

## Condensed exciton polaritons in a two-dimensional trap: Elementary excitations and shaping by a Gaussian pump beam

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An exciton-polariton condensate (EPC) confined in a parabolic two-dimensional trap is considered theoretically. In the realistic limit of weakly interacting polaritons, the nonlinear term in the Gross-Pitaevskii equation describing the properties of the condensate can be considered as a perturbation with respect to the trapping potential, which allows for a convenient analytical description of the EPC ground state and Bogolyubov-type elementary excitations around it. The excitation modes with the energies and wave functions depending on the polariton-polariton coupling strength are derived for the condensate, neglecting interaction with uncondensed polaritons can be neglected. The energies of these modes are shown to be almost equidistant, even for a rather strong polariton-polariton interaction inside the condensate. This makes lateral parabolic traps promising candidates for realization of bosonic cascade lasers based on exciton polaritons. Another physical scenario is also considered where the interaction with a reservoir of uncondensed polaritons is more important than that inside the EPC. In this case, it is shown that the condensate is “reshaped” by the repulsive interaction with the reservoir, namely, pushed out from the center of the trap in real space and blue-shifted in energy, in agreement with the results obtained in a number of recent experiments.

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### I. INTRODUCTION

In recent years, the Bose-Einstein condensation of microcavity exciton polaritons has been demonstrated<sup>1</sup> and its dramatic effects on the properties of this two-dimensional matter-light system have been shown.<sup>2–10</sup> Free electrons and holes in a semiconductor quantum well (QW) are created by pumping with a beam of photons, usually of an energy well above the QW band gap. The electrons and holes relax into lower energy states, form excitons that couple to the microcavity photons, and occupy the lowest exciton-polariton states. These mixed light-matter bosons can eventually condense.<sup>1</sup> Even though the short polariton lifetime permits only the formation of a quasiequilibrium steady state, in which photons escaping from the cavity must be continuously replenished by the external pump,<sup>8</sup> exciton-polariton condensates (EPCs) can propagate over macroscopic distances outside the excitation area while preserving their spatial coherence.<sup>6</sup> The repulsive exchange interaction between QW excitons of the same spin orientation results in a repulsive potential acting on the condensate, induced by photogenerated excitons within the excitation area.<sup>1</sup> If the pumped spot is located within the condensate, the polaritons feel an outward force and the EPC expands.<sup>10</sup> On the contrary, if the reservoir of uncondensed excitons is spatially separated from the condensate, the repulsive potential allows for Bose-Einstein condensate (BEC) localization in a trap with optically controlled dimensions.<sup>6,9</sup>

Alternatively, the Bose-Einstein condensate of exciton polaritons in a semiconductor microcavity can be confined in a lateral trap.<sup>11–14</sup> A two-dimensional (2D) parabolic trap can be induced by local elastic strain, shifting excitonic states downwards in energy.<sup>11,12</sup> Interestingly, the repulsive

potential induced by uncondensed excitons created by two laser spots also appears parabolic along the line between the spots, thus forming a (one-dimensional) parabolic trap for EPC.<sup>10</sup> Moreover, symmetric parabolic traps can be created for exciton polaritons by a ring-shaped optical excitation: repulsive interaction between the optically injected excitons on the ring and the condensate of exciton polaritons is expected to induce the Bose-Einstein condensation in the bottom of the trap. Full optical control of the condensate would be achieved if the injection of polaritons into the trap could be provided by a second laser beam, additional to that creating the trap. In particular, such configuration is important for realization of bosonic cascade lasers recently described theoretically.<sup>15</sup> Such experiments have not yet been realized, to the best of our knowledge. Here we aim at providing a theoretical analysis that would help in designing new experiments on the realization of bosonic cascades in lateral polariton traps.

In this work we consider a polariton condensate excited nonresonantly in an external parabolic trap. We analyze the Bogolyubov-type elementary excitations of the condensate and its shape modified by a Gaussian pump beam focused in the center of the trap. Because of the parabolic confinement of the condensate, the elementary excitations are also localized, i.e., they are not characterized by a certain wave number, in contrast with those considered for unconfined EPCs (see, for example, Ref. 16). While in most of the previous works the theoretical approach was based on numerical solution of a generalized Gross-Pitaevskii equation,<sup>3,5,7,8,10,16</sup> here we use a semianalytical perturbation theory approach. In the realistic limit of weakly interacting polaritons, the nonlinear term in the Gross-Pitaevskii equation<sup>17</sup> (GPE) can be considered as a perturbation with respect to the trap potential, which allows

for a convenient analytical description of the EPC ground state.<sup>18</sup> In the next section, the generalized time-dependent GPE and its simplified limiting forms are discussed. In Sec. III, Bogolyubov-type elementary excitations<sup>19</sup> in a parabolically confined EPC are considered. The effect of the repulsive potential produced by uncondensed excitons generated by a pump beam focused in the center of the trap is considered and discussed in Sec. IV. The last section is devoted to conclusions.

## II. MEAN-FIELD DESCRIPTION OF EPC

Within the mean-field theory approach, the steady-state and dynamics of an exciton-polariton condensate confined in a potential trap are determined by the time-dependent Gross-Pitaevskii equation.<sup>17</sup> In the past, several theoretical works were published, devoted to the description of the collective excitations in atomic condensates within the GPE framework.<sup>20–24</sup> For instance, in Ref. 22 Bogolyubov excitations in two-species condensates were considered and it was shown that the interspecies interaction leads to a metastable state of the alkali-metal condensate. Ruprecht *et al.*<sup>23</sup> studied the matter wave response and the resonance conditions of the cooled alkali-metal atoms under a sinusoidal perturbation of the trapping potential; the phenomenological damping method<sup>24</sup> was employed to describe the damping of excitations at thermal equilibrium. Applying the GPE to the condensates formed by lower branch exciton polaritons, several problems have been tackled, such as the formation of vortices,<sup>25</sup> solitons,<sup>26</sup> and fluid dynamics,<sup>27</sup> among others. A generalized GPE, which includes local interactions between the excitons, was derived in Ref. 28 within the local density approximation, and the nonlinear dynamic effects in coupled quantum wells were studied, showing the appearance of a turbulent state. This approach seems to be valid at temperatures much lower than the Bose-Einstein condensation transition temperature under small trap potential.

We shall consider the time-dependent complex GPE which includes the loss ( $F$ ) and generation ( $R$ ) terms, and the repulsive interaction with uncondensed excitons (reservoir):<sup>16</sup>

$$i\hbar\partial_t\Psi = \left[ -\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) + i\hbar(R - F) + g|\Psi|^2 + V_{\text{res}}(\mathbf{r}) \right] \Psi. \quad (1)$$

Here  $g$  is the polariton-polaritons interaction parameter,  $m$  is the polariton mass, and  $V(\mathbf{r}) = \frac{1}{2}m\omega_0^2(x^2 + y^2)$  is the parabolic trap potential.<sup>29</sup> The last term in Eq. (1) describes the repulsive interaction of the condensate with the reservoir close to the pump spot, which is proportional to the number of uncondensed polaritons ( $N_r$ ). We assume that they are generated by a laser beam focused in the center of the trap, so this potential is axial-symmetric.

Assuming the cw excitation regime, i.e., putting  $(R - F) = \text{const}$ , and using the transformation

$$\Psi = \exp[(R - F)t]\psi,$$

we obtain

$$i\hbar\partial_t\psi = \left[ -\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}) + g \exp[2(R - F)t]|\psi|^2 + V_{\text{res}} \right] \psi. \quad (2)$$

It is important to note that the dynamic stability condition of the condensate described by Eq. (2) requires that  $R \leq F$ . Equation (2) can be conveniently rescaled by introducing  $l_0 = \sqrt{\hbar/m\omega_0}$ ,  $\Lambda = gmN_p/\hbar^2$ ,  $(x, y) = l_0(\xi, \eta)$ , and  $\psi = \sqrt{N_p}\Phi/l_0$ , with  $N_p$  denoting the number of polaritons in the condensate.

Starting from Eq. (2), we can investigate different physical scenarios of the evolution of the condensate by considering two limiting cases described below.

(I) Under cw conditions,  $R \approx F$ , and for low pump power, such that the number of polaritons in the reservoir is small compared to the condensate,  $N_r \ll N_p$ , the repulsive interaction with the reservoir can be neglected,  $V_{\text{res}} \approx 0$ . Thus we recover the usual stationary GPE by substituting  $\psi(\rho, t) = \exp(-i\mu t/\hbar)\Phi_0(\rho)$ :

$$[\hat{H}_0 + \Lambda|\Phi_0|^2 - \bar{\mu}]\Phi_0 = 0. \quad (3)$$

Here  $\hat{H}_0 = -\frac{1}{2}\nabla^2 + \frac{1}{2}\rho^2$ , the operator  $\nabla$  is defined in terms of the dimensionless coordinates  $\xi$  and  $\eta$ ,  $\rho^2 = \xi^2 + \eta^2$ ,  $\bar{\mu} = \mu/(\hbar\omega_0)$ , and  $\mu$  is the chemical potential. This situation will be considered in the next section where the Bogolyubov-type elementary excitations in EPC uncoupled from the reservoir will be derived in terms of the dimensionless interaction parameter  $\Lambda = g/(\hbar\omega_0)$ . For exciton polaritons in a semiconductor microcavity, this coefficient can be expressed as<sup>30</sup>  $\Lambda = (\alpha_1 + \alpha_2)mN_p/(2\hbar^2)$ , where  $\alpha_2 = -\alpha_1(E_{\text{ex-ph}} + \delta)/(2E_{\text{ex-ph}})$  and  $\alpha_1 = 6E_b a_b^2 |X|^2$  denote the interaction constants for excitons with antiparallel and parallel spins, respectively. Here,  $X = 1/\sqrt{1 + [\hbar\Omega_R/(\varepsilon_{LP} - \varepsilon_{\text{ph}})]^2}$  is the excitonic Hopfield coefficient,<sup>28,31</sup>  $\Omega_R$  denotes the Rabi splitting,  $\varepsilon_{LP}$  ( $\varepsilon_{\text{ph}}$ ) is the lower polariton branch energy (cavity photon energy),  $E_{\text{ex-ph}}$  stands for the exciton-photon coupling energy,  $E_b$  is the exciton binding energy,  $a_b$  is the exciton Bohr radius, and  $\delta$  denotes the exciton-photon detuning. In Ref. 18 the range of applicability of the perturbation theory has been reported for the GPE (3):

$$|(\alpha_1 + \alpha_2)N_p| < 6\hbar^2/m, \quad (4)$$

i.e., the model is valid in the weak interaction limit. Using typical parameters of a GaAs/AlGaAs microcavity,<sup>32</sup> the variation of the coefficient  $\Lambda/N_p$  on exciton-photon detuning  $\delta$  is displayed in Fig. 1(a). From this figure, by comparison with Eq. (4) one can conclude that the perturbation theory can be applied in the detuning interval,  $-10 \text{ meV} < \delta < 3$  if the number of particles  $N_p$  is smaller than  $10^4$ , but for  $3 \text{ meV} < \delta < 7 \text{ meV}$   $N_p$  it can be as large as  $10^5$ – $10^6$ .

(II) A qualitatively different situation occurs when the intensity of the beam generating the polariton reservoir is so high that the number of uncondensed polaritons becomes large and the interaction between the EPC and the reservoir cannot be neglected anymore. The limiting case here is the domination of the condensate-reservoir repulsion over the internal interactions within the EPC. Under the conditions

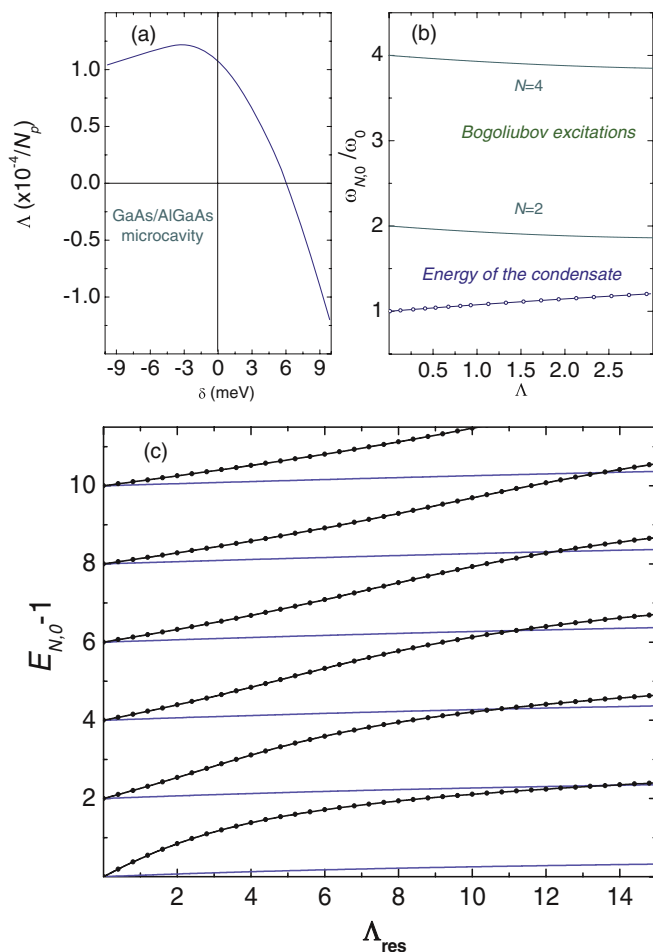


FIG. 1. (Color online) (Panel a) Dependence of the dimensionless polariton-polariton interaction parameter  $\Lambda$ , normalized to the number of particles  $N_p$ , on the detuning  $\delta$  in a GaAs/GaAlAs microcavity. (Panel b) Bogoliubov excitation's spectrum  $\omega_{N,0}$  dependence on  $\Lambda$  for  $N = 2$  and  $4$ . Solid line with open dots shows the reduced ground-state energy of the condensate  $\epsilon$ . (Panel c) Normalized energy levels  $E_{N,0}$  are the solutions of Eq. (10) for the EPC coupled to uncondensed polaritons created at the center of the trap as a function of the laser excitation power ( $\Lambda_{\text{res}}$ ). Solid lines are for  $a = 0.2l_0$  and solid lines with full circles correspond to  $a = l_0$ .

$N_r \gg N_p$  and  $R < F$ , the nonlinear term in Eq. (2) can be neglected in comparison with  $V_{\text{res}}$ , after a certain time,  $t \sim |R - F|^{-1}$ .<sup>33</sup> In this case the EPC can be described by a linear partial differential equation, including the repulsion and the 2D parabolic confinement. The effect of the interaction with the reservoir on the EPC shape will be considered in Sec. IV. The main purpose of the present work is to describe both limiting cases within the same approach and to compare the excitation spectrum dependence on the EPC parameters of the microcavity.

### III. BOGOLIUBOV EXCITATIONS

First, we consider the cw regime with  $V_{\text{res}} = 0$ , where Eq. (2) is transformed into the time-dependent GPE with  $R = F$ . Applying Bogolyubov's method,<sup>19</sup> the collective excitations with frequencies  $\omega$  are obtained by expressing the

solution as

$$\psi(\rho, t) = \exp\left(-\frac{i\mu t}{\hbar}\right) \{\Phi_0(\rho) + \delta\Phi[u, v^*]\} \quad (5)$$

and linearizing it in terms of the amplitudes  $u(\rho, t)$  and  $v^*(\rho, t)$ .

Owing to the axial symmetry, the excitation modes can be classified according to the  $z$  component of the angular momentum  $m_z$  and the principal quantum number  $N$ . Also, by virtue of the inversion symmetry, the space of solutions of the linearized equations can be split into two independent subspaces,  $\mathcal{I}$  and  $\mathcal{II}$  for  $m_z$  even and odd, respectively. Since in real experiments exciton-polariton condensates can be considered as a weakly interacting gas, one can obtain approximate solutions of the stationary GPE (3) and the corresponding Bogolyubov–de Gennes equations for the  $u$  and  $v^*$  components using a perturbation theory approach. In the following, the term  $\Lambda|\Phi_0|^2$  is taken as a perturbation with respect to the harmonic trap potential  $V(\rho) = \frac{1}{2}\rho^2$ , the ground-state wave function (or order parameter)  $\Phi_0(\rho)$ , and the amplitudes  $u(\rho, t)$  and  $v^*(\rho, t)$  can be sought in terms of the complete set of the 2D harmonic oscillator wave functions  $\varphi_{N, m_z}$  (see Appendix A). In Ref. 18, compact solutions for the reduced chemical potential  $\bar{\mu}$  and the dimensionless order parameter  $\Phi_0(\rho)$  are presented up to the second and the first order in  $\Lambda$ , respectively. Following the procedure described in Ref. 18, it is possible to obtain the Bogolyubov excitation mode's frequencies  $\omega_{N, m_z}$ . From the calculation it follows, in particular, that the dipole ( $m_z = 1$ ) mode is harmonic, that is,  $\omega_{N=0, m_z=1} = \omega_0$ .<sup>34</sup> Notice that this result (corresponding to Kohn's theorem<sup>35</sup>) is quite general and independent of the approximation considered (it has been demonstrated also in the hydrodynamic limit<sup>21</sup>).

Below we limit ourselves by considering the EPC states with zero angular momentum  $m_z = 0$ , which refers to the *macroscopic* motion of the condensate. In experiments, EPCs are probed through the (spatially resolved) emission that escapes from the cavity. That related to the  $m_z = 0$  states corresponds to a Gaussian beam and can be distinguished from the higher order (Laguerre-Gaussian) modes [which have been used to detect vortices in EPC (Ref. 5)]. For this case, we have

$$\frac{\omega_{N,0}}{\omega_0} = N + \left[ \frac{N!}{2^{N-1} \left[ \left( \frac{N}{2} \right)! \right]^2} - 1 \right] \frac{\Lambda}{2\pi} + \beta_{N,0} \Lambda^2, \quad (6)$$

where  $N = 2, 4, \dots$  and  $\beta_{N,0}$  are some numbers.<sup>36</sup>

The perturbation of the order parameter  $\delta\Phi_{N,0}$  to the first order in  $\Lambda$  is given by

$$\delta\Phi_{N,0} \equiv \begin{pmatrix} u_{N,0}(\rho) \exp(-i\omega_{N,0}t) \\ v_{N,0}^*(\rho) \exp(i\omega_{N,0}t) \end{pmatrix}$$

with

$$\begin{pmatrix} u_{N,0} \\ v_{N,0}^* \end{pmatrix} = \frac{\exp\left(-\frac{\rho^2}{2}\right)}{\sqrt{\pi}} \begin{pmatrix} L_{N/2}(\rho^2) + 2\Lambda\mathcal{F}_N^-(\rho) \\ -\Lambda\mathcal{F}_N^+(\rho) \end{pmatrix}, \quad (7)$$

and the functions  $\mathcal{F}_N^\pm(\rho)$  are given in Appendix A.

In Fig. 1(b), the eigenfrequencies of the Bogolyubov collective oscillations for  $N = 2$  and  $4$  are plotted against the interaction parameter  $\Lambda$ . Also, the ground-state energy of the condensate per particle  $\epsilon$  is displayed. It should be pointed out that the chemical potential and the ground-state energy (per

particle) are not the same.<sup>37</sup> The (dimensionless) energy per particle can be calculated as

$$\epsilon = \int_V \Phi_0 \left( \hat{H}_0 + \frac{1}{2} \Lambda |\Phi_0|^2 \right) \Phi_0 d\mathbf{r},$$

which differs from  $\bar{\mu}$  by the 1/2 factor in the nonlinear term. Thus  $\bar{\mu}$  and  $\epsilon$  coincide only when the particles do not interact. It is possible to obtain an identity relating the energy per particle and the chemical potential, that is,<sup>38</sup>

$$\epsilon = \frac{1}{\Lambda} \int_0^\Lambda \bar{\mu}(z) dz. \quad (8)$$

Substituting in Eq. (8) the previously obtained expression<sup>18</sup> for the chemical potential, we have

$$\epsilon = 1 + \frac{1}{4\pi} \Lambda - \frac{\ln 4/3}{8\pi^2} \Lambda^2.$$

From Fig. 1(b) it follows that the excitation modes are only weakly dependent on  $\Lambda$ ; thus the total energy of the excited state,  $\mathcal{E}_{N,0}(\Lambda) = N_p \epsilon(\Lambda) + \omega_{N,0}(\Lambda)/\omega_0$ , shows almost the same blue-shift dependence on  $\Lambda$  as the ground-state energy  $N_p \epsilon(\Lambda)$ . Using Eq. (6), a direct calculation of the level spacing between the EPC excited states  $(N+2,0)$  and  $(N,0)$  yields  $\Delta \mathcal{E}_{N,m=0} \simeq 2$ , only very slightly decreasing with  $\Lambda$ . Therefore the spectrum of the Bogolyubov-type excitations is nearly equidistant, even for rather large values of the polariton-polariton interaction parameter.

As is known,<sup>16,19</sup> the spectrum of elementary excitations around the stationary state of a spatially homogeneous Bose-Einstein condensate can be described as a function of their wave vector multiplied by so-called healing length,  $\sqrt{\hbar/(m\mu)}$ . For small values of this product ( $\ll 1$ ), a linear (phonon-type) dispersion of the Bogolyubov's excitations is expected and its observation for a very weakly confined EPC has been reported.<sup>2</sup> In the present case of laterally confined EPC, the wave vector is not a good quantum number and the Bogolyubov's excitation spectrum is described by the ‘‘cylindrical’’ quantum numbers  $N$  and  $m_z$ . The equidistant spectrum of Fig. 1(b) is just characteristic of the phononlike regime for these localized excitations.

#### IV. EPC COUPLED TO UNCONDENSED POLARITONS

As defined in Sec. II,  $V_{\text{res}}$  is the power-dependent potential resulting from the repulsive interactions with uncondensed excitons.<sup>9</sup> As the exciton diffusion coefficient is very small,<sup>16</sup> we can assume that the interaction with the reservoir is proportional to the pump profile, i.e., the dimensionless potential can be cast as  $v_{\text{res}} = V_{\text{res}}/\hbar\omega_0 = \Lambda_{\text{res}} \exp(-\rho^2/\bar{a}^2)$  with  $\Lambda_{\text{res}} = g_{\text{res}}/\hbar\omega_0$ . Here  $\bar{a}$  is an effective spatial size of the polariton reservoir created by the pumping laser beam (measured in units of  $l_0$ ) and  $g_{\text{res}}$  is the coupling constant describing the repulsive interaction with uncondensed excitons. As explained at the end of Sec. II, we neglect the nonlinear term and then Eq. (2) becomes

$$i\hbar \partial_t \psi = [\hat{H}_0 + v_{\text{res}}] \psi. \quad (9)$$

We seek the axial-symmetric solution of Eq. (9) with  $m_z = 0$  in the form  $\psi(\rho, t) = \exp(-iE\omega_0 t) \Phi_{\text{con-res}}(\rho)$  and arrive at the following stationary differential equation for the

order parameter function ‘‘reshaped’’ by the interaction with the reservoir:

$$[\hat{H}_0 + v_{\text{res}}] \Phi_{\text{con-res}} = E \Phi_{\text{con-res}}. \quad (10)$$

The external potential fulfills the condition  $v_{\text{res}}(\rho) \rightarrow 0$  as  $\rho \rightarrow \infty$ ; thus it is possible to show that the eigenfunctions  $\varphi_{N,0}(\rho)$ , which are the solutions of  $\hat{H}_0$  with  $m_z = 0$  and  $N = 0, 1, \dots$  (see Appendix A), represent a complete set for the Hamiltonian  $\hat{H} = \hat{H}_0 + v_{\text{res}}$ .<sup>39</sup> Therefore the solution of Eq. (10) can be cast in the form

$$\Phi_{\text{con-res}}(\rho) = \sum_{N=0}^{\infty} c_N \varphi_{N,0}(\rho). \quad (11)$$

The set  $\{\varphi_{N,0}(\rho)\}$  ensures the convergence, at least in mean, of the series (11) to the solution  $\Phi_{\text{con-res}}$ .<sup>40</sup> It takes place if and only if the coefficients  $\{c_N\}$  obey the relation, which is obtained by inserting Eq. (11) into Eq. (10),

$$[(N+1-E)\mathbf{I} + \Lambda_{\text{res}}\mathbf{M}]\mathbf{C} = 0, \quad (12)$$

where  $\mathbf{I}$  is the unity matrix,  $\mathbf{C} = (c_0, c_1, \dots)$ , and  $\mathbf{M}$  is defined in Appendix B.

We solved Eq. (12) in a finite basis of dimension  $N_{\text{max}}$ , requiring that  $|E_N^{(N_{\text{max}})} - E_N^{(N_{\text{max}}-1)}| < \eta_N^{(N_{\text{max}})}$ , where  $\eta_N^{(N_{\text{max}})}$  is the desired accuracy for the energy  $E_N$ . To warrant the accuracy of  $\eta = 10^{-6}$  for all considered excited states in the range of values  $0 \leq \Lambda_{\text{res}} \leq 15$ , a basis set smaller than 50 oscillator wave functions  $\{\varphi_{N,0}(\rho)\}$  is sufficient. Thus the order of the symmetric matrix  $\mathbf{M}$  is lower than  $50 \times 50$ .

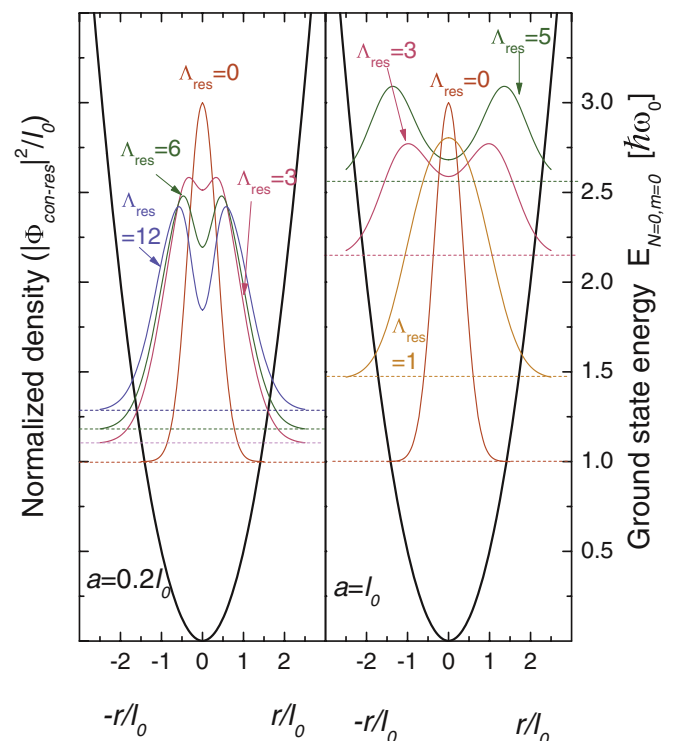


FIG. 2. (Color online) Ground-state condensate density profiles plotted against the energy (right vertical axis) for several values of  $\Lambda_{\text{res}}$  and the dimensionless reservoir spot size  $\bar{a} = 0.2$  (left) and 1 (right).

Figure 1(c) presents the renormalized energy  $E$  versus  $\Lambda_{\text{res}}$  for the nondegenerate states ( $m_z = 0$ ) of the condensate with the principal quantum number  $N$  ranging from 0 to 8 and for two different values of the dimensionless beam radius  $\bar{a}$ . (Remember that  $\Lambda_{\text{res}} = 0$  corresponds to the 2D harmonic oscillator energy levels.) The influence of the pumping spot size is clearly seen in the figure. Two limiting cases of Eq. (12) can be compared. If  $\bar{a} \rightarrow 0$ , the matrix  $\mathbf{M} \rightarrow \mathbf{0}$  and the energy levels tend to the harmonic oscillator eigenvalues,  $E_N = N + 1$ . In the opposite case of  $\bar{a} \rightarrow \infty$  we have  $\mathbf{M} \rightarrow \delta_{N,N_1}$  (see Appendix B) and  $E_N \approx N + 1 + \Lambda_{\text{res}}$ . In both cases, the level spacings,  $\Delta E_N = E_{N+2} - E_N$ , are the same, even though they show a strong dependence on the laser spot size in the intermediate regime ( $\bar{a} \sim 1$ ). For example, if  $\bar{a} = 0.2$ ,  $\Delta E_N \approx 2$ , as for the 2D harmonic potential. On the contrary, if  $\bar{a}$  increases to unity,  $\Delta E_N \neq 2$  [solid lines with circles in Fig. 1(c)] and it depends on the number of polaritons in the reservoir.

Figure 2 shows the influence of the reservoir size  $\bar{a}$  on the EPC density  $|\Phi_{\text{con-res}}(\rho)|^2$  for several values of the dimensionless coupling constant  $\Lambda_{\text{res}}$ . From this figure it can be seen that the position of the density maximum is pushed away from the origin as  $\Lambda_{\text{res}}$  increases. This effect is linked to the repulsive interactions produced by the Gaussian density profile of uncondensed polaritons created in the trap. The condensate is repelled from the origin as the number of uncondensed excitons  $N_r$  (proportional to the pumping beam intensity) increases. This “push-out” effect has clearly been demonstrated experimentally in Ref. 9 for a condensate confined in a micropillar cavity. Moreover, the blue shift of the condensate emission associated with the increase of the EPC ground state due to the action of  $V_{\text{res}}$  has been observed in laterally confined condensates.<sup>9,14</sup> It is also clear from Fig. 2 that the condensate becomes more delocalized as the spot width increases because its ground-state energy grows, an effect characteristic of “soft” parabolic confinement.

Using Eqs. (11) and (12), we can also obtain the dependence of the condensate density profile on  $\Lambda_{\text{res}}$  for the excited states. These profiles are shown in Fig. 3 for the states with  $N = 1, 2$ , and 3, for the states with  $N = 1, 2$ , and 3 ( $m_z = 0$ ).

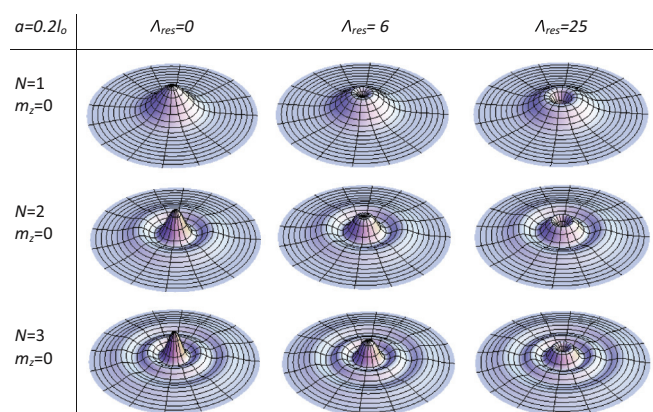


FIG. 3. (Color online) Spatial distribution of the density  $|\Phi_{\text{con-res}}(\rho)|^2$  of EPC coupled to the reservoir for the excited states  $N = 1, 2$ , and 3 ( $m_z = 0$ ). The influence of  $\Lambda_{\text{res}}$  on the density profile is shown for the values of 0 (no interaction), 6, and 25 ( $\bar{a} = 0.2l_0$ ).

2, and 3. Indeed, the condensation of polaritons in several quantized states of a trap has been observed by taking their snapshots in real and reciprocal space.<sup>9,10,14</sup> The shape of the emission pattern observed in these works qualitatively corresponds to the dependence of the condensate density on  $\rho$  (oscillations in real space) and  $E$  (several spectral peaks under intense pumping) that comes out from our calculated results, even though the form of the lateral confinement potential is different.

## V. CONCLUSIONS

In summary, we applied the perturbation theory approach to the Gross-Pitaevskii equation describing a Bose-Einstein condensate of exciton polaritons in a semiconductor microcavity confined in a parabolic lateral trap. This approach allows for a convenient analytical description of the condensate’s ground state and Bogolyubov-type elementary excitations around it. We derived phonon-type modes with the energies and wave functions depending on the polariton-polariton interaction parameter. This set of states can be used to describe the dynamics of the polariton BEC, for instance, vortices and their interaction,<sup>4,6</sup> which is planned for a future work. We point out that the spectrum of these Bogolyubov-type excitations in a condensate whose interaction with uncondensed polaritons can be neglected is almost equidistant, even for rather larger values of the polariton-polariton interaction parameter inside the condensate. This makes polariton parabolic traps promising candidates for realization of bosonic cascade lasers.

We also considered a qualitatively different physical situation where the interaction with a reservoir of uncondensed polaritons is more important than that inside it. (To avoid confusion, note that our model is valid if no dynamic process is described by a characteristic time shorter than  $|R - F|^{-1}$ .) In this “quasi-steady-state” case, we obtained a semianalytical solution for the ground and excited states of the condensate, which shows how it is “reshaped” by the repulsive interaction with the reservoir, namely, pushed out from the center of the trap in real space and blue-shifted in energy. Our results are in agreement with those obtained in a number of recent experiments and numerical simulation studies.<sup>9,10,14</sup> In particular, we show that the level spacings between the condensate states increase with the pump power [Fig. 1(c)], similar to the recent experimental observation.<sup>10</sup> It can imply that the experimentally observed emission patterns in confined condensates pumped through the polariton reservoir are not related to the Bogolyubov-type elementary excitations in the condensate itself; rather, they are determined by the repulsive condensate-reservoir interaction reshaping the density profile of the former.

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### APPENDIX A: COMPLETE BASIS SET AND RELATED FUNCTIONS

The 2D harmonic oscillator functions, solutions of the equation  $\hat{H}_0\varphi_{N,m_z} = \varepsilon_N\varphi_{N,m_z}$ , in cylindrical coordinates are

$$\varphi_{N,m_z} = \frac{1}{\sqrt{2\pi}} e^{im_z\theta} R_{N,m}(\rho). \quad (\text{A1})$$

Here

$$R_{N,m}(\rho) = \frac{1}{\sqrt{\mathcal{N}_{N,m}}} e^{-\frac{\rho^2}{2}} \rho^m L_{n_r}^{(m)}(\rho^2),$$

where  $m_z = 0, \pm 1, \pm 2, \dots$ ,  $m = |m_z|$ ,  $n_r = 0, 1, 2, \dots$ , is the radial quantum number,  $N = 2n_r + m$ ,  $\varepsilon_N = N + 1$  are the dimensionless energy levels,  $\mathcal{N}_{N,m} = (\frac{N+m}{2})!/2(\frac{N-m}{2})!$  (independent of  $m_z$ ), and  $L_{n_r}^{(m)}(z)$  denotes the generalized Laguerre polynomials.<sup>42</sup> Notice that for a given energy  $\varepsilon_N$ , the solutions  $\varphi_{N,m_z}(\rho, \theta)$  are  $\mathcal{D}_N = N + 1$  times degenerate.

The functions  $\mathcal{F}_N^\pm(\rho)$  can be expressed as<sup>34</sup>

$$\mathcal{F}_N^\pm(\rho) = \sum' \frac{C_{N,N_2}^{(0)}}{N \pm N_2} L_{N/2}(\rho^2),$$

where the sum  $\sum'$  is taken for  $N - N_2 \neq 0$  and

$$\begin{aligned} C_{N,N_2}^{(0)} &= \frac{1}{\pi} \times \int_0^\infty L_{N/2}(t) L_{N_2/2}(t) \exp(-2t) dt \\ &= \frac{1}{\pi} \frac{(N/2 + N_2/2)!}{(N/2)!(N_2/2)!} 2^{-N/2 - N_2/2 - 1}. \end{aligned} \quad (\text{A2})$$

Thus

$$\begin{aligned} \mathcal{F}_N^\pm &= \frac{1}{\pi} \frac{1}{2^{N/2+1}(N/2)!} \\ &\times \sum' \frac{(N/2 + N_2/2)!}{2^{N_2/2+1}(N_2/2)!} \frac{1}{N \pm N_2} L_{N/2}(\rho^2). \end{aligned} \quad (\text{A3})$$

Using the identity

$$\sum_{k=0}^{\infty} z^k L_k(t) = (1-z)^{-1} \exp\left(\frac{tz}{z-1}\right); \quad |z| < 1, \quad (\text{A4})$$

it is possible to show that

$$\mathcal{F}_N^+ = \frac{1}{\pi} \frac{1}{(N/2)!} \frac{d^{N/2-1}}{dz^{N/2-1}} \left[ \frac{z^{N/2-1}}{1-z} \exp\left(\frac{tz}{z-1}\right) \right] \Big|_{z=1/2}$$

and

$$\mathcal{F}_N^- = \int_{1/2}^1 \frac{\mathcal{F}_N^+(t, z)}{2^{N/2} z^{N/2+1}} dz - \frac{N!}{\pi 2^{N+1} [(N/2)!]^2} L_n(t) \ln 2,$$

with  $N = 2, 4, \dots$

### APPENDIX B: MATRIX ELEMENTS

The elements of the matrix  $\mathbf{M}$  introduced in Eq. (12) are defined as

$$\begin{aligned} M_{NN_1} &= \langle N | \exp(-\rho^2/\bar{a}^2) | N_1 \rangle \\ &= \int_0^\infty L_{N/2}(t) L_{N_1/2}(t) \exp[-(1+\bar{a}^{-2})t] dt \\ &= \left( \frac{N+N_1}{2} \right) \frac{\bar{a}^2}{(\bar{a}^2+1)^{\frac{N+N_1}{2}+1}} \\ &\times F\left(-\frac{N}{2}, -\frac{N_1}{2}; -\frac{N+N_1}{2}, 1-\bar{a}^4\right), \end{aligned} \quad (\text{B1})$$

where  $N$  and  $N_1$  are even numbers and  $F(a, b; c, z)$  is the hypergeometric function.<sup>42</sup>

From Eq. (B1) we have the following properties:

- (i)  $M_{NN_1} = M_{N_1N}$ ;
- (ii) For  $N = 0$ ,

$$M_{0N_1} = \frac{\bar{a}^2}{(\bar{a}^2+1)^{\frac{N_1}{2}+1}}; \quad (\text{B2})$$

- (iii) If  $\bar{a} = 0$ ,  $M_{NN_1} = 0$ ;

(iv) If  $\bar{a} \rightarrow \infty$ , the function  $\exp(-\rho^2/\bar{a}^2) \rightarrow 1$  and  $M_{NN_1} = \langle N | N_1 \rangle = \delta_{NN_1}$ .

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