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Review

On the classical limit of quantum mechanics, fundamental graininess and chaos: Compatibility of chaos with the correspondence principle

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ABSTRACT

The aim of this paper is to review the classical limit of Quantum Mechanics and to precise the well known threat of chaos (and fundamental graininess) to the correspondence principle. We will introduce a formalism for this classical limit that allows us to find the surfaces defined by the constants of the motion in phase space. Then in the integrable case we will find the classical trajectories, and in the non-integrable one the fact that regular initial cells become "amoeboid-like". This deformations and their consequences can be considered as a threat to the correspondence principle unless we take into account the characteristic timescales of quantum chaos. Essentially we present an analysis of the problem similar to the one of Omnès (1994,1999), but with a simpler mathematical structure.

1. Introduction

It seems that Einstein was the first one to realize that chaos was a threat to quantum mechanics [1] in a paper that was ignored by forty years [2]. A panoramic view of the this incompatibility of the classical chaos and quantum concepts (up to 1994) can be found in [1] and a recent review in [3]. Our first contribution to the subject was the introduction of a theory of the classical limit for closed quantum systems with Hamiltonian with continuous spectrum based in destructive interference (that we have called the "Self Induced Decoherence" – SID – and where we have used the Riemann–Lebesgue theorem [4]) and later we found a class of quantum chaotic systems (that may not contain all cases but certainly it contains the relevant ones) with chaotic classical limit [5,6]. With this idea in mind we study quantum chaos in papers [5–7] and extended the notions of non-integrable, ergodic and mixing quantum systems in paper [8]. These works were inspired in the landmark paper of Bellot and Earman [9]. The aim of this remarkable paper is precisely to show "how chaos puts some pressure on the correspondence principle (CP)" and the author says that there is not a "quick and convincing argument for the conclusion that the CP fails". Another important source of inspiration for us was the two books of Roland Omnès [10,11], precisely the characterization of quantum chaos as the evolution of a square cell to a distorted "amoeboid" cell (see Fig. 6(B)). In this paper we will essentially follow this idea, with simpler mathematical methods, and we will try to precise the origin of the elongated, distorted and final amoeboid cells which, in fact, we consider the main threat to the CP. It should be noted that the standard approach of the graininess has already been pointed out both in classical discretized systems and in quantum mechanics by looking at the Kolmogorov–Sinai entropy and its quantum variants [12–16]. In these cases, there is no threat to the correspondence principle, but only the emergence of a typical time-scale over (logarithmic in

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$\hbar^{-1}$) that signals the non-commutativity of the limit $t \to \infty$ and $\hbar \to 0$. This fact is not really taken as a threat to the correspondence principle. However, we will see that the amoeboid-like behavior involves a "coarse-grained distribution function", i.e. a point-test-distribution function averaged on rectangular rigid boxes of the phase space (see Section 5). This coarse-grain is used to get rid the complicated structure of the phase space which becomes more and more "scarred" as the relaxation proceeds (see [17, p. 7]). Moreover, for the case of a two dimensional phase space we will show a connection between the characteristic timescales of the quantum chaos and an adimensional parameter Ω which measures the degree of the deformation of the cells as the system evolves.

The paper is organized as follows. Section 2: we introduce the mathematical structures we will use. In the next sections we will see that the classical limit can be obtained using three weapons: decoherence, Wigner transformation and the limit $\frac{\hbar}{S} \to 0$. Section 3: we review the decoherence alla SID for non-integrable quantum systems. Section 4: we obtain the classical statistical limit, using Wigner transformation and the limit $\frac{\hbar}{S} \to 0$, and the classical surfaces defined by the constant of the motion in phase space. Section 5: deals the graininess of quantum mechanics. We find the classical trajectories for the integrable system and estimate the threat to the CP, in the non-integrable case. We show that, up to this point the threat to the CP can be suppressed if we take into account the characteristic timescales of quantum chaos. Moreover, we analyze how the fundamental graininess improves the statistical classical limit of Section 4. Section 6: we present our conclusions.

2. Mathematical background

In this section we will review, following Refs. [5,6], the main mathematical concepts we will use in these papers.

2.1. Weak limit

Our presentation is based on the algebraic formalism of quantum mechanics [18,19]. Let us consider an algebra $\mathcal{A}$ of operators, whose self-adjoint elements $O = O^{\dagger}$ are the observables belonging to the space $\mathcal{O}$. The states ρ are linear functionals belonging to the dual space $\mathcal{O}'$, but they must satisfy the usual conditions: self-adjointness, positivity and normalization and therefore the state ρ belongs to a convex $\mathcal{S}$. If $\mathcal{A}$ is a C*-algebra, it can be represented by a Hilbert space (GNS theorem see [19]). If $\mathcal{A}$ is a nuclear algebra, it can be represented by a rigged Hilbert space, as proved by a generalization of the GNS theorem [20,21]. In this case, the van Hove states with a singular diagonal can be properly defined (see [22]; for a rigorous presentation of the formalism, see also [23]).

If we write the action of the functional ρ on the space $\mathcal{O}$ as $(\rho|O)$, then we can say that:

- The evolution $U_t\rho = \rho(t)$ has a *Weak-limit* if, for any $O \in \mathcal{O}$ and any $\rho \in \mathcal{S}$, there is a unique $\rho_* \in \mathcal{S}$ such that

$$\lim_{t\to\infty}(\rho(t)|O) = (\rho_*|O), \quad \forall O \in \mathcal{O} \tag{1}$$

We will symbolize this limit as

$$W - \lim_{t\to\infty}\rho(t) = \rho_* \tag{2}$$

- A particular useful weak limit can be obtained using the Riemann–Lebesgue theorem. The idea of destructive interference is embodied in this theorem, according to which, if $f(v) \in \mathbb{L}_1$, then

$$\lim_{t\to\infty}\int_a^b f(v)\, e^{-ivt}\, dv = 0 \tag{3}$$

If we can express the action of a functional $\rho(t) \in \mathcal{S}$ on the operator $O \in \mathcal{O}$ as

$$(\rho(t)|O) = \int_a^b [A\,\delta(v) + f(v)]\, e^{-ivt}\, dv \tag{4}$$

with $f(v) \in \mathbb{L}_1$, then

$$\lim_{t\to\infty}(\rho(t)|O) = \lim_{t\to\infty}\int_a^b [A\,\delta(v) + f(v)]\, e^{-ivt}\, dv = A = (\rho_*|O), \quad \forall O \in \mathcal{O} \tag{5}$$

We will call this result "*Weak Riemann–Lebesgue limit*".

2.2. Generalized projections

As it is well known, in order to describe an irreversible process in terms of an unitary evolution it is necessary to break the underlying unitary evolution. The usual tool to do this is to introduce a coarse graining, that restricts the information of the system. But generically any information restriction can be obtained using a projection, which retains the "relevant" information and discards the "irrelevant" one of the considered system.

In fact, in its traditional form, the action of a projection is to eliminate some components of the state vector corresponding to the finest description (see [42]) to obtain a coarse grained one. If this idea is generalized, any restriction of information can be conceived as the result of a convenient projection. In fact, we can define a projector Π belonging to the space $\mathcal{O} \otimes \mathcal{O}'$ such that

$$\Pi \doteq \sum_j |O_j)(\rho_j| \tag{6}$$

where $(\rho_j| \in \mathcal{O}'$ satisfies $(\rho_j|O_k) = \delta_{jk}$ where $|O_k)\epsilon\mathcal{O}$.[1] Therefore, the action of Π on $\rho \in \mathcal{O}'$ involves a projection leading to a state ρ_P such that

$$\rho_P \doteq \rho\,\Pi = \sum_j (\rho|O_j)(\rho_j| \tag{7}$$

where in ρ_P only contains the information that we can obtain from the observables $|O_k)\epsilon\mathcal{O}$.

2.3. Weyl–Wigner–Moyal mapping

Let $\Gamma = \mathcal{M}_{2(N+1)} \equiv \mathbb{R}^{2(N+1)}$ be the phase space.The functions over Γ will be called $f(\phi)$, where ϕ symbolizes the coordinates of $\Gamma, \phi = (q^1, \ldots, q^{N+1}, p_q^1, \ldots, p_q^{N+1})$. If we

[1] In fact, Π is a projector since $\Pi^2 = \sum_{jk}|O_j)(\rho_j|O_k)(\rho_k| = \sum_{jk}|O_j)\delta_{jk}(\rho_k| = \sum_j|O_j)(\rho_j| = \Pi$.

consider the operators $\widehat{f},\widehat{g},\ldots\in\widehat{\mathcal{A}}$ and the candidates to be the corresponding distribution functions $f(\phi),g(\phi),\ldots\in\mathcal{A}$, where $\widehat{\mathcal{A}}$ is the quantum algebra of operators and $\mathcal{A}$ is the classical algebra of distribution functions, the *Wigner transformation* reads (see [43–45])

$$symb\widehat{f}\doteq f(\phi)=\int\langle q+\Delta|\widehat{f}\,|q-\Delta\rangle e^{2i\frac{p\Delta}{\hbar}}d^{N+1}\Delta \tag{8}$$

We can also introduce the *star product* (see [46]),

$$\begin{aligned}symb(\widehat{f}\;\widehat{g})&=symb\widehat{f}*symb\widehat{g}=(f*g)(\phi)\\&=f(\phi)\exp\left(-\frac{i\hbar}{2}\overleftarrow{\partial}_a\omega^{ab}\overrightarrow{\partial}_b\right)g(\phi)\end{aligned} \tag{9}$$

and the *Moyal bracket*,that is, the symbol corresponding to the quantum commutator

$$\{f,g\}_{mb}=\frac{1}{i\hbar}(f*g-g*f)=symb\left(\frac{1}{i\hbar}[f,g]\right) \tag{10}$$

It can be proved that (see [43])

$$\begin{aligned}(f*g)(\phi)&=f(\phi)g(\phi)+0(\hbar),\{f,g\}_{mb}\\&=\{f,g\}_{pb}+0(\hbar^2)\end{aligned} \tag{11}$$

To define the inverse $symb^{-1}$, we will use the *symmetrical* or*Weyl* ordering prescription, namely,

$$symb^{-1}[q^i(\phi),p^j(\phi)]\doteq\frac{1}{2}\left(\widehat{q}^i\widehat{p}^j+\widehat{p}^j\widehat{q}^i\right) \tag{12}$$

Therefore, by means of the transformations *symb* and $symb^{-1}$, we have defined an isomorphism between the quantum algebra $\widehat{\mathcal{A}}$ and the "classical-like" algebra $\mathcal{A}_q$,

$$symb^{-1}:\mathcal{A}_q\rightarrow\widehat{\mathcal{A}},\quad symb:\widehat{\mathcal{A}}\rightarrow\mathcal{A}_q \tag{13}$$

The mapping so defined is the *Weyl–Wigner–Moyal symbol*.[2]

The Wigner transformation for states is

$$\rho(\phi)=symb\,\widehat{\rho}=(2\pi\hbar)^{-(N+1)}symb_{(\text{for operators})}\,\widehat{\rho} \tag{14}$$

As it is well known, an important property of the Wigner transformation is that:

$$\begin{aligned}\langle\widehat{O}\rangle_{\widehat{\rho}}&=(\widehat{\rho}|\widehat{O})=(symb\,\widehat{\rho}\,|symb\,\widehat{O})\\&=\int d\phi^{2(N+1)}\rho(\phi)O(\phi)\end{aligned} \tag{15}$$

This means that the definition of $\widehat{\rho}\in\widehat{\mathcal{A}'}$ as a functional on $\widehat{\mathcal{A}}$ is equivalent to the definition of $symb\,\rho\in\mathcal{A}'_q$ as a functional on $\mathcal{A}_q$.

3. Decoherence in non-integrable systems

3.1. Local CSCO

This subsection is a short version of the corresponding subsection of paper [5].

a. In [5] we have proved that, when the quantum system is endowed with a CSCO of $N+1$ observables containing $\widehat{H}$, that defines an eigenbasis in terms of which the state of the system can be expressed, the corresponding classical system is *integrable*. In fact, if the CSCO is $\{\widehat{H},\widehat{G}_1,\ldots,\widehat{G}_N$, the Moyal brackets of its elements are

$$\{G_I(\phi),G_J(\phi)\}_{mb}=symb\left(\frac{1}{i\hbar}[\widehat{G}_I,\widehat{G}_J]\right)=0 \tag{16}$$

where $I,J=0,1,\ldots,N$, $\widehat{G}_0=\widehat{H}$, and $\phi\in\mathcal{M}\equiv\mathbb{R}^{2(N+1)}$. Then, when $\hbar\rightarrow0$, from Eq. (11) we know that

$$\{G_I(\phi),G_J(\phi)\}_{pb}=0 \tag{17}$$

Thus, since $H(\phi)=G_0(\phi)$, the set $\{G_I(\phi)\}$ is a complete set of $N+1$ constants of motion in involution, *globally* defined all over $\mathcal{M}$; as a consequence, the system is *integrable*.

b. We have also proved (see [5]) that, when the CSCO has $A+1<N+1$ observables, a local CSCO $\{\widehat{H},\widehat{G}_1,\ldots,\widehat{G}_A,\widehat{O}_{i(A+1)},\ldots,\widehat{O}_{iN}\}$ can be defined for a maximal domain $\mathcal{D}_{\phi_i}$ around any point $\phi_i\in\Gamma\equiv\mathbb{R}^{2(N+1)}$, where Γ is the phase space of the system. In this case the system is *non-integrable*.

In order to prove this assertion, we have to recall the *Carathèodory–Jacobi theorem* (see [47], Theorem 16.29) according to which, when a system with $N+1$ degrees of freedom has $A+1$ global constants of motion in involution $\{G_0(\phi),G_1(\phi),\ldots,G_A(\phi)\}$, then $N-A$ local constants of motion in involution $\{A_{i(A+1)}(\phi),\ldots,A_{iN}(\phi)\}$ can be defined in a maximal domain $\mathcal{D}_{\phi_i}$ around ϕ_i, for any $\phi_i\in\Gamma\equiv\mathbb{R}^{2(N+1)}$ (see also Section 3.2 below).

Let us consider the particular case of a classical system with $N+1$ degrees of freedom, and whose only global constant of motion (for simplicity) is the Hamiltonian $H(\phi)$. The Carathèodory–Jacobi theorem states that, in this case, the system has N local constants of motion $A_{il}(\phi)$, with $I=0,\ldots,N$, in the maximal domain $\mathcal{D}_{\phi_i}$ around ϕ_i, for any $\phi_i\in\Gamma$.

If we want to translate these phase space functions into the quantum language, we have to apply the transformation $symb^{-1}$; this can be done in the case of the Hamiltonian, $\widehat{H}=symb^{-1}H(\phi)$, but not in the case of the $A_{il}(\phi)$ because they are defined in a maximal domain $\mathcal{D}_{\phi_i}\subset\Gamma$ and the Weyl–Wigner–Moyal mapping can only be applied on phase space functions defined on the whole phase space Γ. To solve this problem, we can introduce a positive partition of the identity (see [48]),

$$1=I(\phi)=\sum_i I_i(\phi) \tag{18}$$

where each $I_i(\phi)$ is the *characteristic* or *index* function

$$I_i(\phi)=\begin{cases}1\text{ if }\phi\in D_{\phi_i}\\0\text{ if }\phi\notin D_{\phi_i}\end{cases} \tag{19}$$

and $D_{\phi_i}\subset\mathcal{D}_{\phi_i},D_{\phi_i}\cap D_{\phi_j}=\emptyset,\bigcup_i D_{\phi_i}=\Gamma$. Then we can define the functions $O_{il}(\phi)$ as

$$O_{il}(\phi)=A_{il}(\phi)I_i(\phi) \tag{20}$$

[2] When $\hbar\rightarrow0$, we have $\mathcal{A}_q\rightarrow\mathcal{A}$, where $\mathcal{A}$ is the classical algebra of observables over phase space.

Now the $O_{il}(\phi)$ are defined for all $\phi \in \Gamma$; so, we can obtain the corresponding quantum operators as

$$\widehat{O}_{il} = symb^{-1} O_{il}(\phi) \tag{21}$$

Since the original functions $A_{il}(\phi)$ are local constants of motion in the maximal domain $\mathcal{D}_{\phi_i}$, they make zero the corresponding Poisson brackets, with H, in such a domain and, a fortiori, in the non-maximal domain $D_{\phi_i} \subset \mathcal{D}_{\phi_i}$. This means that the $O_{il}(\phi)$ makes zero the corresponding Poisson brackets in the whole space Γ. In fact, for $\phi \in D_{\phi_i}$, because $O_{il}(\phi) = A_{il}(\phi)$, and trivially for $\phi \notin D_{\phi_i}$. We also know that, in the macroscopic limit $\hbar \to 0$, the Poisson brackets can be identified with the Moyal brackets, that is, the phase space counterpart of the quantum commutator (see Eq. (11)).[3] Therefore, we can guarantee that all the observables of the set $\left\{\widehat{H}, \widehat{O}_{il}\right\}$ commute with each other:

$$\left[\widehat{H}, \widehat{O}_{il}\right] = 0, \quad \left[\widehat{O}_{il}, \widehat{O}_{iJ}\right] = 0 \tag{22}$$

for $I, J = 1$ to N and in all the D_{ϕ_i}. As a consequence, we will say that the set $\left\{\widehat{H}, \widehat{O}_{i1}, \ldots, \widehat{O}_{iN}\right\}$ is the *local CSCO* of $N+1$ observables corresponding to the domain $D_{\phi_i} \subset \Gamma$. If $\widehat{H}$ has a continuous spectrum $0 \leqslant \omega < \infty$, and the $\widehat{O}_{il}$ a discrete one (just for simplicity) a generic observable $\widehat{O}$ can be decomposed as

$$\widehat{O} = \sum_{im_{il}\, m'_{il}} \int_0^\infty d\omega \int_0^\infty d\omega' \; \widetilde{O}_{im_{il}\, m'_{il}}(\omega, \omega')\, |\omega, m_{il}\rangle\langle\omega', m'_{il}| \tag{23}$$

where the $|\omega, m_{il}\rangle = |\omega, m_{i1}, \ldots, m_{iN}\rangle$ are the eigenvectors of the local CSCO $\left\{\widehat{H}, \widehat{O}_{il}\right\}$ corresponding to D_{ϕ_i}. Since it can be proved that (see [5]), for $i \neq j$,

$$\langle\omega, m_{il}\,|\omega, m_{jl}\rangle = 0 \tag{24}$$

the decomposition of Eq. (23) is orthonormal, and it generalizes the usual eigen-decomposition of the integrable case to the non-integrable case. Therefore, any $\widehat{O}_{il}$ corresponding to the domain D_{ϕ_i} commutes with any $\widehat{O}_{jl}$ corresponding to the domain D_{ϕ_j} with $i \neq j$,[4]

$$\left[\widehat{O}_{il}, \widehat{O}_{jJ}\right] = \delta_{ij}\,\delta_{IJ} \tag{25}$$

3.2. Continuity and differentiability.

In paper [8], we have used a "bump" smooth function $B_i(\phi)$, in each domain D_{ϕ_i} surrounded by a frontier zone F_{ϕ_i}, such that $D_i(\phi) \cup F_{\phi_i} \subset \mathcal{D}_i(\phi)$, and we have defined a new partition of the identity (compare with (18)),

$$1 = I(\phi) = \sum_i B_i(\phi) \tag{26}$$

where each $B_i(\phi) \geqslant 0$ satisfies (compare with (19))

$$B_i(\phi) = \begin{cases} 1 \text{ if } \phi \in D_{\phi_i} \\ \epsilon[0,1] \text{ if } \phi \notin F_{\phi_i} \\ 0 \text{ if } \phi \notin D_{\phi_i} \cup F_{\phi_i} \end{cases} \tag{27}$$

and $F_{\phi_i} \subset \mathcal{F} = \bigcup_i F_{\phi_i}$ is the union of all the joining zones (see Fig. 1(A)).[5] Then if we change the definition $O_{il}(\phi) = A_{il}(\phi) I_i(\phi)$ (compare (20)) by

$$O_{il}(\phi) = A_{il}(\phi) B_i(\phi)$$

we would have smooth connections between $\mathcal{F}$ through the functions $O_{il}(\phi)$.[6] Namely to work with continuos and differential functions force us to introduce continuity zones $\mathcal{F}$ and functions $B_i(\phi)$ in the frontier of the domains D_{ϕ_i} (Fig. 1(A)). Then we can use $C^r -$ *functions* (and eventually $C^\infty -$ *functions*) in the whole treatment (see [8]). For simplicity, up to now, we have not considered these $\mathcal{F}-$ zones, nevertheless we will be forced to use them in Section 6 (Fig. 6(A)).

Another kind of joining zones are used in the decomposition, in small square boxes, of a "cell" [10], i.e. the small boxes distributed in the "boundary of C" in figure 6.1 of the quoted book (see also between Eqs. (6.6) and (6.79) of this book). This figure corresponds to our Fig. 1(B). But, as the D_{ϕ_i} are neither boxes nor cells (that will be introduce in Section 5), $\mathcal{F}$ and the "boundary of C" are completely different concepts.

3.3. Decoherence

Let us consider a quantum system with a globally defined Hamiltonian $\widehat{H}$. In order to complete the CSCO, we can add constants of the motion locally defined as in the previous subsection. Thus, we have the CSCO $\left\{\widehat{H}, \widehat{O}_{il}\right\}$, with $I = 1$ to N and i corresponding to all the necessary domains D_{ϕ_i} obtained from the partition of the phase space Γ.

a. In paper [5] we have considered the case with continuous and discrete spectrum for $\widehat{H}$ and for the $\widehat{O}_{il}$. For the sake of simplicity in this paper we will only consider the continuous spectrum $0 \leqslant \omega < \infty$ for $\widehat{H}$ and discrete spectra $m_{il} \in \mathbb{N}$ for the $\widehat{O}_{il}$. Then in the eigenbasis of $\widehat{H}$, the elements of any local CSCO can be expressed as (see Eq. 23)

$$\widehat{H} = \sum_{im_{il}} \int_0^\infty \omega\, |\omega, m_{il}\rangle\langle\omega, m_{il}|\, d\omega \tag{28}$$

$$\widehat{O}_{iJ} = \sum_{im_{il}} \int_0^\infty m_{il}\, |\omega, m_{il}\rangle\langle\omega, m_{il}|\, d\omega \tag{29}$$

[3] Even if these reasoning is only valid in the limit $\hbar \to 0$ it is enough for our purposes since essentially we are trying to find classical limit.

[4] In this paper we have slightly changed the notation of paper [5], because we consider that the present notation is more explicit than the one.of that paper.

[5] Moreover, as we will discuss in Section 5, quantum phase space has a fundamental graininess. Then the width of $\mathcal{F}$ must be of the order that we will define in that section, i.e. it must contain a box of the size $\Delta x \Delta p = \frac{1}{2}\hbar$.

[6] In some cases it can be shown that the discontinuities in the boundary zones introduces a $0(\hbar^2)$, which vanishes when $\hbar \to 0$ and, therefore, in this cases, the Moyal brackets can be replaced with Poisson brackets in such a limit (see [8]).

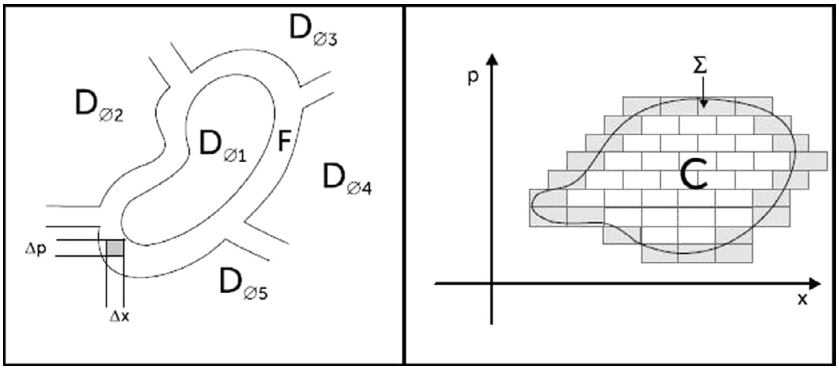

Fig. 1. (A) The domains and the frontier. $\Delta x \Delta p = \frac{1}{2}\hbar$. (B) A cell decomposed in small square boxes.

where m_{il} is a shorthand for $m_{i1}, \ldots, m_{iN}$, and $\sum_{im_{il}}$ is a shorthand for
$\sum_i \sum_{m_{i1}} \cdots \sum_{im_{iN}}$.
With this notation,

$$\widehat{H}\,|\omega, m_{il}\rangle = \omega\,|\omega, m_{il}\rangle, \quad \widehat{O}_{il}\,|\omega, m_{il}\rangle = m_{il}\,|\omega, m_{il}\rangle \tag{30}$$

where the set of vectors $\{|\omega, m_{il}\rangle\}$, with $I = 1$ to N and i corresponding to all the domain D_{ϕ_i}, is an orthonormal basis (see Eq. (24)), i.e.:

$$\langle \omega, m_{il}\,|\,\omega', m'_{il}\rangle = \delta(\omega - \omega')\,\delta_{m_{il}\,m'_{il}} \tag{31}$$

b. Also in the orthonormal basis $\{|\omega, m_{il}\rangle\}$, a generic observable reads (see Eq. 23)

$$\widehat{O} = \sum_{im_{il}\,m'_{il}} \int_0^\infty d\omega \int_0^\infty d\omega' \widetilde{O}_{im_{il}\,m'_{il}}(\omega, \omega')\,|\omega, m_{il}\rangle\langle\omega', m'_{il}| \tag{32}$$

where $\widetilde{O}_{im_{il}\,m'_{il}}(\omega, \omega')$ is a generic *kernel* or *distribution* in ω, ω'. As in paper [5], we will restrict the set of observables (i.e. we make a projection like those of Section 2.2 namely a generalized coarse-graining) by only considering the van Hove observables (see [22]) such that[7]

$$\widetilde{O}_{im_{il}\,m'_{il}}(\omega, \omega') = O_{im_{il}\,m'_{il}}(\omega)\,\delta(\omega - \omega') + O_{im_{il}\,m'_{il}}(\omega, \omega') \tag{33}$$

The first term in the r.h.s. of Eq. (33) is the *singular term* and the second one is the *regular term* since the $O_{im_{il}\,m'_{il}}(\omega, \omega')$ are "regular", i.e. $\mathbb{L}_2$, functions of the variable ω- ω'. Then we will call $\widehat{\mathcal{O}}$ the subspace of observable, of our algebra $\widehat{\mathcal{A}}$, with these characteristics. Moreover we can define a projector Π, as those of Section 2.2, that projects on $\widehat{\mathcal{O}}$. This projection will be our generalized coarse graining.
Therefore, the observables will read

$$\widehat{O} = \sum_{im_{il}\,m'_{il}} \int_0^\infty d\omega O_{im_{il}\,m'_{il}}(\omega)\,|\omega, m_{il}\rangle\langle\omega, m'_{il}| + \sum_{im_{il}\,m'_{il}} \int_0^\infty d\omega \int_0^\infty d\omega' O_{im_{il}\,m'_{il}}(\omega, \omega')\,|\omega, m_{il}\rangle\langle\omega', m'_{il}| \tag{34}$$

Since the observables are the self-adjoint operators of the algebra, $\widehat{O}^\dagger = \widehat{O}$, they belong to a space $\widehat{\mathcal{O}} \subset \widehat{\mathcal{A}}$ whose basis $\{|\omega, m_{il}, m'_{il}), |\omega, \omega', m_{il}, m'_{il})\}$ is defined as

$$|\omega, m_{il}, m'_{il}) \doteq |\omega, m_{il}\rangle\langle\omega, m'_{il}|,$$

$$|\omega, \omega', m_{il}, m'_{il}) \doteq |\omega, m_{il}\rangle\langle\omega', m'_{il}| \tag{35}$$

c. The states belong to a convex set included in the dual of the space $\widehat{\mathcal{O}}, \widehat{\rho} \in \widehat{\mathcal{S}} \subset \widehat{\mathcal{O}'}$. The basis of $\widehat{\mathcal{O}'}$ is $\{(\omega, m_{il}, m'_{il}|, (\omega, \omega', m_{il}, m'_{il}|\}$, whose elements are defined as functionals by the equations

$$\begin{aligned} (\omega, m_{il}, m'_{il}\,|\,\eta, n_{il}, n'_{il}) &\doteq \delta(\omega - \eta)\,\delta_{m_{il}\,n_{il}}\,\delta_{m'_{il}\,n'_{il}} \\ (\omega, \omega', m_{il}, m'_{il}\,|\,\eta, \eta', n_{il}, n'_{il}) &\doteq \delta(\omega - \eta)\,\delta(\omega' - \eta')\,\delta_{m_{il}\,n_{il}}\,\delta_{m'_{il}\,n'_{il}} \\ (\omega, m_{il}, m'_{il}\,|\,\eta, \eta', n_{il}, n'_{il}) &\doteq 0 \end{aligned} \tag{36}$$

and the remaining $(\bullet\,|\bullet)$ are zero. Then, a generic state reads

$$\widehat{\rho} = \sum_{im_{il}\,m'_{il}} \int_0^\infty d\omega \overline{\rho_{im_{il}\,m'_{il}}(\omega)}\,(\omega, m_{il}, m'_{il}| + \sum_{im_{il}\,m'_{il}} \int_0^\infty d\omega \int_0^\infty d\omega' \overline{\rho_{im_{il}\,m'_{il}}(\omega, \omega')}\,(\omega, \omega', m_{il}, m'_{il}| \tag{37}$$

where the functions $\overline{\rho_{im_{il}\,m'_{il}}(\omega, \omega')}$ are "regular", i.e. $\mathbb{L}_2$ functions of the variable ω- ω'. We also require that $\widehat{\rho}^\dagger = \widehat{\rho}$, i.e.,

$$\overline{\rho_{im_{il}\,m'_{il}}(\omega, \omega')} = \rho_{im'_{il}\,m_{il}}(\omega', \omega) \tag{38}$$

and that the $\rho_{im_{il}\,m_{il}}(\omega, \omega) \doteq \rho_{im_{il}}(\omega)$ would be real and non-negative, satisfying the total probability condition,

$$\rho_{im_{il}}(\omega) \geqslant 0, \quad tr\widehat{\rho} = (\widehat{\rho}|\widehat{I}) = \sum_{im_{il}} \int_0^\infty d\omega \rho_{im_{il}}(\omega) = 1 \tag{39}$$

where $\widehat{I} = \sum_{im_{il}} \int_0^\infty d\omega |\omega, m_{il}\rangle\langle\omega, m_{il}|$ is the identity operator in $\widehat{\mathcal{O}}$.

d. On the basis of these characterizations, the expectation value of any observable $\widehat{O} \in \widehat{\mathcal{O}}$ in the state $\widehat{\rho}(t) \in \widehat{\mathcal{S}}$ can be computed as

[7] In papers [7] we have shown that this choice does not diminish the physical generality of the model.

$$\langle\widehat{O}\rangle_{\widehat{\rho}(t)} = (\widehat{\rho}(t)|\widehat{O}) = \sum_{im_{il}\,m'_{il}} \int_0^\infty d\omega \overline{\rho_{im_{il}\,m'_{il}}(\omega)}\, O_{im_{il}\,m'_{il}}(\omega) + \sum_{im_{il}\,m'_{il}} \int_0^\infty d\omega \int_0^\infty d\omega' \overline{\rho_{im_{il}\,m'_{il}}(\omega,\omega')}\, e^{i(\omega-\omega')t/\hbar} \times O_{im_{il}\,m'_{il}}(\omega,\omega') \quad (40)$$

The requirement of "regularity", in variables $\omega-\omega'$, for the involved functions, i.e. $O_{im_{il}\,m'_{il}}(\omega,\omega') \in \mathbb{L}_2$ and $\overline{\rho_{im_{il}\,m'_{il}}(\omega,\omega')} \in \mathbb{L}_2$,. As a consequence of Schwartz inequality, it means that $\overline{\rho_{im_{il}\,m'_{il}}(\omega,\omega')}\, O_{im_{il}\,m'_{il}}(\omega,\omega') \in \mathbb{L}^1$ in the variable $\nu = \omega-\omega'$, a property that we will use below.

Now, for reasons that will be clear further on, it is convenient to choose a new basis $\{|\omega,p_{il}\rangle\}$ that diagonalize the m-variables of ρ (of Eq. (38)), for the case $\omega=\omega'$, through a unitary matrix U, which performs the transformation

$$\rho_{im_{il}\,m'_{il}}(\omega) \to \rho_{ip_{il}\,p'_{il}}(\omega)\,\delta_{p_{il}\,p'_{il}} \doteq \rho_{ip_{il}}(\omega) \quad (41)$$

Such transformation defines the new orthonormal basis $\{|\omega,p_{il}\rangle\}$, where p_{il} is a shorthand for $p_{i1},\ldots,p_{iN}$, and $p_{il}\in\mathbb{N}$. This basis corresponds to a new local CSCO $\left\{\widehat{H},\widehat{P}_{il}\right\}$. Therefore, in each D_{ϕ_i} we can deduce, from Eqs. (40) and (41), that the basis $\{|\omega,p_{il}\rangle\}$ corresponds to the basis of observables. i.e. $\{|\omega,p_{il}),|\omega,\omega',p_{il},p'_{il})\}$, defined as in Eq. (35) but with the indices p instead of m, and also to the corresponding basis for the states is $\{(\omega,p_{il}|,(\omega,\omega',p_{il},p'_{il}|\}$.

Then when the observables $\widehat{P}_{il}$ have discrete spectra, in the new basis the van Hove observables of our algebra $\widehat{\mathcal{A}}$ will read

$$\widehat{O} = \sum_{ip_{il}} \int_0^\infty d\omega O_{ip_{il}}(\omega)\,|\omega,p_{il}) + \sum_{ip_{il}\,p'_{il}} \int_0^\infty d\omega \int_0^\infty d\omega' O_{ip_{il}\,p'_{il}}(\omega,\omega')\,|\omega,\omega',p_{il},p'_{il}) \quad (42)$$

where the first term of the r.h.s is the *singular part* and the second terms the *regular part* of $\widehat{O}$. The states, in turn, will have the following form

$$\widehat{\rho} = \sum_{ip_{il}} \int_0^\infty d\omega \overline{\rho_{ip_{il}}(\omega)}\,(\omega,p_{il}| + \sum_{ip_{il}\,p'_{il}} \int_0^\infty d\omega \int_0^\infty d\omega' \overline{\rho_{ip_{il}\,p'_{il}}(\omega,\omega')}\,(\omega,\omega',p_{il},p'_{il}| \quad (43)$$

where, again, the first term of the r.h.s. is the *singular part* and the second one is the *regular part* of $\widehat{\rho}$.

From the last two equations we have

$$(\widehat{\rho(t)}|\widehat{O}) = \sum_{ip_{il}} \int_0^\infty d\omega \overline{\rho_{ip_{il}}(\omega)}\, O_{ip_{il}}(\omega) + \sum_{ip_{il}\,p'_{il}} \int_0^\infty d\omega \int_0^\infty d\omega' \overline{\rho_{ip_{il}\,p'_{il}}(\omega,\omega')}\, e^{i(\omega-\omega')t/\hbar}\, O_{ip_{il}\,p'_{il}}(\omega,\omega')$$

Then we can make the Riemann–Lebesgue limit to $(\widehat{\rho}|\widehat{O})$ since from the Schwartz inequality $O_{ip_{il}\,p'_{il}}(\omega,\omega'\overline{\rho_{ip_{il}\,p'_{il}}(\omega,\omega')}$ in $\mathbb{L}_1$ in $\nu=(\omega-\omega')$, the regular part vanishes and only the singular part remains:

$$W-\lim_{t\to\infty}\widehat{\rho}(t) = \sum_{ip_{il}} \int_0^\infty d\omega \overline{\rho_{ip_{il}}(\omega)}\,(\omega,p_{il}| = \widehat{\rho}_* \quad (44)$$

and we have decoherence in all the variables (ω,p_{il}) .

Here we have considered the case of observables $\widehat{P}_{il}$ with discrete spectra; the case of $\widehat{P}_{il}$ with continuos spectra is very similar (see [5]).

3.4. Comment

A comment is in order: Usually decoherence is studied in the case of open system surrounded by an environment, up to the point that some people believe that decoherence takes place in open systems. But also several authors have introduced, for different reasons, decoherence formalisms for closed system [24–33]. Related with the method used in this paper two important examples are given:

1. In paper [34], where a system that decoheres at high energy at the Hamiltonian basis is studied.
2. In paper [35], where complexity produces decoherence in a closed triangular box (in what we could call a Sinai–Young model).
   Also we have developed our own theory for decoherence of closed systems, SID (see [36–39]). In paper [40] we show how our formalism explains the decoherence of the Sinai–Young model above. Recently it has been shown that also the gravitational field produces decoherence in the Hamiltonian basis [41].

4. The classical statistical limit

In order to obtain the classical statistical limit, it is necessary to compute the Wigner transformation of observables and states. For simplicity and symmetry we will consider all the variables (ω,p_{il}) continuous in this section. If we do this substitution, Eq. (43), reads

$$\widehat{\rho}(t) = \sum_i \int_{p_{il}} dp_{il}^N \int_0^\infty d\omega \overline{\rho_i(\omega,p_{il})}\,(\omega,p_{il}| + \quad (45)$$

$$\sum_i \int_{p_{il}} dp_{il}^N \int_{p'_{il}} dp_{il}'^N \int_0^\infty d\omega \times \int_0^\infty d\omega' \overline{\rho_i(\omega,\omega',p_{il},p'_{il})}\, e^{i(\omega-\omega')t/\hbar}\,(\omega,\omega',p_{il},p'_{il}| \quad (46)$$

Therefore, Eq. (44) can be written as

$$W-\lim_{t\to\infty}\widehat{\rho}(t) = \widehat{\rho}_* = \sum_i \int_{p_{il}} dp_{il}^N \int_0^\infty d\omega \overline{\rho_i(\omega,p_{il})}\,(\omega,p_{il}| \quad (47)$$

where $\widehat{\rho}_*$ is simply the singular component of $\widehat{\rho}(t)$, where the regular part has vanished as a consequence of the Riemann–Lebesgue theorem.

Now, the task is to find the classical distribution $\rho_*(\phi)$ resulting from the Wigner transformation of $\widehat{\rho}_*$ in the limit $\hbar\to 0$,

$$\rho_*(\phi) = symb\, \widehat{\rho}_* \tag{48}$$

where

$$\rho_*(\phi) = symb\, \widehat{\rho}_* = \sum_i \int_{p_{il}} dp_{il}^N \int_0^\infty d\omega\, \overline{\rho_i(\omega, p_{il})} \quad symb\, (\omega, p_{il}| \tag{49}$$

So, the problem is reduced to compute $symb\ (\omega, p_{il}|$.

As it is well known, in its traditional form the Wigner transformation yields the correct expectation value of any observable in a given state when we are dealing with regular functions (see Eq. (15)). In previous papers [5,49] we have extended the Wigner transformation to singular functions in order to use it in functions like $(\omega, p_{il}|$. Here we will briefly resume the results of these papers in two steps: first, we will consider the transformation of observables and, second, we will study the transformation of states.

4.1. Transformation of observables

As we have seen (see Eq. (42)), our van Hove observables $\widehat{O} \in \widehat{\mathcal{O}}$ have a singular part, i.e. $\widehat{O}_S$, and a regular part, i.e. $\widehat{O._R}$. We will direct our attention to the singular operators $\widehat{O}_S$, since the regular operators $\widehat{O}_R$ "disappear" from the expectation values after decoherence, as explained in Section 2.3. $\widehat{O}_S$ reads:

$$\widehat{O}_S = \sum_i \int_{p_{il}} dp_{il}^N \int_0^\infty d\omega O_i(\omega, p_{il})\, |\omega, p_{il}) \tag{50}$$

Then, the Wigner transformation of $\widehat{O}_S$ can be computed as

$$O_S(\phi) = symb\, \widehat{O}_S \tag{51}$$

where

$$O_S(\phi) = symb\, \widehat{O}_S = \sum_i \int_{p_{il}} dp_{il}^N \int_0^\infty d\omega\, O_i(\omega, p_{il}) \quad symb\, |\omega, p_{il}) \tag{52}$$

Now if we consider that the functions $O_i(\omega, p_{il})$ are polynomials of functions of a certain space where the polynomials are dense it can be probed that

$$\widehat{O}_S = \sum_i O_{\phi_i}(\widehat{H}, \widehat{P}_{il}) = \sum_i \widehat{O_{S\phi_i}}$$

where $\widehat{O_{S\phi_i}} = O_{S\phi_i}(\widehat{H}, \widehat{P}_{il})$, and where $symb\widehat{O_{S\phi_i}} = symbO_{S\phi_i}(\widehat{H}, \widehat{P}_{il}) = O_{S\phi_i}(H(\phi), P_{il}(\phi)) + O(\hbar^2)$. Then if $O_{S\phi_i}(H(\phi), P_{il}(\phi)) = \delta(\omega - \omega')\, \delta^N(p_{il} - p'_{il})$ we have (see paper [8] for details) that the function $symb\,|\omega, p_{il})$ in the limit $\frac{\hbar}{S} \to 0$, is

$$symb\,|\omega, p_{il}) = \delta(H(\phi) - \omega)\, \delta^N(P_{il}(\phi) - p_{il}) \tag{53}$$

where $H(\phi) = symb\widehat{H}$ and $P_{il}(\phi) = symb\widehat{P_{il}}$

4.2. Transformation of states

As in papers [5,49], in order to compute the $symb$ $(\omega, p_{il}|$, we will define the Wigner transformation of the singular operator $\widehat{\rho}_S = \widehat{\rho}_*$ on the base of the only reasonable requirement that such a transformation would lead to the correct expectation value of any observable. Then we must postulate that it is (see Eq. 15),

$$(symb\, \widehat{\rho}_S\, |\, symb\, \widehat{O}_S) \doteq (\widehat{\rho}_S\, |\, \widehat{O}_S) \tag{54}$$

These equations must also hold in the particular case in which $\widehat{O}_S = |\omega', p'_{il})$, $\widehat{\rho}_S = (\omega, p_{il}|$, for some D_{ϕ_i} (see Eq. (24)) i.e.:

$$(symb(\omega, p_{il}|\ |\ symb\,|\omega', p'_{il})) = (\omega, p_{il}\,|\,\omega', p'_{il}) \tag{55}$$

and all the remaining cross terms are zero for any domain D_{ϕ_j}, with $j \neq i$. But from Eq. (53) we know how to compute $symb\,|\omega', p'_{il})$. Moreover, from the definition of the cobasis (see Eq. (36)) we know that

$$(\omega, p_{il}\,|\,\omega', p'_{il}) = \delta(\omega - \omega')\, \delta^N(p_{il} - p'_{il}) \tag{56}$$

Therefore in the limit $\frac{\hbar}{S} \to 0$ we have,

$$(symb(\omega, p_{il}|\ |\ \delta(H(\phi) - \omega')\, \delta^N(P_{il}(\phi) - p'_{il})) = \delta(\omega - \omega')\, \delta^N(p_{il} - p'_{il}) \tag{57}$$

Then in paper [5] we have proved that (always in the $\frac{\hbar}{S} \to 0$ limit)

$$symb(\omega, p_{il}| = \frac{\delta(H(\phi) - \omega)\, \delta^N(P_{il}(\phi) - p_{il})}{C_i(H, P_{il})} \tag{58}$$

where $C_i(H, P_{il})$ is the configuration volume of the region $\Gamma_{H,P_{il}} \cap D_{\phi_i}$, being $\Gamma_{H,P_{il}} \subset \Gamma$ the hypersurface defined by $H = const.$ and $P_{il} = const.$ In this way we have obtained the $symb$ of $|\omega, p_{il})$ and $(\omega, p_{il}|$ so the classical statistical limit is completed.

4.3. Convergence in phase space

Finally, we can introduce the results of Eq. (58) into Eq. (49), in order to obtain the classical distribution $\rho(\phi)$:

$$\rho_*(\phi) = \rho_S(\phi) = \sum_i \int_{p_{il}} dp_{il}^N \int_0^\infty d\omega\, \frac{\overline{\rho_i(\omega, p_{il})}}{C_i(H, P_{il})} \times \delta(H(\phi) - \omega)\, \delta^N(P_{il}(\phi) - p_{il}) \tag{59}$$

As a consequence, the Wigner transformation of the limits of Eq. (44) can be written as

$$W - \lim_{t\to\infty} \rho(\phi, t) = \rho_S(\phi) = \rho_*(\phi) = \sum_i \int_{p_{il}} dp_{il}^N \int_0^\infty d\omega\, \frac{\overline{\rho_i(\omega, p_{il})}}{C_i(H, P_{il})} \quad \delta(H(\phi) - \omega)\, \delta^N(P_{il}(\phi) - p_{il}) \tag{60}$$

Remember that all this is only valid in a domain D_ϕ defined in Eq. (19) and that it would completely change if we change to another domain through a continuity zone $\mathcal{F}$ of Section 3.2.

Then we have obtained a convincing classical limit of the states, that decomposed as in Eq. (60), it turns out to be sums of states peaked in the classical hypersurfaces of constant energy, $H(\phi) = \omega$, and where also the other constants of motions are constant, $P_{il}(\phi) = p_{il}$. This is an important step forward, to have obtained these classical surfaces as a limit of the quantum mechanics formalism.

Up to here chaos has not produced any problem to the CP, even if the system is not integrable. The real problems will begin in the next section.

5. Graininess

This section is devoted to a brief review of the graininess in quantum mechanics and some of its approaches. First we start with the standard approach from the viewpoint of the Kolmogorov–Sinai entropy and its quantum variants. Then we introduce our alternative approach using cells of the phase space that are deformed as the system evolves and where are all physical magnitudes are coarse-grained on domains of minimal size in the sense of the Indetermination Principle.

5.1. The standard approach

The two properties of classical mechanics necessary for chaos to occur are a continuous spectrum and a continuous phase space [17]. On the other hand, the most quantum systems which present chaotic features in its classical limit have discrete spectrum. In addition, the Correspondence Principle CP implies the transition from quantum to classical mechanics for all phenomena including the chaos. However, by the Indetermination Principle in quantum mechanics we have a discretized, and therefore non-continuous, phase space divided in elementary cells of finite size $\Delta x \Delta p \geqslant \hbar$ (per freedom). Then the natural question that arises is: How can we provide a quantum formalism consistent with the Indetermination principle and the CP to explain the emergence of chaos in the classical limit?

This is where the treatment of the graininess arises as a possible answer to the problem. The graininess has several approaches that try to solve the problem without being a threat to the CP. The "natural" possibility of accomplishing this could be the quantization of chaotic systems, but due the compactness of its phase space the quantization yields discrete energy spectrum. Then the situation does not seem so simple at first glance and one must look for other indicators that somehow capture the main properties related to the continuous spectrum of chaotic systems. The Kolmogorov–Sinai entropy [50] (KS-entropy) is perhaps the most significant and robust indicator, both in theory and applications. Roughly speaking, one reason why this is so is due one can model the behavior of classical chaotic systems of continuous spectrum from classical discretized models such that the KS-entropies of the continuous system and of the discrete ones tend to coincide for a certain appropriate range. We recall that the KS-entropy assigns measures to bunches of trajectories and computes the Shannon-entropy per time-step of the ensemble of bunches in the limit of infinitely many time-steps and the Pesin theorem [51] links the KS-entropy with the Lyapunov coefficients. For a quantum description of the chaotic systems, we would need a quantum extension of the KS-entropy. There are several non-commutative candidates [52–56] and the presence of a finite time interval where these KS-extensions yield the KS-entropy is considered as the main peculiarity of quantum chaos [17]. Therefore the issue of graininess is intimately related to quantum chaos timescales and must necessarily be compatible with the restriction to these. Three time scales characterizing the classically chaotic quantum motion are distinguished: The relaxation time scale, the random time scale and the logarithmic breaking time. Only for regular classical limits classical and quantum mechanics are expected to overlap over times t such that

$$t \lesssim t_R \propto \hbar^{-\alpha} \quad \text{for some} \quad \alpha > 0 \tag{61}$$

where t_R is the relaxation time scale which determines the so-called semi-classical regime, i.e. the time scale where the phenomena like the exponential localization and the relaxation can occur. Moreover the discrete spectrum cannot be solved if $t \lesssim t_R$ (see p. 12 of [17]). The breaking time scale (random time scale) τ is much shorter than t_R and is related to a stronger chaotic property, the exponential instability. Basically, τ determines the time interval where the wave-packet motion is as random as the classical trajectory and the time for the spreading of the packet is given by

$$\tau \sim \ln \frac{q}{\hbar} \propto -\log \hbar \tag{62}$$

where q is the quasiclassical parameter which is of the order of the characteristic value of the action value (see p. 14 of [17]). The importance of the logarithmic breaking time τ is that this indicates the typical scaling for a joint time-classical limit suited to classically chaotic quantum systems. We should mention that some authors, see [17], consider that τ is a satisfactory resolution between of the apparent contradiction between the CP and the quantum transient (finite-time) given by t_r and the evidence that time and classical limits do not commute. That is,

$$\lim_{|t|\to\infty} \lim_{q\to\infty} \neq \lim_{q\to\infty} \lim_{|t|\to\infty} \tag{63}$$

where the first order leads to classical chaos and the second one represents a quantum behavior with no chaos at all (see p. 17 of [17]).

Then if we define[8] $q = \frac{S}{\hbar}$ we could claim that the classical statistical limit of the Section 4 (i.e. $t \to \infty, \frac{\hbar}{S} \to 0$) is quite similar to the double limit of the right hand of Eq. (63). In Section 5.4 we will discuss this situation taking into account the graininess and the quantum chaos timescales. In the next two sections we introduce our graininess approach considering phase space cells as the starting point.

5.2. Our approach: fundamental graininess with cells

In this section we describe our approach of the graininess. As we mentioned in the introduction the key is to average a point-test-distribution function on minimal rectangular boxes of the phase space. The motivation of this approach lies in the fact that we can obtain a classical limit (and its limitations) searching the trajectories of the rectangular boxes (and later of the cells) we will consider as "points", integrating the Heisenberg equation, and then

[8] Where S denotes the action.

studying the deformations of the cells under the motion (as in [10]).

In Section 4.3 we have found the hypersurfaces where the classical trajectories lay. Now we want to find the classical motions in these trajectories. Thus we need to define the notion of "*a point that moves*". But in quantum mechanics there is not such a thing. In fact it is well known that the commutation relations and its consequence, the indetermination principle, establishes a fundamental graininess in the "quantum phase space". Precisely if we call $\widehat{J}$ and $\widehat{\Theta}$ two generic conjugated operators (e. g. in our case $\widehat{J}$ will be the constants of the motion $\widehat{H}, \widehat{P}_{il}$ and $\widehat{\Theta}$ the corresponding configuration operators) we have

$$[\widehat{\Theta}, \widehat{J}] = i\hbar\widehat{I} \tag{64}$$

and therefore

$$\Delta_\Theta \Delta_J \geqslant \frac{\hbar}{2} \tag{65}$$

where, from now on, Δ_Θ and Δ_J are defined as the variances of some *typical* state $\widehat{\rho}$ the one with the smallest dimensions we can "determinate" (in the sense of Ballentine chapter 8 [59]) in our experiment. With different choices for this $\widehat{\rho}$ we will obtain different ratios Δ_Θ/Δ_J but the qualitative results will be the same. Then we will consider that the rectangular box $\Delta_\Theta\Delta_J$ of volume $\hbar$ (or the polyhedral box of volume $\hbar^{(n+1)}$ in the many dimensions case) will be the smallest volume that we can determinate with our measurement apparatus, precisely:

$$vol\Delta_\Theta\Delta_J = \hbar \text{ (or eventually } N_0\hbar)$$

for a phase space of two dimensions or

$$vol\prod\Delta_\Theta\prod\Delta_J = \hbar^{(n+1)} \text{ (or eventually } N_0\hbar^{(n+1)})$$

for a phase space of $2(n+1)$ dimensions, where N_0 is not a very large natural number (cf. [10]). This is the new feature of the "quantum phase space": its graininess and this fact will be the origin of the threat to the CP.[9]

In Omnès book [10] the cells produced by the fundamental graininess are described in the (x,p) coordinates, using a mathematical theory, the microlocal analysis, based in the work [60]. In our formalism we will change these $(,p)$ for the (J,Θ) coordinates where J are the constants of the motion and Θ the corresponding configuration variables and where the commutation relations (64) and their consequence the indetermination principle (65) will play the main role.

To see how the fundamental graininess works let us consider a closed simply connected set of a two dimensional phase space that we will call a cell C^T, with its continuous boundary B, (Fig. 1(B), or Fig. 6.1 of [10]). The coordinates (J,Θ) and a lattice of rectangular boxes $\Delta_\Theta\Delta_J$ (eventually $2(n+1)$ polyhedral boxes) define the two domains related with $C^T : \Sigma$, set of boxes that intersect B, and C, the set of the interior rectangular boxes of the cell C^T. Volume is well defined in phase space of any dimension while (hyper) surfaces are not defined, so in order to compare the size of the frontier with the size of the interior we can define the adimensional parameter

$$\Omega = \frac{vol\Sigma}{volC}$$

It is quite clear that $\Omega \ll 1$ corresponds to a bulky cell while $\Omega \gg 1$ corresponds to an elongated and maybe deformed cell. It is also almost evident that if we want that a cell would somehow represent a real point it is necessary that $\Omega < 1$, because if $\Omega > 1$ the volume of the interior C is smaller than the volume of the "frontier" Σ, where we do not know for certain if its points belong or not to C^T since $B \subset \Sigma$. Thus in the case $\Omega \gg 1$ we completely lose the notion of real point and the description of the classical trajectories, as the motion of C^T, becomes impossible.

Analogously Omnès defines semiclassical projectors for each cell and shows that if Ω is very large the definition of these projectors lose all its meaning and the classicality is lost, namely he obtains a similar conclusion.

In the next section we will consider the cells and their evolution in several cases and we will estimate their corresponding Ω. From now, in all cases where the quasiclassical parameter $q = \frac{S}{\hbar}$ is finite it should be noted that we mean "a threat to the CP" to the outside time range of validity of our graininess approach according to the CP and should not be necessarily associated with the emergence of the non-commutative two limits given by Eq. (63). Furthermore, since the timescales considered in our graininess approach will be finite then there is no way that any of the two limits of Eq. (63) appear. All this will be discussed in the next section.

5.3. The classical trajectories

Up to this point we have obtained the classical distribution $\rho_*(\phi) = \rho_S(\phi)$ to which the system converges in phase space. This distribution defines hypersurfaces $H(\phi) = \omega, P_{il}(\phi) = p_{il}$ corresponding to the constant of the motion i.e. our the "momentum" variables. But such a distribution does not define the trajectories of "points" on those hypersurfaces, i.e., it does not fix definite values for the "configuration" variables (the variables canonically conjugated to $H(\phi)$ and $P_{il}(\phi)$). This is reasonable to the extent that definite trajectories would violate the uncertainty principle. In fact we know that, if $\widehat{H}$ and $\widehat{P}_{il}$ have definite values, then the values of the observables that do non-commute with them will be completely undefined.

As in Section 5.2, let us call, $\widehat{J}$ the "momentum" variables $\widehat{H}$ and $\widehat{P}_{il}$ (constants of the motion), and $\widehat{\Theta}$ the corresponding conjugated "configuration" variables, all of them defined in the domain D_{ϕ_i}. The equations of motion, in the Heisenberg picture, read

$$\frac{d\widehat{J}}{dt} = \frac{i}{\hbar}[\widehat{H}, \widehat{J}] \qquad \frac{d\widehat{\Theta}}{dt} = \frac{i}{\hbar}[\widehat{H}, \widehat{\Theta}] \tag{66}$$

where as $[\widehat{H}, \widehat{J}] = 0$

$$\frac{d\widehat{J}}{dt} = 0 \qquad \frac{d\widehat{\Theta}}{dt} = \frac{i}{\hbar}[\widehat{H}, \widehat{\Theta}] \tag{67}$$

Within the domain D_ϕ we know that if we can consider the $\widehat{H}$ as a function (or a convergent sum) of the $\widehat{J}$, i.e.:

[9] Fundamental graininess appears in many other disguises (see [57,58], etc.)

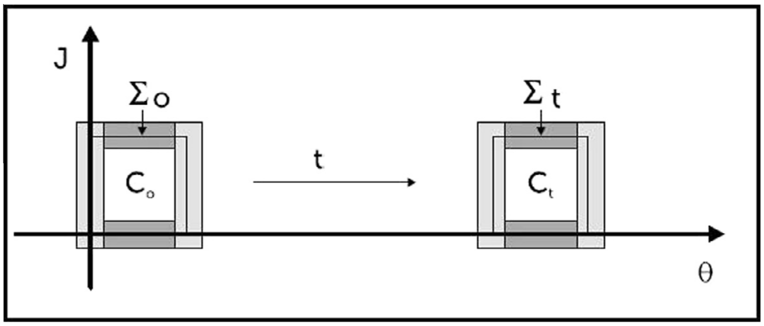

Fig. 2. Evolution of a cell with constant velocity.

$$\widehat{H} = F(\widehat{J}) = \sum_n a_n \widehat{J}^n \tag{68}$$

and since $[\widehat{\Theta}, \widehat{J}] = i\hbar \widehat{I}$ we have $[\widehat{\Theta}, \widehat{J}^n] = in\hbar \widehat{J}^{(n-1)}$ so

$$[\widehat{H}, \widehat{\Theta}] = \frac{d\widehat{H}}{d\widehat{J}}$$

where $\widehat{H}$ and $\widehat{J}$ are constant in time, so calling $\widehat{V}(0) = \frac{d\widehat{H}}{d\widehat{J}}$, which is another constant in time, we have

$$\widehat{J}(t) = \widehat{J}(0), \widehat{\Theta}(t) = \widehat{\Theta}(0) + \widehat{V}(0)t$$

Then we can make the Wigner transformation from these equations and, since this transformation is linear, we have

$$J(\phi, t) = J(\phi, 0), \Theta(\phi, t) = \Theta(\phi, 0) + V(\phi, 0)t \tag{69}$$

We will use this equation to follow the motion of the boxes and the cells in the phase space:

Let us first consider a *rectangular* (eventually $2(n+1)$ polyhedral)moving box of size $\Delta_\Theta \Delta_J$ with $\Delta_\Theta \Delta_J \sim \hbar$ (eventually $\hbar^{(n+1)}$), that we will symbolize by a small square in Figs. 2–4 (and just by a point in the Figs. 6(A) and 6.B) and let us also consider the *typical* point-test-distribution function $symb\widehat{\rho} = \rho(\phi) = \rho(j, \theta)$, (see under Eq. (65), also from now on $\phi = (j, \theta)$) with support contained in $\Delta_\Theta \Delta_J$, then let us define the mean values

$$\overline{j(t)} = \int_{\Delta_\Theta \Delta_J} J(j, \theta, t)\rho(j, \theta) dj d\theta; \overline{\theta(t)} = \int_{\Delta_\Theta \Delta_J} \Theta(j, \theta, t)\rho(j, \theta) dj d\theta,$$

$$\overline{v(t)} = \int_{\Delta_\Theta \Delta_J} V(j, \theta, t)\rho(j, \theta) dj d\theta; \tag{70}$$

where the $\rho(j, \theta)$ is not a function of the time since we are in the Heisenberg picture and

$$\begin{aligned} J(j, \theta, t) &= symb\ \widehat{J}(t) \\ \Theta(j, \theta, t) &= symb\ \widehat{\Theta}(t) \\ V(j, \theta, t) &= symb\ \widehat{V}(0) \end{aligned} \tag{71}$$

Now using Eq. (69) we have

$$\bar{j}(\phi, t) = \bar{j}(\phi, 0), \overline{\theta}(\phi, t) = \overline{\theta}(\phi, 0) + \overline{v}(\phi, 0)t \tag{72}$$

so our minimal rectangular box moves along a classical trajectory of our system.

Now our rectangular boxes are so small that we can not even consider their possible deformation. Precisely the Indetermination Principle makes this deformation merely hypothetical. Thus, from now on, we will consider that the rectangular boxes *are not in motion (and therefore they can not be deformed by motion)* and that they are the most elementary theoretical fixed notion of a point at $(\bar{j}, \overline{\theta})$.[10] In this way we have obtained the classical trajectories of theoretical points (i.e. Eq. (72)) and we would have completed our quantum to classical limit (apparently CP is safe up to now).

But remember that the real physical points are not these rectangular boxes but the cells with $\Omega < 1$ that we must also consider, because real measurement devices cannot see the elementary rectangular boxes but bigger cells of dimensions far bigger than the Planck ones. In the next examples we will see what happens with these cells that we will consider as real points: the cells can be deformed by the motion (while the rectangular boxes always remain rigid). We will show the interplay of these theoretical points (boxes) and physical real points (cells) in some examples bellow:

1. Then, as a first example, let us consider a two dimensional space within a domain D_ϕ (much larger than the cell that we will define below) and let as also consider the system of coordinates (J, Θ) and the corresponding trajectories when the Hamiltonian is a linear function, $\widehat{H} = a_0 \widehat{I} + a_1 \widehat{J}$, Then $\widehat{V} = a_1 \widehat{I}$ so

$$J(\phi, t) = J(\phi, 0), \Theta(\phi, t) = \Theta(\phi, 0) + a_1 It$$

and, with the same reasoning as above the trajectories of the boxes (theoretical points) are

$$\bar{j}(t) = \bar{j}(0), \overline{\theta}(t) = \overline{\theta}(0) + a_1 t \tag{73}$$

Namely we obtain the Fig. 2 and we have a uniform translation motion with constant velocity $\overline{v}[\bar{j}(0)]$ along all the trajectories. Let us then consider two parallel lines with constant velocities $\overline{v}[\bar{j}_1] = \overline{v}[\bar{j}_2]$, thus the difference of velocities is

$$\overline{v}(\bar{j}_1) - \overline{v}(\bar{j}_2) = 0 \tag{74}$$

Then if we consider an initial rectangular cell the motion will not deform the cell. Since there is no deformation of the cell Ω is rigid, thus if $\Omega < 1$ in the initial cell Ω will

[10] The rectangular moving cell defined after Eq. (69) will be the only rectangular objects that moves in this paper.

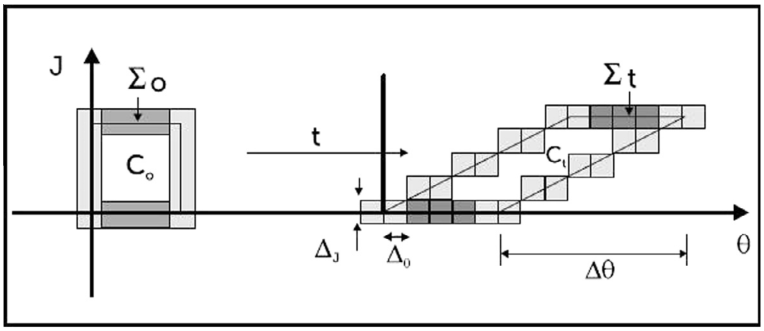

Fig. 3. Evolution of the cell with linear velocity.

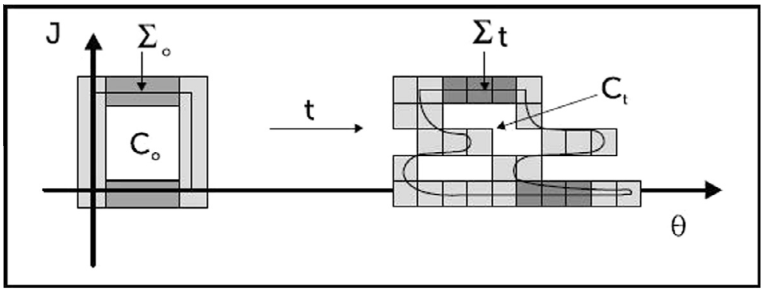

Fig. 4. Evolution of the cell with non linear velocity.

be < 1 in any transferred cell. Therefore, in this trivial case the cell will represent a physical real point moving according to Eq. (73). Thus in this case we have completed our classical limit and the CP is safe.

2. As a further example let us consider the same two dimensional space within a D_ϕ and let as consider the system of coordinates (J, Θ) and the corresponding trajectories when $\widehat{H} = a_0\widehat{I} + a_1\widehat{J} + a_2\widehat{J}^2$. Then $\widehat{V} = a_1\widehat{I} + a_2\widehat{J}$ so
$$J(\phi, t) = J(\phi, 0), \Theta(\phi, t) = \Theta(\phi, 0) + [a_1 I + 2a_2 J(\phi, 0)]t$$
and, with the same reasoning as above
$$\bar{j}(t) = \bar{j}(0), \overline{\theta}(t) = \overline{\theta}(0) + [a_1 + 2a_2\bar{j}(0)]t$$

Namely we obtain the Fig. 3 and we have a uniform motion with constant velocity $\overline{v}[\bar{j}(0)] = a_1 + 2a_2\bar{j}(0)$ along straight lines parallel to the axis θ. That is,
$$\overline{\theta}(t) = \overline{\theta}(0) + \overline{v}[\bar{j}(0)]t$$

Let us then consider two parallel lines with constant velocities $\overline{v}(\bar{j}_1) \neq \overline{v}(\overline{j_2})$, thus the difference of velocities is
$$\overline{v}(\bar{j}_1) - \overline{v}(\overline{j_2}) = 2a_2(\bar{j}_1 - \overline{j_2}) = v \tag{75}$$

Let J, Θ be the dimension of the initial cell and Δ_J, Δ_Θ the dimension of the fix rectangular boxes. Then the length of the basis is constant and so $volC$ also is constant. Then if we consider an initial rectangular box the motion will deform this cell in a parallelogram, where the height continue to be J and the base will now be $\Theta + \Delta\theta$, i.e. there is "elongation" $\Delta\theta$ (see Fig. 3), precisely
$$\Delta\theta = vt$$

Let us compute the evolution of Ω in this case: the number of new boxes that appears at time t will be
$$n = 2\frac{\Delta\theta}{\Delta_\Theta} = 2\frac{vt}{\Delta_\Theta} \tag{76}$$
Now
$$\Omega = \frac{vol\Sigma}{volC} = \frac{vol\Sigma + \Delta vol\Sigma}{volC} = \frac{vol\Sigma + n\Delta_J\Delta_\Theta}{volC} \tag{77}$$
so
$$\Delta\Omega = \frac{n\Delta_J\Delta_\Theta}{volC} = \frac{n\hbar}{volC} = 2\frac{v}{\Delta_\Theta}\frac{\hbar}{volC}t$$
$$= 2\frac{\Delta\theta}{\Delta_\Theta}\frac{\hbar}{volC} > 0 \tag{78}$$
Then:

a. The increment $\Delta\Omega$ is proportional to the time t.
b. It is also proportional to the product of the ratio of the elongation $\Delta\theta$ measured in units of Δ_Θ.
c. Finally it is proportional to $\frac{\hbar}{volC}$ so in the macroscopic limit $\frac{\hbar}{volC} \to 0$ we have $\Delta\Omega \to 0$ and the threat to CP disappears.

But the most important conclusion is that, in a generic case, even if $\frac{\hbar}{volC}$ would be small but if it is far from the limit $\frac{\hbar}{volC} \to 0$, after enough time we will have $\Omega \gg 1$. Then the cell ceases to be a good model for a point and it surely is the beginning of threat to the CP. This happens even if the system is integrable, namely, $D_\phi = \Gamma$ the phase space, and the Hamiltonian $\widehat{H} = a_0\widehat{I} + a_1\widehat{J} + a_2\widehat{J}^2$, e. g., simply be $\widehat{H} = \frac{1}{2m}\widehat{P}^2$, namely the one of a free particle. So fundamental graininess alone (with no chaos) can be a threat to the CP, in the case $\frac{\hbar}{volC} > 0$.

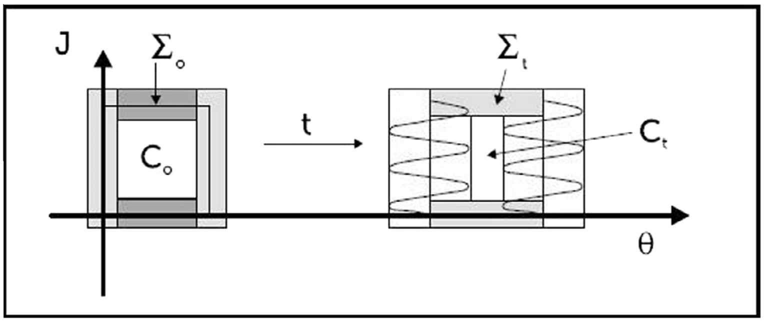

Fig. 5. Evolution of the cell with periodical velocity.

3. In the most general case the Hamiltonian is $\widehat{H} = a_0\widehat{I} + a_1\widehat{J} + a_2\widehat{J}^2 + a_3\widehat{J}^3 + \ldots$ and Eq. (74) becomes

$$\overline{v}(\bar{j}_1) - \overline{v}(\bar{j}_2) = 2a_2(\bar{j}_1 - \bar{j}_2) + 3a_3(\bar{j}_1^2 - \bar{j}_2^{\,2}) + \ldots \tag{79}$$

as described in Fig. 4 where there are not vertical deformations but there are strong horizontal ones. Then for Hamiltonians with power bigger than 2 the threat of chaos begins.
In fact, let us consider the case

$$\widehat{H} = \sum_{n=0}^{\infty} A_n(\widehat{J}) e^{in\frac{\widehat{I}}{\Delta_J}}$$

then

$$\overline{v(\bar{j})} = \sum_{n=0}^{\infty}\left[A'_n(\bar{j}) + \frac{in}{\Delta_J}A_n(\bar{j})\right] e^{in\frac{\bar{j}}{\Delta_J}} = \sum_{n=0}^{\infty} B_n(\bar{j}) e^{in\frac{\bar{j}}{\Delta_J}}$$

and

$$\overline{\theta}(j,t) = \overline{\theta}(j,0) + \overline{v}(\bar{j})t = \overline{\theta}(\bar{j},0) + t\sum_{n=0}^{\infty} B_n(\bar{j}) e^{in\frac{\bar{j}}{\Delta_J}}$$

Then the elongation will be

$$\Delta\theta = t\sum_{n=0}^{\infty} B_n(\bar{j}) e^{in\frac{\bar{j}}{\Delta_J}}$$

Let us consider the simple case $B_m(\bar{j}) = const \neq 0$ and all other $B_n(\bar{j}) = 0$ (Fig. 5), then

$$\Delta\theta = tB_m e^{\frac{\bar{j}}{\Delta_J}} \quad so \quad Re(\Delta\theta) = tB_m \cos\left(m\frac{\bar{j}}{\Delta_J}\right)$$

and the wave longitude of the oscillation of the vertical boundary curves is $\lambda = \frac{\Delta_J}{m}$ and we can have $\lambda \ll \Delta_J$ if $m \gg 1$. Then we have

$$\Delta\Omega = \frac{\Delta vol\Sigma}{volC} = 2\frac{J\Delta\theta}{J\Theta} = 2\frac{B_m}{\Theta}t$$

So when $t \to \infty$ then $\Delta\Omega \to \infty$, and we have a real threat to the CP with no redemption in the classical limit. And this can happen even in a not chaotic case since we can have $D_\phi = \Gamma$.[11]

4. But things get really worst if, instead of one D_ϕ, we consider two D_{ϕ_1} and D_{ϕ^2} and their joining zone $\mathcal{F}$, as in Fig. 6(A). Precisely let us suppose that in $D_{\phi 1}$ we have two parallel motions and only a parallelogram deformation as in point 2, and we use the (θ, j) coordinate of D_{ϕ_1}. But neither in $\mathcal{F}$ nor in D_{ϕ_2} the just quoted coordinate j is a constant of the motion, so in D_{ϕ_2} the motion becomes completely deformed as shown in the Fig. 6(A). Then if the motion goes through several joining zones $\mathcal{F}$ it is clear that the initial regular cell will become the amoeboid object of Fig. 6(B), where of course $\Omega \gg 1$. Remember that, for the sake of simplicity, the points of all these Fig. 6(A) and (B) have a volume $\hbar$(or really $\hbar^{(n+1)}$ in the general case). Then when, as a consequence of chaos, the volume of the complex details of the amoeboid figure becomes of the order of $\hbar$(or $\hbar^{(n+1)}$ in the general case) the classical limit representing the notion the original cell becomes meaningless as a result of chaos. Moreover in this case we could speculate that the square box becomes strongly deformed. But this kind of reasonings are forbidden by the Indetermination Principle and because in our treatment square boxes are considered rigid.
Another way to see that there is a real problem is to consider that the classical motion of the center of the initial cell (where the probabilities to find the particle are different from zero) as the real classical motion of a classical particle. Then in the chaotic case it may happen that at time t, the cell would get the amoeboid shape of Fig. 6(B). Now the center of the original cell turns out to be outside of the amoeboid figure. Then this center is in a zone of zero probability and cannot represent the motion of a real point-like classical particle anymore.
So chaos and fundamental graininess are a real threat to the classical limit of quantum mechanics and so for its interpretation.

Example (*The Henon–Heiles system and the high energy problem*). In the case of Henon–Heiles classical system [61, p. 121] with Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2 + x^2 + y^2) + x^2y - \frac{1}{2}y^3$$

We can observe that:

[11] In if the cases 1, 2, and 3 we would take the $\widehat{H}$ as the free variable we would have $\widehat{J} = F^{-1}(\widehat{H})$, and, in the corresponding figures, $H = const.$ would appear in the vertical axis, and t, in the horizontal one, with the same qualitative results.

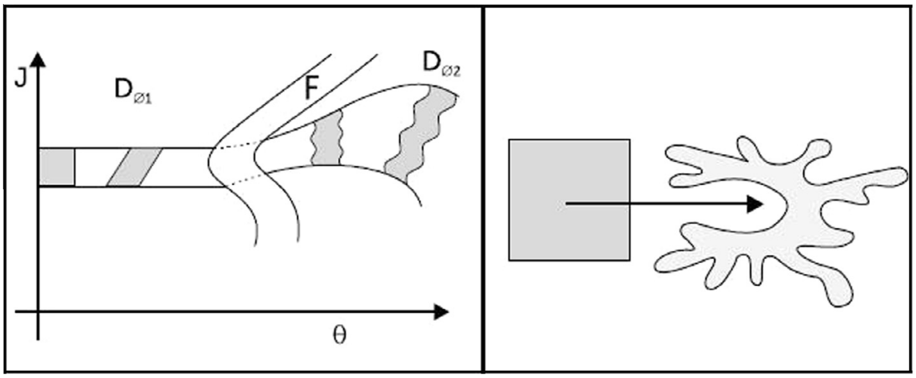

Fig. 6. (A) A square cell scattered by a frontier. (B) A square cell becomes an ameoboidal cell.

a. The Hamiltonian is non integrable so in the whole phase space we will find something like Fig. 6(A).
b. For energies $E = \frac{1}{12}$ (figure 44a of [61]) the tori are practically unbroken, as in case 3 above. But in large D_ϕ and in a physical case most likely $volD_\phi \gg \hbar^2$ and CP could be far from having practical problems with chaos at least for short periods of time. These D_ϕ become smaller for $E = \frac{1}{8}$ (figure 44b of [61]) and probably very tiny for $E = \frac{1}{6}$ (figure 44c of [61]) so in such cases we may have serious problems with chaos (i.e. those of case 4) since for real high energy we could have $volD_\phi \approx \hbar^2$. We can obtain these conclusions because our method allows us to evaluate the $volD_\phi$ on the surface defined by the constant of motion (tori) from the Poincaré sections.

So we conclude that when the D_ϕ in phase space are of the order of $\hbar$ CP has real problems. But also we see that for high energy there is not a generic well defined "high energy limit". The threat of chaos to the CP is thus explained. Moreover this example introduces the threat of chaos to the high energy limit. In the next section we analyze how the threat of the chaos to the CP can be suppressed taken into account the relationship between our graininess approach and the characteristic timescales of quantum chaos.

5.4. Timescales and graininess

As we mentioned in Section 5.1 the graininess must be compatible with the quantum chaos timescales within which the typical phenomena as the statistical relaxation, the exponential localization and more generally, the instability of motion can occur. These timescales are an attempt to reconcile the discrete spectrum with the CP where the distinction between the discrete and continuous spectrum becomes relevant only for large times $t \to \infty$, see p. 9 of [17].

In this section, from our graininess approach we study the relations that can be obtained for the quantum chaos timescales. In Section 5.3 we have seen that the condition $\Omega < 1$ represents the allowed range where the notion of real point and the description of the classical trajectories become possible. The main idea is that $\Omega \ll 1$ (bulky cell) implies a temporal range of validity of the fundamental graininess which can be identified with some of the characteristic timescales of quantum chaos. As in the first example of Section 5.3, let us consider a two dimensional space within a domain D_ϕ and the conjugated coordinates (J, Θ) with a Hamiltonian

$$\widehat{H} = \sum_{n=0}^{\infty} a_n \widehat{J}^n \tag{80}$$

In such case the difference of velocities is (see Eq. (79))

$$v = \overline{v}(\bar{j}_1) - \overline{v}(\overline{j_2}) = \sum_{n=1}^{\infty} n a_n \widehat{J}^{(n-1)}$$
$$= 2a_2(\bar{j}_1 - \overline{j_2}) + 3a_3(\bar{j}_1^2 - \overline{j_2}^2) + \dots \tag{81}$$

On the other hand the evolution of Ω can be given in terms of the number of new boxes $n = n(t)$ that appear at time t (see Eqs. 77 and (78))

$$\Delta\Omega = \frac{n(t)\Delta_J \Delta_\Theta}{volC} = \frac{n(t)\hbar}{volC} \tag{82}$$

We initially assume we have a bulky cell, i.e. $\Omega \ll 1$. In order to obtain the characteristic timescales we only need to consider two cases: 1) Linear velocity and 2) nonlinear velocity. Let Ω' the value of Ω at time t. Then by Eq. (77) we have

$$\Omega' = \Omega + \Delta\Omega \tag{83}$$

Since $\Omega \ll 1$ if we impose that $\Omega' = \Omega + \Delta\Omega \lesssim 1$, i.e. the allowed range of the graininess, then this condition becomes into

$$\Delta\Omega \lesssim 1 \tag{84}$$

That is,

$$\frac{n(t)\hbar}{volC} \lesssim 1 \tag{85}$$

Let us see that Eq. (85) contains the different timescales according to the form of the Hamiltonian of Eq. (80). When the velocity is linear we have $a_n = 0$ for all $n \geqslant 3$ in the Hamiltonian given by Eq. (80). In such case we can replace Eq. (76) in Eq. (85) to obtain

$$2\frac{v}{\Delta_\Theta}\frac{\hbar}{volC}t \lesssim 1 \tag{86}$$

Now since v, Δ_Θ and $volC$ are fixed, from Eq. (86) we have

$$t \lesssim \left(\frac{\Delta_\Theta}{2v}volC\right)\hbar^{-1} = t_R \propto \hbar^{-1} \tag{87}$$

Therefore we have obtained the relaxation timescale $t_R = (\frac{\Delta_\Theta}{2v}volC)\hbar^{-1}$ for the case of a Hamiltonian $\widehat{H} = a_0\widehat{I} + a_1\widehat{J} + a_2\widehat{J}^2$ which is consistent with the so-called "semiclassical regime" of the regular classical limits (with no chaos). In other words, for two dimensional systems our approach of the graininess plus the condition $\Delta\Omega \lesssim 1$ implies a temporal range of validity of the graininess given by the relaxation timescale $t_R = (\frac{\Delta_\Theta}{2v}volC)\hbar^{-1}$ for the quadratic Hamiltonian case and viceversa. Moreover, from these arguments and Section 5.3 it follows that there is threat to the CP only for times $t > t_R$ which are outside of the range of validity of the fundamental graininess.

Let us see what happens in the other case, i.e. when the Hamiltonian is $\widehat{H} = \sum_{n=0}^{\infty} a_n\widehat{J}^n$ with $a_n \neq 0$ for some $n \geqslant 3$. As we mentioned in the Example 3 of the Section 5.3 there are only strong horizontal deformations of the cells (see Figs. 4 and 5). This case includes the exponential instability where the wave-packet motion is as random as the classical trajectory and the packet is exponentially spreading with a classical rate h (see p. 14 of [17]). So we can reasonably assume,[12] hypothetically, that the number of new boxes $n = n(t)$ that appear at time t is proportional to $\exp(\frac{t}{h})$ as the packet spreads, i.e.

$$n = n(t) \propto e^{\frac{t}{h}} \tag{88}$$

Then if we replace Eq. (88) in (85) we have

$$\frac{e^{\frac{t}{h}}\hbar}{volC} \lesssim 1 \tag{89}$$

Now applying logarithm to both sides of Eq. (89) we obtain

$$\frac{t}{h} + \log\left(\frac{\hbar}{volC}\right) \lesssim 0 \tag{90}$$

That is,

$$t \lesssim -h\log\left(\frac{\hbar}{volC}\right) = \tau \propto -\log\hbar \tag{91}$$

The time scale $\tau = -h\log(\frac{\hbar}{volC})$ corresponds to the logarithmic breaking time where classical and quantum mechanics agree for quantum systems with a chaotic classical behavior. In this case fundamental graininess is a real threat to CP as $t > \tau$. Given that $\tau < t_R$ then we see that the nonlinear velocity case (i.e. $a_n \neq 0$ for some $n \geqslant 3$) restricts the time range more than the linear velocity case (with no chaos). Therefore we conclude that the chaos increases the threat to the CP.

5.5. Classical statistical limit and graininess

We conclude with a brief discussion about the classical statistical limit of the Section 4 and its relation with the non-commutative double limit of Eq. (63). According to Eq. (47) the classical statistical limit requires the asymptotic limit $t \to \infty$ and the limit $\frac{\hbar}{S} \to 0$ (see Eq. (58)) plus the "graininess compatibility relation" (see Eq. (84) or (85)) to guarantee that there is no threat to the CP. However, we have seen that the graininess compatibility relation leads to the different timescales of quantum chaos. Therefore, following the research line of [17] p. 18 we should be take the two limits simultaneously but keeping the ratio $\frac{t}{t_R(q)}$ or $\frac{t}{\tau(q)}$ fixed where q is the quasiclassical parameter given by $q = \frac{volC}{\hbar}$.[13] From Eqs. (87) and (91) we have

$$t_R(q) = \left(\frac{\Delta_\Theta}{2v}\right)q^{-1} \tag{92}$$

and

$$\tau(q) = -h\log q \tag{93}$$

In other words, if we take into account the graininess we must to rewrite the classical statistical limit of Eq. (60) according to

$$W - \lim_{t,q\to\infty,\ t\lesssim t_R(q)\ \text{or}\ \tau(q)} \rho(\phi, t) = \rho_S(\phi) = \rho_*(\phi) = \tag{94}$$

$$\sum_i \int_{p_{il}} dp_{il}^N \int_0^\infty d\omega\, \frac{\overline{\rho_i(\omega, p_{il})}}{C_i(H, P_{il})} \quad \delta(H(\phi) - \omega)\,\delta^N(P_{il}(\phi) - p_{il}) \tag{95}$$

In this manner the classical statistical limit is always compatible with the graininess and the CP is safe in all cases, regular and chaotic. On the other hand if we only take the limit $t, q \to \infty$ then we fall into the ambiguity of the non-commutative double limit given by Eq.(63) which, as we have seen in the Sections 5.3 and 5.4, represents a threat to the CP for times that are outside of the time range of the graininess, i.e. when $t > t_R(q)$ or $t > \tau$.

6. Conclusions

In this paper we have:

1. Presented a new formalism to study the classical limit of quantum mechanics.
2. Showed that somehow fundamental graininess alone is a threat to the CP unless the timescales of quantum chaos are taken into account (Section 5.4).
3. Demonstrated how chaos increases this threat.
4. Proved that these threats which compromise the high energy limit of quantum mechanics can be suppressed if we identify the bulky cell condition $\Omega \ll 1$ with the quantum chaos timescales (Section 5.4).
5. Found a non trivial connection between the characteristic timescales of quantum chaos and the fundamental graininess that allowed us to redefine a statistical classical limit that is compatible with the CP and the fundamental graininess (Section 5.5).

[12] Here we are considering that the exponential spreading implies an exponential elongation of the cell as it evolves, see Fig. 4.

[13] We assume the action S proportional to $volC$ which is the volume of a given initial cell.

Table 1
Fundamental graininess, statistical classical limit, and their relationships.[a]

Statistical classical limit (only)	Fundamental graininess (only)	Fundamental graininess + Statistical classical limit
$\lim_{q\to\infty}\lim_{t\to\infty}$ (quantum behavior with no chaos at all)	Undefined classical limit	$\lim_{t,q\to\infty,\ t\lesssim t_R(q)\ or\ \tau(q)}$ (double limit taken simultaneously, chaotic quantum motion)
Infinite relaxation time $t_R = \infty$	Finite relaxation time $t_R(q) = \left(\frac{\Delta_\Theta}{2v}\right)q^{-1}$ (two dimensional phase space)	Finite relaxation time $t_R(q) = \left(\frac{\Delta_\Theta}{2v}\right)q^{-1}$ (two dimensional phase space)
Undefined timescales	Defined quantum chaos timescales $t_R(q)$ and $\tau(q)$ (two dimensional phase space)	Defined quantum chaos timescales $t_R(q)$ and $\tau(q)$ (two dimensional phase space)
Threat to the CP	No threat to the CP	No threat to the CP
$\rho_*(\phi) = \lim_{q\to\infty}\lim_{t\to\infty}\rho(\phi,t)$ non compatible weak limit with the CP	Undefined weak limit	$\rho_*(\phi) = \lim_{t,q\to\infty,\ t\lesssim t_R(q)\ or\ \tau(q)}\rho(\phi,t)$ compatible weak limit with the fundamental graininess and the CP

[a] By "undefined" we mean the absence of this element within the formalism.

We conclude that to avoid the threat of chaos and fundamental graininess to the CP is necessary to take into account the characteristic timescales of quantum chaos. As we mentioned before, these timescales are an alternative solution to the ambiguity of the non commutative double limit

$$\lim_{t\to\infty}\lim_{q\to\infty} \neq \lim_{q\to\infty}\lim_{t\to\infty} \tag{96}$$

where q is the quasiclassical parameter (see p. 17 of [17]). More precisely, the mathematical need to take the limit $t\to\infty$ in the statistical classical limit (see Section 4) and in asymptotic theories (e.g. ergodic theory) imply a simultaneously and conditional double limit that solves the apparent contradiction between the CP and the quantum transient pseudochaos (see p. 18 of [17]). In our fundamental graininess approach this contradiction emerged in a geometrical way studying the domains of definition of the constants of the motion (in the considered non-integrable system), the corresponding broken tori at different energies and the behavior of the cells for different Hamiltonians (as in case 1,2, and 3 of Section 5.3). In Section 5.5 considering an initial bulky cell $\Omega \ll 1$, the compatibility condition $\Delta\Omega \lesssim 1$ and taking into account the statistical classical limit of Section 4 we translated these finite time intervals of "quantum pseudochaos" to a classical limit that is compatible with the CP and the general structure of classically chaotic quantum motion (see Fig. 5 of [17]). In this sense we conclude that the fundamental graininess plus the statistical classical limit provide a new formalism to study the classical limit that is compatible with the CP and the quantum chaos timescales. In the next table we summarize these results.

From the Table 1 we can see how the fundamental graininess and the statistical classical limit complement their indefinite sectors (rows) to give rise to a better classical limit that is compatible with the fundamental graininess and where the CP is safe (third column). Also, it should be noted that at least for the quadratic Hamiltonian case $\widehat{H} = a_0\widehat{I} + a_1\widehat{J} + a_2\widehat{J}^2$ (linear velocity, see example 2 of Section 5.3), the results of the 1 can be generalized for an phase space of any finite dimension. Consider that the dimension of the phase space is $2D$. Let $\Delta J_1,\ldots,\Delta J_D$, $\Delta\Theta_1,\ldots,\Delta\Theta_D$ be the size of the fix rectangular boxes. Then following the arguments of the Example 2 of Section 5.3 we have an "elongation" at time t

$$\Delta\theta_1\Delta\theta_2\ldots\Delta\theta_D = vt \tag{97}$$

where $v = 2a_2(\bar{j}_1 - \bar{j}_2) = \overline{v}(\bar{j}_1) - \overline{v}(\bar{j}_2)$, see Eq. (75). In this case we have "elongations" in each of the D directions, i.e. for each direction j with $j = 1,\ldots,D$ we have a stretch like Fig. 3. Therefore, the increment $\Delta\Omega$ at time t will be[14]

$$\Delta\Omega = n\frac{\Delta J_1\ldots\Delta J_D\Delta\Theta_1\ldots\Delta\Theta_D}{volC} = n\frac{\hbar^D}{volC} \tag{98}$$

where $n = n(D)$ is the number of new boxes that appears at time t which depends on the dimension D of the phase space. Moreover, as in the Example 2 of Section 5.3. since the velocity is linear then n is proportional to the time t. Then we have

$$n = \alpha(D,\Delta J_1,\ldots,\Delta J_D,\Delta\Theta_1,\ldots,\Delta\Theta_D)t \quad \textit{for some } \alpha \in \mathbb{R} \tag{99}$$

Now, by replacing Eq. (99) in Eq. (98) and assuming the graininess condition $\Delta\Omega \lesssim 1$ (see Eq. (84)) we obtain

$$\Delta\Omega = \alpha t\frac{\hbar^D}{volC} \lesssim 1 \quad \Rightarrow \quad t \lesssim (\alpha^{-1}volC)\hbar^{-D} = t_R \propto \hbar^{-D} \tag{100}$$

which is the relaxation timescale t_R for the case of a quadratic Hamiltonian with a phase space of dimension 2D.

Finally, taking into account the fundamental graininess and based in these results we could go on with the following speculation: In the classical level, the KAM theorem was the solution of the problem of the scarcity of chaos in the solar system, since the tori were broken but not badly broken. In the same way we could consider that the study of the size of the D_{ϕ_i}, for different levels of energy, could also explain the behavior of chaotic quantum systems and may be the scarcity of chaos in these systems. I.e. it may be that, many cases, the D_{ϕ_i} would be large enough to endow these systems with a quasi-integral chaotic behavior Along these lines we will continue our research.

[14] Here we use that $\Delta J_1\ldots\Delta J_D\Delta\Theta_1\ldots\Delta\Theta_D \sim \hbar^D$ which represents a small square in a 2D-dimensional phase space.

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