# Exact solutions for the fluctuations in flat FRW universe coupled to a scalar fields.

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In this work we rigorously study the fluctuations in FRW models coupled with $n$ neutral scalar fields, minimally coupled to the gravitational field. We find the exact solutions and the asypmtotic expansions for the fluctuation around the critical point for an arbitrary potential.

## I. INTRODUCTION

Once upon a time to find exact solutions of Einstein equation was one of the most important tasks in General Relativity: The universes of Einstein, De-Sitter, and Schwarzshild were discovered in this heroic period. Since then the number of exact solutions has increased but also other urgent problems have appeared and captured the attention of the general relativity community. Nevertheless the importance of exact solutions always remains the same [1]. Precisely, only when we have an exact solution we have truly mastered the problem since only in this case we have a mathematical model that we can completely understand. In fact: Einstein solution was our first model for the universe, Friedmann-Lemaïtre -Robertson-Walker solution turns out to be the model for the expanding universe, De-Sitter solution the model for inflation, Schwarzshild solution the model for stars, Kerr solution de model for rotating black holes, Vaidya solution allows to study the dynamics of spherical shells, and so forth. Following this line in this paper we present exact solutions for the fluctuations of a FRW cosmology minimally coupled to a set of scalar fields with an arbitrary potential, around the equilibrium points of the background.

Using the results of papers [4], [3], [2] we have already made an analytic study of the fluctuation in the simple case of constant potential [5]. As the background dynamics and the first order fluctuations are represented by strongly non linear equations [2] the analytic solutions are very difficult, if not impossible, to find and their properties must be obtained by numerical experiments. In this paper we present the beginning of an alternative view. We consider that the background is in a singular point of its dynamic which can be stable or unstable and we find the behavior of the fluctuations around these points. Moreover, since in cosmology the motions of the fields are usually damped by the $H$ term, the system naturally finishes in one stable point (an atractor) and its asymptotic behavior can be heuristically foreseen (see section V).

We hope that both new exact solutions, the one of paper [5] and the one presented in this paper would be a solid base for future researches and may be as useful those quoted at the beginning of this introduction.

The paper is organized as follows:

In section II we introduce our model: fluctuations in a FRW universe with $n$ minimally coupled massless scalar fields $\phi_i$with arbitrary potential $V$.

In section III we present our exact solutions for the motion around the equilibrium points.

In section IV we study the asymptotic behavior of perturbations around different kind of equilibrium points.

In section V we draw our conclusions. As a first future simple application of the obtained exact solutions we will try to find some results in the eventual chaotic behavior of the system, in the cases where the non linearity of the system is increased, e.g. the case of fluctuation around arbitrary (not fixed) solutions. In turn chaos is extremely important in cosmology as a way to explain the homogeneity and isotropy of the universe [1] and we have already studied the subject in FRW cosmologies with no fluctuations [6]

## II. THE COSMOLOGICAL MODEL.

Our metric is the flat FRW metric:

$$ds^2 \;=\; dt^2 - a(t)^2\,(dx^2+dy^2+dz^2). \tag{1}$$

where $t$ is the proper time and $a(t)$ is the scale factor. The equation corresponding to a gravitational field minimally coupled to $n$ neutral massless scalar fields $\phi_i$, , with arbitrary potential $V(\phi_1, ...., \phi_n)$ are well-known [7], [8]

The background equations, namely the Klein Gordon equations for each field are:

$$\ddot{\phi}_i + 3\,H\;\dot{\phi}_i + \frac{\partial\,V}{\partial\,\phi_i} \;=\; 0,\;\; i=1,2,...,n, \tag{2}$$

where $H = \dot{a}\,/\,a$ is the Hubble function and:

$$\dot{H} \;=\; -3\sum_{i=1}^{n}\dot{\psi}_i^{\,2},$$

and the Hamiltonian constraint is:

$$\mathcal{H}^2 = \frac{8\pi}{3\,m_{pl}^2}\,(\,V + \sum_{i=1}^{n}\frac{1}{2}\;\dot{\phi}_i^{\,2}\,) \;=0 \tag{3}$$

The perturbed metric reads:

$$ds^2 = (1-2\,\Phi)\;dt^2 - a(t)^2\,(1+2\,\Phi)\,(dx^2+dy^2+dz^2), \tag{4}$$

while the field's perturbations of $\phi_i$ are symbolized as $\delta\phi_i$. Then, the $k$–Fourier transform of the equations for the perturbations is given in [2], [3] and [4],

$$3\,H\;\dot{\Phi}^{\,k} + \left(\frac{k^2}{a^2}+3\,H^2\right)\Phi^k = -\frac{3\lambda}{2}\sum_{i=1}^{n}(\dot{\phi}_i\;\delta\;\dot{\phi}_i - \Phi^k\;\dot{\phi}_i^{\,2} + \frac{\partial\,V}{\partial\,\phi_i}\,\delta\phi_i)\,, \tag{5}$$

$$\dot{\Phi}^{\,k} + H\,\Phi^k = \frac{3\lambda}{2}\sum_{i=1}^{n}\dot{\phi}_i\;\;\delta\phi_i, \tag{6}$$

$$\delta\;\ddot{\phi}_i + 3\,H\,\delta\;\dot{\phi}_i + \sum_{j=1}^{n}\frac{\partial^2\,V}{\partial\,\phi_i\,\partial\,\phi_j}\;\delta\phi_j \;= 4\;\dot{\Phi}^{\,k}\;\dot{\phi}_i - 2\,\frac{\partial\,V}{\partial\,\phi_i}\,\Phi^k - \frac{k^2}{a^2}\;\delta\phi_i, \quad i=1,2,\ldots,n, \tag{7}$$

where $\bullet = d\;()\,/$and $\lambda = 8\pi\,/\,3\,m_{pl}^2$.

## III. THE EXACT SOLUTION FOR THE FLUCTUATION.

In this work we will study the behavior of the fluctuations around singular points as in papers [10], [9] when the Hubble function $H$ is positive since it is the physically more interesting and mathematically more complex case (when $H=0$ the solution are easy to find),

Let $\phi^0 = (\phi_1^0, \phi_2^0, ..., \phi_n^0)$ and $H_0$ be a singular point of the background dynamics. Then $\partial V\,/\,\partial\phi_i(\phi^0) = 0$ and $\phi_i = \phi_i^0$ is a solution of Eq. (2) and the Einstein conditions impose the condition $H_0^2 = \lambda\,V(\phi^0) > 0$ . So the $k-$ mode fluctuation equations read:

$$\delta\;\ddot{\phi}_i^{\,k} + 3\,H_0\,\delta\;\dot{\phi}_i^{\,k} + \frac{k^2}{a^2}\;\delta\phi_i^k + \sum_{j=1}^{n}V_{ij}\;\,\delta\phi_j^{\;k} = 0,\;\; i=1,2,\ldots,n, \tag{8}$$

where $V_{ij} = \partial\,V/\partial\,\phi_i\,\partial\,\phi_j\;\,(\phi^0)$ and

$$3\,H_0\;\dot{\Phi}^{\,k} + \left(\frac{k^2}{a^2}+3\;H_0^2\right)\Phi^k = 0, \tag{9}$$

$$\overset{\bullet}{\Phi}^{k} + H_0\, \Phi^k = 0. \tag{10}$$

From these equations we see that $k^2\, \Phi^k \,/\, a^2 = 0$ and therefore either $k = 0$ or $\Phi^k = 0$. Moreover from the last equation we obtain that $\Phi^0 = \Phi^0_0\, e^{\mp H_0\, t}$, and $\Phi^k = 0$ for $k > 0$.

Let us now define the symmetric Hessian matrix of potential $V$: $\mathbb{A} = (V_{ij})$. It is a symmetric matrix therefore its eigenvalues are real. Let as further define the diagonal matrix

$$\mathbb{D} = \mathbb{T}^{-1} \mathbb{A} \mathbb{T} \tag{11}$$

so $\mathbb{T}$ is the matrix that diagonalizes $\mathbb{A}$. Let us introduce the "diagonal" fluctuations $\delta\psi^k$ such that $\delta\phi^k = \mathbb{T}\delta\psi^k$. If we $\mathbb{T}-$transform Eq.. (8) we have

$$\delta \overset{\bullet\bullet}{\psi}{}_i^{k} + 3\, H_0\, \delta\, \overset{\bullet}{\psi}{}_i^{k} + \frac{k^2}{a^2}\, \delta\psi_i^k + \lambda_i\, \delta\psi_i{}^k = 0,\; i = 1, 2, \ldots, n, \tag{12}$$

and since $H_0^2 = cons\tan t$ because we are in a fixed point it is $a = a_0\, e^{\pm\, H_0\, t}$ and therefore the last equation reads

$$\delta\, \overset{\bullet\bullet}{\psi}{}_i^{k} \pm 3\, H_0\, \delta\, \overset{\bullet}{\psi}{}_i^{k} + \left(\lambda_i + \frac{k^2}{a_0^2}\, e^{\mp\, H_0\, t}\right) \delta\psi_i{}^k = 0,\; i = 1, 2, \ldots, n, \tag{13}$$

where $\lambda_i$ is the $i - th$ eigenvalue of $\mathbb{A}$. So in the particular case $k = 0$ we have

$$\delta\, \overset{\bullet\bullet}{\psi}{}_i^{0} \pm 3\, H_0\, \delta\, \overset{\bullet}{\psi}{}_i^{0} + \lambda_i\, \delta\psi_i{}^0 = 0,\; i = 1, 2, \ldots, n, \tag{14}$$

so

$$\delta\psi_i{}^0 = C^0_{i1}\, e^{\frac{1}{2}\left(\pm\, 3\, H_0 + \sqrt{9\, H_0^2 - 4\, \lambda_i}\right)\, t} + C^0_{i2}\, e^{\frac{1}{2}\left(\pm\, 3\, H_0 - \sqrt{9\, H_0^2 - 4\, \lambda_i}\right)\, t},\; i = 1, 2, \ldots, n, \tag{15}$$

and if $9\, H_0^2 - 4\, \lambda_i = 0$

$$\delta\psi_i{}^0 = (C^0_{i1} + C^0_{i2}\, t)\, e^{\mp\, \frac{3\, H_0}{2}\, t},\; i = 1, 2, \ldots, n. \tag{16}$$

To simplify the notation let us make the change of variables

$$z = \delta\psi_i{}^k,\;\; c^2 = \frac{k^2}{a_0^2},\;\;\; x = \frac{2\, c}{H_0}\, e^{\mp\, H_0\, t},\;\; \Lambda = \frac{4\, \lambda_i}{H_0^2}, \tag{17}$$

so Eq. (13) reads

$$x^2\, z'' - 5\, x\, z' + (\Lambda - x^2)\, z = 0, \tag{18}$$

where the prime symbolizes the $x-$derivative. Now we introduce a new variable $u = z \,/\, x^3$ obtaining the Bessel equation

$$x^2\, u'' + x\, u' + (x^2 - \nu^2)\, u = 0, \tag{19}$$

where $\nu^2 = 9 - \Lambda$. Then we have found the exact solution that, in the primitive variables, reads:

$$\delta\psi_i{}^k = e^{\mp \frac{3}{2}\, H_0\, t} \left[c^k_{i1}\;\; J^1_{\nu_i}\left(\frac{2\, k}{H_0\, a_0}\, e^{\mp \frac{1}{2}\, H_0\, t}\right) + c^k_{i2}\;\; J^2_{\nu_i}\left(\frac{2\, k}{H_0\, a_0}\, e^{\mp \frac{1}{2}\, H_0\, t}\right)\right], \tag{20}$$

where $k \neq 0$, $\nu_i = \sqrt{9H_0^2 - 4\, \lambda_i}\; /\; H_0$. Now using $\delta\phi^k = \mathbb{T}\; \delta\psi^k$, when all the eigenvalues are positive definite (we will not study the case with negative eigenvalues for simplicity), we arrive to the final solution

$$\delta\phi_i{}^k = \sum_{j=1}^{n} e^{\mp \frac{3}{2}\, H_0\, t} \left[\widehat{c^k_{ij1}}\;\; J^1_{\nu_j}\left(\frac{2\, k}{H_0\, a_0}\, e^{\mp \frac{1}{2}\, H_0\, t}\right) + \widehat{c^k_{ij2}}\;\; J^2_{\nu_j}\left(\frac{2\, k}{H_0\, a_0}\, e^{\mp \frac{1}{2}\, H_0\, t}\right)\right], \tag{21}$$

when $\nu_j$ is different to a positive integer, then $J^1_{\nu_i} = J_{\nu_i}$ and $J^2_{\nu_i} = J_{-\nu_i}$ which are Bessel functions [11] of first kind and $\widehat{c^k_{ij1,2}}$ are integration constants. In the particular case when $\nu_j$ are positive integers we have $J^1_{\nu_i} = J_{\nu_i}$ and $J^2_{\nu_i} = Y_{\nu_i}$ where the last function is a Bessel function of the second kind.

## IV. THE ASYMPTOTIC BEHAVIOR OF THE FLUCTUATIONS.

In the relevant case $H_0 > 0$ we can compute the limits when $t \to \infty$. In the case $k = 0$, using the Eqs. (7, 16) when the eigenvalues are positive $\delta\phi_i{}^0 \to 0$ and when the eigenvalues are negative, for some initial conditions it turns out that $\delta\phi_i{}^0 \to 0$ while for other conditions we have $\delta\phi_i{}^0 \to \pm\infty$ . In the case $k > 0$, we study the limit $\varsigma \to 0$, in the new variable $\varsigma = e^{-\frac{H_0}{2}\,t}$, which is equivalent to the limits $t \to \infty$. Then as we are interested in the case $\varsigma << 1$ we can expand the Bessel functions, when $\nu$ is not a positive integer, using the $\gamma$ functions, as $J_\nu\,(\alpha\,\varsigma) = \frac{1}{2}\,\alpha\,\varsigma\;/\;\gamma\,(\nu+1)$ where $\alpha = 2\,k\;/(H_0\;a_0)$. In this case Eq. (21) reads

$$\delta\phi_i{}^k(\varsigma) = \varsigma^3 \sum_{j=1}^{n} \left[d_{ij1}^k\,\varsigma^{\nu_j} + \,d_{ij2}^k\,\varsigma^{-\nu_j}\right] \tag{22}$$

where $d_{ij\,1,2}^k$ are integration constants. When we are considering the case of non integer and positive eigenvalues $3+\nu_j$ has a positive real part and therefore $\lim_{\varsigma\to 0}\delta\phi_i{}^k\,(\varsigma) = 0$. In the case, when the eigenvalues are negative, for some initial conditions we obtain the equal limits. But in general $\lim_{\varsigma\to 0}\delta\phi_i{}^k\,(\varsigma) = \,\pm\infty$.

When $\nu_j$ is a positive integer then $\lambda_j = (9-n^2)\;/\,4$ where $n$ is an integer $n = 0,\,1,\,2,\,3$, since we are considering the positive eigenvalue case. Then $J^1_{\nu_{\;i}}\;(\varsigma) = J_{\nu_{\;i}}\;(\varsigma)$ and $J^2_{\nu_{\;i}}\;(\varsigma) = Y_{\nu_{\;i}}\;(\varsigma)$ which in the case $\varsigma << 1$ can be expanded, if $\nu = 1,\,2,\,3$ as $Y_\nu\;(\alpha\,\varsigma) = -\,\left(\frac{1}{2}\,\alpha\;\varsigma\right)^{-\nu}\;\gamma\;(\nu)\;/\,\pi$ , and if $\nu = 0$ it is $Y_0 = \;2\;\ln\varsigma\,/\pi$ [11]. Then let us consider the cases i and iii of the introduction:

i.- If $\nu_j \neq 3$, namely $\lambda_j \neq 0$, we have $\lim_{\varsigma\to 0}\delta\phi_i{}^k\,(\varsigma) = 0$,since $\lim_{\zeta\to 0}\varsigma^3\;Y_{\nu\;j}\,(\alpha\,\varsigma) = 0$

ii- If $\nu_j = 3$, namely $\lambda_j = 0$, we have $\lim_{\varsigma\to 0}\delta\phi_i{}^k\,(\varsigma) = const. \neq 0$ since $\lim_{\zeta\to 0}\varsigma^3\;Y_3\,(\alpha\;\varsigma) = -\,\gamma(3)\;(\frac{1}{2}\,\alpha)^{-3}\;/\,\pi$.

When $\nu_j$ is a positive integer and $\lambda_j$ are negative we obtaining the same results for very special initial conditions , in general $\lim_{\varsigma\to 0}\delta\phi_i{}^k\,(\varsigma) = \,\pm\infty$.

## V. CONCLUSION.

We have presented the exact solution for the fluctuations around the fixed point of a generic potential of a cosmological model where the mater energy tensor comes from $n$ scalar fields minimally coupled.

In the cosmological interesting case $H > 0$ the motions is dumped and we have reached to the following conclusions:

i.- If the eigenvalues of the Hessian of the potential at $\phi^0 = (\phi^0_1, \phi^0_2, ..., \phi^0_n)$ are positive the limit $t \to \infty$ of the fluctuations vanish.

ii.- If the eigenvalues of the Hessian of the potential at $\phi^0 = (\phi^0_1, \phi^0_2, ..., \phi^0_n)$ negative the limit $t \to \infty$ of the fluctuation will diverge (being the solution only reliable up to the moment when the fluctuations become very big, in such a way that the linear approximation we are using is not valid anymore), but in some very peculiar initial conditions the fluctuation can also vanish.

iii.-If all the eigenvalues of the Hessian of the potential at $\phi^0 = (\phi^0_1, \phi^0_2, ..., \phi^0_n)$ are positive or they vanish we are in a flat saddle point and the motion can follow the curve defined by the vanishing eigenvalues, so the limit $t \to \infty$ of the fluctuations along this curve can take a finite not zero value.

We can foresee other similar cases. Our exact computation confirms reasonable heuristics predictions. The only real gain is that we will past from reasonable conjectures to rigorous theorems. Moreover it is clear that in all these cases the have not chaos. In fact, it is quite amazing that in such complex models as those presented in this paper chaos is absent. Nevertheless we can foresee the presence of chaos if we further perturb the model around the solution corresponding to the vanishing eigenvalues of case iii. This will be our next step in the search of chaos in cosmological models.

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