Cambridge International
Science Publishing
Cambridge CB1 6AZ
Great Britain

Journal of Computational Methods in Sciences and Engineering
vol. 2, no. 3, 2, pp. 1–14
ISSN 1472–7978

An Interface Strip Preconditioner for Domain Decomposition Methods

M. Storti[1], L. Dalcín, R. Paz, A. Yommi, V. Sonzogni and N. Nigro

Centro Internacional de Métodos Computacionales en Ingeniería (CIMEC)
CONICET - INTEC - U.N.L.
Güemes 3450, (3000) Santa Fe, Argentina
http://www.cimec.org.ar/

Abstract: A preconditioner for iterative solution of the interface problem in Schur Complement Domain Decomposition Methods is presented. This preconditioner is based on solving a problem in a narrow strip around the interface. It requires much less memory and computing time than classical Neumann-Neumann preconditioner and its variants, and handles correctly the flux splitting among subdomains that share the interface. The performance of this preconditioner is assessed with an analytical study of Schur complement matrix eigenvalues. Results in a production parallel finite element code are given in a companion paper [1].

Keywords: domain decomposition, preconditioner, parallel computing.

PACS: 02.70.Dh, 83.85.Pt

Mathematics Subject Classification:

1 INTRODUCTION

Linear systems obtained from discretization of PDE's by means of Finite Difference or Finite Element Methods are normally solved in parallel by iterative methods [2] because they are much less coupled than direct solvers. The Schur complement domain decomposition method leads to a reduced system better suited for iterative solution than the global system, since its condition number is lower ($\propto 1/h$ vs. $\propto 1/h^2$ for the global system, h being the mesh size) and the computational cost per iteration is not so high once the subdomain matrices have been factorized. In addition, it has other advantages over global iteration. It solves bad "inter-equation" conditioning, it can handle Lagrange multipliers and in a sense it can be thought as a mixture between a global direct solver and a global iterative one.

The efficiency of iterative methods can be further improved by using preconditioners [3]. We propose a preconditioner based on solving a problem in a *"strip"* of nodes around the interface. When the width of the strip is narrow, the computational cost and memory requirements are low and the iteration count is relatively high, when the strip is wide, the converse is verified. This preconditioner performs better than the standard Neumann-Neumann one for non-symmetric operators and does not suffer from the *rigid body modes* for *internal floating subdomains*. A detailed computation of the eigenvalue spectra for simple cases is shown.

[1]Corresponding author, mstorti@intec.unl.edu.ar

2 SCHUR COMPLEMENT DOMAIN DECOMPOSITION METHOD

To obtain analytical expressions for Schur complement matrix eigenvalues and also the influence of several preconditioners, we consider a simplified problem, namely the solution to the Poisson problem in a unit square with Dirichlet boundary conditions

$$\begin{aligned} &\Delta\phi = g, \quad \text{in } \Omega = \{0 < x, y < 1\}; \\ &\phi = \bar{\phi}, \quad \text{at } \Gamma = \{x, y = 0, 1\}. \end{aligned} \tag{1}$$

where ϕ is the unknown, $g(x, y)$ is a given source term and Γ is the boundary. Consider now the partition of Ω in 2 non-overlapping subdomains Ω_1, Ω_2, such that $\Omega = \Omega_1 \bigcup \Omega_2$. For the sake of simplicity, we assume that the subdomains are rectangles of unit height and width L_j, $L_1 + L_2 = 1$. Let Γ_{int} be the interior interface ($x = L_1, 0 \le y \le 1$). Given a guess ψ for the trace of ϕ in the interface $\phi|_\Gamma$, we can solve each interior problem independently as

$$\begin{aligned} \Delta\phi &= g, \quad \text{in } \Omega_j, \\ \phi &= \begin{cases} \psi, & \text{at } \Gamma, \\ \bar{\phi}, & \text{at the other boundaries} \end{cases} \end{aligned} \tag{2}$$

Not any trace value ψ give the solution of the original problem (1). Indeed, the solution to (1) is obtained when the trace value are chosen in such a way that the flux balance condition at the internal interfaces is satisfied,

$$f = \left.\frac{\partial\phi}{\partial x}\right|_{\Gamma_{\text{int}}}^{-} - \left.\frac{\partial\phi}{\partial x}\right|_{\Gamma_{\text{int}}}^{+} = 0, \tag{3}$$

where the $\pm$ superscripts stand for the derivative taken from the left and right sides of the interface. We can think of the correspondence between the trace value ψ and the flux imbalances f as an interface operator $\mathcal{S}$ such that

$$\mathcal{S}\psi = f - f_0, \tag{4}$$

where all inhomogeneities coming from the source term and Dirichlet boundary conditions are concentrated in the constant term f_0. $\mathcal{S}$ is the *Steklov operator* that, roughly speaking, relates the unknown values and fluxes at boundaries when the internal domain is in equilibrium.

It can be shown by standard operational methods [1] that the eigenvalues ω_n and eigenfunctions ϕ_n of the Steklov operator are,

$$\begin{aligned} \omega_n &= \text{eig}(\mathcal{S})_n = \text{eig}(\mathcal{S}^-)_n + \text{eig}(\mathcal{S}^+)_n = k_n \left[\coth(k_n L_1) + \coth(k_n L_2)\right], \\ \phi_n(x, y) &= \begin{cases} [\sinh(k_n x)/\sinh(k_n L_1)] \sin(k_n y), & 0 \le x \le L_1, \\ [\sinh(k_n(L - x))/\sinh(k_n L_2)] \sin(k_n y), & L_1 \le x \le L, \end{cases} \end{aligned} \tag{5}$$

where the wave number k_n and the wavelength λ_n are defined as $k_n = 2\pi/\lambda_n$, $\lambda_n = 2L/n$, for $n = 1, \ldots, \infty$ (i.e. such that an integer number of half-wavelengths enter in the vertical length). In fact, we can compute the eigenvalues of the Steklov operator for each subdomain, resulting in $\text{eig}(\mathcal{S}^{\mp})_n = k_n \coth(k_n L_{1,2})$.

For large n, the hyperbolic cotangents in (5) both tend to unity. This shows that the eigenvalues of the Steklov operator grow proportionally to n for large n, and then its condition number is infinity. However, when considering the discrete case the wave number k_n is limited by the largest frequency that can be represented by the mesh, which is $k_{\max} = \pi/h$ where h is the mesh spacing. The maximum eigenvalue is $\omega_{\max} = 2k_{\max} = \frac{2\pi}{h}$, which grows proportionally to $1/h$. As the lowest eigenvalue is independent of h, this means that the condition number of the Schur complement matrix grows as $1/h$. Note that the condition number of the discrete Laplace operator typically grows as $1/h^2$.

3 PRECONDITIONERS FOR THE SCHUR COMPLEMENT MATRIX

In order to further improve the efficiency of iterative methods, a preconditioner has to be added so that the condition number of the Schur complement matrix is lowered. The most known preconditioners for mechanical problems are Neumann-Neumann and its variants [4, 5] for Schur complements methods, and Dirichlet for FETI methods and its variants [6, 7, 8, 9]. It can be proved that they reduce the condition number of the preconditioned operator to $O(1)$ (i.e. independent of h) in some special cases.

A key point about the Steklov operator is that its high frequency eigenfunctions decay very strongly far from the interface, so that a preconditioning that represents correctly the high frequency modes can be constructed if we solve a problem on a narrow strip around the interface. In fact, the n-th eigenfunction with wave number k_n given by (5) decays far from the interface as $\exp(-k_n|s|)$ where s is the distance to the interface. Then, this high frequency modes will be correctly represented if we solve a problem on a strip of width b around the interface, provided that the interface width is large with respect to the mode wave length λ_n.

The *"Interface Strip Preconditioner"* (ISP) is defined as

$$\begin{aligned}
\mathcal{P}_{\mathrm{IS}} v = f, \quad & f = \left.\frac{\partial w}{\partial x}\right|_{x=L_1^-} - \left.\frac{\partial w}{\partial x}\right|_{x=L_1^+} \\
\Delta w = 0 \quad & \text{in } 0 < |x - L_1| < b \text{ and } 0 \leq x \leq 1, \\
w = 0 \quad & \text{at } |x - L_1| = b \text{ or } y = 0, 1, \\
w = v \quad & \text{at } x = L_1.
\end{aligned} \tag{6}$$

Please note that for high frequencies (i.e. $k_n b$ large) the eigenfunctions of the Steklov operator are negligible at the border of the strip, so that the boundary condition at $|x - L_1| = b$ is justified. The eigenfunctions and eigenvalues for this preconditioner are again given by (5), replacing $L_{1,2}$ by b, i.e. $\mathrm{eig}(\mathcal{P}_{\mathrm{IS}})_n = 2\,\mathrm{eig}(\mathcal{S}_b)_n = 2k_n \coth(k_n b)$, where $\mathcal{S}_b$ is the Steklov operator corresponding to a strip of width b.

For the preconditioned Steklov operator, we have

$$\mathrm{eig}(\mathcal{P}_{\mathrm{IS}}^{-1}\mathcal{S})_n = {}^1\!/_2 \tanh(k_n b)\,[\coth(k_n L_1) + \coth(k_n L_2)] \tag{7}$$

We note that $\mathrm{eig}(\mathcal{P}_{\mathrm{IS}}^{-1}\mathcal{S})_n \to 1$ for $n \to \infty$, so that the preconditioner is optimal, independently of b. Consider now the advective-diffusive case,

$$\kappa \Delta \phi - u \phi_{,x} = g \quad \text{in } \Omega, \tag{8}$$

where κ is the thermal conductivity of the medium and u the advection velocity. The problem can be treated in a similar way, and the Steklov operators are defined as

$$\mathcal{S}^{\mp} \psi = \pm \left.\phi_{,x}\right|_{L_1^{\mp}}, \tag{9}$$

For a symmetric operator and a symmetric partition (see figure 1), the symmetric flux splitting is exact and the Neumann-Neumann preconditioner is optimal. The largest discrepancies between the IS preconditioner and the Steklov operator occur at low frequencies and yield a condition number less than two.

On the other hand, for an important advection term, given by a global Pèclet number of 50 (see figure 2), the asymmetry in the flux splitting ($\mathcal{S}^+ \neq \mathcal{S}^-$) is evident, mainly for small wave numbers, and this results in a large discrepancy between the Neumann-Neumann (wich basically assumes an equal splitting) preconditioner and the Steklov operator.

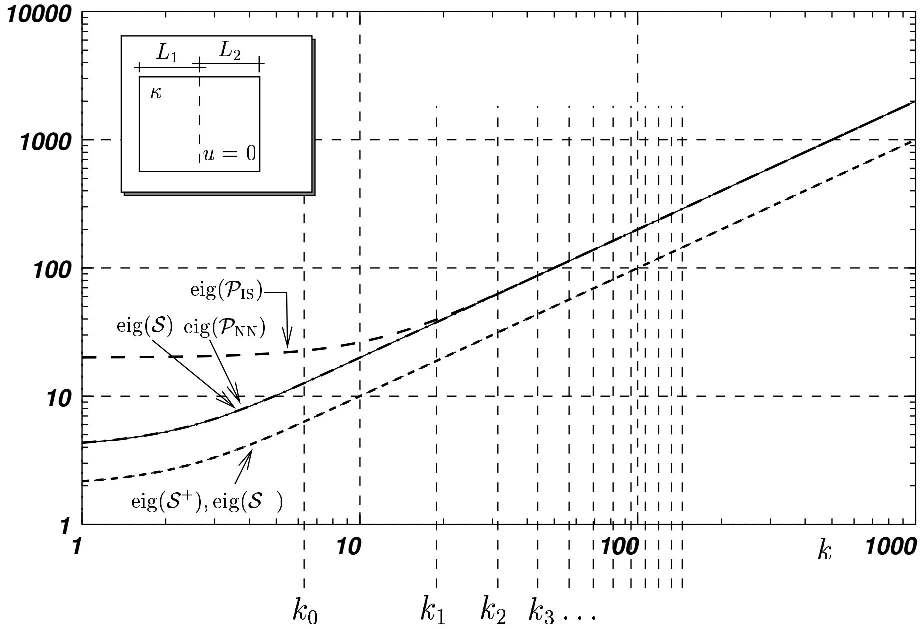

Figure 1: Eigenvalues of Steklov operators and preconditioners for the Laplace operator ($Pe = 0$) and symmetric partitions ($L_1 = L_2 = L/2$, $b = 0.1L$).

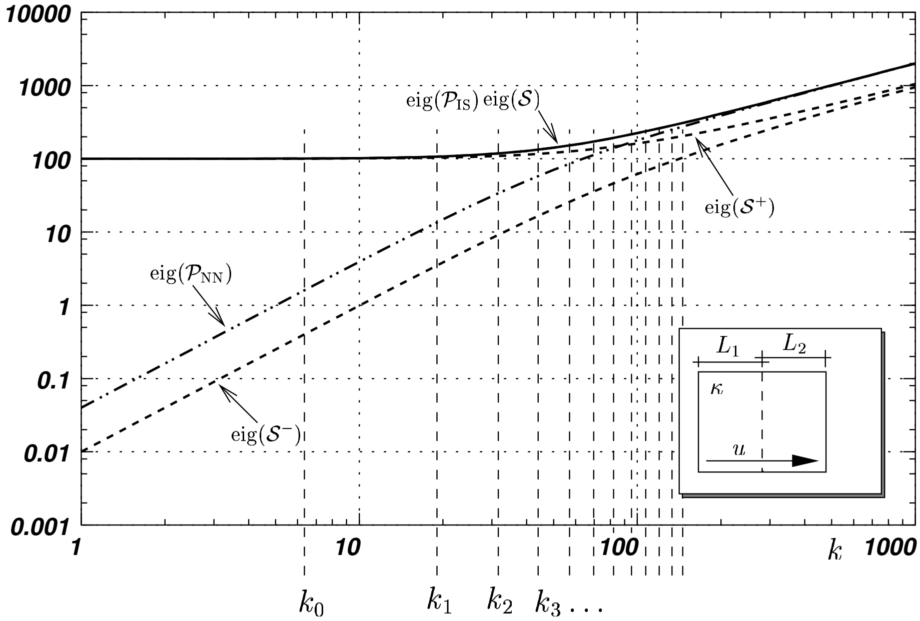

Figure 2: Eigenvalues of Steklov operators and preconditioners for the advection-diffusion operator (Pe = 50) and symmetric partitions ($L_1 = L_2 = L/2$, $b = 0.1L$).

4 CONCLUSIONS

We have presented a new preconditioner for Schur complement domain decomposition methods. This preconditioner is based on solving a problem posed in a narrow strip around the inter-subdomain interfaces. Some analytical results have been derived to present its mathematical basis. In advective-diffusive real-life problems, where the Pèclet number can vary on the domain between low and high values, the proposed preconditioner outperforms classical ones in advection-dominated regions while it is capable to handle reasonably well diffusion-dominated regions.

Acknowledgment. This work has received financial support from Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET, Argentina, grants PIP 0198/98), Universidad Nacional del Litoral (Argentina, grants CAI+D 2000/43) and ANPCyT (Argentina, grants PICT 6973/99, PID-74/99, PICT Lambda/2003, PME 209/2003). We made extensive use of freely distributed software as GNU/Linux OS, gcc compilers, Octave and Xfig among many others.

References

[1] R. Paz and M. Storti. An interface strip preconditioner for domain decomposition methods: Application to hydrology. *Int. J. Numer. Meth. Eng.*, 62(13):1873–1894, 2005.

[2] Y. Saad. *Iterative Methods for Sparse Linear Systems.* PWS Publishing Co., 2000.

[3] P. Le Tallec and M. Vidrascu. Solving large scale structural problems on parallel computers using domain decomposition techniques. In M. Papadrakakis, editor, *Parallel Solution Methods in Computational Mechanics*, chapter 2, pages 49–85. John Wiley & Sons Ltd., 1997.

[4] J. Mandel. Balancing domain decomposition. *Comm. Appl. Numer. Methods*, 9:233–241, 1993.

[5] J.M. Cros. A preconditioner for the Schur complement domain decomposition method. In *14th International Conference on Domain Decomposition Methods*, 2002.

[6] C. Farhat and F.X. Roux. A method of finite element tearing and interconnecting and its parallel solution algorithm. *Int. J. Numer. Meth. Eng.*, 32:1205–1227, 1991.

[7] C. Farhat, J. Mandel, and F.X. Roux. Optimal convergence properties of the FETI domain decomposition method. *Comput. Meth. Appl. Mech. Engrg.*, 115:365–385, 1994.

[8] C. Farhat and J. Mandel. The two-level FETI method for static and dynamic plate problems. *Comput. Meth. Appl. Mech. Engrg.*, 155:129–152, 1998.

[9] C. Farhat, M. Lesoinne, P. Le Tallec, K. Pierson, and D. Rixen. FETI-DP: a dual-primal unified FETI method-part I: A faster alternative to the two-level FETI method. *Int. J. Numer. Meth. Eng.*, 50:1523–1544, 2001.