Conformal Bounds in Three Dimensions from Entanglement Entropy

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The entanglement entropy of an arbitrary spacetime region A in a three-dimensional conformal field theory (CFT) contains a constant universal coefficient, $F(A)$. For general theories, the value of $F(A)$ is minimized when A is a round disk, F_0, and in that case it coincides with the Euclidean free energy on the sphere. We conjecture that, for general CFTs, the quantity $F(A)/F_0$ is bounded above by the free scalar field result and below by the Maxwell field one. We provide strong evidence in favor of this claim and argue that an analogous conjecture in the four-dimensional case is equivalent to the Hofman-Maldacena bounds. In three dimensions, our conjecture gives rise to similar bounds on the quotients of various constants characterizing the CFT. In particular, it implies that the quotient of the stress-tensor two-point function coefficient and the sphere free energy satisfies $C_T/F_0 \leq 3/(4\pi^2 \log 2 - 6\zeta[3]) \simeq 0.14887$ for general CFTs. We verify the validity of this bound for free scalars and fermions, general $O(N)$ and Gross-Neveu models, holographic theories, $\mathcal{N}=2$ Wess-Zumino models and general ABJM theories.

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The ratio of the trace-anomaly coefficients characterizing any unitary conformal field theory (CFT) in four dimensions is bounded, above and below, by the free scalar and Maxwell field results, respectively [1,2],

$$\left.\frac{c}{a}\right|_{\text{Maxwell}} \leq \frac{c}{a} \leq \left.\frac{c}{a}\right|_{\text{free scalar}}, \tag{1}$$

the numerical values being 18/31 and 3, respectively. Roughly, these "Hofman-Maldacena" (HM) bounds follow from imposing the positivity of the energy flux escaping at null infinity for states resulting from a local insertion of the stress tensor on the vacuum. Analogous constructions in general spacetime dimensions $d \geq 3$ give rise to constraints involving correlators of the stress tensor [3,4].

For odd-dimensional CFTs there is no trace anomaly and the coefficients a, c are not defined. A somewhat canonical general-dimension version of c is provided by the stress-tensor two-point function coefficient, C_T, which is proportional to c in $d=4$. On the other hand, a generalization of a which departs from stress-tensor correlators follows from the entanglement entropy (EE) universal coefficient across a round (hyper-)spherical entangling surface, which we denote by F_0. Again, in $d=4$ one finds $F_0 \propto a$, and hence the analogy. In odd-dimensional theories, this quantity coincides with the Euclidean free energy on the round sphere, $F_0 = -\log Z_{\mathbb{S}^d}$ [5,6]. Also, in $d=3$ it defines a renormalization group monotone for general theories [7–9].

In this Letter we present strong evidence that the quotient C_T/F_0 satisfies bounds analogous to (1) for general three-dimensional CFTs, namely,

$$0 \leq \frac{C_T}{F_0} \leq \left.\frac{C_T}{F_0}\right|_{\text{free scalar}} = \frac{3}{4\pi^2 \log 2 - 6\zeta[3]} \simeq 0.14887\ldots \tag{2}$$

These are particular cases of more general conjectural bounds involving the EE of arbitrary regions in $d=3$. Given some entangling region A, this is given, for a general CFT, by $S^{3d}(A) = c_0 \cdot \text{perimeter}(\partial A)/\delta - F(A) + \mathcal{O}(\delta)$, where c_0 is a nonuniversal coefficient, δ is a UV cutoff, and $F(A)$ is a dimensionless universal coefficient of nonlocal nature. Naturally, the round-disk case anticipated above corresponds to $F_0 \equiv F|_{\partial A = \mathbb{S}^1}$. Recently, it has been proved in [10] that F_0 minimizes $F(A)$ for any given CFT, namely, $F(A)/F_0 \geq 1$ with $F(A) = F_0 \Leftrightarrow A = \text{round disk}$. Consequently, F_0 provides a canonical normalization for $F(A)$. With these provisos in mind, we are ready to formulate the conjecture which is the central proposal of this Letter.

Conjecture.—For general CFTs in three dimensions, the universal coefficient in the entanglement entropy, $F(A)$, of an arbitrary region A normalized by the disk result F_0 is bounded above by the free scalar result and below by the free Maxwell field one. Namely, we conjecture that

$$\left.\frac{F(A)}{F_0}\right|_{\text{Maxwell}} \leq \frac{F(A)}{F_0} \leq \left.\frac{F(A)}{F_0}\right|_{\text{free scalar}} \tag{3}$$

holds for general entangling regions and CFTs.
The lower bound is in fact equivalent to the number n of connected components in the boundary of A, and was proved in [10]. The Maxwell field saturates the lower bound and so do topological theories.

The rest of the Letter will be devoted to provide evidence in support of the above conjecture and to extract various consequences.

Hints from four dimensions.—In $d=4$, the EE universal term is local in nature and appears as the coefficient of a logarithmic divergence. It is given by a linear combination of two theory-independent local integrals over the corresponding entangling surfaces, which appear weighted by the trace-anomaly coefficients a, c. The expression reads [11,12]

$$\frac{S^{4d}_{\log}(A)}{a} = \frac{1}{\pi}\left[\mathcal{W}_{\partial A} + \left(\frac{c}{a} - 1\right)\frac{\mathcal{K}_{\partial A}}{2}\right], \tag{4}$$

where $\mathcal{W}_{\partial A}$ is the so-called Willmore energy [13] of ∂A and $\mathcal{K}_{\partial A}$ is an integral involving a quadratic combination of extrinsic curvatures of ∂A. Observe that we normalized the expression by a following the analogy with the three-dimensional case [14]. Now, $\mathcal{W}_{\partial A}$ and $\mathcal{K}_{\partial A}$ are positive definite and positive semidefinite respectively, so it is straightforward to prove that a conjecture analogous to Eq. (3) in four dimensions, namely,

$$\left.\frac{S^{4d}_{\log}(A)}{a}\right|_{\text{Maxwell}} \leq \frac{S^{4d}_{\log}(A)}{a} \leq \left.\frac{S^{4d}_{\log}(A)}{a}\right|_{\text{free scalar}}, \tag{5}$$

is trivially equivalent to the HM bounds of Eq. (1). Since these have been rigorously proven in [2], Eq. (5) is also true in general. In this case, the local nature of $S^{4d}_{\log}(A)$ limits the content of Eq. (5) to be exactly equivalent to the one of the HM bounds. On the other hand, our three-dimensional conjecture (3) contains much more information, as $F(A)$ does not have a closed geometric expression dependent on just a few coefficients which may be valid for general theories.

On the definition of $F(A)$.—Going back to three dimensions, let us start by observing that a direct computation of $F(A)$ from the EE formula using a lattice regularization does not produce unambiguous results in general. This has to do with the fact that it is not possible to resolve the characteristic scales of region A with a precision better than the UV cutoff, e.g., we cannot distinguish R from $R(1+a\delta)$ with $a\sim\mathcal{O}(1)$. This uncertainty pollutes $F(A)$ via the area-law term, $F\to F - a\cdot c_0\cdot \text{perimeter}(\partial A)$, and the situation cannot be improved by making R larger in the lattice.

In order to define $F(A)$ rigorously we can make use of mutual information [16–19]. Given some region A with characteristic scale R, consider two concentric regions A^- and $\overline{A^+}$ with the same shape as A, defined by moving a distance $\varepsilon/2$ inwards and outwards, respectively, along the normal direction to ∂A. Then, the mutual information $I(A^+,A^-)$ tends, in the $\varepsilon/R \ll 1$ limit, to twice the EE of A, providing a well-defined notion of F in the continuum, namely,

$$I(A^+,A^-) = \kappa\frac{\text{perimeter}(\partial A)}{\varepsilon} - 2F(A) + \mathcal{O}(\varepsilon). \tag{6}$$

The robustness of this way of defining $F(A)$ has been previously exploited in several papers [10,19,20] and we will use it henceforth.

Orbifold theories and multicomponent regions.—Let us consider the case of orbifold theories O—namely, theories obtained from the quotient of some complete parent theory C by some finite symmetry group G. For them, the mutual information is given by [21]

$$\begin{aligned} I^{\mathrm{O}}(A^+,A^-) = {} & I^{\mathrm{C}}(A^+,A^-) - n\log|G| \\ & + \Delta\mathcal{S}(A^+) + \Delta\mathcal{S}(A^-), \end{aligned} \tag{7}$$

where n is the number of connected boundaries of A and $|G|$ is the number of elements of G. For $A^\pm$ formed by more than one connected components $A^\pm = A_1^\pm \cup A_2^\pm \cup \ldots \cup A_k^\pm$, the quantities $\Delta\mathcal{S}(A^\pm) \equiv \mathcal{S}(\rho_{A^\pm}|\otimes_i^k \rho_{A_i^\pm})|_C - \mathcal{S}(\rho_{A^\pm}|\otimes_i^k \rho_{A_i^\pm})|_O$ are the differences of the relative entropies between the reduced density matrix on the region, and the tensor product of the density matrices reduced on each of its components. By monotonicity these differences are positive semi-definite.

Hence, we can obtain $F(A)$ for a given orbifold theory in terms of the complete theory one using Eq. (6). One finds

$$\begin{aligned} F(A)|_{\mathrm{O}} &= F(A)|_{\mathrm{C}} + \frac{n}{2}\log|G| - \frac{1}{2}[\Delta\mathcal{S}(A^+) + \Delta\mathcal{S}(A^-)], \\ F_0|_{\mathrm{O}} &= F_0|_{\mathrm{C}} + \frac{1}{2}\log|G|. \end{aligned} \tag{8}$$

From this, it is easy to argue that orbifolding tends to decrease the value of $F(A)/F_0$, in agreement with our conjecture. Indeed, consider a region with arbitrary topology. In that case, we have

$$n \leq \left.\frac{F(A)}{F_0}\right|_{\mathrm{O}} \leq \frac{F(A)|_{\mathrm{C}} + \frac{n}{2}\log|G|}{F_0|_{\mathrm{C}} + \frac{1}{2}\log|G|} \leq \left.\frac{F(A)}{F_0}\right|_{\mathrm{C}}. \tag{9}$$

The third inequality follows from $n \leq (F(A)/F_0)|_{\mathrm{C}}$, proved in [10], the second is a consequence of Eq. (8), and the first follows from the semi-positivity of $\mathcal{S}(\rho_{A^\pm}| \otimes_i^k \rho_{A_i^\pm})|_O$. Hence, the quotient for the parent theory is always greater than the one of the orbifold. Similarly, in all cases the lower bound is provided by the number of connected boundaries of the region. Therefore, as far as our conjecture is concerned, any conclusions which hold for complete theories are also valid for orbifolds of such theories.

The same happens for infinite symmetry groups. In that case, $\log|G|$ is replaced by a divergent contribution, and the quotient saturates the lower bound appearing in Eq. (9), namely, $(F(A)/F_0)|_{\mathrm{O}} = n$. This implies, in particular, that the Maxwell field, which is an orbifold of the free scalar theory under the group $\mathbb{R}$ implementing $\phi \to \phi + \lambda$ [22] has

$$\left.\frac{F(A)}{F_0}\right|_{\mathrm{Maxwell}} = n \quad \forall\ A, \tag{10}$$

where n is the number of connected boundaries of the region. More precisely, for the Maxwell field we get $F(A)|_{\mathrm{Maxwell}} \sim F(A)|_{\text{free scalar}} + n/4\log[-\log(\delta)]$ that diverges with the regularization scale δ [23]. The same saturation (10) holds for topological models, for which $F(A) = \gamma n$ for some constant γ. Hence, the lower bound in our general conjecture (3) is not only consistent but fully equivalent to the general inequality $n \leq F(A)/F_0$. Let us now try to motivate the upper bound.

Regions with disconnected components and large separations.—In case there is a theory which provides an upper bound for $F(A)/F_0$ for general CFTs and arbitrary regions, this must be given by the free scalar. Indeed, consider an entangling region A consisting of two disconnected subregions $A = A_1 \cup A_2$. Then, we have $S(A_1 \cup A_2) = S(A_1) + S(A_2) - I(A_1, A_2)$, where I is the mutual information. Now, assume that A_1 and A_2 are both disk regions. Then, dividing both sides by F_0 and noticing that divergences cancel in both sides of the equality, one is left with $F(A_1 \cup A_2)/F_0 = 2 + I(A_1, A_2)/F_0$. Now, notice that in the long-distance regime the free scalar provides the greatest possible value of $I(A_1, A_2)$. Indeed, on general grounds, one has $I \sim \ell^{-4\Delta}$ where ℓ is the separation between regions and Δ is the smallest scaling dimension of the model. This is minimized by the free scalar in general dimensions, $\Delta_{\text{free scalar}} = (d-2)/2$, which saturates the corresponding unitarity bound—see, e.g., [24]. This means that $F(A)/F_0$ is absolutely maximized by the free scalar in that case. If one replaces now A_1 and A_2 by arbitrary shapes with characteristic lengths much smaller than ℓ, the inequality Eq. (3) also holds provided it holds for A_1 and A_2 individually.

Regions with disconnected components and thin deformations on a null cone.—Additional evidence follows from the so-called *pinching property*. Consider the causal cone H associated to some disk region A. Parametrizing the cone by some angular coordinate θ and an affine parameter $s \in (0, L)$, where L is the position of A, the region $f(\delta, \epsilon) \equiv \{(\theta, s)/|\theta - \theta_0| < \epsilon, s > \delta\}$ around some arbitrary direction θ_0 is a sector of a conical frustrum—see Fig. 4 in [25] for a drawing. Then, the region $A_1(\delta, \epsilon) \equiv H - f(\delta, \epsilon)$ corresponds to the causal cone of A with such frustrum removed. A_1 has a boundary which corresponds to the original disk boundary for all θ except for $|\theta - \theta_0| < \epsilon$, in which case it is given by the boundary of f. Now, the pinching property establishes that, given some other arbitrary region B [25–28],

$$\begin{aligned} &\lim_{\epsilon\to 0}\lim_{\delta\to 0} I[A_1(\delta, \epsilon), B] = 0, \qquad \text{(interacting CFT)}, \\ &\lim_{\epsilon\to 0}\lim_{\delta\to 0} I[A_1(\delta, \epsilon), B] = I(A, B), \qquad \text{(free CFT)}, \end{aligned} \tag{11}$$

namely, for interacting CFTs—including generalized free fields [29]—the mutual information vanishes identically when we make the tip of f approach the tip of the cone and then we make the conical sector arbitrarily thin. On the other hand, taking the same limits in the case of free CFTs—in the sense of being fields satisfying a local linear equation of motion—we are instead left with the mutual information of the original disk region with B. Hence, considering the entanglement entropy for $A \equiv A_1(\delta, \epsilon) \cup A_2$ where A_2 is a disk region, we have

$$\begin{aligned} &F(\lim_{\epsilon\to 0}\lim_{\delta\to 0} A)/F_0 = 2, \qquad \text{(interacting CFT)}, \\ &F(\lim_{\epsilon\to 0}\lim_{\delta\to 0} A)/F_0 = 2 + \frac{I(A_1, A_2)}{F_0}, \qquad \text{(free CFT)}, \end{aligned} \tag{12}$$

where by A_1 in the second line we mean the disk region which results from fully removing f. This holds regardless of the relative separation between A_1 and A_2. Hence, it is obvious that in this case $F(A)/F_0$ is smaller for any interacting CFT than for any free one. If the construction is repeated using regions other than disks, the same conclusion holds again as long as the individual regions satisfy Eq. (3). Now, Eq. (12) does not say anything about the hierarchy between the free theories themselves. However, strong numerical evidence suggests that

$$\left.\frac{I(A_1, A_2)}{F_0}\right|_{\text{free fermion}} < \left.\frac{I(A_1, A_2)}{F_0}\right|_{\text{free scalar}} \tag{13}$$

for arbitrary spatial regions A_1, A_2 [30]. Hence, once again we find that the free scalar provides an absolute maximum for $F(A)/F_0$ in this case.

Small deformations of a disk region.—Let us now consider regions with a single connected component. The first obvious case is the one of slightly deformed disks. We can parametrize their boundary by the radial equation

$$\frac{r(\theta)}{R} = 1 + \frac{\epsilon}{\sqrt{\pi}} \sum_{\ell} [a_{\ell,(c)} \cos(\ell\theta) + a_{\ell,(s)} \sin(\ell\theta)], \qquad (14)$$

where $\epsilon \ll 1$. Then, at leading order in ϵ, we have [31–33]

$$\frac{F(A)}{F_0} = 1 + \frac{\pi^3}{24} \frac{C_T}{F_0} \sum_{\ell} \ell(\ell^2 - 1) \left[a^2_{\ell,(c)} + a^2_{\ell,(s)}\right] \epsilon^2, \qquad (15)$$

where C_T is the coefficient which controls, for a general CFT, the flat-space stress-tensor two-point function charge, namely,

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle_{\mathbb{R}^3} = \frac{C_T}{x^6} \left[I_{\mu(\rho} I_{\sigma)\nu} - \frac{\delta_{\mu\nu}\delta_{\rho\sigma}}{3} \right], \qquad (16)$$

where $I_{\mu\nu} \equiv \delta_{\mu\nu} - 2x_\mu x_\nu / x^2$. Now, noting that the coefficient which accompanies ϵ^2 is positive semidefinite, applying our conjecture (3) to the deformed-disks case we are left with a conjecture for the quotient of charges C_T/F_0, namely, with Eq. (2). In that expression, the lower bound becomes trivial, as for the three-dimensional Maxwell field this quotient simply vanishes. As anticipated in the introduction, an inequality of this type is highly reminiscent of the four-dimensional HM bounds for the trace-anomaly coefficients c/a—see Fig. 1.

As it turns out, both C_T and F_0 have been computed for a plethora of three-dimensional CFTs and we can test the validity of Eq. (2) in all those cases. In the Supplemental Material [34]—which also includes [35–56]—we have gathered the results, and in Fig. 1 we have plotted them together. We observe that all considered theories satisfy the conjectural bounds. In particular, one finds a similar hierarchy as in the four-dimensional c/a case, with the free scalar [9,57–59] representing the upper bound, the free fermion [9,57–59] taking a lower value, holographic theories [60,61] an even lower one, and the Maxwell field providing the lowest possible one (zero in the three-dimensional case). Explicitly, we have: $(C_T/F_0)|_{\text{free scalar}} = 3/[4\pi^2 \log 2 - 6\zeta(3)] \simeq 0.148\,87$, $(C_T/F_0)|_{\text{free fermion}} = 3/[4\pi^2 \log 2 + 6\zeta(3)] \simeq 0.086\,764$, $(C_T/F_0)|_{\text{EMI}} = 8/\pi^4 \simeq 0.082\,128$, $(C_T/F_0)|_{\text{holography}} = 6/\pi^4 \simeq 0.061\,596$, $(C_T/F_0)|_{\text{Maxwell}} = 0$, where we have also included the result corresponding to the so-called "extensive mutual information" (EMI) model [17,62].

Among the interacting theories considered, we have the Gross-Neveu $O(N)$ UV fixed points models [9,63,64], for which it is possible to find values of N which are both greater and smaller than the free fermion one, the whole

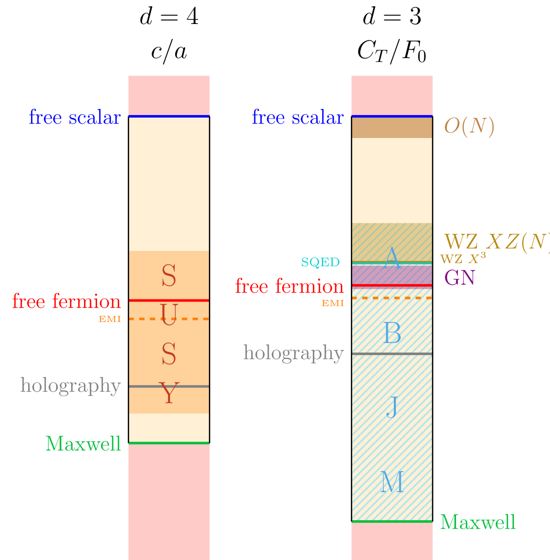

FIG. 1. Quotients c/a in $d=4$ and C_T/F_0 in $d=3$ for various CFTs. For visual clarity, each diagram is normalized by the free scalar result. For $d=4$, the unitarity bounds are known to be saturated by the free scalar and the Maxwell field, respectively. The holographic result, the free fermion and the EMI model (dashed orange) are also shown. For supersymmetric theories, the band of allowed values is smaller and appears displayed in pale orange. In $d=3$ the theories saturating the conjectural bounds are also the free scalar and the Maxwell field, for which $C_T/F_0|_{\text{Maxwell}} = 0$. Besides the free fermion, the EMI model, and holography, we also present the range of values covered by various other theories: the $O(N)$ models for general N (brown band), the Gross-Neveu models for general N (purple band), the $\mathcal{N}=2$ Wess-Zumino model with superpotentials X^3 (orange line), $X\sum_i^N Z_i Z_i$ for general N (pale brown band), SQED (green line), and general ABJM models (pale green band with diagonal lines). Red bands correspond to nonallowed values.

range being $0.0854 \lesssim C_T/F_0|_{\text{GN},O(N)} \lesssim 0.094 \;\forall\; N$. On the other hand, for the Wilson-Fisher fixed points of the scalar $O(N)$ models [9,63,65–67], one finds that the free scalar result is always an upper bound, for arbitrary values of N. The range is $0.1409 \lesssim C_T/F_0|_{O(N)} \leq C_T/F_0|_{\text{free scalar}} \;\forall\; N$. Additional theories considered include various supersymmetric $\mathcal{N}=2$ Wess-Zumino models as well as general $U(N)_k \times U(N)_{-k}$ ABJM models [68], for which we find—using results from [69–77]—that $0 \leq C_T/F_0|_{U(N)_k \times U(N)_{-k}\text{ABJM}} \leq 3/(2\pi^2 \log 4) \simeq 0.109\,63$ for all N and k—see the figure presented in the Supplemental Material [34]. In all cases, the conjectural bounds are respected. It would be certainly interesting to test the conjecture for additional theories.

Ellipses and corners.—Moving from the perturbed-disks regime, values of $F(A)/F_0$ for more complicated regions exist in some cases, at least for a few theories. In particular, there exist results for free scalars and fermions, the EMI model, as well as for holographic theories in the case of

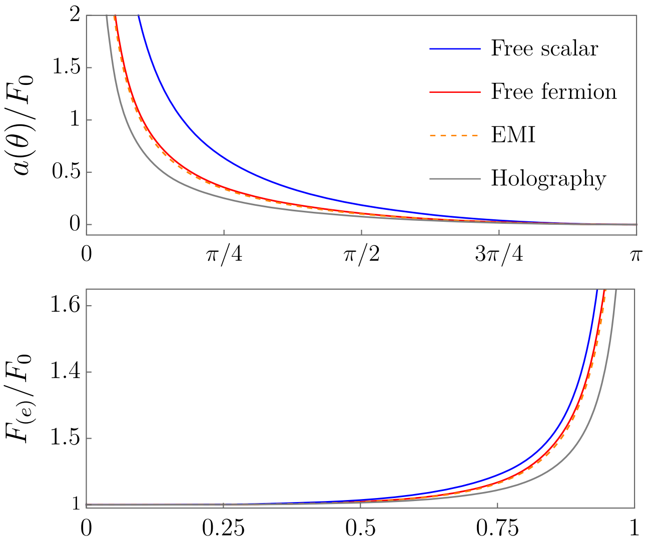

FIG. 2. We plot the EE universal coefficients corresponding, respectively, to a corner region of opening angle θ, and an ellipse of eccentricity e—both normalized by F_0—as functions of those parameters for a free scalar (blue), a free fermion (red), the EMI model (dashed orange) and holographic Einstein gravity (gray). In all cases, the free scalar one lies above the curves of all the rest of the theories. The Maxwell field is a trivial lower bound of constant value (0 and 1, respectively) in both cases.

ellipses of arbitrary eccentricity [10,78]. The results are shown in Fig. 2, where it is clear that the free scalar always takes the greatest value. The lower bound provided by the Maxwell theory is always trivially satisfied by all theories since one has $(F_{(e)}/F_0)|_{\mathrm{Maxwell}} = 1\ \forall\ e$.

In Fig. 2 we have also presented results for the same set of theories in the case of corner regions of opening angle θ [59,79–85]. In that case, $F(A)$ builds up a logarithmic divergence weighted by some function $a(\theta)$ which, normalized by F_0, inherits the same hierarchies as in Eq. (3). Again, the free scalar curve—which also coincides with the one of the large-N limit of the Wilson-Fisher $O(N)$ model [86]—is above all others. On the other hand, there exists a general lower bound for $a(\theta)$ constructed in [87] and given by $a(\theta)/F_0 \geq (\pi^2 C_T/3F_0)\log[1/\sin(\theta/2)]$. In the case of the Maxwell field, the right-hand-side is just zero, so again we find consistency with the lower bound in Eq. (3). Computations of the corner function with $\theta = \pi/2$ have been performed using numerical methods for the $O(N)$ models with $N = 1, 2, 3$ [88–92]. In all cases, the result is very close to the free scalar one, but the precision achieved does not seem to allow for a trustworthy quantitative comparison [93].

Both for ellipses and corners, the behavior in the regime in which the region becomes very sharp—i.e., for $e, \theta \to 0$, respectively—is controlled by the universal coefficient characterizing the EE of a thin strip. Given such a strip of dimensions L, r with $L \gg r$, one finds

$$\frac{F(A)}{F_0} = \frac{\kappa}{F_0}\frac{L}{r} + \mathcal{O}(r^0). \tag{17}$$

The coefficient κ is yet another quantity characterizing any given three-dimensional CFT. It is not known to be related with any other coefficient defined beyond EE, so our general conjecture (3) predicts additional independent bounds on the possible values of κ/F_0. Using the free scalar values of κ computed in [59], one finds

$$0 \leq \frac{\kappa}{F_0} \leq \frac{\kappa}{F_0}\bigg|_{\text{free scalar}} \simeq 0.6223\ldots \tag{18}$$

The values of κ are also known for free fermions [59], the EMI model [17], as well as for holographic theories dual to Einstein gravity [94]. In each of those cases, one finds $(\kappa/F_0)|_{\text{free fermion}} \simeq 0.3297$, $(\kappa/F_0)|_{\mathrm{EMI}} = 1/\pi \simeq 0.3183$, $(\kappa/F_0)|_{\text{holography}} = 2\Gamma[\frac{3}{4}]^2/\Gamma[\frac{1}{4}]^2 \simeq 0.2285$, always in agreement with Eq. (18). Naturally, using Eq. (18) we can obtain putative bounds for κ for any CFT for which F_0 is known. Evaluating this coefficient for additional theories would be another way of testing our general conjecture.

Discussion.—In this Letter we have presented evidence in favor of a new conjecture for the EE universal coefficient of general three-dimensional CFTs. As we have seen, the conjecture fits very well with previous results like the HM bounds in $d = 4$ as well as with the fact that $F(A)/F_0$ is bounded below by the number of connected boundaries of A for general theories. Naturally, it would be very interesting to find a proof (or a counterexample) to our conjectures. This would entail a better understanding of what makes the free scalar theory special from an entropic point view.

An obvious question is whether our conjecture may also extend to higher dimensions. In $d = 5$, an analogous putative upper bound corresponding to a free scalar would imply—via the perturbed spheres EE [32],

$$\frac{C_T}{F_0}\bigg|^{d=5} \leq \frac{C_T}{F_0}\bigg|^{d=5}_{\text{free scalar}} \simeq 0.314\,221\ldots \tag{19}$$

The analogous bound on the strip coefficient would be

$$\frac{\kappa}{F_0}\bigg|^{d=5} \leq \frac{\kappa}{F_0}\bigg|^{d=5}_{\text{free scalar}} \simeq 0.228\,104\ldots \tag{20}$$

In both cases, the lower bound provided by the Maxwell field would always be trivially satisfied, since F_0 diverges for that theory [95]. It is easy to check that both Eq. (19) and Eq. (20) are satisfied for free fermions as well as for holographic theories. A related question is whether or not the round $\mathbb{S}^3$ is the entangling surface with the smallest $F(A)$ in $d = 5$. A study of the $d = 6$ case would also be interesting. This would be trickier than in $d = 4$ since there are four trace-anomaly coefficients rather than two, and the geometric expression of $S_{\log}^{6d}(A)$ is considerably more involved [96,97].

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