

ON THE TOPOLOGICAL ENTROPY OF THE IRREGULAR PART OF V -STATISTICS MULTIFRACTAL SPECTRA

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ABSTRACT. Let (X, d) be a compact metric space and $f : X \rightarrow X$, if X^r is the product of r -copies of X , $r \geq 1$, and $\Phi : X^r \rightarrow \mathbf{R}$, then the multifractal decomposition for V -statistics is given by

$E_{\Phi}(\alpha) = \left\{ x : \lim_{n \rightarrow \infty} \frac{1}{n^r} \sum_{0 \leq i_1 \leq \dots \leq i_r \leq n-1} \Phi(f^{i_1}(x), \dots, f^{i_r}(x)) = \alpha \right\}$. The irregular part, or historic set, of the spectrum is the set points $x \in X$, for which the limit does not exist.

In this article we prove that for dynamical systems with specification, the irregular part of the V -statistics spectrum has topological entropy equal to that of the whole space X .

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1. INTRODUCTION

Motivated by the problems on convergence of multiple ergodic averages Fan, Schmeling and Wu[5], treated the problem of multifractal analysis of V -statistics.

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In the present paper, we would like to study the irregular part of the multifractal decomposition.

Let us consider a topological dynamical system (X, f) , with X a compact metric space and f a continuous map. Let $X^r = X \times \dots \times X$ be the product of r -copies of X with $r \geq 1$, if $\Phi : X^r \rightarrow \mathbf{R}$ is a continuous map, then let

$$(1) \quad V_{\Phi}(n, x) = \frac{1}{n^r} \sum_{1 \leq i_1, \dots, i_r \leq n} \Phi(f^{i_1}(x), \dots, f^{i_r}(x)).$$

These averages are called the V -statistics of order r with kernel Φ . For the idea of V -statistics from a Statistical point of view and its relationship with the U -statistics see section 2 of [5]

Ergodic limits of the form

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \Phi(f^{i_1}(x), \dots, f^{i_r}(x)),$$

were studied among others by Furstenberg [8], Bergelson [2] and Bourgain [3].

The multifractal spectra of V -statistics are specified by the decomposition sets

$$E_{\Phi}(\alpha) = \left\{ x : \lim_{n \rightarrow \infty} V_{\Phi}(n, x) = \alpha \right\}.$$

Fan, Schemeling and Wu [5] treated the problem of measuring the sizes of the multifractal sets $E_{\Phi}(\alpha)$. They established the following variational principle:

$$h_{top}(E_{\Phi}(\alpha)) = \sup \left\{ h_{\mu}(f) : \int \Phi d\mu^{\otimes r} = \alpha \right\},$$

where h_{μ} is the measure-theoretic entropy of μ . This formula is valid for dynamical systems with the specification property. This generalizes the variational formula obtained by Takens and Verbitski for $r = 1$ [9].

The *irregular part* of the spectrum, or *historic set*, is the set of points x for which $\lim_{n \rightarrow \infty} V_{\Phi}(n, x)$ does not exist. We denote this set by E_{Φ}^{∞} , so that the space X can be decomposed as

$$X = \bigcup_{\alpha \in \mathbf{R}} E_{\Phi}(\alpha) \cup E_{\Phi}^{\infty}$$

An important problem in Multifractal Analysis is to determine the dimension of the irregular part. For $r = 1$ the irregular part of the spectrum has been extensively studied. Fan, Feng and Wu, in reference [6], did it for topological mixing subshifts. Barreira and Schmeling[1] obtained a similar result than [6] but for Hölder continuous maps. More recently the irregular part was studied by Thompson [11] and by Zhou and Chen [13]. Here we propose the study of the irregular part of the spectrum for multiple ergodic averages. The result to be proved is

Theorem: Let (X, f) be a dynamical system with the property of specification, let $\Phi \in C(X^r)$, $r \geq 1$, if the irregular part E_Φ^∞ of the spectrum of multiple ergodic averages $V_\Phi(n, x)$ is non-empty then it has the same topological entropy as the whole space X .

The case $E_\Phi^\infty = \emptyset$ can occur in situations like for instance Φ cohomologous to 0, or when the ergodic limits $V_\Phi(n, x)$ have the same value for any x .

2. PRELIMINARY DEFINITIONS

Firstly let us recall the Bowen definition of topological entropy of sets: Let $f : X \rightarrow X$, with X a compact metric space, for $n \geq 1$ the dynamical metric, or Bowen metric, is $d_n(x, y) = \max \{d(f^i(x), f^i(y)) : i = 0, 1, \dots, n - 1\}$. We denote by $B_{n,\varepsilon}(x)$ the ball of centre x and radius ε in the metric d_n . Let $Z \subset X$ and let $\mathcal{C}(n, \varepsilon, Z)$ be the collection of finite or countable coverings of the set Z by balls $B_{m,\varepsilon}(x)$ with $m \geq n$. Let

$$M(Z, s, n, \varepsilon) = \inf_{\mathcal{B} \in \mathcal{C}(n, \varepsilon, Z)} \sum_{B_{m,\varepsilon}(x) \in \mathcal{B}} \exp(-sm),$$

and set

$$M(Z, s, \varepsilon) = \lim_{n \rightarrow \infty} M(Z, s, n, \varepsilon).$$

There is an unique number \bar{s} such that $M(Z, s, \varepsilon)$ jumps from $+\infty$ to 0. Let

$$H(Z, \varepsilon) = \bar{s} = \sup \{s : M(Z, s, \varepsilon) = +\infty\} = \inf \{s : M(Z, s, \varepsilon) = 0\},$$

and

$$(2) \quad h_{top}(Z) = \lim_{\varepsilon \rightarrow 0} H(Z, \varepsilon).$$

The number $h_{top}(Z)$ is the *topological entropy* of Z .

A dynamical system (X, f) has the *specification property* if the following condition holds: for $\varepsilon > 0$, there is an integer $M(\varepsilon)$ such that for any finite disjoint collection of integer intervals $I_1 = [a_1, b_1], \dots, I_k = [a_k, b_k]$, of length $\geq M(\varepsilon)$ and for any points $x_1, x_2, \dots, x_k \in X$, there is a point $z \in X$ which ε -shadows the sequence $\{x_1, x_2, \dots, x_k\}$, i.e. $d(f^{a_j+n}(z), f^n(x_j)) \leq \varepsilon$, for any $n = 0, \dots, b_j - a_j$ and $j = 0, 1, \dots, k$.

By $\mathcal{M}(X)$ we denote the space of measures in X , and by $\mathcal{M}_{inv}(X, f)$ the space of f -invariant measures on X . The space $\mathcal{M}(X)$ can be endowed with a metric D compatible with the metric in X , in the sense that $D(\delta_x, \delta_y) = d(x, y)$, where δ is the point mass measure. More precisely the metric considered in $\mathcal{M}(X)$ will be

$$D(\mu, \nu) = \lim_{n \rightarrow \infty} \frac{|\int \varphi_n d\mu - \int \varphi_n d\nu|}{2^n \|\varphi_n\|_\infty},$$

where $\{\varphi_n\}$ is a dense set in $C(X)$. We denote by $B_R(\mu)$ the ball of center μ and radius R in the above metric. The topology induced by this metric is the weak star topology, and if X is compact then $\mathcal{M}(X)$ is compact in the weak topology. The weak star convergence is the convergence in the metric which induces the weak star topology.

The so called empirical measures on X associated to the dynamical system (X, f) are

$$\mathcal{E}_n(x) = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}.$$

We denote the weak limits of the sequence $\{\mathcal{E}_n(x)\}$ by $V(x)$. Since X is compact, $V(x) \neq \emptyset$. If μ is a measure on X then a point $x \in X$ is μ -generic if $V(x) = \{\mu\}$, by $G(\mu)$ is denoted the set of μ -generic points. A result by Bowen[4] is that if μ is ergodic then

$$h_{top}(G(\mu)) = h_\mu(f).$$

For general measures, not necessarily ergodic, the equality holds for dynamical systems with the specification property[7]. This result is the key point in the proof of variational theorem of Fan, Schemeling and Wu[5].

3. PROOF OF THE THEOREM

Let

$$\mathcal{M}_{\Phi}(\alpha) = \left\{ \mu \in \mathcal{M}_{inv}(X) : \int \Phi d\mu^{\otimes r} = \alpha \right\},$$

and let

$$G_{\Phi}(\alpha) = \left\{ x : \text{there is } \{n_k\} \text{ such that } w^* - \lim_{k \rightarrow \infty} \mathcal{E}_{n_k}(x) = \mu \in \mathcal{M}_{\Phi}(\alpha) \right\},$$

here w^* – means weak star convergence.

For $\alpha_1 \neq \alpha_2 \in \mathbf{R}$, we shall find a set $G \subset G_{\Phi}(\alpha_1) \cap G_{\Phi}(\alpha_2)$.

Before proving the theorem we give some lemmas.

Lemma 1: If $\alpha_1 \neq \alpha_2$ then $G_{\Phi}(\alpha_1) \cap G_{\Phi}(\alpha_2) \subset E_{\Phi}^{\infty}$.

Proof: In [5] was established, as a consequence of the Stone-Weierstrass theorem, that for any $\Phi \in C(X^r)$ and for any $\varepsilon > 0$ there is a map $\tilde{\Phi} : X^r \rightarrow \mathbf{R}$ of the form

$$\tilde{\Phi} = \sum_j \varphi_j^{(1)} \otimes \dots \otimes \varphi_j^{(r)},$$

with $\varphi_j^{(i)} \in C(X)$ such that $\|\Phi - \tilde{\Phi}\|_{\infty} < \varepsilon$. Let $x \in G_{\Phi}(\alpha_1) \cap G_{\Phi}(\alpha_2)$, so there are sequences $\{n_k\}, \{m_k\}$ such that

$$(3) \quad \begin{aligned} \mu &= w^* - \lim_{k \rightarrow \infty} \mathcal{E}_{n_k}(x); \mu \in \mathcal{M}_{\Phi}(\alpha_1) \\ \nu &= w^* - \lim_{k \rightarrow \infty} \mathcal{E}_{m_k}(x); \nu \in \mathcal{M}_{\Phi}(\alpha_2), \end{aligned}$$

We have

$$(4) \quad V_{\tilde{\Phi}}(n, x) = \sum_j \prod_{i=1}^r \frac{1}{n} S_n(\varphi_j^{(i)}(x)),$$

where $S_n(\varphi_j^{(i)}(x)) = \sum_{k=0}^{n-1} \varphi_j^{(i)}(f^k(x))$. Therefore, by Eqs.(3-4)

$$\begin{aligned}\lim_{k \rightarrow \infty} V_{\tilde{\Phi}}(n_k, x) &= \int \tilde{\Phi} d\mu^{\otimes r} \\ \lim_{k \rightarrow \infty} V_{\tilde{\Phi}}(m_k, x) &= \int \tilde{\Phi} d\nu^{\otimes r}.\end{aligned}$$

By the above argument of approximation we get in the same way of [5] that

$$\lim_{k \rightarrow \infty} V_{\Phi}(n_k, x) = \int \Phi d\mu^{\otimes r} = \alpha_1 \text{ and } \lim_{k \rightarrow \infty} V_{\Phi}(m_k, x) = \int \Phi d\nu^{\otimes r} = \alpha_2, \text{ with } \alpha_1 \neq \alpha_2.$$

Then $x \in E_{\Phi}^{\infty}$. ■

We have that

$$G_{\Phi}(\alpha) \subset \left\{ x : \exists \mu \in V(x), \text{ such that } h_{\mu}(f) \leq \sup \left\{ h_{\mu}(f) : \int \Phi d\mu^{\otimes r} = \alpha \right\} \right\},$$

and so, by the Bowen lemma

$$h_{top}(G_{\Phi}(\alpha)) \leq \sup \left\{ h_{\mu}(f) : \int \Phi d\mu^{\otimes r} = \alpha \right\}$$

For $\rho_1, \rho_2, \dots, \rho_k \in \mathcal{M}(X)$ and positive numbers R_1, R_2, \dots, R_k , let $x_1, x_2, \dots, x_k \in X$, $n_1, n_2, \dots, n_k \in \mathbf{N}$ such that $\mathcal{E}_{n_j}(x_j) \in B_{R_j}(\rho_j)$, $j = 1, 2, \dots, k$, for a given $\rho_1, \rho_2, \dots, \rho_k \in \mathcal{M}(X)$ and R_1, R_2, \dots, R_k . Let $\varepsilon_1 > 0$, $\varepsilon_2 > 0, \dots, \varepsilon_k > 0$, if $n_i > M(\varepsilon_i)$ (the number of specification), $i = 1, 2, \dots, k$, then by specification property

$$\bigcap_{j=1}^k f^{-M_j-1}(B_{n_j, \varepsilon_j}(x_j)) \neq \emptyset, \text{ with } M_j = n_1 + n_2 + \dots + n_j.$$

Lemma 2: Let $z \in \bigcap_{j=1}^k f^{-M_j-1}(B_{n_j, \varepsilon_j}(x_j))$, then for any $\rho \in \mathcal{M}(X)$ holds

$$D(\mathcal{E}_{M_k}(z), \rho) \leq \frac{1}{M_k} \sum_{j=1}^k n_j (\overline{R_j} + D(\rho_j, \rho)),$$

where $\overline{R_j} = R_j + \varepsilon_j$, $j = 1, 2, \dots, k$,

Remark: It can replace an uniform ε for all balls $B_{n_j, \varepsilon}(x_j)$, by the $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k$, following a trick used in the proof of the proposition 2.1 in [10].

Proof: We have

$$\mathcal{E}_{M_k}(z) = \frac{1}{M_k} \sum_{j=1}^k n_j \mathcal{E}_{n_j}(f^{M_{j-1}}(z)),$$

and

$$D(\mathcal{E}_{n_j}(x_j), \mathcal{E}_{n_j}(f^{M_{j-1}}(z))) \leq \frac{1}{n_j} \sum_{l=0}^{n_j-1} d(f^l(x_j), f^{-M_{j-1}-l}(z)).$$

Therefore

$$\begin{aligned} & D(\mathcal{E}_{M_k}(z), \rho) \\ & \leq \frac{1}{M_k} \sum_{j=1}^k [D(\mathcal{E}_{n_j}(x_j), \mathcal{E}_{n_j}(f^{M_{j-1}}(z))) + D(\mathcal{E}_{n_j}(x_j), \rho_j) + D(\rho_j, \rho)] \\ & \leq \frac{1}{M_k} \sum_{j=1}^k [R_j + \varepsilon_j + D(\rho_j, \rho)] \end{aligned}$$

■.

Lemma 3: Let $\alpha_1 \neq \alpha_2$ with $\mathcal{M}_\Phi(\alpha_1) \neq \emptyset, \mathcal{M}_\Phi(\alpha_2) \neq \emptyset$ then

$$h_{top}(G_\Phi(\alpha_1) \cap G_\Phi(\alpha_2)) = \min \{h_{top}(G_\Phi(\alpha_1)), h_{top}(G_\Phi(\alpha_2))\}.$$

Proof: Since $G_\Phi(\alpha_1) \cap G_\Phi(\alpha_2) \subset G_\Phi(\alpha_1)$ and $G_\Phi(\alpha_1) \cap G_\Phi(\alpha_2) \subset G_\Phi(\alpha_2)$, by the monotonicity of the entropy we have

$$h_{top}(G_\Phi(\alpha_1) \cap G_\Phi(\alpha_2)) \leq \min \{h_{top}(G_\Phi(\alpha_1)), h_{top}(G_\Phi(\alpha_2))\}$$

. To prove the other inequality we shall find a set $G \subset G_\Phi(\alpha_1) \cap G_\Phi(\alpha_2)$ with $h_{top}(G) \geq \min \{h_{top}(G_\Phi(\alpha_1)), h_{top}(G_\Phi(\alpha_2))\}$.

To construct G , let us choose sequences $\{n_k\}, \{R_k\}, \{\varepsilon_k\}$ with $R_k \searrow 0$ and $\varepsilon_k \searrow 0$ and, for a given sequence $\{\rho_1, \rho_2, \dots, \rho_k\} \subset \mathcal{M}(X)$, for $\bar{\varepsilon} > \varepsilon_1$.let us consider $(n_k, \bar{\varepsilon})$ -sets $\Gamma_k \subset \{x : \mathcal{E}_{n_k}(x) \in B_{R_k}(\rho_k)\}$, so that (by the Lemma 2)

$$x \in \Gamma_k, z \in B_{n_k, \varepsilon_k}(x) \implies \mathcal{E}_{n_k}(z) \in B_{R_k + \varepsilon_k}(\rho_k).$$

Let us choose now a strictly increasing sequence $\{N_k\}$ such that

$$n_{k+1} \leq R_k \sum_{j=1}^k n_j N_j$$

and

$$\sum_{j=1}^{k-1} n_j N_j \leq R_k \sum_{j=1}^k n_j N_j.$$

We consider stretched sequences $\{n'_j\}$, $\{\varepsilon'_j\}$, $\{\Gamma'_j\}$ such that if $j = N_1 + \dots + N_{k-1} + q$ with $1 \leq q \leq N_k$ then $n'_j = n_k$, $\varepsilon'_j = \varepsilon_k$ and $\Gamma'_j = \Gamma_k$.

Finally, we can define

$$(5) \quad G_k := \bigcap_{j=1}^k \left(\bigcup_{x_j \in \Gamma'_j} f^{-M_{j-1}} \left(B_{n'_j, \varepsilon'_j}(x_j) \right) \right),$$

with $M_j = n'_1 + n'_2 + \dots + n'_j$ and

$$(6) \quad G := \bigcap_{k \geq 1} G_k.$$

Any element of G can be labelled by a sequence $x_1 x_2 \dots$, with $x_j \in \Gamma'_j$. According to Pfister and Sullivan [10] the following holds: Let $x_j, y_j \in \Gamma'_j$, $x_j \neq y_j$, if $x \in B_{n_j, \varepsilon_j}(x_j)$, $y \in B_{n_j, \varepsilon_j}(y_j)$ then $\max \{d(f^k(x), f^k(y)) : k = 0, \dots, n_j - 1\} > 2\varepsilon$, with $\varepsilon > \varepsilon_1/4$.

We see that $G \subset G_\Phi(\alpha_1) \cap G_\Phi(\alpha_2)$. Let $z \in G$, and let $\mu_0 \in \mathcal{M}_\Phi(\alpha_1)$, $\nu_0 \in \mathcal{M}_\Phi(\alpha_2)$, it can be considered sequences [13] $\{\mu_k\}$, $\{\nu_k\}$ such that $D(\mu_0, \mu_k) < R_k$ and $D(\nu_0, \nu_k) < R_k$, then form the sequence

$$\{\rho_k\} = \{\mu_1, \mu_1, \nu_1, \nu_1, \mu_2, \mu_2, \nu_2, \nu_2, \dots\}.$$

Let $\rho \in \{\mu_0, \nu_0\}$, and $\sum_{l=1}^j n_l N_l \leq M_k \leq \sum_{l=1}^{j+1} n_l N_l$, thus

$$D(\mathcal{E}_{M_k}(z), \rho) \leq \frac{1}{M_k} \sum_{l=1}^{j-1} n_l N_l D \left(\mathcal{E}_{\sum_{i=1}^{j-1} n_i N_i}(z), \rho \right) + \frac{n_j N_j}{M_k} D(\mathcal{E}_{n_j N_j}(z), \rho) + \frac{M_k - \sum_{l=1}^j n_l N_l}{M_k} D(\mathcal{E}_{n_{j+1} N_{j+1}}(z), \rho).$$

Therefore

$$\begin{aligned}
 & D(\mathcal{E}_{M_k}(z), \rho) \\
 & \leq R_j + D(\mathcal{E}_{n_j N_j}(z), \rho_j) + D(\rho_j, \rho) + D(\mathcal{E}_{n_{j+1} N_{j+1}}(z), \rho) + D(\rho_{j+1}, \rho) \\
 & \leq 2R_j + \varepsilon_j + D(\rho_j, \rho) + D(\rho_{j+1}, \rho).
 \end{aligned}$$

Thus, choosing subsequences $t_k = 4k + 1$ and $s_k = 4k + 3$, we get

$$\begin{aligned}
 \mu_0 &= w^* - \lim_{k \rightarrow \infty} \mathcal{E}_{M_{t_k}}(z) \\
 \nu_0 &= w^* - \lim_{k \rightarrow \infty} \mathcal{E}_{M_{s_k}}(z),
 \end{aligned}$$

so that $z \in G_\Phi(\alpha_1) \cap G_\Phi(\alpha_2)$.

To complete the proof it must be proved that

$$h_{top}(G) \geq \min \{h_{top}(G_\Phi(\alpha_1)), h_{top}(G_\Phi(\alpha_2))\}.$$

For this, we follow [10]. Let $s < \bar{h} := \min \{h_{top}(G_\Phi(\alpha_1)), h_{top}(G_\Phi(\alpha_2))\}$, the set G is closed, and so it is compact, let us consider a finite covering \mathcal{U} by balls $B_{m,\varepsilon}(x)$ having non-empty intersection with G . Now

$$M(G, s, N, \varepsilon) = \inf_{\mathcal{U} \in \mathcal{C}(n,\varepsilon,G)} \sum_{B_{m,\varepsilon}(x) \in \mathcal{U}} \exp(-sm).$$

For any finite covering \mathcal{U} of G , we can construct a covering \mathcal{U}_0 in the following way: each ball $B_{m,\varepsilon}(x)$ is replaced by a ball $B_{M_r,\varepsilon}(x)$ with $M_r \leq m \leq M_{r+1}$. Thus

$$M(G, s, N, \varepsilon) = \inf_{\mathcal{U} \in \mathcal{C}(n,\varepsilon,G)} \sum_{B_{m,\varepsilon}(x) \in \mathcal{U}} \exp(-sm) \geq \inf_{\mathcal{U} \in \mathcal{C}(N,\varepsilon,G)} \sum_{B_{M_r,\varepsilon}(x) \in \mathcal{U}_0} \exp(-sM_{r+1}).$$

Now we can consider a covering \mathcal{U}_0 in which

$$m = \max \{r : \text{there is a ball } B_{M_r,\varepsilon}(x) \in \mathcal{U}_0\}.$$

We set

$$W_k := \prod_{i=1}^k \Gamma_i, \quad \overline{W}_m = \bigcup_{k=1}^m W_k.$$

Let $x_j, y_j \in \Gamma_j$, $x_j \neq y_j$, as we pointed out earlier, if $x \in B_{N_j, \varepsilon_j}(x_j)$, $y \in B_{N_j, \varepsilon_j}(y_j)$ then $d(f^l(x), f^l(y)) > 2\varepsilon$

for any $l = 0, \dots, N_j - 1$, and with $\varepsilon > \varepsilon_1/4$. Now for any $x \in B_{M_r, \varepsilon}(z) \cap G$ there is a, uniquely determined $z = z(x) \in W_r$. A word $\bar{w} \in W_j$, with $j = 1, 2, \dots, k$, is called a prefix of a word $w \in W_k$ if the first j -letters of \bar{w} agree with the first j -letters of w . The number of times that each $w \in W_k$ is a prefix of a word in W_m is

$\text{card}W_m/\text{card}W_k$, thus if W is a subset of $\overline{W_m}$ then

$$\sum_{k=1}^m \frac{\text{card}(W \cap W_k)}{\text{card}(W_k)} \geq \text{card}(W_m).$$

If each word in W_m has a prefix contained in a $W \subset \overline{W_m}$ then

$$\sum_{k=1}^m \frac{\text{card}(W \cap W_k)}{\text{card}(W_k)} \geq 1,$$

and since \mathcal{U}_0 is a covering each point of W_m has a prefix associated to a ball in \mathcal{U}_0 .

By this and because $\text{card}W_k \geq \exp(\bar{h}M_r)$, we obtain

$$\sum_{B_{M_r, \varepsilon} \in \mathcal{U}_0} \exp(-sM_r) \geq 1.$$

Thus if r is taken such that $k \geq r$ then $sM_{k+1} \leq \bar{h}M_k$, for $N \geq M_r$, $\mathcal{U} \in \mathcal{G}(N, \varepsilon, G)$.

Therefore

$$\sum_{B_{m, \varepsilon}(x) \in \mathcal{U}} \exp(-sm) \geq 1,$$

and so

$$M(G, s, N, \varepsilon) \geq 1.$$

By this $h_{\text{top}}(G) \geq \bar{h}$.

■

We are now in condition of giving the proof of the theorem. Let

$$\Psi = \Psi_{r,\Phi} : \mathcal{M}(X) \rightarrow \mathbf{R}$$

$$\Psi(\mu) = \int \Phi d\mu^{\otimes r}$$

and let

$h = h_{top}(X)$ be the topological entropy of the whole space X . By the classical variational principle and by the variational principle of [5]

$$h = \sup \{h_\mu(f) : \mu \in \mathcal{M}_{inv}(X, f)\} = \sup_{\alpha \in Im(\Psi)} \{h_\mu(f) : \mu \in \mathcal{M}_\Phi(\alpha)\} =$$

$$\sup_{\alpha \in Im(\Psi)} \{h_{top}(E_\Phi(\alpha))\}.$$

We must show that $h_{top}(E_\Phi^\infty) \geq h$. For any $\gamma > 0$, there is an $\alpha_1 \in Im\Psi$ such that $h_{top}(E_\Phi(\alpha_1)) > h - \gamma$, let $\alpha_2 \in Im\Psi$ and let $\mu_1, \mu_2 \in \mathcal{M}(X, f)$ with $\Psi(\mu_1) = \alpha_1$, $\Psi(\mu_2) = \alpha_2$. The map $\lambda \mapsto \Psi((1 - \lambda)\mu_1 + \lambda\mu_2)$ is continuous. Recall that

$$h_{top}(G_\Phi(\alpha_1) \cap G_\Phi((1 - \lambda)\alpha_1 + \lambda\alpha_2))$$

$$= \min \{h_{top}(G_\Phi(\alpha_1)), h_{top}(G_\Phi((1 - \lambda)\alpha_1 + \lambda\alpha_2))\},$$

then, by the continuity of Ψ as a function of λ , we have

$$h_{top}(E_\Phi^\infty) \geq \lim_{\lambda \rightarrow 0} h_{top}(G_\Phi(\alpha_1) \cap G_\Phi((1 - \lambda)\alpha_1 + \lambda\alpha_2)) \geq$$

$$h_{top}(G_\Phi(\alpha_1)) \geq h_{top}(E_\Phi(\alpha_1)) > h - \gamma.$$

Since γ is arbitrary the result follows. ■

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