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A Wigner Quasiprobability Distribution of Work

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Abstract: In this article, we introduce a quasiprobability distribution of work that is based on the Wigner function. This proposal rests on the idea that the work conducted on an isolated system can be coherently measured by coupling the system to a quantum measurement apparatus. In this way, a quasiprobability distribution of work can be defined in terms of the Wigner function of the apparatus. This quasidistribution contains the information of the work statistics and also holds a clear operational definition that can be directly measured in a real experiment. Moreover, it is shown that the presence of quantum coherence in the energy eigenbasis is related with the appearance of features related to non-classicality in the Wigner function such as negativity and interference fringes. On the other hand, from this quasiprobability distribution, it is straightforward to obtain the standard two-point measurement probability distribution of work and also the difference in average energy for initial states with coherences.

Keywords: quantum thermodynamics; work statistics; quantum coherence

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1. Introduction

The notion of work is one of the most basic and fundamental concepts in physics, particularly in thermodynamics. During the last decades, several attempts have been made to obtain the work statistics for non-equilibrium thermodynamic transformations in the quantum regime. These definitions were motivated by the idea of extending classical fluctuation theorems [1–5] to quantum operations. In order to describe the thermodynamics of general non-equilibrium quantum processes, it is necessary to provide a general definition of work valid for any quantum system and process. However, this task presents serious difficulties. This is due to the fact that many concepts belonging to the classical definition of work cannot be directly translated to quantum mechanics. For example, the basic definition of the work that a force performs on a particle along a trajectory cannot be used in quantum mechanics because of the lack of a ubiquitous meaning of trajectories in the theory, although recently a definition of quantum work was made by considering Bohmian trajectories [6]. A great advancement came in the area with the definition of the two-point measurement protocol (TPM) to define work in driven isolated quantum systems [3,4,7–9]. This definition is based on the simple observation that, for an isolated system, work is a random variable associated to the difference in energy along the process. Thus, in order to determine this random value. one should make an energy measurement at the beginning and another at the end of the process. This definition is not only straightforward in an operational sense, but it also recovers the results of the fluctuation theorems for quantum systems [3–5,7,8,10] and was verified experimentally in different platforms [11–16].

However, there is a caveat with the TPM when one considers initial states that have coherences in the energy basis. This is because the first energy measurement destroys these coherences, and therefore the TPM scheme is insensitive to quantum coherence between different energy subspaces. This leads to undesirable consequences, for instance, related with the fact that the average work performed in the process is different from the change in the average energy of the system. Moreover, it has been shown that it is impossible to define a probability distribution of work that satisfies at the same time the fluctuation theorems and whose mean value of work equals the average energy change for states with coherences [17]. This has led the community to consider different approaches to generalize the work distribution [6,18–20], including some proposals for quasiprobability distributions of work [21–32].

In this article, we propose a distribution based on the Wigner function. This definition relies on the fact that the work probability distribution can also be coherently measured by coupling the system to a quantum apparatus and making a single measurement over the apparatus, i.e., a single-measurement protocol (SM) [13,33,34]. In this way, the final state of the apparatus contains the information about the work distribution and one can define a quasiprobability distribution [35]. This approach provides a clear operational definition with an immediate experimental implementation. In addition, the Wigner function is represented using coordinates that have an intuitive interpretation in terms of time and energy associated with the work. Moreover, it can be shown that the presence of quantum coherence is related with the appearance of features related to non-classicality in the Wigner function, such as negativity and interference fringes. On the other hand, for coherence-free states, this definition agrees with the standard two-point measurement probability distribution of work.

The paper is organized as follows. In Section 2, we briefly discuss the two-point measurement scheme and the single-measurement protocol. In Section 3, we introduce the quasiprobability distribution of work based on the Wigner function, showing how it works for initial states of the system with and without coherence. In Section 4, we discuss experimental implementations, and we end with discussions and conclusion in Section 5.

2. Work Statistics

We are interested in the work distribution for isolated quantum systems that are subjected to an external driving. In this way, the external work can be associated to the energy change of the system. The typical scenario consists of a system $\mathcal{S}$ that starts in a given initial state, $\rho_{\mathcal{S}}$, and is subjected to an external driving, represented by a unitary evolution $\mathcal{U}$. The driving is such that it changes the Hamiltonian from an initial H to a final one $\tilde{H}$, such that

$$H = \sum_n E_n \Pi_n, \qquad \tilde{H} = \sum_m \tilde{E}_m \tilde{\Pi}_m, \tag{1}$$

where Π_n ($\tilde{\Pi}_m$) are the projectors on each energy subspace of the initial (final) Hamiltonian. In this case, what we know is that the average change of energy in the system is

$$\Delta E = \mathrm{tr}\left[\tilde{H}\,\mathcal{U}\,\rho_{\mathcal{S}}\,\mathcal{U}^\dagger\right] - \mathrm{tr}[H\,\rho_{\mathcal{S}}], \tag{2}$$

where $\mathcal{U}\,\rho_{\mathcal{S}}\,\mathcal{U}^\dagger$ is the final state after the driving. Clearly, it would be desirable that the average work obtained from the corresponding probability distribution equals this average energy change. This requisite is equivalent to asking that the first law of thermodynamics for mean values is satisfied for an isolated system. However, it can be shown that, if one imposes that the statistics of work is consistent with the standard fluctuation theorems, the distribution of work should be defined by the two-point measurement protocol [3,4,7,8]. In this case, although the resulting work average coincides with the mean energy difference for initial stationary states (i.e., diagonal in the initial energy eigenbasis), it is different for initial states with coherences.

2.1. The Two-Point Measurement Protocol

The two-point measurement protocol allows us to define a probability distribution of work consistent with fluctuation theorems. In order to do so, one should define a stochastic work value for each realization of the given driving protocol. This is conducted in terms of the difference of two energy values that are obtained by making two projective energy measurements: one at the beginning, and the other one at the end of the driving. In this way, the corresponding probability distribution can be written as

$$P_{\text{TPM}}(w) = \sum_{n,m} p_n p_{m|n} \, \delta\big(w - (\tilde{E}_m - E_n)\big), \tag{3}$$

where p_n is the probability of obtain E_n in the first energy measurement, and $p_{m|n}$ is the conditional probability of obtaining $\tilde{E}_m$ at the end given that E_n was obtained at the beginning. Therefore, if the initial state is already diagonal in the energy eigenbasis, the first measurement does not modify the state, and it is straightforward to verify that the mean value of work equals the average energy difference. Indeed, from (3), we have that, in general, the mean value of work is

$$\begin{aligned} \langle w \rangle &= \int \mathrm{d}w \, P_{\text{TPM}}(w) \, w = \sum_{n,m} p_n p_{m|n} (\tilde{E}_m - E_n) \\ &= \sum_{n,m} \mathrm{tr}\Big[\tilde{\Pi}_m \, \mathcal{U} \, \Pi_n \, \rho_{\mathcal{S}} \, \Pi_n \, \mathcal{U}^\dagger\Big] (\tilde{E}_m - E_n) \\ &= \mathrm{tr}\Big[\tilde{H} \, \mathcal{U} \, \bar{\rho}_{\mathcal{S}} \, \mathcal{U}^\dagger\Big] - \mathrm{tr}[H \, \bar{\rho}_{\mathcal{S}}], \end{aligned} \tag{4}$$

where $\bar{\rho}_{\mathcal{S}} = \sum_n \Pi_n \rho_{\mathcal{S}} \Pi_n$ is the dephased initial state. This state is obtained by removing all the coherences between different energy subspaces of the initial Hamiltonian, and it is equivalent to the state resulting the following asymptotic temporal average

$$\bar{\rho}_{\mathcal{S}} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \mathrm{d}\tau \, e^{-\frac{i}{\hbar} H\tau} \rho_{\mathcal{S}} e^{\frac{i}{\hbar} H\tau}. \tag{5}$$

Therefore, unless the initial state is diagonal in the basis of the initial Hamiltonian, the work average given by the TPM is different from the difference of average energy of the system. In fact, if the initial state is diagonal in this basis, then $\rho_{\mathcal{S}}$ can be interpreted as a 'classical' probability distribution over the different energies. In that case, the first measurement is not invasive, in the sense that it only 'reveals' the value of the energy in each realization of the experiment. On the other hand, for an initial state with coherences, the initial energy is not well defined and this interpretation is not straightforward.

2.2. The Single-Measurement Protocol

Another method for assessing the work probability distribution was introduced in [33]. The method is based on the idea that the work measurement can be described in terms of a generalized measurement (POVM). That is, by coupling the system to an ancilla, which is finally subjected to a 'single measurement' (SM). In this way, it can be shown that one can obtain the same probability distribution provided the ancilla is properly initialized.

Let us now describe briefly the general method that is summarized in the circuit of Figure 1. Initially, the system is in the state $\rho_{\mathcal{S}}$ and there is an auxiliary system (ancilla) $\mathcal{A}$ whose state is described terms of a continuous degree of freedom. In the ancilla's space, one can consider two canonically conjugated operators, $\mathcal{W}_{\mathcal{A}}$ and $\mathcal{T}_{\mathcal{A}}$, such that $[\mathcal{W}_{\mathcal{A}}, \mathcal{T}_{\mathcal{A}}] = i\hbar$. Thus, the evolution contains two coherent interactions between $\mathcal{S}$ and $\mathcal{A}$: one before, $\mathrm{e}^{iH\otimes\mathcal{T}_{\mathcal{A}}/\hbar}$, and another, $\mathrm{e}^{-i\tilde{H}\otimes\mathcal{T}_{\mathcal{A}}/\hbar}$, at the end of the driving. Each interaction can be viewed either as a coherent translation in the variable w of the ancilla in an amount that depends on the energy of the system or, conversely, as a coherent time-translation (free evolution) of the system whose time interval is proportional to the variable τ of the ancilla. Therefore, one can immediately associate the variables w and τ to energy (work) and time, respectively.

This analogy between the variables of the ancilla with work and time will become clearer after analyzing some examples of our proposed distribution.

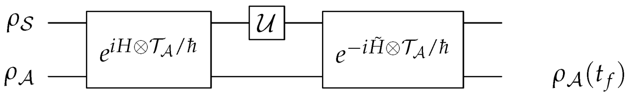

Figure 1. Circuit that describes the single-measurement protocol from which the work probability distribution can be obtained.

Following the protocol of the circuit, if the initial state of $\mathcal{A}$ is $\rho_{\mathcal{A}}$, then at time t_f after both interactions with the system, its reduced state is

$$\rho_{\mathcal{A}}(t_f) = \sum_{n,n',m} \mathrm{tr}\left[\tilde{\Pi}_m \mathcal{U} \Pi_n \rho_{\mathcal{S}} \Pi_{n'} \mathcal{U}^{\dagger}\right] \times \mathrm{e}^{-i w_{nm} \mathcal{T}_{\mathcal{A}}/\hbar} \rho_{\mathcal{A}} \mathrm{e}^{i w_{n'm} \mathcal{T}_{\mathcal{A}}/\hbar}, \tag{6}$$

where $w_{nm} = \tilde{E}_m - E_n$ are the different work values. The SM protocol finishes by performing a projective measurement of the observable $\mathcal{W}_{\mathcal{A}}$. In this case, for highly localized initial pure states of $\mathcal{A}$, the resulting probability distribution is equivalent to the work distribution of the TPM protocol [33]. Notably, within this formulation, one can associate work to an observable that is acting over the ancillary system. Of course, work is not an observable acting on the system's space [4].

It is important to stress at this point that the entangling interaction between system and apparatus establishes a coherent record of the different values of work. Therefore, the reduced state of the ancilla contains information not only about the probability distribution given by the TPM, but also about the initial state of the system. At the end, the type of measurement that is conducted over the ancilla determines which information is extracted from the protocol. It is also interesting to note that this type of interaction appears in a very related task: the work extraction from a quantum system. This can be modeled by adding an interaction between the system and an auxiliary system that acts as a battery in which work is stored [36–38]. In general, the battery can be thought of as a continuous variable system, an ideal weight, with a Hamiltonian like the operator $\mathcal{W}_{\mathcal{A}}$. The work extraction process consists on some unitary evolution on the joint system (where the driving on the system is included) that can change the system Hamiltonian from H to $\tilde{H}$. The extracted work, in this way, is stored in the battery. There are a few conditions that should be imposed in this framework in order to ensure that the weight does not provide any thermodynamical resource to the work extraction process [37], one of them is of course energy conservation. It has been shown in [37] that the unitary operations that satisfy these conditions are of the form $e^{iH\otimes\mathcal{T}_{\mathcal{A}}/\hbar}(\mathcal{U}\otimes\mathbb{I}_{\mathcal{A}})e^{-i\tilde{H}\otimes\mathcal{T}_{\mathcal{A}}/\hbar}$ where $\mathcal{U}$ is the driving of the system. Therefore, it is straightforward to see that these are the same operations (up to a sign) used in the SM protocol for measuring work. Thus, there is also a clear operational interpretation of the state of the ancilla as the state of a battery where work is stored.

3. The Wigner Distribution of Work

In the following, we will define a generalized work distribution. The general idea is inspired by the SM protocol. As we just mentioned, the state of the ancilla after the interaction not only holds information about work, but also about the coherences present in the initial state. In order to extract such information, we will evaluate their Wigner function [39,40], P_W. The Wigner function is a quasiprobability distribution that is used to represent quantum states in phase space. This is a real-valued function that, unlike their classical counterparts, can be negative for generic quantum states. This property has been widely used as an indicator of quantumness in different contexts, for instance in the study of the quantum-classical transition [41,42].

In our case, we will define it for the final state of the ancilla and in terms of the conjugate variables w and τ as

$$P_W(w,\tau) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \mathrm{d}y \left\langle w + \frac{y}{2}\middle|\rho_{\mathcal{A}}(t_f)\middle|w - \frac{y}{2}\right\rangle \mathrm{e}^{-i\tau y/\hbar}$$
$$= \frac{1}{2\pi\hbar}\sum_{n,n',m} \mathrm{tr}\left[\tilde{\Pi}_m \mathcal{U}\Pi_n \rho_{\mathcal{S}}\Pi_{n'}\mathcal{U}^\dagger\right]\int_{-\infty}^{\infty} \mathrm{d}y \left\langle w + \frac{y}{2} - w_{nm}\middle|\rho_{\mathcal{A}}\middle|w - \frac{y}{2} - w_{n'm}\right\rangle \mathrm{e}^{-i\tau y/\hbar} \quad (7)$$

This expression is valid for a generic initial state of the ancilla. In order to evaluate it, we will assume that the initial state of the ancilla is a coherent Gaussian state. This assumption not only will allow us to easily perform analytical calculations, but is also an appropriate choice for the description of typical experimental situations. Moreover, Gaussian states are classical, in the sense that they have a positive Wigner function. This guarantees that any negativity appearing in the Wigner function of the ancilla comes exclusively from their interaction with the system. Thus, we consider $\rho_{\mathcal{A}} = |0,\sigma\rangle\langle 0,\sigma|$ as a coherent Gaussian state with zero mean and variance σ^2 in $\mathcal{W}_{\mathcal{A}}$ (and hence zero mean and variance $\frac{\hbar^2}{2\sigma^2}$ in $\mathcal{T}_{\mathcal{A}}$). After replacing this in Equation (7) (see Appendix A) and using that $\Pi_n \rho_{\mathcal{S}} \mathrm{e}^{i\tau E_n/\hbar} = \Pi_n \mathrm{e}^{i\tau H/\hbar}\rho_{\mathcal{S}}$, we obtain an expression for the quasidistribution of work for a generic process

$$P_W(w,\tau) = \sum_{n,n',m} \mathrm{tr}\left[\tilde{\Pi}_m \mathcal{U}\Pi_n \rho_{\mathcal{S}}(-\tau)\Pi_{n'}\mathcal{U}^\dagger\right]$$
$$\times \mathcal{N}\left(w \,\middle|\, \frac{w_{nm} + w_{n'm}}{2}, \sigma\right)\mathcal{N}\left(\tau \,\middle|\, 0, \frac{\hbar}{\sqrt{2}\sigma}\right), \quad (8)$$

where $\rho_{\mathcal{S}}(-\tau)$ is the state obtained after performing a free evolution of the initial state of the system for a time $-\tau$, and $\mathcal{N}(w \mid \mu, \sigma)$ is a normal probability density in w with mean μ and variance σ^2 (analogously for τ). The fact that the evolved state of the system appears in the distribution is a consequence of quantum coherence. If the initial state has quantum coherence in the energy basis then it is not a steady state, and it will evolve with its free Hamiltonian; therefore, it becomes important the amount of time τ that passes between the preparation of the state and the beginning of the work measurement protocol. From this expression, we can appreciate again the operational interpretation of the variables w and τ that characterize the state of the ancilla.

In the following, we will introduce some notation that will be useful to simplify forthcoming expressions. First, let us recall that the distribution $P_{\mathrm{TPM}}(w)$ does not take into account any coherence between the different energy subspaces of H in the initial state $\rho_{\mathcal{S}}$. Therefore, we can associate this probability distribution to the dephased state $\bar{\rho}_{\mathcal{S}}$. It would then be convenient to define the probability distribution $P_{\mathcal{N}}(w|\sigma)$ that is the convolution of $P_{\mathrm{TPM}}(w)$ with a normal distribution with zero mean and variance σ^2

$$P_{\mathcal{N}}(w|\sigma) = \int_{-\infty}^{\infty} \mathrm{d}u\, P_{\mathrm{TPM}}(w-u)\mathcal{N}(u \mid 0,\sigma) \quad (9)$$
$$= \sum_{n,m} \mathrm{tr}\left[\tilde{\Pi}_m \mathcal{U}\Pi_n \rho_{\mathcal{S}}\Pi_n \mathcal{U}^\dagger\right]\mathcal{N}(w \mid w_{nm},\sigma).$$

Notice that $P_{\mathcal{N}}(w|\sigma)$ is simply the TPM distribution, Equation (3), with the Dirac deltas replaced by a normal distribution with the corresponding mean values of work and variance σ^2. Thus, for a highly localized normal distribution, it satisfies $P_{\mathcal{N}}(w|\sigma) \xrightarrow[\sigma\to 0]{} P_{\mathrm{TPM}}(w)$.

In order to illustrate the effect of initial coherences, let us consider Equation (8), and split it into diagonal ($n = n'$) and non-diagonal ($n \neq n'$) contributions

$$P_W(w,\tau) = P_{\mathcal{N}}(w|\sigma)\mathcal{N}\left(\tau \,\middle|\, 0, \frac{\hbar}{\sqrt{2}\sigma}\right) + P_W^{(\mathrm{c})}(w,\tau). \quad (10)$$

The non-diagonal one corresponds to the contribution of the so-called initial coherences and it is easy to see that

$$\begin{aligned} P_W^{(c)}(w,\tau) &= \sum_{n\neq n',m} \mathrm{tr}\Big[\tilde{\Pi}_m \mathcal{U} \Pi_n \rho_{\mathcal{S}}(-\tau) \Pi_{n'} \mathcal{U}^\dagger\Big] \\ &\times \mathcal{N}\left(w \,\middle|\, \frac{w_{nm}+w_{n'm}}{2}, \sigma\right) \mathcal{N}\left(\tau \,\middle|\, 0, \frac{\hbar}{\sqrt{2}\sigma}\right). \end{aligned} \tag{11}$$

3.1. Quasidistribution for Initial Dephased States

When the initial state of the system is diagonal in the energy basis ($\Pi_n \rho_{\mathcal{S}} \Pi_{n'} = 0$ for $n \neq n'$), then $P_W^{(c)}(w,\tau) = 0$ and the Wigner function is just

$$P_W(w,\tau) = P_{\mathcal{N}}(w|\sigma) \mathcal{N}\left(\tau \,\middle|\, 0, \frac{\hbar}{\sqrt{2}\sigma}\right), \tag{12}$$

that is, it is proportional to the convoluted TPM distribution for every value of τ. Moreover, if we calculate the marginal $P_W(w)$,

$$P_W(w) = \int_{-\infty}^{\infty} \mathrm{d}\tau\, P_W(w,\tau) = P_{\mathcal{N}}(w|\sigma), \tag{13}$$

we recover the probability distribution that would be obtained if one measures the observable $\mathcal{W}_{\mathcal{A}}$. This expression reflects a characteristic property of the Wigner function: The partial integration provides the probability distribution corresponding to the other variable. Therefore, for initial states without coherences in the initial energy eigenbasis, $P_W(w)$ is exactly $P_{\mathcal{N}}(w|\sigma)$. If, in addition, $\sigma \ll (w_{nm} - w_{n'm'}), \forall n, n', m, m'$, then we recover the probability distribution of work given by the TPM protocol.

In Figure 2b, upper panel, we show the distribution $P_W(w,\tau)$ for a two-level system $\mathcal{S}$ without initial coherences. In the lower panel, we show the marginal of the distribution in w, $P_W(w)$, and compare it with the discrete probabilities associated to the TPM. Notice that the area under each Gaussian in the marginal is equal to the corresponding probability in the TPM protocol. We can further see that it effectively reproduces the ideal TPM distribution. On the other hand, in Figure 2a, we show the distribution of work obtained for the same system but using an ancilla that has an initial state with a standard deviation five times smaller. One can easily note that this case is much closer to the ideal projective measurement regime. In this case, the Wigner function is invariant under translations in τ, as expected, since the initial state of the system commutes with the initial Hamiltonian. In Figure 2c, we show the distribution for a standard deviation even bigger than the one in Figure 2b. As we can see, while the position of the peaks matches the correct work values, there is a significant overlap between the different Gaussians.

3.2. Effects of Quantum Coherences

Let us now consider a system with initial coherences. From Equation (10), we can notice that, in this case, the Wigner function also has Gaussian peaks on each work value w_{nm}, just as it happens for the dephased state. However, there are some additional terms centered around the average of two work values with different initial energy, $(w_{nm} + w_{n'm})/2$. These terms are the ones that hold the non-trivial dependence on the variable τ and, as we will see, they can be negative. This can be easily seen from the following argument. If we look at Equation (10), we can see that $\int_{-\infty}^{\infty} \mathrm{d}\tau\, \mathrm{d}w\, P_W(w,\tau) = 1$, and, in addition, also the integral over the phase space of the first term is equal to one, as it is the Wigner function of the initial dephased state. Therefore, the integral of the second term must be zero. In order for it to be so, some of the terms in the sum must be negative. In these terms, besides the global Gaussian modulation, the variable τ appears as a time evolution of the state.

The fact that time appears explicitly only for initial states with coherences has a clear interpretation. If the initial state ρ_S is diagonal, then it is a steady state of the initial Hamiltonian, and the state is the same for every instant in time before the driving is applied. On the other hand, if ρ_S has coherences, the state evolves due to the free evolution induced by the initial Hamiltonian. This time, of course, is irrelevant at the moment of performing the first projecting energy measurement for the TPM distribution. However, it appears in our approach due to the fact that we are performing coherent operations between system and ancilla. Notably, one can also observe that the mean energy difference Equation (2) is not invariant under initial time translations for states with coherences. Thus, given a reference state ρ_S, the calculated distribution contains, in principle, information about every initial state that is unitarily connected with ρ_S by the initial Hamiltonian. Nevertheless, the amplitude of the Wigner function decays exponentially to zero when $\tau \to \pm\infty$ due to the Gaussian modulation, putting in practice some cut-off to the maximum time for which such information can be obtained. At the same time, given the complementary nature of the variables work w and time τ, when localizing the Gaussian in w, we are delocalizing it in τ. We will come back to this issue when we consider the marginals of the distribution.

We have already shown that, if the initial state does not have coherence in the energy basis, the resulting quasidistribution of work function is positive, because the diagonal terms in Equation (10) are all positive. Therefore, if the distribution $P_W(w,\tau)$ has some negativities, it is a signature of the presence of coherences in the initial state. This can be clearly seen in the upper panel of Figure 3, where the quasiprobability distribution of work $P_W(w,\tau)$ of a two-level system is plotted. The Hamiltonians, drivings, and ancilla parameters are identical to those of Figure 2. Moreover, in both cases, the initial state of the system has the same probability distribution in the energy basis. The only difference between Figures 2 and 3 is that, in the latter, the initial state has coherence between the two energy levels. Comparing both figures, we can notice that we effectively have the same Gaussian distributions over the same work values. The key difference lies in the fact that, for the initial coherent state, the quasidistribution displays additional oscillations that become negative. This interference fringes indicate the presence of non-classicality in the Wigner function and in the initial state of the system.

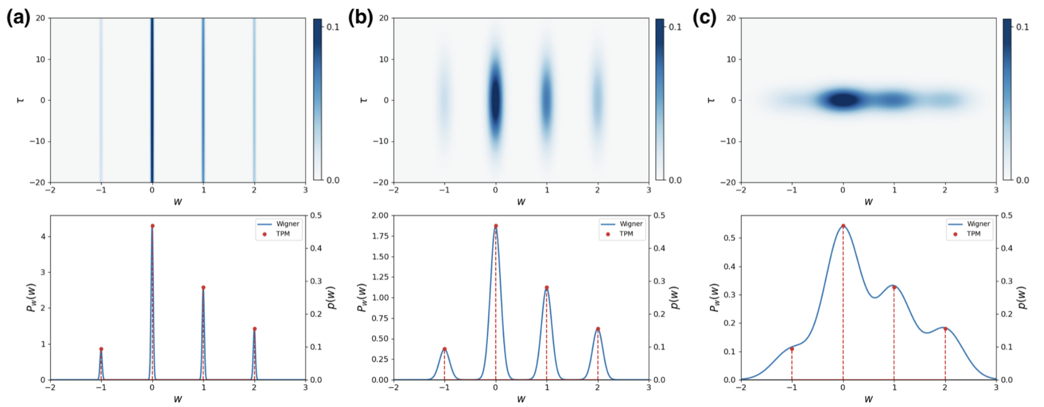

Figure 2. Wigner function of work for a two-level system using a Gaussian ancilla. The initial Hamiltonian is $H = E\sigma_+\sigma_-$, with $\sigma_\pm$ the Pauli creation and annihilation operators. The unitary driving is given by $\mathcal{U} = (\sqrt{2}\mathbb{I} + i\sigma_x + i\sigma_z)/2$ and the final Hamiltonian is $\tilde{H} = 2E\sigma_+\sigma_-$. The initial state of the system is $\rho_S = (\mathbb{I} + \sigma_z/4)/2$ and the variance of the initial Gaussian packets of the ancilla are (**a**) $\sigma = 0.02E$, (**b**) $\sigma = 0.1E$, and (**c**) $\sigma = 0.35E$. The upper panel shows the distribution $P_W(w,\tau)$ of Equation (10) based on the Wigner function. In the lower panel, we show the marginal of w, given by Equation (13), along with the discrete probabilities $p(w)$ corresponding to the usual TPM distribution.

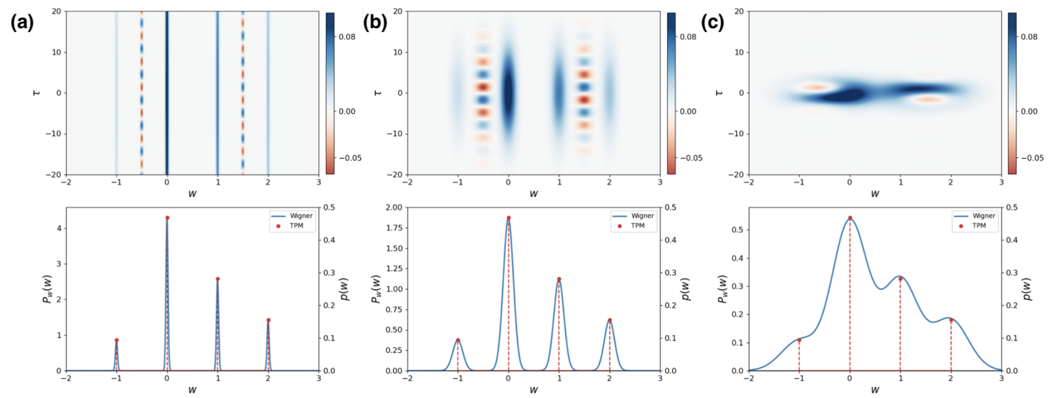

Figure 3. Wigner function for work of a two-level system using a Gaussian ancilla with with variance (**a**) σ = 0.02E, (**b**) σ = 0.1E and (**c**) σ = 0.35E. The parameters used are the same as in Figure 2, except that now the initial density matrix has non-diagonal elements, $\rho_{\mathcal{S}} = (\mathbb{I} + \sigma_x/2 + \sigma_y/2 + \sigma_z/4)/2$. The upper panel shows the distribution $P_W(w,\tau)$ Equation (10) based on the Wigner function. We notice now, because of of the initial coherences, the appearance of negative values in the distribution. The lower panel shows the marginal of w, given by Equation (13), and it is compared to the work values and respective discrete probabilities $p(w)$ that appear in the usual TPM distribution.

In the lower panel of Figure 3, we show the marginal probability distribution for w. Comparing them with Figure 2, we can notice that the marginal distributions are equivalent. In Figure 3, we also show the work distribution for the same system but using an initial state of the ancilla $\mathcal{A}$ with different standard deviations. In the case of the smaller standard deviation (corresponding to an ideal projective measurement), we can see that the marginal probability recovers that of the TPM distribution. Notably, in the case of a bigger standard deviation, the interference between different Gaussian peaks modifies the distribution of w, and there are corrections due to coherences, as shown in Equation (A2). In this limit, the marginal distribution may not even coincide with that of the corresponding dephased state. This behavior is similar to what happen when one makes a weak measurement [43]. To understand when it is possible to observe these differences, lets note that, when the marginal for w is calculated from Equation (10), the diagonal terms give exactly the convoluted distribution $P_{\mathcal{N}}(w|\sigma)$. For the non-diagonal contributions, we have time-averages of the form

$$\int_{-\infty}^{\infty} \mathrm{d}\tau\, \rho_{\mathcal{S}}(-\tau)\mathcal{N}\left(\tau \,\middle|\, 0, \frac{\hbar}{\sqrt{2}\sigma}\right). \tag{14}$$

This operation is similar to a dephasing map on the energy basis, but there is a significant difference since this average is weighted with a normal distribution with variance $\hbar^2/(2\sigma^2)$ centered in the origin. The bigger the variance of the Gaussian (and therefore the smaller σ), more values of τ enter in the time-average. Therefore, in the limit of small σ, we expect the non-diagonal terms to average to zero. Hence, one can show that, if $\sigma \ll (E_n - E_{n'})/2$, $\forall n, n'$, independent of the initial state,

$$\begin{aligned} P_W(w) &= \int_{-\infty}^{\infty} \mathrm{d}\tau\, P_W(w,\tau) \\ &\approx P_{\mathcal{N}}(w|\sigma). \end{aligned} \tag{15}$$

Thus, the marginal of the quasiprobability distribution reproduces the TPM distribution.

3.3. Calculation of Mean Values

Given the formalism associated to the Wigner function [40], one can easily obtain average values from this quasidistribution. In fact, using the Wigner–Weyl representation [40] of an operator $\mathcal{A}$ acting on the ancillary space,

$$A(w,\tau) = \int_{-\infty}^{\infty} \mathrm{d}y \left\langle w + \frac{y}{2} \middle| \mathcal{A} \middle| w - \frac{y}{2} \right\rangle \mathrm{e}^{-i\tau y/\hbar}, \tag{16}$$

their mean value is just

$$\mathrm{tr}\Big[\mathcal{A}\, \rho_{\mathcal{A}}(t_f)\Big] = \int_{-\infty}^{\infty} \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}w\, P_W(w,\tau)\; A(w,\tau). \tag{17}$$

For instance, the mean value of work is just the mean value of the operator $\mathcal{W}_{\mathcal{A}}$, and it is obtained by integrating the function w over the phase space

$$\begin{aligned} \langle w \rangle &\equiv \mathrm{tr}\Big[\mathcal{W}_{\mathcal{A}}\, \rho_{\mathcal{A}}(t_f)\Big] \\ &= \int_{-\infty}^{\infty} \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}w\, P_W(w,\tau)\; w. \end{aligned} \tag{18}$$

The other typical average that is calculated in the context of fluctuation theorems, where the system is initially in thermal equilibrium at inverse temperature β, is $\left\langle \mathrm{e}^{-\beta w} \right\rangle$. This is easily conducted by integration of the function $\mathrm{e}^{-\beta w}$. In all cases, the calculated mean values depend on the initial state of the ancilla. As it can be easily proven, for any observable of the type $f(\mathcal{W}_{\mathcal{A}})$, in the limit of $\sigma \to 0$, their averages converge to the values associated with the TPM distribution.

3.4. Energy Difference in the Presence of Coherences

Finally, we will show another interesting property of the work quasidistribution we have defined. As previously discussed, unless the initial state of the system is diagonal in the energy eigenbasis, the difference in mean energy and the mean value of work (Equations (2) and (4)) do not coincide. Thus, the TPM distribution does not provide any information about the initial coherences. Notably, as we will show, this information is also contained in the quasiprobability distribution.

In order to do so, let us consider the average in phase space of the function $g_{\tau_0}(w,\tau) = w\,\delta(\tau - \tau_0)$ (see Appendix A). Using the Wigner–Weyl transform [40], it corresponds to the expectation value of the operator $\hat{G}_{p_0} = (\mathcal{W}_{\mathcal{A}}|\tau_0\rangle\langle\tau_0| + |\tau_0\rangle\langle\tau_0|\mathcal{W}_{\mathcal{A}})/2$ measured over the ancilla. It can be easily shown that this average, which is equivalent to the integral of the function w weighed by the Wigner function along an horizontal line at τ_0, is proportional to

$$\int_{-\infty}^{\infty} \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}w\, P_W(w,\tau) g_{\tau_0}(w,\tau) \propto \Delta E_{\tau_0}, \tag{19}$$

where $\Delta E_{\tau_0} = \mathrm{tr}\big[\tilde{H}\mathcal{U}\rho_{\mathcal{S}}(-\tau_0)\mathcal{U}^\dagger\big] - \mathrm{tr}[H\rho_{\mathcal{S}}(-\tau_0)]$ is the mean energy difference for a situation where the driving $\mathcal{U}$ is turned on at time $-\tau_0$, and the proportionality constant is just equal to the Gaussian modulation at τ_0, $\mathcal{N}\left(\tau_0 \,\middle|\, 0, \frac{\hbar}{\sqrt{2}\sigma}\right)$ (see Appendix A). Therefore, when $\tau_0 = 0$, this is just proportional to the 'initial' energy difference ΔE in Equation (2). As we have shown, from this quasiprobability distribution, we can calculate not only the energy difference corresponding to the actual initial state, but also for the set of states $\rho_{\mathcal{S}}(\tau)$, $\tau \in \mathbb{R}$. This set can be viewed as different 'initial times' at which the driving is turned on starting from a reference state $\rho_{\mathcal{S}}$ at time zero. This is so because this set of initial states is connected with $\rho_{\mathcal{S}}$ by a free Hamiltonian evolution.

Interestingly, for a Gaussian initial state of the ancilla, one obtains the correct value ΔE independent of their initial variance σ. However, since there is a Gaussian modulation centered around $\tau_0 = 0$ (the proportionality constant), the error in its determination increases as one localizes the initial state of the ancilla in the variable w. However, if one reduces the value of σ, the estimation of $P_{\mathrm{TPM}}(w)$ becomes worse. Hence, one can again appreciate in this case the complementary nature of the variables w and τ.

4. Possible Experimental Implementations

For any proposal of a generalized work distribution to be of practical interest, it should be experimentally accessible and measurable. Here, we discuss how the Wigner work distribution can be measured. The measurement of the quasiprobability distribution that we propose requires two fundamental ingredients: (i) coherent control of two degrees of freedom of system and ancilla in order to implement the interactions of the SM protocol; (ii) being able to measure the Wigner function of the ancilla. In particular, implementing the SM requires the ability of performing translations of the ancilla conditioned on the energy of the degree of freedom on which the work is performed. There is a great variety of systems where this sort of interaction can be implemented, and an experimental realization of the SM protocol has been realized using cold atoms [13]. However, it is not clear how one can implement the measurement of the Wigner function in such platform. Nevertheless, there are systems where both requirements are, in principle, satisfied, and in what follows we will briefly describe two of them.

The first example is given by superconducting qubits coupled to a cavity, e.g., circuit quantum electrodynamics (cQED) [44]. Here, the qubit circuit can be coupled to a waveguide that acts as a microwave cavity where coherent states or states with a well defined number of photons can be stored [45]. For instance, in Ref. [46], they generate coherent displacements of the state of the cavity depending on the state of the qubit. This interaction is exactly what is needed for implementing the protocol where the qubit acts as the system and the cavity as the ancilla. On the other hand, in a different coupling regime between qubit and cavity, this same scheme has been used to measure the Wigner function of the state of the field in the cavity [47].

The second possible platform are trapped ions. In this case, ions are trapped in an electric potential such that the motion degrees of freedom of the ion are subjected to an effective harmonic oscillator potential [48]. At the same time, using the interactions between the electronic degree of freedom and the position of the ion, it is possible to generate coherent, squeezed, and Fock states of the oscillator [48]. In particular, in different experiments [49,50], it has been shown that one can apply forces on the ion depending on its electronic state, and in this way, displacements in phase space depending on the qubit state can be coherently implemented. Again, this is the interaction needed to perform the protocol. The Wigner function of the motion degree of freedom of trapped ions has been successfully measured [51].

5. Conclusions

In this work, we introduced a generalization of the probability distribution of work based on the Wigner function. The starting point is the single-measurement protocol proposed in [33], where an ancilla is coupled to the system whose work one wants to measure in order to keep a coherent record of all possible work values. Following this idea, we define the Wigner function of the final state of the ancilla. This quasiprobability distribution contains all the information regarding both work and coherence in the initial state of the system. In fact, initial quantum coherence in the system results in negativities in the quasiprobability distribution of work, a clear signature of non-classicality. In this case, we can also recover the mean value of energy, which is different from the average work for states with coherences. Moreover, we show that, from this quasiprobability distribution, one can easily recover the standard TPM distribution simply by integrating over the time variable. In addition, we show that, given that the average work and other quantities of interest can be obtained as the mean value of an operator acting on the ancillary space, it is easy to calculate mean values using the formalism of the Wigner function. The quasiprobability distribution here defined has certain similarities with the one proposed in [22,23,52]. The way in which the distribution is defined there is also inspired by the SM scheme [53] and requires the preparation of a coherent superposition of the ancilla between two momentum eigenstates, $|p\rangle + |-p\rangle$, together with the implementation of an interaction analogous to that of the SM. At the end of the protocol, the relative phase between these states is measured and a quasiprobability distribution that contains information about work and coherence is obtained [53]. In contrast, our proposal has a clear operational interpretation and direct experimental application, as it is simply the Wigner function of the final state of the measurement apparatus. Moreover, our protocol not only contains all the information of Refs. [22,23,52], but for coherent initial states, it has additional information on the dependence of the time variable, τ. From a practical point of view, our protocol does not need ideal (non-physical) states and it is easy to adapt to any initial state of the ancilla. Here, we have just developed the case of Gaussian states given that they are easy to treat analytically and are typically appropriate to model experimental conditions. However, this whole analysis can be repeated for any initial state. We hope that this approach to the work distribution can shed some light to elucidate the effects of quantum coherences in thermodynamic transformations.

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Appendix A. Quasiprobability Distribution for Gaussian States

We start with the general expression in Equation (7) for the Wigner function and replace the initial state of the ancilla with a coherent (squeezed) Gaussian state $\rho_{\mathcal{A}} = |0,\sigma\rangle\langle 0,\sigma|$ centered in the origin of coordinates of the phase space and with variance σ^2 in $\mathcal{W}_{\mathcal{A}}$ and $\frac{\hbar^2}{2\sigma^2}$ in $\mathcal{T}_{\mathcal{A}}$. Then, we obtain

$$\begin{aligned}
P_W(w,\tau) &= \frac{1}{2\pi\hbar}\sum_{n,n',m}\mathrm{tr}\left[\tilde{\Pi}_m\mathcal{U}\Pi_n\rho_{\mathcal{S}}\Pi_{n'}\mathcal{U}^\dagger\right]\times \\
&\quad\times\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\mathrm{d}y\,\exp\left[-\frac{\left(w+\frac{y}{2}-w_{nm}\right)^2}{4\sigma^2}\right]\exp\left[-\frac{\left(w-\frac{y}{2}-w_{n'm}\right)^2}{4\sigma^2}\right]\mathrm{e}^{-i\tau y/\hbar} \\
&= \frac{1}{2\pi\hbar}\sum_{n,n',m}\mathrm{tr}\left[\tilde{\Pi}_m\mathcal{U}\Pi_n\rho_{\mathcal{S}}\Pi_{n'}\mathcal{U}^\dagger\right]\exp\left[-\frac{\left(w-\frac{w_{nm}+w_{n'm}}{2}\right)^2}{2\sigma^2}\right]\times \\
&\quad\times\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\mathrm{d}y\,\exp\left[-\frac{\left(y-(w_{nm}-w_{n'm})\right)^2}{2\sigma^2}\right]\mathrm{e}^{-i\tau y/\hbar} \\
&= \sum_{n,n',m}\mathrm{tr}\left[\tilde{\Pi}_m\mathcal{U}\Pi_n\rho_{\mathcal{S}}\Pi_{n'}\mathcal{U}^\dagger\right]\mathcal{N}\left(w\,\middle|\,\frac{w_{nm}+w_{n'm}}{2},\sigma\right)\mathcal{N}\left(\tau\,\middle|\,0,\frac{\hbar}{\sqrt{2}\sigma}\right)\mathrm{e}^{i\tau(E_n-E_{n'})/\hbar},
\end{aligned} \tag{A1}$$

Now, let us calculate the marginal of Equation (A1) in order to see that it gives us the convoluted probability distribution of work of Equation (9). First, we split the function into diagonal and non-diagonal terms as in Equation (10), and then integrate each of them:

$$
\begin{aligned}
P_W(w) &= \int_{-\infty}^{\infty} \mathrm{d}\tau\, P_W(w,\tau) \\
&= P_{\mathcal{N}}(w|\sigma) \int_{-\infty}^{\infty} \mathrm{d}\tau\, \mathcal{N}\left(\tau \,\middle|\, 0, \frac{\hbar}{\sqrt{2}\sigma}\right) \\
&\quad + \sum_{n\neq n',m} \mathrm{tr}\left[\tilde{\Pi}_m \mathcal{U} \Pi_n \rho_{\mathcal{S}} \Pi_{n'} \mathcal{U}^\dagger\right] \mathcal{N}\left(w \,\middle|\, \frac{w_{nm}+w_{n'm}}{2}, \sigma\right) \int_{-\infty}^{\infty} \mathrm{d}\tau\, \mathrm{e}^{i\tau(E_n - E_{n'})/\hbar} \mathcal{N}\left(\tau \,\middle|\, 0, \frac{\hbar}{\sqrt{2}\sigma}\right) \\
&= P_{\mathcal{N}}(w|\sigma) + \sum_{n\neq n',m} \mathrm{tr}\left[\tilde{\Pi}_m \mathcal{U} \Pi_n \rho_{\mathcal{S}} \Pi_{n'} \mathcal{U}^\dagger\right] \mathcal{N}\left(w \,\middle|\, \frac{w_{nm}+w_{n'm}}{2}, \sigma\right) \mathrm{e}^{-\frac{(E_n - E_{n'})^2}{4\sigma^2}}.
\end{aligned} \tag{A2}
$$

Thus, if $\sigma \ll (E_n - E_{n'})/2$, meaning that the dispersion is much smaller than all the energy gaps of the initial Hamiltonian, then the non-diagonal terms become exponentially small and we have

$$
P_W(w) = \int_{-\infty}^{\infty} \mathrm{d}\tau\, P_W(w,\tau) \approx P_{\mathcal{N}}(w|\sigma), \qquad \text{if } \sigma \ll \frac{E_n - E_{n'}}{2}\ \forall n, n'. \tag{A3}
$$

Finally, let us consider the mean value of the $g_{\tau_0}(w,\tau) = w\delta(\tau - \tau_0)$; this is equivalent to an average (weighted by the Wigner function) of work variable w at a given fixed time $\tau = \tau_0$:

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{d}w\, P_W(w,\tau) g_{\tau_0}(w,\tau) = \int_{-\infty}^{\infty} \mathrm{d}w\, P_W(w,\tau_0) w \\
&= \mathcal{N}\left(\tau_0 \,\middle|\, 0, \frac{\hbar}{\sqrt{2}\sigma}\right) \sum_{n,n',m} \mathrm{tr}\left[\tilde{\Pi}_m \mathcal{U} \Pi_n \rho_{\mathcal{S}}(-\tau_0) \Pi_{n'} \mathcal{U}^\dagger\right] \int_{-\infty}^{\infty} \mathrm{d}w\, \mathcal{N}\left(w \,\middle|\, \frac{w_{nm}+w_{n'm}}{2}, \sigma\right) w \\
&= \mathcal{N}\left(\tau_0 \,\middle|\, 0, \frac{\hbar}{\sqrt{2}\sigma}\right) \sum_{n,n',m} \mathrm{tr}\left[\tilde{\Pi}_m \mathcal{U} \Pi_n \rho_{\mathcal{S}}(-\tau_0) \Pi_{n'} \mathcal{U}^\dagger\right] \left(\frac{w_{nm}+w_{n'm}}{2}\right) \\
&= \mathcal{N}\left(\tau_0 \,\middle|\, 0, \frac{\hbar}{\sqrt{2}\sigma}\right) \sum_{n,n',m} \mathrm{tr}\left[\tilde{\Pi}_m \mathcal{U} \Pi_n \rho_{\mathcal{S}}(-\tau_0) \Pi_{n'} \mathcal{U}^\dagger\right] \frac{1}{2}(2\tilde{E}_m - E_n - E_{n'}) \\
&= \mathcal{N}\left(\tau_0 \,\middle|\, 0, \frac{\hbar}{\sqrt{2}\sigma}\right) \left(\mathrm{tr}\left[\tilde{H}\, \mathcal{U} \rho_{\mathcal{S}}(-\tau_0) \mathcal{U}^\dagger\right] - \mathrm{tr}[H \rho_{\mathcal{S}}(-\tau_0)]\right) \\
&\equiv \mathcal{N}\left(\tau_0 \,\middle|\, 0, \frac{\hbar}{\sqrt{2}\sigma}\right) \Delta E_{\tau_0}
\end{aligned}
$$

That is, this integral is proportional to the mean energy difference for an initial state, which may have coherences, when the driving is turned on at time $-\tau_0$. Thus, for $\tau_0 = 0$, it is the usual mean energy difference ΔE of (2).

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