,\omega}$, for each i. Thus the MV-reduct of $\mathbf{S}^{k,\mathbf{P}}_{n,\omega}$ is isomorphic to $\mathbf{S}^{p_1,1}_{n,\omega}\times\cdots\times\mathbf{S}^{p_s,1}_{n,\omega}$.

If the cardinal of $\mathbf{P}$ is 1, then the subalgebra $\mathbf{S}^{k,\mathbf{P}}_{n,\omega}$ is isomorphic to $\mathbf{S}^{k,1}_{n,\omega}$, and if it is equal to k, then $\mathbf{S}^{k,\mathbf{P}}_{n,\omega} = \mathbf{S}^k_{n,\omega}$.

As a consequence of Lemma 3.2, we have that $\mathbf{A}_i = [\mathbf{b}_i)\cap\mathbf{S}^k_{n,\omega}$ is an MMV-algebra, and it is straightforward to see that $\mathbf{A}_i$ is isomorphic to the MMV-algebra $\mathbf{S}^{p_i,1}_{n,\omega}$. From Lemma 5.21, we know also that $(f_n^{k,\mathbf{P}})$ holds in $\mathbf{S}^{k,\mathbf{P}}_{n,\omega}$.

Proposition 5.22. *Let $\mathbf{A}$ be a subdirectly irreducible algebra that satisfies* $\operatorname{rank}(\forall\mathbf{A}) = n$, $\operatorname{width}(\mathbf{A}) = k$, *and* $\operatorname{bwidth}(\mathbf{A}) = s$ *with* $s > 1$. *Let* $\mathbf{P} = \{P_1,\dots,P_s\}$ *be the partition of* $\{1,\dots,k\}$ *associated to* $\mathbf{B}(\mathbf{A})$ *and let* $(f_n^{k,\mathbf{P}})$ *hold in* $\mathbf{A}$. *Then* $\mathcal{V}(\mathbf{S}^{k,\mathbf{P}}_{n,\omega})\subseteq\mathcal{V}(\mathbf{A})$.

Proof. We know that $\langle\mathbf{A};\forall\rangle$ is isomorphic to a subalgebra of $\langle(\forall\mathbf{A})^{k,\mathbf{P}};\forall_\wedge\rangle$. Considering Proposition 5.18, it is straightforward to see that there exists a subalgebra $\mathbf{B}$ of $\mathbf{A}$ isomorphic to $(\forall\mathbf{B})^{k,\mathbf{P}}$, where $\operatorname{rank}(\forall\mathbf{B}) = n$, $\operatorname{width}(\mathbf{B}) = k$, and $\operatorname{bwidth}(\mathbf{B}) = s$. Then $\mathcal{V}((\forall\mathbf{B})^{k,\mathbf{P}}) = \mathcal{V}(\mathbf{S}^{k,\mathbf{P}}_{n,\omega})\subseteq\mathcal{V}(\mathbf{A})$. $\square$

Let $\mathbf{A}$ be a subdirectly irreducible algebra that satisfies $\mathrm{rank}(\forall\mathbf{A}) = n$, $\mathrm{width}(\mathbf{A}) = k$, and $\mathrm{bwidth}(\mathbf{A}) = s$ with $s > 1$. Let $\mathbf{P} = \{P_1, \dots, P_s\}$ be the partition of $\{1, \dots, k\}$ associated to $\mathbf{B}(\mathbf{A})$ and let $(f_n^{k,\mathbf{P}})$ hold in $\mathbf{A}$. From Theorem 5.8, we know that $\mathbf{A} \in \mathrm{I\,S\,P_U}(\mathbf{S}^k_{n,\omega})$, and from Lemma 5.15, we know that $\mathbf{A} \in \mathrm{I\,S\,P_U}(\mathbf{S})$ where $\mathbf{S}$ is a subalgebra of $\mathbf{S}^k_{n,\omega}$ that satisfies $(f_n^{k,\mathbf{P}})$. In the following, we prove that $\mathbf{S}$ is a subalgebra of $\mathbf{S}^{k,\mathbf{P}}_{n,\omega}$. As a consequence of this, we will have that $\mathcal{V}(\mathbf{A}) \subseteq \mathcal{V}(\mathbf{S}^{k,\mathbf{P}}_{n,\omega})$.

Given two partitions

$$\mathbf{P} = \{P_i : i = 1, \dots, |P|\} \quad \text{and} \quad \mathbf{Q} = \{Q_i : i = 1, \dots, |P| = |Q|\}$$

of the same set, we say that $\mathbf{P}$ is *equivalent* to $\mathbf{Q}$, denoted $\mathbf{P} \sim \mathbf{Q}$, if there exists a permutation σ of the set $\{1, \dots, |P|\}$ such that $P_i = Q_{\sigma(i)}$ for all $i = 1, \dots, |P|$.

Lemma 5.23. *If $\mathbf{S}$ is a subalgebra of $\mathbf{S}^k_{n,\omega}$ that satisfies* $\mathrm{width}(\mathbf{S}) = k$, $\mathrm{bwidth}(\mathbf{S}) = s$, *and $(f_n^{k,\mathbf{P}})$ holds in $\mathbf{S}$ for some partition $\mathbf{P} = \{P_1, \dots, P_s\}$ of the set $\{1, \dots, k\}$, then the partition $\mathbf{Q}$ associated to $\mathbf{B}(\mathbf{S})$ is equivalent to $\mathbf{P}$.*

Proof. If $\mathbf{S}$ is a subalgebra of $\mathbf{S}^k_{n,\omega}$ such that $\mathrm{width}(\mathbf{S}) = k$ and $\mathrm{bwidth}(\mathbf{S}) = s$, from Proposition 5.13, we have that the cardinal of $\mathbf{Q} = \{Q_i\}$ is exactly s. Let us denote $|Q_i| = q_i$ and $|P_i| = p_i$, for all i.

Let $a_i \in S$ be such that

$$\Big(\big(\bigwedge_{i=1}^{s} 2a_i^{n+1}\big) \leftrightarrow 0\Big) \wedge \Big(\bigwedge_{1 \le i < j \le s} \big((2a_i^{n+1} \vee 2a_j^{n+1}) \leftrightarrow 1\big)\Big)$$
$$\wedge \Big(\bigwedge_{i=1}^{s} (\forall 2a_i^{n+1} \leftrightarrow 0)\Big) \wedge \Big(\bigwedge_{i=1}^{s} (\exists 2a_i^{n+1} \leftrightarrow 1)\Big) = 1$$

We know that the set $\{2a_i^{n+1}\}$ is the set of coatoms of $\mathbf{S}$. Since $(f_n^{k,\mathbf{P}})$ holds in $\mathbf{S}$, there exists $\sigma \in \mathbb{P}(\{1, \dots, s\})$ with $\bigwedge_{i=1}^{s} \alpha^{p_{\sigma(i)}}_{\forall_{2a^{n+1}_{\sigma(i)}}}(z_1^{\sigma(i)}, \dots, z_{p_{\sigma(i)}+1}^{\sigma(i)}) = 1$. This means that the width of $[\mathbf{a}_{\sigma(i)})$ in $\mathbf{S}$ is less than or equal to $p_{\sigma(i)}$. But the width of $[\mathbf{a}_{\sigma(i)})$ is q_i. Then $q_i \le p_{\sigma(i)}$ for all i. In addition, $\sum_{i=1}^{s} q_i = \sum_{i=1}^{s} p_{\sigma(i)} = k$. Then $p_{\sigma(i)} = q_i$ for all i. Hence, $\mathbf{Q}$ is equivalent to $\mathbf{P}$. $\square$

Lemma 5.24. *If $\mathbf{S}$ is a subalgebra of $\mathbf{S}^k_{n,\omega}$ that satisfies* $\mathrm{width}(\mathbf{S}) = k$, $\mathrm{bwidth}(\mathbf{S}) = s$, *and $(f_n^{k,\mathbf{P}})$ holds in $\mathbf{S}$ for some partition $\mathbf{P} = \{P_1, \dots, P_s\}$ of the set $\{1, \dots, k\}$, then $\mathbf{S}$ is isomorphic to a subalgebra of $\mathbf{S}^{k,\mathbf{P}}_{n,\omega}$.*

Proof. From Lemma 5.23, we have that the partition $\mathbf{Q}$ associated to $\mathbf{B}(\mathbf{S})$ is equivalent to $\mathbf{P}$. So there exists a permutation $\sigma \in \mathbb{P}(\{1, \dots, s\})$ such that $P_i = Q_{\sigma(i)}$. If $b_{\sigma(i)}$, for $1 \le i \le s$, are the coatoms of $\mathbf{B}(\mathbf{S})$, then $[\mathbf{b}_{\sigma(i)})$ are

MMV-algebras by Lemma 3.2 and satisfy

$$\forall_{b_{\sigma(i)}}\Big(\bigvee_{r=1}^{p_{\sigma(i)}+1}\bigwedge Z_r^-\Big)\to\bigvee_{j=1}^{p_{\sigma(i)}+1}\forall_{b_{\sigma(i)}} z_j^{\sigma(i)} =$$

$$\Big(\forall\Big(\bigvee_{r=1}^{p_{\sigma(i)}+1}\bigwedge Z_r^-\Big)\vee b_{\sigma(i)}\Big)\to\bigvee_{j=1}^{p_{\sigma(i)}+1}\big(\forall z_j^{\sigma(i)}\vee b_{\sigma(i)}\big)\approx 1,$$

where $Z=\{z_1^{\sigma(i)},\dots,z_{p_{\sigma(i)}+1}^{\sigma(i)}\}$ and $Z_r^- = Z-\{z_r^{\sigma(i)}\}$. Then $[\mathbf{b}_{\sigma(i)})$ is isomorphic to a subalgebra of $\mathbf{S}_{n,\omega}^{p_{\sigma(i)},1}$. Thus, $\mathbf{S}\cong[\mathbf{b}_{\sigma(1)})\times\cdots\times[\mathbf{b}_{\sigma(s)})$ is isomorphic to a subalgebra of $\mathbf{S}_{n,\omega}^{p_{\sigma(1)},1}\times\cdots\times\mathbf{S}_{n,\omega}^{p_{\sigma(s)},1}$. In addition, $\forall_A=\forall_\wedge$. Then $\mathbf{S}$ is isomorphic to a subalgebra of $\mathbf{S}_{n,\omega}^{k,\mathbf{P}}$. □

From Theorem 5.8, the above result, and Lemma 5.24, we have the following corollary.

Corollary 5.25. *If* $\mathbf{A}$ *is a subdirectly irreducible algebra with* $\mathrm{rank}(\forall\mathbf{A}) = n$, $\mathrm{width}(\mathbf{A})=k$, $\mathrm{bwidth}(\mathbf{A})=s$, *and* $(f_n^{k,\mathbf{P}})$ *holds in* $\mathbf{A}$, *then* $\mathbf{A}\in \mathrm{ISP_U}(\mathbf{S}_{n,\omega}^{k,\mathbf{P}})$. *In particular,* $\mathcal{V}(\mathbf{A})\subseteq\mathcal{V}(\mathbf{S}_{n,\omega}^{k,\mathbf{P}})$.

From Corollary 5.25 and Proposition 5.22, we have the following theorem.

Theorem 5.26. *Let* $\mathbf{A}$ *be a subdirectly irreducible MMV-algebra such that* $\forall\mathbf{A}$ *is non-simple of rank* n, $\mathrm{width}(\mathbf{A})=k$, $\mathrm{bwidth}(\mathbf{A})=s$, *and such that* $(f_n^{k,\mathbf{P}})$ *holds in* $\mathbf{A}$. *Then* $\mathcal{V}(\mathbf{S}_{n,\omega}^{k,\mathbf{P}})=\mathcal{V}(\mathbf{A})$.

The next theorems, which are consequences of the above results, characterize by identities the subvarieties generated by $\mathbf{S}_{n,\omega}^{k,\mathbf{P}}$.

Theorem 5.27. *Let* s *be a integer such that* $s>1$. *If* $n=1$ *or* $n=2$, *then* $(f_n^{k,\mathbf{P}})$, (α^k), (β_n^s), *and* (ρ_n) *characterize the subvariety generated by* $\mathbf{S}_{n,\omega}^{k,\mathbf{P}}$, *where* $\mathbf{P}=\{P_1,\dots,P_s\}$. *If* $n\geq 3$, *then* $(f_n^{k,\mathbf{P}})$, (α^k), (β_n^s), (ρ_n), *and*

$$\big(px^{p-1}\big)^{n+1}\approx(n+1)x^p, \tag{γ_{np}}$$

for each natural number $1<p<n$ *such that* p *does not divide* n, *characterize the subvariety generated by* $\mathbf{S}_{n,\omega}^{k,\mathbf{P}}$.

Let us see now the inclusion relation between the proper subvarieties of the variety generated by $\mathbf{S}_{n,\omega}^{k}$.

Let $\mathbf{P}=\{P_1,\dots,P_s\}$ and $\mathbf{P}'=\{P_1',\dots,P_{s'}'\}$ be two partitions of the set $\{1,\dots,k\}$, and let us consider the subalgebras $\mathbf{S}_1=\mathbf{S}_{n,\omega}^{k,\mathbf{P}}$ and $\mathbf{S}_2=\mathbf{S}_{n,\omega}^{k,\mathbf{P}'}$ of $\mathbf{S}_{n,\omega}^{k}$ associated to each partition. If $\mathbf{S}_1$ is a subalgebra of $\mathbf{S}_2$, then $\mathbf{P}'$ is a refinement of $\mathbf{P}$ and $\mathbf{B}(\mathbf{S}_1)$ is a subalgebra of $\mathbf{B}(\mathbf{S}_2)$. Conversely, let us suppose that $\mathbf{B}(\mathbf{S}_1)$ is a subalgebra of $\mathbf{B}(\mathbf{S}_2)$. Then the partition $\mathbf{P}'$ associated to $\mathbf{B}(\mathbf{S}_2)$ is a subpartition of the partition $\mathbf{P}$ associated to $\mathbf{B}(\mathbf{S}_1)$. That is, each element of $\mathbf{P}$ is a union of elements of $\mathbf{P}'$, i.e., $\mathbf{P}'$ is a refinement of $\mathbf{P}$. Let $a\in S_{n,\omega}^{k,\mathbf{P}}$ and let $i,j\in P_t'\in\mathbf{P}'$. Since $\mathbf{P}'$ is a refinement of $\mathbf{P}$, then there is $P_h\in\mathbf{P}$

such that $P'_t \subseteq P_h$. Then a satisfies that $a(i)/\operatorname{Rad}(\mathbf{S}_{n,\omega}) = a(j)/\operatorname{Rad}(\mathbf{S}_{n,\omega})$. That is, $a \in S_{n,\omega}^{k,\mathbf{P}'}$. Then the following lemma follows.

Lemma 5.28. *Let $\mathbf{P} = \{P_1, \ldots, P_s\}$ and $\mathbf{P}' = \{P'_1, \ldots, P'_{s'}\}$ be two partitions of $\{1, \ldots, k\}$. Let us consider the subalgebras $\mathbf{S}_{n,\omega}^{k,\mathbf{P}}$ and $\mathbf{S}_{n,\omega}^{k,\mathbf{P}'}$ of $\mathbf{S}_{n,\omega}^{k}$ associated to each partition. Then $\mathbf{S}_{n,\omega}^{k,\mathbf{P}}$ is a subalgebra of $\mathbf{S}_{n,\omega}^{k,\mathbf{P}'}$ if and only if $\mathbf{P}'$ is a refinement of $\mathbf{P}$.*

Corollary 5.29. *Let $\mathbf{P} = \{P_1, \ldots, P_s\}$ and $\mathbf{P}' = \{P'_1, \ldots, P'_{s'}\}$ be two partitions of the set $\{1, \ldots, k\}$. Then $\mathcal{V}(\mathbf{S}_{n,\omega}^{k,\mathbf{P}}) \subseteq \mathcal{V}(\mathbf{S}_{n,\omega}^{k,\mathbf{P}'})$ if and only if $\mathbf{P}'$ is a refinement of $\mathbf{P}$.*

Given two partitions $\mathbf{P}' = \{P'_1, \ldots, P'_s\}$ and $\mathbf{P} = \{P_1, \ldots, P_r\}$ of the set $\{1, \ldots, k'\}$ and the set $\{1, \ldots, k\}$, respectively, we say that $\mathbf{P}$ is less than or equal to $\mathbf{P}'$, and we denote $\mathbf{P} \leq \mathbf{P}'$, if there exists a subset of $\mathbf{P}'$ that it is equivalent to a refinement of $\mathbf{P}$.

We know that $\mathbf{S}_m$ and $\mathbf{S}_{m,\omega}$ are subalgebras of $\mathbf{S}_{n,\omega}$ if and only if m divides n. As a consequence, the following lemma holds.

Lemma 5.30. (1) *$\mathcal{V}(\mathbf{S}_{m,\omega}^{t,\mathbf{P}}) \subseteq \mathcal{V}(\mathbf{S}_{n,\omega}^{k,\mathbf{P}'})$ if and only if m divides n, $t \leq k$, and $\mathbf{P} \leq \mathbf{P}'$.*
(2) *$\mathcal{V}(\mathbf{S}_{m}^{t}) \subseteq \mathcal{V}(\mathbf{S}_{n,\omega}^{k,\mathbf{P}})$ if and only if m divides n and $t \leq |\mathbf{P}|$.*

We know from Lemma 4.11 that a class $\{\mathbf{S}_i^k : i \in I\}$ generates $\mathcal{V}([\mathbf{0},\mathbf{1}]^k)$ if and only if I is infinite.

Lemma 5.31. *Let $\{\mathbf{S}_{n,\omega}^{k,\mathbf{P}_s} : n \in \mathbb{N}\}$ be an infinite set of algebras such that the cardinal of each partition $\mathbf{P}_s$ is s. Then $\mathcal{V}(\{\mathbf{S}_{n,\omega}^{k,\mathbf{P}_s} : n \in \mathbb{N}\}) = \mathcal{V}([\mathbf{0},\mathbf{1}]^s)$.*

Proof. Let us note that $\mathbf{S}_{n,\omega}^{s}$ is a subalgebra of $\mathbf{S}_{n,\omega}^{k,\mathbf{P}_s}$ and $\mathcal{V}(\mathbf{S}_{n,\omega}^{k,\mathbf{P}_s}) \subseteq \mathcal{V}([\mathbf{0},\mathbf{1}]^s)$ since $\mathbf{S}_{n,\omega}^{k,\mathbf{P}_s}$ satisfies the identity (α^s). Then

$$\mathcal{V}(\{\mathbf{S}_{n}^{s} : n \in \mathbb{N}\}) \subseteq \mathcal{V}(\{\mathbf{S}_{n,\omega}^{k,\mathbf{P}_s} : n \in \mathbb{N}\}) \subseteq \mathcal{V}([\mathbf{0},\mathbf{1}]^s).$$

But, from Lemma 4.11, we know that $\mathcal{V}(\{\mathbf{S}_{n}^{s} : n \in \mathbb{N}\}) = \mathcal{V}([\mathbf{0},\mathbf{1}]^s)$, and from this we have the lemma. $\square$

6. Subvarieties of $\mathcal{V}([\mathbf{0},\mathbf{1}]^k)$

In this section, we describe the general forms of a non-trivial subvariety contained in $\mathcal{V}([\mathbf{0},\mathbf{1}]^k)$. We also give the identity that characterizes each proper subvariety.

In the following, $\{m_1, \ldots, m_r\}$ is a finite subset of $\mathbb{N}$. If $r = 0$, then $\{m_1, \ldots, m_r\} = \emptyset$ and similarly for the set $\{s_1, \ldots, s_p\}$.

Theorem 6.1. *If $\mathcal{V}$ is a non-trivial subvariety of MMV-algebras such that $\mathcal{V} \subseteq \mathcal{V}([\mathbf{0},\mathbf{1}]^k)$, then $\mathcal{V}$ has one of the following forms:*

(F1) $\mathcal{V} = \mathcal{V}(\mathbf{S}_{m_1}^{t_1}, \dots, \mathbf{S}_{m_r}^{t_r})$ *where* $r \geq 1$ *and* $t_i \leq k$ *for all* i,

(F2) $\mathcal{V} = \mathcal{V}(\mathbf{S}_{m_1}^{t_1}, \dots, \mathbf{S}_{m_r}^{t_r}, \mathbf{S}_{n_1,\omega}^{s_1,\mathbf{P}_1}, \dots, \mathbf{S}_{n_p,\omega}^{s_p,\mathbf{P}_p})$ *where* $r \geq 0$, $p \geq 1$, $t_i \leq k$ *and* $s_i \leq k$ *for all* i,

(F3) $\mathcal{V} = \mathcal{V}(\mathbf{S}_{m_1}^{t_1}, \dots, \mathbf{S}_{m_r}^{t_r}, \mathbf{S}_{n_1,\omega}^{s_1,\mathbf{P}_1}, \dots, \mathbf{S}_{n_p,\omega}^{s_p,\mathbf{P}_p}, [\mathbf{0},\mathbf{1}]^{k_1})$ *where* $r \geq 0$, $p \geq 0$, $t_i \leq k$ *and* $s_i \leq k$ *for all* i, *and* $k_1 \leq k$.

Proof. If $\mathcal{V} = \mathcal{V}([\mathbf{0},\mathbf{1}]^k)$, then $\mathcal{V}$ is of the form (F3) and we have nothing to prove. Let $\mathcal{V} \subsetneq \mathcal{V}([\mathbf{0},\mathbf{1}]^k)$. Suppose first that for some $t \leq k$, we have that $\mathcal{V}([\mathbf{0},\mathbf{1}]^t) \subseteq \mathcal{V}$. Then there exists $k_1 = \max\{r : \mathcal{V}([\mathbf{0},\mathbf{1}]^r) \subseteq \mathcal{V}\}$. If $\mathcal{V} = \mathcal{V}([\mathbf{0},\mathbf{1}]^{k_1})$, then $\mathcal{V}$ is of the form (F3). Suppose that $\mathcal{V}([\mathbf{0},\mathbf{1}]^{k_1}) \subsetneq \mathcal{V}$. Let $I = \{m : \mathbf{S}_m^t \in \mathcal{V} \setminus \mathcal{V}([\mathbf{0},\mathbf{1}]^{k_1})\}$ and $J = \{n : \mathbf{S}_{n,\omega}^{t,\mathbf{P}} \in \mathcal{V} \setminus \mathcal{V}([\mathbf{0},\mathbf{1}]^{k_1})\}$. If $I \cup J = \emptyset$, then $\mathcal{V} = \mathcal{V}([\mathbf{0},\mathbf{1}]^{k_1})$, and this case has already been considered. Then $I \cup J \neq \emptyset$. From Lemma 4.11 and Lemma 5.31, we have that I and J are finite subsets of $\mathbb{N}$. If they were not, k_1 would not be a maximal element in the set $\{r : \mathcal{V}([\mathbf{0},\mathbf{1}]^r) \subseteq \mathcal{V}\}$. Let $\mathcal{W}$ be the subvariety of $\mathcal{V}$ generated by

$$\{[\mathbf{0},\mathbf{1}]^{k_1}\} \cup \{\mathbf{S}_m^t : m \in I\} \cup \{\mathbf{S}_{n,\omega}^{t,\mathbf{P}} : n \in J\}.$$

Let us see that $\mathcal{W} = \mathcal{V}$. Let $\mathbf{A} \in si(\mathcal{V})$ where $si(\mathcal{V})$ is the family of subdirectly irreducible members of $\mathcal{V}$. In particular, $\mathrm{width}(\mathbf{A}) \leq k$.

Suppose that $\mathbf{A}$ is finite. Then $\mathbf{A} \cong \mathbf{S}_m^t$, and since it is in $\mathcal{V}$, we have that $t \leq k_1$ or $m \in I$. Then $\mathcal{V}(\mathbf{A}) = \mathcal{V}(\mathbf{S}_m^t) \subseteq \mathcal{W}$.

If $\mathbf{A}$ is not finite and $\mathrm{rank}(\forall \mathbf{A}) = n$, then $\mathbf{A} \in \mathcal{V}(\mathbf{S}_{n,\omega}^{t,\mathbf{P}})$ for some $t \leq k_1$ or $n \in J$. Then $\mathcal{V}(\mathbf{A}) \subseteq \mathcal{W}$.

Finally, if $\mathrm{rank}(\forall \mathbf{A}) = \omega$ and $\mathrm{width}(\mathbf{A}) = t$, then $\mathcal{V}(\mathbf{A}) = \mathcal{V}([\mathbf{0},\mathbf{1}]^t)$ with $t \leq k_1$. Thus, $\mathcal{V}(\mathbf{A}) \subseteq \mathcal{W}$.

If $\mathcal{V}([\mathbf{0},\mathbf{1}]^t) \subsetneq \mathcal{V}$ for any t, choosing the set $I = \{m : \mathbf{S}_m^t \in \mathcal{V}\}$ and $J = \{n : \mathbf{S}_{n,\omega}^{t,\mathbf{P}} \in \mathcal{V}\}$ and reasoning as before, we have that $\mathcal{V}$ is of the form (F1) or (F2). $\square$

Let us recall that $\mathcal{MMV}$ is a congruence distributive variety. Then if $\mathbf{A}, \mathbf{B}_1, \dots, \mathbf{B}_l$ are subdirectly irreducible algebras in

$$\{[\mathbf{0},\mathbf{1}]^k : k \in \mathbb{N}\} \cup \{\mathbf{S}_m^t : m, t \in \mathbb{N}\} \cup \{\mathbf{S}_{n,\omega}^{t,\mathbf{P}} : n, t \in \mathbb{N}\},$$

then by Jónsson's theorems, we have $\mathbf{A} \in \mathcal{V}(\mathbf{B}_1, \dots, \mathbf{B}_l) = \mathcal{V}(\mathbf{B}_1) \vee \cdots \vee \mathcal{V}(\mathbf{B}_l)$ if and only if $\mathbf{A} \in \mathcal{V}(\mathbf{B}_i)$ for some i. Taking Theorem 6.1 into account, we have that if $\mathcal{V}$ is a non-trivial subvariety contained in $\mathcal{V}([\mathbf{0},\mathbf{1}]^k)$, then $\mathcal{V}$ has one of the following forms:

(F1) $\mathcal{V} = \bigvee_{i=1}^r \mathcal{V}(\mathbf{S}_{m_i}^{t_i})$ such that $t_i \leq k$ for each i,

(F2) $\mathcal{V} = \bigvee_{i=0}^r \mathcal{V}(\mathbf{S}_{m_i}^{t_i}) \vee \bigvee_{i=1}^p \mathcal{V}(\mathbf{S}_{n_i,\omega}^{s_i,\mathbf{P}_i})$ such that $t_i \leq k$ and $s_i \leq k$ for each i,

(F3) $\mathcal{V} = \bigvee_{i=0}^r \mathcal{V}(\mathbf{S}_{m_i}^{t_i}) \vee \bigvee_{i=0}^p \mathcal{V}(\mathbf{S}_{n_i,\omega}^{s_i,\mathbf{P}_i}) \vee \mathcal{V}([\mathbf{0},\mathbf{1}]^{k_1})$ such that $t_i \leq k$ and $s_i \leq k$ for all i, and $k_1 \leq k$.

In the following proposition we resume the inclusion properties between subvariety of width less than or equal to k.

Proposition 6.2. *Let* $\mathcal{V} \subseteq \mathcal{V}([\mathbf{0},\mathbf{1}]^k)$.

(1) $\mathcal{V}([\mathbf{0},\mathbf{1}]^t) \subseteq \mathcal{V}$ *if and only if* $\mathcal{V}$ *is of the form* (F3) *and* $t \leq k_1$.

(2) $\mathcal{V}(\mathbf{S}_{n,\omega}^{t,\mathbf{P}'}) \subseteq \mathcal{V}$ *if and only if one of the following conditions is satisfied:*

  (2a) $\mathcal{V}$ *is of the form* (F2) *and* n *divides some* $n_i \in \{n_1,\ldots,n_p\}$, $t \leq s_i$, *and* $\mathbf{P}' \leq \mathbf{P}_i$,

  (2b) $\mathcal{V}$ *is of the form* (F3) *and* $t \leq k_1$, *or* n *divides some* $n_i \in \{n_1,\ldots,n_p\}$, $t \leq s_i$, *and* $\mathbf{P}' \leq \mathbf{P}_i$.

(3) $\mathcal{V}(\mathbf{S}_m^t) \subseteq \mathcal{V}$ *if and only if one of the following conditions is satisfied:*

  (3a) m *divides* m_i *for some* $m_i \in \{m_1,\ldots,m_r\}$ *and* $t \leq t_i$,

  (3b) m *divides* n_i *for some* $n_i \in \{n_1,\ldots,n_p\}$ *and* $t \leq s_i$,

  (3c) $t \leq k_1$.

We have already given identities that characterize each of the subvarieties $\{\mathcal{V}(\mathbf{S}_m^t) : t \leq k\} \cup \{\mathcal{V}(\mathbf{S}_{n,\omega}^{t,\mathbf{P}}) : t \leq k\} \cup \{\mathcal{V}([\mathbf{0},\mathbf{1}]^t) : t \leq k\}$. Now, we give identities characterizing a proper subvariety of the variety generated by $[\mathbf{0},\mathbf{1}]^k$.

First note that every identity $\tau_1 \approx \tau_2$ is equivalent to $(\tau_1 \rightarrow \tau_2) \wedge (\tau_2 \rightarrow \tau_1) \approx 1$. In addition, $\eta_1(x_{11},\ldots,x_{1n_1}) \approx 1$, $\ldots$, $\eta_r(x_{r1},\ldots,x_{rn_r}) \approx 1$ hold in $\mathcal{V}$ if and only if $\eta_1(x_{11},\ldots,x_{1n_1}) \wedge \cdots \wedge \eta_r(x_{r1},\ldots,x_{rn_r}) \approx 1$ holds in $\mathcal{V}$. Therefore, we can assume that each subvariety

$$\mathcal{V}_i \in \{\mathcal{V}(\mathbf{S}_m^t) : t \leq k\} \cup \{\mathcal{V}(\mathbf{S}_{n,\omega}^{t,\mathbf{P}}) : t \leq k\} \cup \{\mathcal{V}([\mathbf{0},\mathbf{1}]^t) : t \leq k\}$$

has one identity of the form $\lambda_{\mathcal{V}}(x_1,\ldots,x_n) \approx 1$ that characterizes it.

Theorem 6.3. *If* $\mathcal{V} = \bigvee_{i=1}^s \mathcal{V}_i$, *where*

$$\mathcal{V}_i \in \{\mathcal{V}(\mathbf{S}_m^t) : t \leq k\} \cup \{\mathcal{V}(\mathbf{S}_{n,\omega}^{t,\mathbf{P}}) : t \leq k\} \cup \{\mathcal{V}([\mathbf{0},\mathbf{1}]^t) : t \leq k\},$$

then the identity that characterizes $\mathcal{V}$ *is*

$$\lambda_{\mathcal{V}}(x_{11},\ldots,x_{1n_1},x_{21},\ldots,x_{2n_2},x_{s1},\ldots,x_{sn_s}) = \bigvee_{i=1}^{s} \forall\,(\lambda_{\mathcal{V}_i}(x_{i1},\ldots,x_{in_i})) \approx 1,$$

where $\lambda_{\mathcal{V}_i}(x_{i1},\ldots,x_{in_i}) \approx 1$ *characterizes the subvariety* $\mathcal{V}_i$ *for each* $i = 1,\ldots,s$.

Proof. Let $\mathbf{A}$ be a subdirectly irreducible MMV-algebra. Suppose first that $\mathbf{A} \in si(\mathcal{V})$. Then $\mathbf{A} \in si(\mathcal{V}_i)$ for some $i = 1,\ldots,s$. So $\lambda_{\mathcal{V}_i}(x_{i1},\ldots,x_{in_i}) \approx 1$ holds in $\mathbf{A}$, and it follows that $\forall\,(\lambda_{\mathcal{V}_i}(x_{i1},\ldots,x_{in_i})) \approx 1$ also holds in $\mathbf{A}$. Finally, $\bigvee_{i=1}^s \forall\,(\lambda_{\mathcal{V}_i}(x_{i1},\ldots,x_{in_i})) \approx 1$ holds in $\mathbf{A}$. Now let $\mathbf{A} \notin si(\mathcal{V})$. Then $\mathbf{A} \notin si(\mathcal{V}_i)$ for all i. For each i, we choose elements $a_{i1},\ldots,a_{in_i} \in A$ such that $\lambda_{\mathcal{V}_i}(a_{i1},\ldots,a_{in_i}) < 1$. This leads to $\forall\,(\lambda_{\mathcal{V}_i}(a_{i1},\ldots,a_{in_i})) < 1$ for all i. Since $\forall A$ is a chain, there is some $t \in \{1,\ldots,s\}$ such that

$$\bigvee_{i=1}^{s} \forall\,(\lambda_{\mathcal{V}_i}(a_{i1},\ldots,a_{in_i})) = \forall\,(\lambda_{\mathcal{V}_t}(a_{i1},\ldots,a_{in_i})) < 1$$

So $\bigvee_{i=1}^s \forall(\lambda_{\mathcal{V}_i}(x_{i1},\ldots,x_{in_i})) \approx 1$ does not hold in $\mathbf{A}$. $\square$

7. Subvarieties generated by algebras of infinite width

In this section, we prove that the variety generated by a functional MMV-algebra $[\mathbf{0},\mathbf{1}]^X$, for X infinite, is the variety generated by $\{[\mathbf{0},\mathbf{1}]^k : k \in \mathbb{N}\}$. As a consequence, we give a finite set of generators for the subvarieties $\mathcal{K}_n$ and for $\mathcal{V}(\{\mathbf{S}_n^k : k \in \mathbb{N}\})$, which we denote by $\mathcal{MMV}_n$.

Theorem 7.1. *If $[\mathbf{0},\mathbf{1}]^X$ is a functional MMV-algebra such that X is infinite, then $\mathcal{V}_{\mathcal{MMV}}([\mathbf{0},\mathbf{1}]^X) \subseteq \mathcal{V}_{\mathcal{MMV}}\left(\{[\mathbf{0},\mathbf{1}]^k : k \in \mathbb{N}\}\right)$.*

Proof. Consider the MMV-algebra whose universe is the infinite product of MMV-algebras $\langle [\mathbf{0},\mathbf{1}]^Y; \forall_\wedge\rangle$ indexed by the set $I = \{Y \in Su(X) : |Y| \text{ is finite}\}$, where $Su(X)$ is the set of all subsets of X and $|Y|$ indicates the cardinal of the set Y. Let us define the MV-homomorphism $\phi\colon [\mathbf{0},\mathbf{1}]^X \longrightarrow \prod_{Y\in I} [\mathbf{0},\mathbf{1}]^Y$ by $\phi(a)_Y = \langle a_k\rangle_{k\in Y}$ for $a = \langle a_k\rangle_{k\in X} \in [0,1]^X$, and where $\phi(a)_Y$ denotes the Yth coordinate in the product.

Let us note first that for each $a = \langle a_k\rangle_{k\in X} \in [0,1]^X$, $\forall_\wedge(a)$ is the constant X-tuple $\langle \bigwedge_{k\in X} a_k\rangle_{k\in X} \in [0,1]^X$. Let us observe also that $\forall$ is defined pointwise in $\prod_{Y\in I} [\mathbf{0},\mathbf{1}]^Y$. In particular, for each $a \in [0,1]^X$, we have that $\forall(\phi(a)_Y) = \langle \bigwedge_{k\in Y} a_k, \ldots, \bigwedge_{k\in Y} a_k\rangle$ is a constant $|Y|$-tuple.

Let us consider now that monadic filter F in $\prod_{Y\in I} [\mathbf{0},\mathbf{1}]^Y$ generated by all elements of the form $\forall(\phi(a)) \to \phi(\forall_\wedge(a))$, $a \in [0,1]^X$. Let $\bar{\phi}$ be the canonical MMV-epimorphism $\bar{\phi}\colon [\mathbf{0},\mathbf{1}]^X \longrightarrow \left(\prod_{Y\in I} [\mathbf{0},\mathbf{1}]^Y\right)/F$. We claim that $\bar{\phi}$ is one-to-one. Suppose $\bar{\phi}(b) = \langle b_k\rangle_{k\in X} \in F$. Then there exist $a_1, \cdots, a_n \in [0,1]^X$ such that

$$b_k \geq \bigodot_{j=1}^{n} \Big(\bigwedge_{k\in Y} a_{jk} \to \bigwedge_{k\in X} a_{jk}\Big)$$

for all $Y \in I$.

We denote $\bigwedge_{k\in X} a_{jk}$ by d_j. For $m \in \mathbb{N}$, choose $a_{jk_{mj}} \in [0,1]$ such that $d_j \leq a_{jk_{mj}} \leq d_j + \frac{1}{m}$, and consider the sets $Y_m = \{k_{rj} : j = 1, \ldots, n, r = 1, \ldots, m\}$. Then

$$\begin{aligned} b_k &\geq \bigodot_{j=1}^{n} \Big(\bigwedge_{k\in Y} a_{jk} \to d_j\Big) \geq \bigodot_{j=1}^{n} \Big(a_{jk_{mj}} \to d_j\Big) = \bigodot_{j=1}^{n} \Big(1 - a_{jk_{mj}} + d_j\Big) \\ &= \bigodot_{j=1}^{n} \Big(1 - a_{jk_{mj}} + d_j - (1-1)\Big) = 1 + \Sigma_{j=1}^{n} \Big(d_j - a_{jk_{mj}}\Big) \geq 1 - \frac{n}{m}, \end{aligned}$$

for all Y_m. Then $b_k = 1$ for all $k \in X$. $\square$

The following result follows from Theorem 7.1 and the fact that $[\mathbf{0},\mathbf{1}]^k$ is a subalgebra of $[\mathbf{0},\mathbf{1}]^X$, for each integer k.

Corollary 7.2. *Let X be an infinite set. Then*

$$\mathcal{V}\big([\mathbf{0},\mathbf{1}]^X\big) = \mathcal{V}\big(\{[\mathbf{0},\mathbf{1}]^k : k \in \mathbb{N}\}\big).$$

In particular, $\mathcal{V}(\mathbf{S}_n^{\mathbb{N}}) = \mathcal{V}\left(\{\mathbf{S}_n^k : k \in \mathbb{N}\}\right)$.

Corollary 7.3. *For each positive integer n, $\mathcal{K}_n = \mathcal{V}(\{\mathbf{S}_1^{\mathbb{N}}, \mathbf{S}_2^{\mathbb{N}}, \ldots, \mathbf{S}_n^{\mathbb{N}}\})$.*

Proof. Let m be a positive integer such that $m \leq n$. Then $\mathbf{S}_m^{\mathbb{N}}$ satisfies (δ_n). Therefore, $\mathcal{V}(\{\mathbf{S}_1^{\mathbb{N}}, \mathbf{S}_2^{\mathbb{N}}, \ldots, \mathbf{S}_n^{\mathbb{N}}\}) \subseteq \mathcal{K}_n$. Since $\mathbf{S}_m^k$ is a subalgebra of $\mathbf{S}_m^{\mathbb{N}}$, for each k, we have $\mathcal{K}_n = \mathcal{V}(\{\mathbf{S}_m^k : k \in \mathbb{N}, 1 \leq m \leq n\}) \subseteq \mathcal{V}(\{\mathbf{S}_1^{\mathbb{N}}, \mathbf{S}_2^{\mathbb{N}}, \ldots, \mathbf{S}_n^{\mathbb{N}}\})$. Finally, $\mathcal{K}_n = \mathcal{V}(\{\mathbf{S}_1^{\mathbb{N}}, \mathbf{S}_2^{\mathbb{N}}, \ldots, \mathbf{S}_n^{\mathbb{N}}\})$. □

If $\mathbf{A} \in \mathcal{MMV}_n$ is subdirectly irreducible, then $\mathbf{A}$ is isomorphic to a subalgebra of $\langle \mathbf{S}_n^X; \forall_\wedge \rangle$ for some non-empty set X [13]. From this and Corollary 7.2, we have the following lemma.

Lemma 7.4. *The subvariety $\mathcal{MMV}_n$ is equal to $\mathcal{V}(\mathbf{S}_n^{\mathbb{N}})$.*

Since $\mathbf{S}_n^{\mathbb{N}}$ is a subalgebra of $\mathbf{S}_m^{\mathbb{N}}$ if and only if n divides m, and from Lemma 7.4, we have the next result.

Corollary 7.5. *Let n and m be positive integers. Then $\mathcal{MMV}_n \subseteq \mathcal{MMV}_m$ if and only if n divides m.*

Acknowledgments. The authors thank Diego N. Castaño for valuable discussions and the referee for useful comments that helped to improve the paper.

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