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Generalized contexts and consistent histories in quantum mechanics

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ABSTRACT

We analyze a restriction of the theory of consistent histories by imposing that a valid description of a physical system must include quantum histories which satisfy the consistency conditions *for all states*. We prove that these conditions are equivalent to imposing the compatibility conditions of our formalism of generalized contexts. Moreover, we show that the theory of consistent histories with the consistency conditions for all states and the formalism of generalized context are equally useful representing expressions which involve properties at different times.

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1. Introduction

In quantum mechanics, the properties of a physical system are represented by closed subspaces of the Hilbert space. The orthocomplemented lattice structure of the set of properties allows defining the conjunction, the disjunction and the negation of properties. The probabilities for the properties at a given value of time are given by the Born rule.

The standard formalism of quantum mechanics does not give a meaning to conjunctions or disjunctions of properties corresponding to different times. However, there are situations in which it is necessary to relate properties at different times. For example, in the measurement process it is necessary to establish a link between the pointer position after the measurement and the previous value of some observable of the measured system. In the double slit experiment it is necessary to argue

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about the impossibility to say through which slit passed the particle before it is detected, producing a spot in a photographic plate.

In order to deal with conjunctions or disjunctions of properties at different times, R. Griffiths [1], R. Omnès [2], M. Gell-Mann and J. Hartle [3] have developed the theory of consistent histories. In this theory the allowed sets of quantum histories included in a description of the system must satisfy some consistency conditions. As the consistency conditions depend on the state of the system, the properties of the system which can be included in a valid description of the system also depend on the state. This is an odd situation compared with the standard formalism of quantum mechanics, where the allowed contexts of properties are all the possible distributive sublattices of the Hilbert space, which do not depend on the state.

Moreover, in the axiomatic approaches of the standard formalism of quantum mechanics, once the possible properties are identified, the states can be defined as functionals acting on the space of the properties, appearing after these observables in a somehow subordinated position [4,5].

We presented in previous papers the generalized context formalism [6,7]. In this formalism, well defined probabilities can be obtained for the conjunction of properties at different times, provided they satisfy some compatibility conditions. Compatible properties at different times are represented by commuting projectors when they are translated to a common time. This formalism was applied to the double slit experiment [7], to the logic of quantum measurements [8] and to quantum decay processes [9].

In this paper we explore the results that can be obtained from the theory of consistent histories by imposing the consistency conditions on all the states of the system. We obtain that the consistency conditions over all the states are equivalent to the compatibility conditions of our generalized contexts formalism. We also show that the theory of state-independent consistent histories and the theory of generalized context are equally useful representing expressions which involve properties at different times. Both formalism can represent the same expressions and the corresponding probabilities have the same value.

In Section 2 we present a brief description of the lattice of properties in standard quantum mechanics, emphasizing the notion of contexts of properties for a fixed value of time. The main features of the theory of consistent histories are given in Section 3, pointing out the motivations for considering the consistency conditions for all states. In Section 4 we relate the consistency conditions for all states with the commutation of the projectors corresponding to the time translation of properties to a common time. In Section 5 we summarize our previously developed formalism of generalized contexts. The consistency conditions for all states and the formalism of generalized contexts are compared in Section 6. The main conclusions are given in Section 7.

2. Quantum contexts

In quantum mechanics, each isolated physical system is associated with a Hilbert space $\mathcal{H}$ and a Hamiltonian operator $H : \mathcal{H} \longrightarrow \mathcal{H}$. The *state* of the system is represented by a nonnegative, normalized and self adjoint *density operator* $\rho : \mathcal{H} \longrightarrow \mathcal{H}$.

The time evolution of the state is generated by the Liouville–von Neumann equation. If ρ_t is the density operator representing the state at time t, the state at a different time t' is represented by

$$\rho_{t'} = U(t', t)\ \rho_t\ U(t', t)^{-1}, \qquad (1)$$

where $U(t', t) = e^{-\frac{i}{\hbar}H(t'-t)}$ is the unitary time translation operator.

The *properties* of a quantum system are represented by closed vector subspaces of the Hilbert space $\mathcal{H}$. As for each closed subspace V there exists only one orthogonal projection operator $\Pi_V : \mathcal{H} \longrightarrow \mathcal{H}$ such that $V = \Pi_V\ \mathcal{H}$, each property V can also be represented by the projector Π_V.

The set of all closed vector subspaces of $\mathcal{H}$, with the *partial order relation* given by the set inclusion ($\subset$), is an orthocomplemented nondistributive lattice. The *supremum* of V and V' is given by $\mathrm{Sup}(V, V') = V + V'$ and the *infimum* is given by $\mathrm{Inf}(V, V') = V \cap V'$. The *universal property* is represented with the whole space $\mathcal{H}$ and the *zero property* is represented with the subspace $\{0_{\mathcal{H}}\}$, where $0_{\mathcal{H}}$ is the zero element of $\mathcal{H}$. The *complement* of a property V is the orthogonal complement $V^{\perp}$ of the subspace V in $\mathcal{H}$.

A relevant difference between quantum mechanics and classical mechanics is that in classical mechanics the lattice of properties is a Boolean lattice. However, in quantum mechanics, as the lattice of properties is nondistributive, it is not Boolean.

In order to define probabilities it is necessary to have a Boolean lattice of properties. Therefore, it is not possible to consider simultaneously the whole set of properties of a system, so it is necessary to choose a Boolean sublattice.

With this purpose it is considered a set B of mutually orthogonal closed subspaces of $\mathcal{H}$ which expand the whole Hilbert space, i.e.

$$B = \left\{ V_i \mid i \in \sigma,\ V_i \text{ is a closed subspace of } \mathcal{H}, V_i \perp V_j \text{ if } i \neq j, \sum_{i\in\sigma} V_i = \mathcal{H} \right\}, \tag{2}$$

where σ is a set of indices. Each closed subspace V_i of the set B represents an *atomic property* and the corresponding projectors Π_i satisfy the relations

$$\sum_{i\in\sigma} \Pi_i = I, \qquad \Pi_i \Pi_j = \delta_{ij} \Pi_j, \quad i, j \in \sigma, \tag{3}$$

where I is the identity operator in $\mathcal{H}$.

From the set B, a *context of properties* C_B can be obtained as the set of all subspaces which are sums and intersections of elements of B

$$C_B = \left\{ V \subset \mathcal{H} \mid V = \sum_i \alpha_i V_i,\ \alpha_i = 0, 1 \right\}. \tag{4}$$

The context of properties C_B *generated* by the set of atomic properties B, with the partial order relation defined by the inclusion ($\subset$), is a Boolean lattice.

If ρ_t is the state operator for the system at time t, the Born rule can be used to compute

$$\mathrm{Pr}_{\rho_t}(V) = \mathrm{Tr}(\rho_t \Pi_V), \tag{5}$$

for each property V. If we restrict the properties V to be elements of a context C_B, the function Pr_t satisfies the Kolmogorov axioms

(i) $\mathrm{Pr}_{\rho_t}(V) \geqslant 0$,
(ii) $\mathrm{Pr}_{\rho_t}(\mathcal{H}) = 1$,
(iii) If $V_1 \cap V_2 = \{0_{\mathcal{H}}\}$, then $\mathrm{Pr}_{\rho_t}(V_1 + V_2) = \mathrm{Pr}_{\rho_t}(V_1) + \mathrm{Pr}_{\rho_t}(V_2)$.

It is interesting to note that for each ρ_t there is a different probability function Pr_{ρ_t}.

The function $\mathrm{Pr}_{\rho_t} : C_B \longrightarrow \mathbb{R}$ is a well defined *probability* on the context of properties C_B. Therefore, ordinary quantum mechanics gives a prescription to compute well defined probabilities for properties of a context at a fixed value of time.

However, ordinary quantum mechanics is unable to assign probabilities to the conjunction of properties corresponding to different values of time. As we pointed out in the introduction, in some physical situations, for example the process of measurement or the double slit experiment, it is necessary to relate properties at different times. This suggest the importance of a generalization of ordinary quantum mechanics for dealing with time dependent properties, as it is realized in the theory of consistent histories [1–3] and in the formalism of generalized contexts [6,7].

In the following sections we give a brief account of both formalisms, we explore the consequences of imposing more restrictive consistency conditions on the theory of consistent histories and we obtain the relation of these new conditions with our formalism of generalized contexts.

3. Consistent histories

In what follows we present the main features of the theory of consistent histories, developed by R. Griffiths [1], R. Omnès [2], M. Gell-Mann and J. Hartle [3].

As we explained in the previous section, each property of a quantum system is represented by a closed subspace of the Hilbert space of the system or by its corresponding projection operator. An *homogeneous history* is defined as a sequence of properties for n consecutive times. The theory of consistent histories represents each homogeneous history with the tensor product of the corresponding projection operators of each property involved in the history, i.e.

$$(p_1, p_2, \dots, p_n) \longleftrightarrow \Pi_1 \otimes \Pi_2 \otimes \cdots \otimes \Pi_n,$$

where Π_i is the projector corresponding to the property p_i.

Not every history is homogeneous. The general form of an n times history is represented by a projection operator $\widetilde{\Pi} : \widetilde{\mathcal{H}} \longrightarrow \widetilde{\mathcal{H}}$ in the Hilbert space $\widetilde{\mathcal{H}}$, which is the tensor product of n copies of the Hilbert space $\mathcal{H}$ ($\widetilde{\mathcal{H}} = \mathcal{H} \otimes \overset{n}{\dots} \otimes \mathcal{H}$). As $\widetilde{\mathcal{H}}$ is a Hilbert space, the set of all its projection operators is an orthocomplemented nondistributive lattice.

It is possible to obtain a distributive lattice of histories by considering at each time t_j ($j = 1, \dots, n$) a different context of properties generated by projectors $\Pi_j^{k_j} : \mathcal{H} \longrightarrow \mathcal{H}$ ($k_j \in \sigma_j$) satisfying

$$\Pi_j^{k_j} \Pi_j^{k'_j} = \delta_{k_j k'_j} \Pi_j^{k_j}, \qquad \sum_{k_j} \Pi_j^{k_j} = I, \quad k_j, k'_j \in \sigma_j, \ j = 1, \dots, n \tag{6}$$

where I is the identity operator in $\mathcal{H}$.

Expressions of the form "property $V_1^{k_1}$ at time t_1 and … and property $V_n^{k_n}$ at time t_n" are called elementary histories. They are represented by the projectors

$$\widetilde{\Pi}^{\mathbf{k}} \equiv \Pi_1^{k_1} \otimes \cdots \otimes \Pi_n^{k_n}, \quad \mathbf{k} \equiv (k_1, \dots, k_n).$$

It is easy to check that the projectors $\widetilde{\Pi}^{\mathbf{k}}$ satisfy

$$\widetilde{\Pi}^{\mathbf{k}} \widetilde{\Pi}^{\mathbf{k}'} = \delta_{\mathbf{k}\mathbf{k}'} \widetilde{\Pi}^{\mathbf{k}}, \qquad \sum_{\mathbf{k}} \widetilde{\Pi}^{\mathbf{k}} = \widetilde{I}, \quad \mathbf{k} \in \sigma_1 \times \cdots \times \sigma_n$$

where $\widetilde{I}$ is the identity operator in $\widetilde{\mathcal{H}}$. The set of projectors given by

$$A = \left\{ \widetilde{\Pi} = \sum_{\mathbf{k}} \alpha_{\mathbf{k}} \widetilde{\Pi}^{\mathbf{k}}, \ \alpha_{\mathbf{k}} = 0, 1, \ \mathbf{k} \in \sigma_1 \times \cdots \times \sigma_n \right\} \tag{7}$$

represent a distributive lattice of histories. Each projector of the set A represents an expression of the form "(property $V_1^{k_1}$ at time t_1 and … and property $V_n^{k_n}$ at time t_n) or (property $V_1^{k'_1}$ at time t_1 and … and property $V_n^{k'_n}$ at time t_n) or …".

In order to define probabilities on A, the theory of consistent histories introduce the *chain operator*. For each elementary history represented by $\widetilde{\Pi}^{\mathbf{k}}$ the chain operator is defined by

$$C(\widetilde{\Pi}^{\mathbf{k}}) \equiv \Pi_{1,0}^{k_1} \Pi_{2,0}^{k_2} \dots \Pi_{n,0}^{k_n}, \tag{8}$$

where

$$\Pi_{j,0}^{k_j} \equiv U(t_0, t_j) \, \Pi_j^{k_j} \, U(t_j, t_0) \tag{9}$$

and t_0 is the time at which the state operator ρ_{t_0} is considered.

For any element of the distributive lattice of histories given in Eq. (7) the chain operator is obtained by the linear extension of the definition given in Eq. (8) for elementary histories, i.e.

$$C(\widetilde{\Pi}) \equiv \sum_{\mathbf{k}} \alpha_{\mathbf{k}} \, C(\widetilde{\Pi}^{\mathbf{k}}), \quad \mathbf{k} \in \sigma_1 \times \cdots \times \sigma_n. \tag{10}$$

The probability of a history represented by $\widetilde{\Pi}$ is defined by the expression

$$\mathrm{Pr}_{ch}(\widetilde{\Pi}) \equiv \mathrm{Tr}(C^\dagger(\widetilde{\Pi}) \rho_{t_0} C(\widetilde{\Pi})), \tag{11}$$

where we incorporated the subindex ch to the probability symbol for consistent histories to make an explicit distinction with the probability Pr_{gc}, which will be defined in the next section for the formalism of generalized contexts.

The probability $\Pr_{ch}$ do not satisfy the additivity condition. In order to have a well defined probability, it is necessary that the elementary histories of A satisfy the *consistency conditions* defined by

$$\mathrm{Re}[\mathrm{Tr}(C^{\dagger}(\tilde{\Pi}^{\mathbf{k}})\rho_{t_0}C(\tilde{\Pi}^{\mathbf{k}'}))] = 0, \quad \text{for all } \mathbf{k} \neq \mathbf{k}'. \tag{12}$$

A set of histories satisfying these conditions is called a *set of consistent histories*. Eq. (12) give the weakest consistency conditions typically considered for the theory of consistent histories, but stronger conditions can also be used (see also the final part of Section 6).

The theory of consistent histories is suitable to include properties at different times and it allows computing well defined probabilities provided we consider properties within a set of consistent histories.

Each set of consistent histories gives a possible description of the physical system. Different authors gave different names to the sets of consistent histories (framework for R. B. Griffiths [10] or universe of discourse for R. Omnès [11]), but all these different names refer to a sort of "perspective" of the quantum system. The main difference of these sets of consistent histories and the usual meaning of the notion of perspective is that in general they cannot be combined in a single more refined description. Different sets of consistent histories are considered complementary descriptions of the physical system.

As the sets of consistent histories must satisfy consistency conditions depending on the state of the system, the properties of the system which can be included in a valid description of the system also depend on the state. This is an odd situation compared with the standard formalism of quantum mechanics, in which the allowed contexts of properties are all the possible distributive sublattices of the Hilbert space, which do not depend on the state.

In what follows we point out some reasons which motivate us to explore a modification of the theory of consistent histories in which the consistency conditions do not depend on the state.

- In the axiomatic theories of quantum mechanics the state is considered as a functional on the space of observables and it appears after these observables in a somehow subordinate position. The importance of the notion of state functionals acting on a previously defined space of observables was stressed by one of us in Refs. [4,5]. As quantum histories play the role of observables, it seems reasonable that the allowed sets of histories satisfy state-independent conditions.
- The consistency conditions allow too many histories and some of them are difficult to interpret [12,13]. If we modify the restriction and we impose the consistency conditions for all the states we limit the number of allowed histories.
- The partial order relation between two properties has a clear probabilistic interpretation in ordinary quantum mechanics. For two properties a and b of the same context, with the corresponding subspaces V_a and V_b of the Hilbert space $\mathcal{H}$, it can be proved that the probability of property b conditional to property a is equal to one for all quantum states, if and only if $V_a \subset V_b$. In the theory of consistent histories there is not such a strong connection unless we imposed the consistency conditions to all the states.

Motivated by these remarks, in the following section we are going to consider the theory of consistent histories with the consistency conditions imposed for all states.

4. Consistency conditions for all states

In this section we are going to consider a restriction to the theory of consistent histories by imposing that a valid description of a physical system must include quantum histories which satisfy the consistency conditions *for all states*.

At each time t_j $(j = 1, \ldots, n)$ we consider a different context of properties generated by projectors $\Pi_j^{k_j} : \mathcal{H} \longrightarrow \mathcal{H} (k_j \in \sigma_j)$ satisfying

$$\Pi_j^{k_j}\Pi_j^{k_j'} = \delta_{k_j k_j'}\,\Pi_j^{k_j}, \qquad \sum_{k_j}\Pi_j^{k_j} = I, \quad k_j, k_j' \in \sigma_j,\ j = 1, \ldots, n.$$

We define the following elementary histories from these n different contexts

$$\widetilde{\Pi}^{\mathbf{k}} \equiv \Pi_1^{k_1} \otimes \cdots \otimes \Pi_n^{k_n}, \quad \mathbf{k} \equiv (k_1, \ldots, k_n)$$

and we consider the set of histories given by

$$A = \left\{ \widetilde{\Pi} = \sum_{\mathbf{k}} \alpha_{\mathbf{k}}\, \widetilde{\Pi}^{\mathbf{k}}, \alpha_{\mathbf{k}} = 0, 1, \mathbf{k} \in \sigma_1 \times \cdots \times \sigma_n \right\}.$$

The consistency conditions are given by

$$\mathrm{Re}[\mathrm{Tr}(C^{\dagger}(\tilde{\Pi}^{\mathbf{k}})\rho_{t_0} C(\tilde{\Pi}^{\mathbf{k}'}))] = 0, \quad \text{for all } \mathbf{k} \neq \mathbf{k}',$$

and for a fixed state ρ_{t_0}.

We propose to replace these consistency conditions by what we will call the *state-independent consistency conditions* defined by

$$\mathrm{Re}[\mathrm{Tr}(C^{\dagger}(\tilde{\Pi}^{\mathbf{k}})\rho_{t_0} C(\tilde{\Pi}^{\mathbf{k}'}))] = 0, \quad \text{for all } \mathbf{k} \neq \mathbf{k}' \text{ and for all } \rho_{t_0}.$$

We will call *theory of state-independent consistent histories* to the theory of consistent histories with the state-independent consistency conditions.

It can be proved that imposing the state-independent consistency conditions on the elementary histories of a set of quantum histories A is equivalent to impose on the projectors $\Pi_i^{k_i}$, representing the atomic properties of the contexts at each time t_i $(i = 1, \ldots, n)$, the commutation condition when they are translated to a common time t_0 $([\Pi_{i,0}^{k_i}, \Pi_{j,0}^{k_j}] = 0, \Pi_{j,0}^{k_j} \equiv U(t_0, t_j)\ \Pi_j^{k_j}\ U(t_j, t_0))$. More specifically, we proved the following theorems (see Appendix B):

Theorem 1. *If the elementary histories of the set A given in Eq. (7) satisfy the consistency conditions for any state ρ_{t_0}, then $[\Pi_{i,0}^{k_i}, \Pi_{j,0}^{k_j}] = 0$, for all $i, j = 1, \ldots, n$, $k_i \in \sigma_i$, $k_j \in \sigma_j$.*

Theorem 2. *If $[\Pi_{i,0}^{k_i}, \Pi_{j,0}^{k_j}] = 0$, for all $i, j = 1, \ldots, n$, $k_i \in \sigma_i$, $k_j \in \sigma_j$, then the elementary histories of A given in Eq. (7) satisfy the consistency conditions for all state ρ_{t_0}.*

We will show in the following section that the commutation relations appearing in the previous theorems are precisely the compatibility conditions of our formalism of generalized contexts.

Moreover, the probability has a simple form on a set of state-independent consistent histories. Using the conditions $[\Pi_{i,0}^{k_i}, \Pi_{j,0}^{k_j}] = 0$ on the Eqs. (8), (10) and (11) we obtain the following expression for the probability of an elementary history, represented by the projector $\widetilde{\Pi}^{\mathbf{k}} \equiv \Pi_1^{k_1} \otimes \cdots \otimes \Pi_n^{k_n}$,

$$\mathrm{Pr}_{ch}(\widetilde{\Pi}^{\mathbf{k}}) = \mathrm{Tr}(\rho_{t_0} \Pi_{n,0}^{k_n} \ldots \Pi_{1,0}^{k_1}). \tag{13}$$

For a non elementary history, represented by the projector $\widetilde{\Pi} = \sum_{\mathbf{k}} \alpha_{\mathbf{k}}\ \widetilde{\Pi}^{\mathbf{k}}$ $(\alpha_{\mathbf{k}} = 0, 1)$, the probability is given by

$$\mathrm{Pr}_{ch}(\widetilde{\Pi}) = \sum_{\mathbf{k}} \alpha_{\mathbf{k}}\ \mathrm{Pr}(\widetilde{\Pi}^{\mathbf{k}}) = \mathrm{Tr}\left(\rho_{t_0} \sum_{\mathbf{k}} \alpha_{\overline{k}} \Pi_{n,0}^{k_n} \ldots \Pi_{1,0}^{k_1}\right). \tag{14}$$

We emphasize that if the consistency conditions are imposed for all states, the histories represented by projectors on $\widetilde{\mathcal{H}} = \mathcal{H} \otimes \overset{n}{\ldots} \otimes \mathcal{H}$ have a probability which can be computed with the "Born rule" in terms of projectors defined on $\mathcal{H}$. In Eq. (13) the projector $\Pi_{n,0}^{k_n} \ldots \Pi_{1,0}^{k_1}$ corresponds to the conjunction in $\mathcal{H}$ of the atomic properties $\Pi_i^{k_i}$ translated to a common time t_0 and the projector in Eq. (14) corresponds to the disjunction of these conjunctions. In the following sections we are going to show that the same expressions for the probabilities are obtained with our formalism of generalized contexts.

5. Generalized contexts

In this section we review the main aspects of the formalism of generalized contexts [6,7]. We define an *event* as a property at a given time and we represent it with the pair (V, t), where V is a closed subspace of the Hilbert space $\mathcal{H}$ and t is the time of the event. The set of all events will be called E.

Given a subspace V we can consider the time translated subspace $U(t', t)V$, where $U(t', t) = e^{-iH(t'-t)/\hbar}$ and H is the Hamiltonian operator. The relation $(V', t') \backsim (V, t)$ defined by $V' = U(t', t)V$

is an equivalence relation. We will denote by $[V, t]$ the class of events which are equivalent to the event (V, t), i.e.

$$[V, t] \equiv \{(V', t') \mid (V', t') \sim (V, t)\}.$$

We also call by $[E] \equiv E/\sim$ to the set of all equivalence classes of events,

$$[E] = E/\sim = \{[V, t] \mid (V, t) \in E\}.$$

We define a partial order relation $\leq$ on $[E]$ in the following way: $[V_1, t_1] \leq [V_2, t_2]$ if and only if $U(t_2, t_1)V_1 \subset V_2$. From this partial order relation the disjunction and the conjunction of equivalence classes are obtained,

$$[V, t] \vee [V', t'] \equiv [U(t_0, t)V + U(t_0, t')V', t_0],$$
$$[V, t] \wedge [V', t'] \equiv [U(t_0, t)V \cap U(t_0, t')V', t_0], \tag{15}$$

where t_0 is an arbitrary fixed time.

We need a complemented and distributive lattice in order to define a probability function. Even though $[E]$ is a complemented lattice, it is not distributive. It is possible to obtain a Boolean sublattice $[E]_B$ starting from an ordinary context of properties C_B having the form given by Eqs. (2), (4) and (15). For a given fixed value t_0 of time, the set $[E]_B \subset [E]$ given by

$$[E]_B \equiv \{[V, t_0] \in [E] \mid V \in C_B\},$$

is a Boolean sublattice of $[E]$ and it will be called a *generalized context of events*. As C_B is generated by B, we will say that $[E]_B$ is generated by B.

Once we have the generalized context $[E]_B \subset [E]$, a well defined probability $\Pr_{gc} : [E]_B \longrightarrow \mathbb{R}$ can be defined as a generalization of the Born rule

$$\Pr_{gc}[V, t_0] \equiv \mathrm{Tr}(\rho_{t_0} \Pi_V), \tag{16}$$

where ρ_{t_0} is the state of the system at time t_0 and Π_V is the projector corresponding to $V \in C_B$. This is a well defined probability, which satisfies the Kolmogorov conditions.

In this way we have obtained a formalism for computing probabilities of classes of events. In what follows we are going to show how to apply this formalism to expressions involving properties at different times.

We consider n times $t_1 < \cdots < t_n$ and for each time t_i a generalized context of events $[E]_{B_i}$, generated by the atomic properties

$$B_i = \left\{ V_i^{k_i} \mid k_i \in \sigma_i,\ V_i^{k_i} \perp V_i^{k_i'} \text{if } k_i \neq k_i',\ \sum_{k_i} V_i^{k_i} = \mathcal{H} \right\},$$

where the projectors $\Pi_i^{k_i}$, corresponding to the atomic properties, $V_i^{k_i}$ satisfy the equations

$$\Pi_i^{k_i} \Pi_i^{k_i'} = \delta_{k_i k_i'} \Pi_i^{k_i}, \qquad \sum_{k_i} \Pi_i^{k_i} = I, \quad k_i, k_i' \in \sigma_i.$$

To consider descriptions which involve classes of events of different generalized contexts $[E]_{B_i}$ we need that they could be included in a common generalized context. The formalism of generalized contexts impose that the time translated projectors of each context commute, i.e.

$$[\Pi_{i,0}^{k_i}, \Pi_{j,0}^{k_j}] = 0, \quad k_i \in \sigma_i,\ k_j \in \sigma_j,\ i, j = 1, \ldots, n$$

where $\Pi_{i,0}^{k_i} = U(t_0, t_i)\Pi_i^{k_i} U^{-1}(t_0, t_i)$ and $\Pi_{j,0}^{k_j} = U(t_0, t_j)\Pi_j^{k_j} U^{-1}(t_0, t_j)$. We will call to these conditions the *compatibility conditions*. If some generalized contexts satisfy these conditions we say that they are *compatible*.

Once we have n compatible generalized contexts we can form a new generalized context including all of them. First, from the sets of atomic properties B_i, we define a new set of atomic properties B in the following way

$$B = \{\cap_i V_{i,0}^{k_i} \mid V_i^{k_i} \in B_i,\ k_i \in \sigma_i,\ i = 1, \ldots, n\},$$

where $V_{i,0}^{k_i} = U(t_0, t_i)V_i^{k_i}$. The projector corresponding to the subspace $\cap_i V_{i,0}^{k_i}$ is $\Pi_{1,0}^{k_1} \dots \Pi_{n,0}^{k_n}$. Then, from B we can define the generalized context generated by it, i.e. $[E]_B$.

Let us consider an expression of the form

"Property $V_1^{k_1}$ at time t_1 **and** property $V_2^{k_2}$ at time t_2 **and** ... **and** property $V_n^{k_n}$ at time t_n".

The formalism of generalized contexts identifies this expression with the following conjunction of equivalence classes

$$[V_1^{k_1}, t_1] \wedge [V_2^{k_2}, t_2] \wedge \cdots \wedge [V_n^{k_n}, t_n] = [V_{1,0}^{k_1} \cap V_{2,0}^{k_2} \cap \cdots \cap V_{n,0}^{k_n}, t_0],$$

and the probability for this expression is given by

$$\begin{aligned} \mathrm{Pr}_{gc}([V_1^{k_1}, t_1] \wedge [V_2^{k_2}, t_2] \wedge \cdots \wedge [V_n^{k_n}, t_n]) &= \mathrm{Pr}_{gc}([V_{1,0}^{k_1} \cap V_{2,0}^{k_2} \cap \cdots \cap V_{n,0}^{k_n}, t_0]) \\ &= \mathrm{Tr}\{\rho_{t_0}(\Pi_{1,0}^{k_1} \dots \Pi_{n,0}^{k_n})\}, \end{aligned}$$

where $\Pi_{1,0}^{k_1}\Pi_{2,0}^{k_2} \dots \Pi_{n,0}^{k_n}$ is the projector associated with the subspace $V_{1,0}^{k_1} \cap V_{2,0}^{k_2} \dots \cap V_{n,0}^{k_n}$.

Moreover, an expression of the form

"(Property $V_1^{k_1}$ at time t_1 **and** ... **and** property $V_n^{k_n}$ at time t_n) **or**

(property $V_1^{k'_1}$ at time t_1 **and** ... **and** property $V_n^{k'_n}$ at time t_n) **or** ...",

is identified with

$$([V_1^{k_1}, t_1] \wedge \cdots \wedge [V_n^{k_n}, t_n]) \vee ([V_1^{k'_1}, t_1] \wedge \cdots \wedge [V_n^{k'_n}, t_n]) \vee \dots$$

and the probability for this expression is given by

$$\begin{aligned} &\mathrm{Pr}_{gc}\{([V_1^{k_1}, t_1] \wedge \cdots \wedge [V_n^{k_n}, t_n]) \vee ([V_1^{k'_1}, t_1] \wedge \cdots \wedge [V_n^{k'_n}, t_n]) \vee \dots\} \\ &\quad = \mathrm{Tr}\{\rho_{t_0}(\Pi_{1,0}^{k_1} \dots \Pi_{n,0}^{k_n} + \Pi_{1,0}^{k'_1} \dots \Pi_{n,0}^{k'_n} + \cdots)\}, \end{aligned}$$

where $\Pi_{1,0}^{k_1} \dots \Pi_{n,0}^{k_n} + \Pi_{1,0}^{k'_1} \dots \Pi_{n,0}^{k'_n} + \cdots$ is the projector associated with the subspace

$$(V_{1,0}^{k_1} \cap \cdots \cap V_{n,0}^{k_n}) + (V_{1,0}^{k'_1} \cap \cdots \cap V_{n,0}^{k'_n}) + \cdots.$$

In the following section we are going to compare the results of the formalism of generalized contexts with the results of the theory of state-independent consistent histories presented in the previous section.

6. State-independent consistent histories and generalized contexts

In this section we will show that the theory of state-independent consistent histories and the theory of generalized context are equally useful representing expressions which involve properties at different times. Both formalism can represent the same expressions and the probabilities are the same.

Let us consider at each time t_j $(j = 1, \dots, n)$ a different context of properties generated by projectors $\Pi_j^{k_j} : \mathcal{H} \longrightarrow \mathcal{H}(k_j \in \sigma_j)$ satisfying

$$\Pi_j^{k_j}\Pi_j^{k'_j} = \delta_{k_j k'_j}\, \Pi_j^{k_j}, \qquad \sum_{k_j} \Pi_j^{k_j} = I, \quad k_j, k'_j \in \sigma_j,\ j = 1, \dots, n.$$

We assume that these projectors also verify the conditions

$$[\Pi_{i,0}^{k_i}, \Pi_{j,0}^{k_j}] = 0, \quad i, j = 1, \dots, n\ k_i \in \sigma_i\ k_j \in \sigma_j \tag{17}$$

where $i, j = 1, \dots, n$, $k_i \in \sigma_i$, $k_j \in \sigma_j$ and $\Pi_{j,0}^{k_j} = U(t_0, t_j)\Pi_j^{k_j} U(t_j, t_0)$.

We consider an expressions of the form

$$\text{“(Property } V_1^{k_1} \text{ at time } t_1 \textbf{ and } \ldots \textbf{ and } \text{property } V_n^{k_n} \text{ at time } t_n) \textbf{ or}$$
$$\text{(property } V_1^{k'_1} \text{ at time } t_1 \textbf{ and } \ldots \textbf{ and } \text{property } V_n^{k'_n} \text{ at time } t_n) \textbf{ or } \ldots\text{”.} \tag{18}$$

This expression can be included in the formalism of generalized contexts, because Eqs. (17) are the compatibility conditions of this formalism. According to Theorems 1 and 2 of Section 4, Eqs. (17) are equivalent to the consistency conditions for all states. Therefore expression (18) can also be included in the theory of consistent histories (and moreover in what we have called the theory of state-independent consistent histories).

We have shown in Section 5 that, in the formalism of generalized contexts, expression (18) is represented by the equivalence class

$$[V_0, t_0] \equiv [(V_{1,0}^{k_1} \cap \cdots \cap V_{n,0}^{k_n}) + (V_{1,0}^{k'_1} \cap \cdots \cap V_{n,0}^{k'_n}) + \cdots, t_0],$$

with the corresponding probability

$$\Pr_{gc}([V_0, t_0]) = \mathrm{Tr}\{\rho_{t_0}(\Pi_{1,0}^{k_1} \ldots \Pi_{n,0}^{k_n} + \Pi_{1,0}^{k'_1} \ldots \Pi_{n,0}^{k'_n} + \cdots)\}. \tag{19}$$

As we mentioned in the Sections 3 and 4, in the theory of state-independent consistent histories, expression (18) is represented by the projector

$$\widetilde{\Pi} = \Pi_1^{k_1} \otimes \cdots \otimes \Pi_n^{k_n} + \Pi_1^{k'_1} \otimes \cdots \otimes \Pi_n^{k'_n} + \cdots,$$

defined in $\widetilde{\mathcal{H}} = \mathcal{H} \otimes \cdots \otimes \mathcal{H}$. In this theory the corresponding probability is given by

$$\Pr_{ch}(\widetilde{\Pi}) = \mathrm{Tr}(C^\dagger(\widetilde{\Pi})\rho_{t_0}C(\widetilde{\Pi})) = \mathrm{Tr}\{\rho_{t_0}(\Pi_{1,0}^{k_1} \ldots \Pi_{n,0}^{k_n} + \Pi_{1,0}^{k'_1} \ldots \Pi_{n,0}^{k'_n} + \cdots)\}. \tag{20}$$

Comparing Eqs. (19) and (20), we find that, if the consistency conditions are valid for all states, the theory of consistent histories and the formalism of generalized contexts can be used to represent the same expressions and they give the same probabilities.

Therefore, the theory of state-independent consistent histories is equivalent to our formalism of generalized contexts. Although these two equivalent formalisms impose stronger requirements than the theory of consistent histories, we have proved in previous publications that our formalism can be successfully applied to describe the time dependent logic of quantum measurements [8], the quantum decay process [9] and the double slit experiment with and without measurement instruments [7].

We based our discussion on the Griffiths consistency conditions

$$\mathrm{Re}[\mathrm{Tr}(C^\dagger(\tilde{\Pi}^{\mathbf{k}})\rho_{t_0}C(\tilde{\Pi}^{\mathbf{k}'}))] = 0, \quad \text{for all } \mathbf{k} \neq \mathbf{k}',$$

already given in Eq. (12). However, several reasons have been given by R. Omnès [11] and L. Diosi [14] in favor of the stronger consistency conditions proposed by Gell-Mann and Hartle, and given by

$$\mathrm{Tr}(C^\dagger(\tilde{\Pi}^{\mathbf{k}})\rho_{t_0}C(\tilde{\Pi}^{\mathbf{k}'})) = 0, \quad \text{for all } \mathbf{k} \neq \mathbf{k}'.$$

It is interesting to notice that the quantum histories obtained from our formalism of generalized contexts not only satisfy the Griffiths consistency conditions for all states, but they also satisfy the Gell-Mann and Hartle conditions for all states (see Appendix B, Eq. (B.1) in the proof of Theorem 2).

These partial results encourage us to continue our research on the formalism of generalized contexts.

7. Conclusions

The theory of consistent histories, developed by R. Griffiths, R. Omnès, M. Gell-Mann and J. B. Hartle [1–3], and our formalism of generalized contexts [6,7] are generalizations of the standard quantum theory, designed to deal with expressions involving properties at different times.

The formalism of consistent histories imposes consistency conditions on the sets of histories to have well define probabilities. These conditions depend on the state of the system. By contrast, the formalism of generalized contexts imposes compatibility conditions which are independent of the quantum states.

In this paper we criticized the dependence on the state of the consistency conditions of the formalism of consistent histories. We consider several reasons in favor of a formalism of consistent histories with consistency conditions valid for all states.

Moreover, we analyzed a restriction of the theory of consistent histories by imposing that a valid description of a physical system must include quantum histories which satisfy the consistency conditions *for all states*. We proved that these conditions are equivalent to impose the compatibility conditions of our formalism of generalized contexts, i.e., that the projectors generating the contexts at each different time commute when they are translated to a common time.

Finally, we showed that the resulting formalisms of quantum histories, i.e. the theory of state-independent consistent histories and the formalism of generalized contexts, are equally useful representing expressions involving properties at different times. Both formalisms can represent the same expressions with the same value of the corresponding probabilities, being therefore equivalent.

Although these two equivalent formalisms impose stronger requirements than the theory of consistent histories, we have proved in previous publications that our formalism can be successfully used in several applications [8,9,7]. These partial results encourage us to continue our research on the formalism of generalized contexts.

Appendix A. Operators with zero mean values

Proposition. *If an operator $O:\mathcal{H}\longrightarrow\mathcal{H}$ satisfies $Tr(\rho O)=0$ for all state ρ, then $O=0$.*

Proof. Let $\{|\phi_i\rangle\}_{i\in\sigma}$ be an orthonormal basis of the Hilbert space $\mathcal{H}$.

Given $i\in\sigma$, we can choose $\rho_i=|\phi_i\rangle\langle\phi_i|$, then $\mathrm{Tr}(\rho_i O)=\langle\phi_i|O|\phi_i\rangle=0$.

Therefore,

$$\langle\phi_i|O|\phi_i\rangle=0\quad\forall i\in\sigma. \tag{A.1}$$

Given two indices $i\neq j$, we can also choose the following normalized vectors

$$|\Psi\rangle=\frac{1}{\sqrt{2}}(|\phi_j\rangle+|\phi_k\rangle),\qquad|\Phi\rangle=\frac{1}{\sqrt{2}}(|\phi_j\rangle+i|\phi_k\rangle).$$

For the corresponding density operators $|\Psi\rangle\langle\Psi|$ and $|\Phi\rangle\langle\Phi|$ we obtain

$$\langle\Psi|O|\Psi\rangle=\frac{1}{2}(\langle\phi_j|O|\phi_k\rangle+\langle\phi_k|O|\phi_j\rangle)=0,$$
$$\langle\Phi|O|\Phi\rangle=\frac{i}{2}(\langle\phi_j|O|\phi_k\rangle-\langle\phi_k|O|\phi_j\rangle)=0,$$

and then

$$\langle\phi_j|O|\phi_k\rangle=0,\quad\forall j,k\in\sigma,j\neq k. \tag{A.2}$$

From Eqs. (A.1) and (A.2) we deduce $O=0$. ■

Appendix B. Compatibility conditions and state-independent consistency conditions

Theorem 1. If the elementary histories of the set A given in Eq. (7) satisfy the consistency conditions for any state ρ_{t_0}, then $[\Pi_{i,0}^{k_i},\Pi_{j,0}^{k_j}]=0$, for all $i,j=1,\ldots,n,\ k_i\in\sigma_i,\ k_j\in\sigma_j$.

Proof. If $i=j$, from Eq. (6) we have $\Pi_j^{k_j}\Pi_j^{k'_j}=\delta_{k_jk'_j}\Pi_j^{k_j}$, then $[\Pi_j^{k_j},\Pi_j^{k'_j}]=0$.

The commutation relations are invariant under the unitary transformation given in Eq. (9), therefore $[\Pi_{j,0}^{k_j},\Pi_{j,0}^{k'_j}]=0$ for all $j=1,\ldots,n$, $k_j,k'_j\in\sigma_j$.

If $i\neq j$, we consider the histories $\tilde{\Pi},\tilde{\Pi}'\in A$ given by

$$\tilde{\Pi}=I\otimes\cdots\otimes I\otimes\Pi_i^{k_i}\otimes I\otimes\cdots\otimes I\otimes\Pi_j^{k_j}\otimes I\otimes\cdots\otimes I,$$
$$\tilde{\Pi}'=I\otimes\cdots\otimes I\otimes(I-\Pi_i^{k_i})\otimes I\otimes\cdots\otimes I\otimes\Pi_j^{k_j}\otimes I\otimes\cdots\otimes I.$$

It is easy to prove that if the elementary histories of A satisfy the consistency conditions for all state ρ_{t_0}, then all the histories of A also satisfy the conditions.

If we apply the consistency conditions given in Eqs. (12) to $\tilde{\Pi}$ and $\tilde{\Pi}'$, we obtain

$$\begin{aligned}\mathrm{Re}[\mathrm{Tr}\{C^\dagger(\widetilde{\Pi})\rho_{t_0}C(\widetilde{\Pi}')\}] &= \frac{1}{2}[\mathrm{Tr}\{C^\dagger(\widetilde{\Pi})\rho_{t_0}C(\widetilde{\Pi}')\} + \overline{\mathrm{Tr}\{C^\dagger(\widetilde{\Pi})\rho_{t_0}C(\widetilde{\Pi}')\}}] \\ &= \frac{1}{2}[\mathrm{Tr}\{C^\dagger(\widetilde{\Pi})\rho_{t_0}C(\widetilde{\Pi}')\} + \mathrm{Tr}\{(C^\dagger(\widetilde{\Pi})\rho_{t_0}C(\widetilde{\Pi}'))^\dagger\}] \\ &= \frac{1}{2}\mathrm{Tr}\{\rho_{t_0}(C(\widetilde{\Pi}')C^\dagger(\widetilde{\Pi}) + C(\widetilde{\Pi})C^\dagger(\widetilde{\Pi}'))\} = 0.\end{aligned}$$

As this equation is valid for all state operator ρ_{t_0}, we can apply the proposition of Appendix A to obtain

$$C(\widetilde{\Pi}')C^\dagger(\widetilde{\Pi}) + C(\widetilde{\Pi})C^\dagger(\widetilde{\Pi}') = 0.$$

The chain operators of $\tilde{\Pi}$ and $\tilde{\Pi}'$ are

$$C(\tilde{\Pi}) = \Pi_{i,0}^{k_i}\Pi_{j,0}^{k_j}, \qquad C(\tilde{\Pi}') = (I - \Pi_{i,0}^{k_i})\Pi_{j,0}^{k_j}.$$

Then, $(I - \Pi_{i,0}^{k_i})\Pi_{j,0}^{k_j}\Pi_{i,0}^{k_i} + \Pi_{i,0}^{k_i}\Pi_{j,0}^{k_j}(I - \Pi_{i,0}^{k_i}) = 0$.

If we apply the projector $\Pi_{i,0}^{k_i}$ to both members of the previous equation, we obtain

$$\Pi_{i,0}^{k_i}\Pi_{j,0}^{k_j} = \Pi_{i,0}^{k_i}\Pi_{j,0}^{k_j}\Pi_{i,0}^{k_i},$$

and the adjoint equation is

$$\Pi_{j,0}^{k_j}\Pi_{i,0}^{k_i} = \Pi_{i,0}^{k_i}\Pi_{j,0}^{k_j}\Pi_{i,0}^{k_i}.$$

Then, from these equations we finally obtain $[\Pi_{i,0}^{k_i}, \Pi_{j,0}^{k_j}] = 0$.

Therefore,

$$[\Pi_{i,0}^{k_i}, \Pi_{j,0}^{k_j}] = 0, \quad \text{for all } i, j = 1, \ldots, n,\ k_i \in \sigma_i,\ k_j \in \sigma_j. \quad \blacksquare$$

Theorem 2. If $[\Pi_{i,0}^{k_i}, \Pi_{j,0}^{k_j}] = 0$, for all $i, j = 1, \ldots, n$, $k_i \in \sigma_i$, $k_j \in \sigma_j$, then the elementary histories of A given in Eq. (7) satisfy the consistency conditions for all state ρ_{t_0}.

Proof. Let be ρ_{t_0} an arbitrary state and let be $\tilde{\Pi}^{\mathbf{k}} = \Pi_1^{k_1} \otimes \cdots \otimes \Pi_n^{k_n}$ and $\tilde{\Pi}^{\mathbf{k}'} = \Pi_1^{k'_1} \otimes \cdots \otimes \Pi_n^{k'_n}$ two different elementary histories of A.

The corresponding chain operators of $\tilde{\Pi}^{\mathbf{k}}$ and $\tilde{\Pi}^{\mathbf{k}'}$ are

$$C(\tilde{\Pi}^{\mathbf{k}}) = \Pi_{1,0}^{k_1}\Pi_{2,0}^{k_2}\ldots\Pi_{n,0}^{k_n}, \qquad C(\tilde{\Pi}^{\mathbf{k}'}) = \Pi_{1,0}^{k'_1}\Pi_{2,0}^{k'_2}\ldots\Pi_{n,0}^{k'_n}.$$

Then,

$$\begin{aligned}\mathrm{Tr}[C^\dagger(\tilde{\Pi}^{\mathbf{k}})\rho_{t_0}C(\tilde{\Pi}^{\mathbf{k}'})] &= \mathrm{Tr}[\rho_{t_0}C(\tilde{\Pi}^{\mathbf{k}'})C^\dagger(\tilde{\Pi}^{\mathbf{k}})] \\ &= \mathrm{Tr}[\rho_{t_0}\Pi_{1,0}^{k'_1}\Pi_{2,0}^{k'_2}\ldots\Pi_{n,0}^{k'_n}\Pi_{n,0}^{k_n}\ldots\Pi_{2,0}^{k_2}\Pi_{1,0}^{k_1}] \\ &= \mathrm{Tr}[\rho_{t_0}\Pi_{1,0}^{k'_1}\Pi_{1,0}^{k_1}\Pi_{2,0}^{k'_2}\Pi_{2,0}^{k_2}\ldots\Pi_{n,0}^{k'_n}\Pi_{n,0}^{k_n}].\end{aligned}$$

In the last member of the previous equation we have used that $[\Pi_{i,0}^{k_i}, \Pi_{j,0}^{k_j}] = 0$ for all $i, j = 1, \ldots, n$, $k_i \in \sigma_i$, $k_j \in \sigma_j$.

As $\tilde{\Pi}^{\mathbf{k}}$ and $\tilde{\Pi}^{\mathbf{k}'}$ are different, $\mathbf{k}' \neq \mathbf{k}$. Then, there is some $1 \leq i \leq n$ for which $k_i \neq k'_i$, hence $\Pi_{i,0}^{k'_i}\Pi_{i,0}^{k_i} = 0$.

Therefore,

$$\mathrm{Tr}[C^\dagger(\tilde{\Pi}^{\mathbf{k}})\rho_{t_0}C(\tilde{\Pi}^{\mathbf{k}'})] = 0, \quad \forall \mathbf{k} \neq \mathbf{k}'\ \forall \rho_{t_0}, \tag{B.1}$$

and

$$\mathrm{Re}\{\mathrm{Tr}[C^{\dagger}(\tilde{\Pi}^{\mathbf{k}})\rho_{t_0}C(\tilde{\Pi}^{\mathbf{k}'})]\} = 0, \quad \forall \mathbf{k} \neq \mathbf{k}' \, \forall \rho_{t_0}.$$

Then, all the elementary histories of A satisfy the consistency conditions for all state ρ_{t_0}. ■

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