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A Convergence Criterion for a Class of Stationary Inclusions in Hilbert Spaces

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Abstract: Here, we consider a stationary inclusion in a real Hilbert space X, governed by a set of constraints K, a nonlinear operator A, and an element $f \in X$. Under appropriate assumptions on the data, the inclusion has a unique solution, denoted by u. We state and prove a covergence criterion, i.e., we provide necessary and sufficient conditions on a sequence $\{u_n\} \subset X$, which guarantee its convergence to the solution u. We then present several applications that provide the continuous dependence of the solution with respect to the data K, A and f on the one hand, and the convergence of an associate penalty problem on the other hand. We use these abstract results in the study of a frictional contact problem with elastic materials that, in a weak formulation, leads to a stationary inclusion for the deformation field. Finally, we apply the abstract penalty method in the analysis of two nonlinear elastic constitutive laws.

Keywords: stationary inclusion; projection operator; convergence criterion; convergence results; penalty method; frictional contact problem; elastic constitutive law

MSC: 47J22; 47J20; 47J30; 74M10; 74C05

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1. Introduction

Besides existence and uniqueness results, convergence results represent an important topic in Functional Analysis and Numerical Analysis, as well as Differential and Partial Differential Equations Theory. They are important in the study of mathematical models that occur in Mechanics and Engineering Sciences. References in the field include [1–3].

For all these reasons, a considerable effort was undertaken to obtain convergence results in the study of various mathematical problems, including nonlinear equations, inequality problems, fixed point problems, and optimization problems. Most of the convergence results obtained in the literature provide sufficient conditions that guarantee that a given sequence $\{u_n\}$ converges to the solution of the corresponding problem, denoted hereafter as $\mathcal{P}$. In other words, these results do not describe all the sequences that have this property. Therefore, we naturally consider the following problem.

Problem 1 ($\mathcal{Q}_{\mathcal{P}}$)**.** *Provided a metric space (X, d), Problem $\mathcal{P}$, which has a unique solution $u \in X$, provides necessary and sufficient conditions that guarantee the convergence of an arbitrary sequence $\{u_n\} \subset X$ to the solution u.*

In other words, Problem $\mathcal{Q}_{\mathcal{P}}$ provides a convergence criterion to the solution of Problem $\mathcal{P}$.

Note that the solution of Problem $\mathcal{Q}_{\mathcal{P}}$ depends on the structure of the initial problem $\mathcal{P}$, which cannot be provided in this abstract framework and requires additional assumptions. The results after solving Problem $\mathcal{Q}_{\mathcal{P}}$ were obtained in [4], where $\mathcal{P}$ represented a variational inequality and a minimization problem.

Stationary and evolutionary inclusions represent an important topic that arises in the study of nonsmooth problems with multivalued operators. They are closely related to the study of variational inequalities, hemivariational inequalities, and fixed point problems, as explained in [2,5,6], as well as in recent papers [7,8]. Moreover, they can be used in the analysis of various mathematical models that describe the contact of a deformable body with an obstacle, the so-called foundation. A reference in this field is the book [3], which includes results on the well-posedness of stationary and history-dependent inclusions, together with some applications in contact mechanics.

In this current paper, we continue our research in [4] regarding the study of Problem $\mathcal{Q}_{\mathcal{P}}$ by considering a case when $\mathcal{P}$ is an inclusion problem of the form

$$-u \in N_K(Au + f). \tag{1}$$

In this paper, K is a nonempty subset of a real Hilbert space X, N_K represents the outward normal cone of K, $A : X \to X$ is a nonlinear operator, and $f \in X$. Our study is motivated by possible applications in solid and contact mechanics, among others. Indeed, a large number of constitutive laws in nonlinear elasticity and plasticity can be cast in the form (1), as well as a number of mathematical models that describe the contact of a deformable body with a foundation. We provide such examples in the last two sections of the current paper. Moreover, we refer the reader to [9], as well as to the recent book [3], where inclusions of the form (1) have been considered, together with various applications in contact mechanics.

The current manuscript is structured in several sections, as follows. In Section 2, we introduce some preliminary material. Then, in Section 3, we state and prove our main result, Theorem 2. Next, in Section 4, we apply Theorem 2 in order to deduce the continuous dependence of the solution with respect to the data and to obtain a convergence result for an associated penalty problem. In Section 5, we use these convergence results in the study of a specific inclusion problem, which describes the frictional contact of an elastic body with a foundation. In Section 6 we provide an application of the abstract results, obtained Section 4, in the study of two elastic constitutive laws. We conclude the results in Sections 5 and 6 with various mechanical interpretations. Finally, in Section 7, we present some concluding remarks.

2. Preliminaries

Most of the preliminary results we present here can be found in many books or surveys. For the convenience of the reader, these are the books [10–14]. There, details on the framework and notation we used, as well as additional results from the field, can be found.

Throughout this paper, unless otherwise specified, we use the functional framework described in Introduction. Therefore, X represents a real Hilbert space endowed with the inner product $(\cdot,\cdot)_X$ and its associated norm $\|\cdot\|_X := \sqrt{(\cdot,\cdot)_X}$. The set of parts of X is denoted by 2^X, and the notations 0_X and I_X represent the zero element and the identity operator of X, respectively. All of the limits below are considered as $n \to \infty$, even if we do not mention it explicitly. We used the symbols "$\rightharpoonup$" and "$\to$" for weak and strong convergence in various spaces, respectively, which will be specified, except in the case when the convergences takes place in $\mathbb{R}$. For sequence $\{\varepsilon_n\} \subset \mathbb{R}_+$, which converges to zero, we use the simple notation $0 \le \varepsilon_n \to 0$. Finally, we denote by $d(u, K)$ the distance between the element $u \in X$ and the set K, that is

$$d(u, K) = \inf_{v \in K} \|u - v\|_X. \tag{2}$$

We now recall the following definition.

Definition 1. *Let $\{K_n\}$ be a sequence of nonempty subsets of X and let K be a nonempty subset of X. We say that the sequence $\{K_n\}$ converges to K in the sense of Mosco ([15]) and we write $K_n \xrightarrow{M} K$, if the following conditions hold:*

(a) for each $u \in K$, there exists a sequence $\{u_n\}$, such that $u_n \in K_n$ for each $n \in \mathbb{N}$ and $u_n \to u$ in X;

(b) for each sequence $\{u_n\}$, such that $u_n \in K_n$ for each $n \in \mathbb{N}$ and $u_n \rightharpoonup u$ in X, we have $u \in K$.

In Problem (1), we consider the following assumptions using the data.

$$K \text{ is a nonempty closed convex subset of } X. \tag{3}$$

$$\begin{cases} A : X \to X \text{ is a strongly monotone and Lipschitz continuous operator,} \\ \text{that is, } m_A > 0 \text{ and } 0 < L_A < \infty \text{ exist, such that} \\ \text{(a)} \quad (Au - Av, u - v)_X \geq m_A \|u - v\|_X^2 \quad \forall\, u, v \in X, \\ \text{(b)} \quad \|Au - Av\|_X \leq L_A \|u - v\|_X \quad \forall\, u, v \in X. \end{cases} \tag{4}$$

$$f \in X. \tag{5}$$

Recall that in (1) and below, $N_K : X \to 2^X$ is the outward normal cone of K in the sense of convex analysis and $P_K : X \to K$ represents the projection operator on K. Then, the following equivalences hold, for all $u, \xi \in X$:

$$\xi \in N_K(u) \iff u \in K, \quad (\xi, v - u)_X \leq 0 \qquad \forall\, v \in K, \tag{6}$$

$$u = P_K \xi \iff u \in K, \quad (\xi - u, v - u)_X \leq 0 \quad \forall\, v \in K. \tag{7}$$

Note that (6) represents the definition of the outward normal cone on K and (7) represents the so-called variational characterization of the projection. Therefore, using (6), it follows that

$$-u \in N_K\big(Au + f\big) \iff Au + f \in K, \quad (Au + f - v, u)_X \leq 0 \quad \forall\, v \in K. \tag{8}$$

This equivalence will be repeatedly used in the rest of the manuscript. Moreover, recall that the projection operator is monotone and nonexpansive, that is,

$$(P_K\xi_1 - P_K\xi_2, \xi_1 - \xi_2)_X \geq 0 \qquad \forall\, \xi_1, \xi_2 \in X, \tag{9}$$

$$\|P_K\xi_1 - P_K\xi_2\|_X \leq \|\xi_1 - \xi_2\|_X \qquad \forall\, \xi_1, \xi_2 \in X. \tag{10}$$

In addition, using assumption (3), we deduce that for each $u \in X$, the following equality holds:

$$d(u, K) = \|u - P_K u\|_X. \tag{11}$$

On the other hand, it is well known that conditions (4) implies that the operator is invertible; moreover, its inverse $A^{-1} : X \to X$ is a strongly monotone Lipschitz continuous operator with constants $\frac{m_A}{L_A^2}$ and $\frac{1}{m_A}$, respectively. A proof of this result can be found in [16]. Therefore, under assumption (4), the following inequalities hold:

$$(A^{-1}u - A^{-1}v, u - v)_X \geq \frac{m_A}{L_A^2} \|u - v\|_X^2 \qquad \forall\, u, v \in X, \tag{12}$$

$$\|A^{-1}u - A^{-1}v\|_X \leq \frac{1}{m_A} \|u - v\|_X \qquad \forall\, u, v \in X. \tag{13}$$

We now recall the following existence and uniqueness result.

Theorem 1. *Assume (3)–(5). Then, a unique element $u \in X$ exists, such that (1) holds.*

Theorem 1 was proven in [9] using a fixed point argument. There, various convergence results to the solution of this inclusion have been proven and an example arising in Contact Mechanics has been presented.

We now proceed with the following elementary result, which will be used in the next section.

Proposition 1. *Let K be a nonempty closed convex subset of X and let $A = I_X$. Then, for each $f \in X$, the solution of the inclusion (1) is provided by*

$$u = P_K f - f. \tag{14}$$

In addition, if K is a closed ball with a radius of 1 centered at 0_X, then

$$u = \begin{cases} \left(\frac{1}{\|f\|_X} - 1\right) f & \text{if } \|f\|_X > 1, \\ 0 & \text{if } \|f\|_X \leq 1. \end{cases} \tag{15}$$

The proof of Proposition 1 can be found in [3], based on equivalences (6) and (7).

Note that the solution of the inclusion (1) depends on the data A, K, and f. For this reason, below we sometimes use the notation $u(f)$ or $u(K)$. This dependence was studied in [3], where the following results were proven.

Proposition 2. *Assume (3)–(5). Then, the solution $u = u(f)$ of inclusion (1) depends continuously on f, that is, if $u_n = u(f_n)$ denotes the solution of (1) with $f = f_n \in X$, for each $n \in \mathbb{N}$, then*

$$f_n \to f \quad \text{in } X \quad \Longrightarrow \quad u_n \to u \quad \text{in } X. \tag{16}$$

Proposition 3. *Assume (3)–(5). Then, the solution $u = u(K)$ of inclusion (1) depends continuously on K, that is, if for each $n \in \mathbb{N}$, K_n is a nonempty closed convex subset of X and $u_n = u(K_n)$ denotes the solution of (1) with $K = K_n$, then*

$$K_n \xrightarrow{M} K \quad \text{in } X \quad \Longrightarrow \quad u_n \to u \quad \text{in } X. \tag{17}$$

Note that Propositions 2 and 3 provide sequences $\{u_n\} \subset X$, which converge to the solution of the inclusion (1). Nevertheless, these proposition do not describe all the sequences that have this property, as it results from the two elementary examples below.

Example 1. *Consider the inclusion (1) in the particular case $X = \mathbb{R}$, $K = [-1, 1]$, $A = I_X$, and $f = 0$. Then, using (14), we deduce that the solution of inclusion (1) is $u = P_K f - f = 0$. Let $\{u_n\} \subset \mathbb{R}$ be the sequence provided by $u_n = -\frac{1}{n}$ for all $n \in \mathbb{N}$. Then, $u_n \to u$, but we cannot find a sequence $\{f_n\} \subset \mathbb{R}$, such that $f_n \to 0$ and $u_n = u(f_n)$. Indeed, assume that $u_n = u(f_n)$ and $f_n \to 0$. Then, $u_n = P_K f_n - f_n = -\frac{1}{n}$ and, using the analytic expression of the function $x \mapsto P_K x - x$, we deduce that either $f_n = \frac{1}{n} + 1$ or $f_n = \frac{1}{n} - 1$, which contradicts the assumption $f_n \to 0$. It follows from here that the convergence $u_n \to u$ above cannot be deduced as a consequence of Proposition 2.*

Example 2. *Keep the same notation as those in Example 1. We claim that we cannot find a sequence $\{K_n\} \subset \mathbb{R}$, such that $K_n \xrightarrow{M} K$ and u_n is the solution of the inclusion (1) with K_n instead of K. Indeed, arguing by contradiction, assume that there exists K_n, such that $u_n = u(K_n)$ and $K_n \xrightarrow{M} K$. Then, $u_n = P_{K_n} f - f = P_{K_n} 0 = -\frac{1}{n}$. Therefore, K_n is an interval of the form $(-\infty, -\frac{1}{n}]$ or $[a, -\frac{1}{n}]$ with $a \in \mathbb{R}$, $a \leq -\frac{1}{n}$. In both cases, we arrive to a contradiction, as the Mosco convergence $K_n \xrightarrow{M} K$ does not hold. We conclude that the convergence $u_n \to u$ above cannot be deduced as a consequence of Proposition 3.*

3. A Convergence Criterion

In this section, we state and prove a convergence criterion for the solution of the inclusion (1). To this end, under the assumption of Theorem 1, the solution of the inclusion (1) is denoted by u. Moreover, provided an arbitrary sequence $\{u_n\} \subset X$, we consider the following statements:

$$u_n \to u \quad \text{in } X. \tag{18}$$

$$\begin{cases} \text{there exists } 0 \le \varepsilon_n \to 0 \text{ such that } d(Au_n + f, K) \le \varepsilon_n \text{ and} \\ (Au_n + f - v, u_n)_X \le \varepsilon_n(\|v\|_X + 1) \quad \forall\, v \in K, \ \forall\, n \in \mathbb{N}. \end{cases} \tag{19}$$

Our main result in this section is the following.

Theorem 2. *Assume (3)–(5). Then, the statements (18) and (19) are equivalent.*

Proof. Assume that (18) holds and let $n \in \mathbb{N}$. Then, the regularity $Au + f \in K$ implies that

$$d(Au_n + f, K) \le \|(Au_n + f) - (Au + f)\|_X = \|Au_n - Au\|_X$$

and, using assumption (4)(b), we find that

$$d(Au_n + f, K) \le L_A\|u_n - u\|_X. \tag{20}$$

Consider now an arbitrary element $v \in K$. Then, using the equality

$$\begin{aligned} &(Au_n + f - v, u_n)_X \\ &\quad = (Au_n - Au, u_n)_X + (Au + f - v, u_n - u)_X + (Au + f - v, u)_X \end{aligned}$$

and inequality

$$(Au + f - v, u)_X \le 0$$

in (8), we find that

$$\begin{aligned} (Au_n + f - v, u_n)_X &\le (Au_n - Au, u_n)_X + (Au + f - v, u_n - u)_X \\ &\le \|Au_n - Au\|_X\|u_n\|_X + \|Au + f\|_X\|u_n - u\|_X + \|v\|_X\|u_n - u\|_X. \end{aligned} \tag{21}$$

Note also that assumption (18) implies that the sequence $\{u_n\}$ is bounded in X. Therefore, using assumption (4)(b), we deduce that there exists a constant C, which does not depend on n, such that

$$\|Au_n - Au\|_X\|u_n\|_X \le C\|u_n - u\|_X. \tag{22}$$

We now combine inequalities (21) and (22) to see that

$$(Au_n + f - v, u_n)_X \le \big(C + \|Au + f\|_X\big)\|u_n - u\|_X + \|v\|_X\|u_n - u\|_X. \tag{23}$$

Denote

$$\varepsilon_n = \max\Big\{(C + \|Au + f\|_X)\|u_n - u\|_X, \|u_n - u\|_X, L_A\|u_n - u\|_X\Big\} \tag{24}$$

and note that, using assumption (18), it follows that

$$0 \le \varepsilon_n \to 0. \tag{25}$$

Finally, we use (25), (24), (20), and (23) to see that (19) holds.

Conversely, assume that (19) holds. Let $n \in \mathbb{N}$ and denote

$$v_n = P_K(Au_n + f), \qquad w_n = Au_n + f - v_n. \tag{26}$$

Then, we have

$$v_n = Au_n + f - w_n \tag{27}$$

and, using (11) and (19), we deduce that

$$w_n \to 0_X, \tag{28}$$

$$(Au_n + f - v, u_n)_X \leq \varepsilon_n(1 + \|v\|_X) \qquad \forall\, v \in K. \tag{29}$$

We now take $v = Au + f$ in (29) and $v = v_n = Au_n + f - w_n$ in (8) to deduce that

$$(Au_n - Au, u_n)_X \leq \varepsilon_n(1 + \|Au + f\|_X), \quad (Au_n - Au, u)_X - (w_n, u)_X \leq 0$$

and, adding these inequalities, we find that

$$(Au_n - Au, u_n - u)_X \leq \varepsilon_n(1 + \|Au + f\|_X) - (w_n, u)_X.$$

Next, we use assumption (4)(a) on the operator A to find that

$$m_A\|u_n - u\|_X^2 \leq \varepsilon_n(1 + \|Au + f\|_X) + \|w_n\|_X\|u\|_X. \tag{30}$$

Finally, we combine inequality (30) with the convergences $\varepsilon_n \to 0$ and $w_n \to 0_X$, guaranteed by (19) and (28). As a result, we deduce $u_n \to u$ in X, which concludes the proof. □

We remark that Theorem 2 provides an answer to Problem $\mathcal{Q}_{\mathcal{P}}$ in the particular case when Problem $\mathcal{P}$ is the inclusion problem (1). Indeed, it provides a convergence criterion to the solution of this problem.

4. Some Applications

Theorem 2 is useful to obtain various convergence results in the study of the inclusion (1). In this section, we present two types of such results: results concerning the continuous dependence of the solution with respect to the data and a result concerning the convergence of the solution of a penalty problem.

(a) *We start with a continuous dependence result of the solution with respect to the data A and f.* To this end, we consider two sequences $\{A_n\}$ and $\{f_n\}$, such that

$$\left\{\begin{array}{l} A_n\colon X \to X \text{ satisfies condition (4) with } m_n > 0 \text{ and } L_n > 0 \\ \text{and, moreover, there exists } a_n \geq 0,\, m_0 > 0 \text{ such that:} \\ \text{(a)}\ \ \|A_n v - Av\|_X \leq a_n(\|v\|_X + 1) \quad \forall\, v \in X,\ n \in \mathbb{N}. \\ \text{(b)}\ \ a_n \to 0 \ \text{ as } \ n \to \infty. \\ \text{(c)}\ \ m_n \geq m_0 \quad \forall\, n \in \mathbb{N}. \end{array}\right. \tag{31}$$

$$f_n \in X, \qquad f_n \to f \quad \text{in} \ \ X. \tag{32}$$

It follows from Theorem 1 that for each $n \in \mathbb{N}$, there exists a unique solution to the inclusion problem.

$$-u_n \in N_K(A_n u_n + f_n). \tag{33}$$

Moreover, the solution satisfies

$$A_n u_n + f_n \in K, \quad (A_n u_n + f_n - v, u_n)_X \leq 0 \quad \forall\, v \in K. \tag{34}$$

Our first result in this section is the following.

Theorem 3. *Assume* (3)–(5) *and* (31), (32). *Then* $u_n \to u$ *in* X.

Proof. Let $n \in \mathbb{N}$ and $v_0 \in K$ be fixed. We use inequality (34) to write

$$(A_n u_n - A_n 0_X, u_n)_X \le (v_0 - f_n, u_n)_X - (A_n 0_X, u_n)_X$$

and, using assumption (31)(a),(c), we deduce that

$$\begin{aligned} m_0 \|u_n\|_X^2 \le m_n \|u_n\|_X^2 &\le \|v_0 - f_n\|_X \|u_n\|_X + \|A_n 0_X\|_X \|u_n\|_X \\ &\le \|v_0 - f_n\|_X \|u_n\|_X + \|A_n 0_X - A0_X\|_X \|u_n\|_X + \|A0_X\|_X \|u_n\|_X \\ &\le \|v_0 - f_n\|_X \|u_n\|_X + a_n \|u_n\|_X + \|A0_X\|_X \|u_n\|_X. \end{aligned}$$

It follows from here that

$$\|u_n\|_X \le \frac{1}{m_0} \Big(\|v_0 - f_n\|_X + a_n + \|A0_X\|_X \Big)$$

and, using assumptions (31)(b), (32), we deduce that there exists $M > 0$, which does not depend on n, such that

$$\|u_n\|_X \le M \qquad \forall\, n \in \mathbb{N}. \tag{35}$$

Next, we use the regularity $A_n u_n + f_n \in K$ in (34), definition (2), and assumption (31)(a) to see that

$$\begin{aligned} d(Au_n + f, K) &\le \|Au_n + f - A_n u_n - f_n\|_X \\ &\le \|Au_n - A_n u_n\|_X + \|f - f_n\|_X \le a_n (\|u_n\|_X + 1) + \|f - f_n\|_X \end{aligned}$$

and, using the bound (35), we deduce that

$$d(Au_n + f, K) \le a_n (M + 1) + \|f - f_n\|_X \qquad \forall\, n \in \mathbb{N}. \tag{36}$$

Consider now an arbitrary element $v \in K$ and let $n \in \mathbb{N}$. Then, using the equality

$$\begin{aligned} &(Au_n + f - v, u_n)_X \\ &\quad = (Au_n - A_n u_n + f - f_n, u_n)_X + (A_n u_n + f_n - v, u_n)_X \end{aligned}$$

and inequality in (34), we find that

$$(Au_n + f - v, u_n)_X \le (Au_n - A_n u_n + f - f_n, u_n)_X$$

and, therefore,

$$(Au_n + f - v, u_n)_X \le \|Au_n - A_n u_n\|_X \|u_n\|_X + \|f_n - f\|_X \|u_n\|_X.$$

We now use assumption (31)(a) and the bound (35) to deduce that

$$(Au_n + f - v, u_n)_X \le a_n (M + 1) M + M \|f_n - f\|_X. \tag{37}$$

Denote

$$\varepsilon_n = \max \Big\{ a_n (M + 1) + \|f_n - f\|_X,\ a_n M (M + 1) + M \|f_n - f\|_X \Big\}. \tag{38}$$

and note that, using assumptions (31)(b), (32), it follows that

$$0 \le \varepsilon_n \to 0. \tag{39}$$

Finally, we use (36)–(39) to see that condition (19) is satisfied. We are now in a position to use Theorem 2 to deduce the convergence $u_n \to u$ in X, which concludes the proof. □

(b) *We proceed with a result that shows the dependence of the solution with respect to the set of constraints.* To this end, we consider two sequences of $\{a_n\} \subset \mathbb{R}$ and $\{b_n\} \subset X$, such that

$$\begin{cases} \text{(a)} \quad a_n \neq 0 \quad \forall\, n \in \mathbb{N}, \quad a_n \to 1 \text{ as } n \to \infty. \\ \text{(b)} \quad b_n \to 0_X \text{ as } n \to \infty. \end{cases} \tag{40}$$

We define the set K_n by equality

$$K_n = a_n K + b_n. \tag{41}$$

Then, it follows from Theorem 1 that for each $n \in \mathbb{N}$, there exists a unique solution u_n to the inclusion problem

$$-u_n \in N_{K_n}(Au_n + f). \tag{42}$$

Moreover, the solution satisfies

$$Au_n + f \in K_n, \quad (Au_n + f - v, u_n)_X \le 0 \quad \forall\, v \in K_n. \tag{43}$$

Our second result in this section is the following.

Theorem 4. *Assume* (3)–(5) *and* (40), (41). *Then,* $u_n \to u$ *in* X.

Proof. We use Theorem 2 and, to this end, we check in what follows that condition (19) is satisfied. Let $n \in \mathbb{N}$. As $Au_n + f \in K_n$, it follows from (41) that there exists $v_n \in K$, such that $Au_n + f = a_n v_n + b_n$, which implies that

$$v_n = \frac{1}{a_n}(Au_n + f - b_n). \tag{44}$$

Therefore,

$$\begin{aligned} d(Au_n + f, K) &\le \|Au_n + f - v_n\|_X = \Big\|Au_n + f - \frac{1}{a_n}(Au_n + f - b_n)\Big\|_X \\ &= \Big\|(1 - \frac{1}{a_n})(Au_n + f) + \frac{1}{a_n} b_n\Big\|_X, \end{aligned}$$

which implies that

$$d(Au_n + f, K) \le \Big|1 - \frac{1}{a_n}\Big| \|Au_n + f\|_X + \frac{1}{|a_n|} \|b_n\|_X. \tag{45}$$

Now, using (41) and arguments similar to those used in the proof of inequality (35), we find that the sequence $\{u_n\}$ is bounded in X and, therefore, there exists $N > 0$, which does not depend on n, such that

$$\|u_n\|_X \le N, \qquad \|Au_n + f\|_X \le N. \tag{46}$$

Thus, it follows from (45) that

$$d(Au_n + f, K) \leq N\Big|1 - \frac{1}{a_n}\Big| + \frac{1}{|a_n|}\|b_n\|_X. \tag{47}$$

Assume now that $v \in K$. We write

$$(Au_n + f - v, u_n)_X = (Au_n + f - a_n v - b_n, u_n)_X + ((a_n - 1)v + b_n, u_n)_X \tag{48}$$

and, as $a_n v + b_n \in K_n$, using (43), we deduce that

$$(Au_n + f - a_n v - b_n, u_n)_X \leq 0. \tag{49}$$

We now combine (48) and (49) to see that

$$(Au_n + f - v, u_n)_X \leq (|a_n - 1|\|v\|_X + \|b_n\|_X)\|u_n\|_X$$

and, using (46), we find that

$$(Au_n + f - v, u_n)_X \leq N(|a_n - 1|\|v\|_X + \|b_n\|_X). \tag{50}$$

Denote

$$\varepsilon_n = \max\Big\{ N\Big|1 - \frac{1}{a_n}\Big| + \frac{1}{|a_n|}\|b_n\|_X, N|a_n - 1|, N\|b_n\|_X \Big\}. \tag{51}$$

and note that, using assumptions (40), it follows that

$$0 \leq \varepsilon_n \to 0. \tag{52}$$

Finally, we use (52), (51), (47) and (50) to see that condition (19) is satisfied. We are now in a position to use Theorem 2 to deduce the convergence of $u_n \to u$ in X, which concludes the proof. □

(c) *We now present a convergence result concerning a penalty method.* To this, end we consider a numerical sequence $\{\lambda_n\}$, such that

$$\lambda_n > 0 \quad \forall\, n \in \mathbb{N}, \quad \lambda_n \to 0 \ \text{ as } \ n \to \infty, \tag{53}$$

together with the problem of finding u_n, such that

$$u_n \in X, \quad u_n + \frac{1}{\lambda_n}\big(Au_n + f - P_K(Au_n + f)\big) = 0_X. \tag{54}$$

Our third result in this section is the following.

Theorem 5. *Assume (3)–(5) and (53). Then, for each $n \in \mathbb{N}$, Equation (54) has a unique solution. Moreover, $u_n \to u$ in X.*

Proof. The proof is obtained in five steps, which we present in the following.

(Step i) We prove the unique solvability of Equation (54). Let $n \in \mathbb{N}$, $u_n \in X$ and denote

$$\sigma_n = Au_n + f. \tag{55}$$

Then, as $A : X \to X$ is invertible, we have

$$u_n = A^{-1}(\sigma_n - f). \tag{56}$$

Using these equalities, it is easy to see that u_n is a solution of Equation (54) if and only if σ_n is a solution of the equation

$$A^{-1}(\sigma_n - f) + \frac{1}{\lambda_n}(\sigma_n - P_K\sigma_n) = 0_X. \tag{57}$$

Consider now the operator $B_n : X \to X$ defined by

$$B_n\sigma = A^{-1}(\sigma - f) + \frac{1}{\lambda_n}(\sigma - P_K\sigma) \qquad \forall\, \boldsymbol{\sigma} \in X. \tag{58}$$

Then, using the properties (9), (10) and (12), (13) of the operators P_K and A, respectively, it is easy to see that the operator B_n is strongly monotone and Lipschitz continuous with constants $\frac{m_A}{L_A^2}$ and $\frac{1}{m_A} + \frac{2}{\lambda_n}$, that is

$$(B_n\sigma_1 - B_n\sigma_2, \sigma_1 - \sigma_2)_X \geq \frac{m_A}{L_A^2}\,\|\sigma_1 - \sigma_2\|_X^2 \qquad \forall\, \sigma_1,\, \sigma_2 \in X, \tag{59}$$

$$\|B_n\sigma_1 - B_n\sigma_2\|_X \leq \Big(\frac{1}{m_A} + \frac{2}{\lambda_n}\Big)\|\sigma_1 - \sigma_2\|_X \qquad \forall\, \sigma_1,\, \sigma_2 \in X. \tag{60}$$

Therefore, it is invertible, and its inverse, denoted by B_n^{-1}, is defined on X with values in X. We conclude that from here, there exists a unique element σ_n, such that $B_n\sigma_n = 0_X$. Using the definition (58), we obtain the unique solvability of the nonlinear Equation (57) and, equivalently, the unique solvability of the nonlinear Equation (54).

(Step ii) We prove the boundedness of the sequences $\{\sigma_n\}$ *and* $\{u_n\}$. Let $n \in \mathbb{N}$ and let v_0 be a fixed element in K. We use (59) to deduce that

$$\frac{m_A}{L_A^2}\,\|\sigma_n - v_0\|_X^2 \leq (B_n\sigma_n - B_n v_0, \sigma_n - v_0)_X$$

and, since $B_n\sigma_n = 0_X$, $B_n v_0 = A^{-1}(v_0 - f)$, we find that

$$\frac{m_A}{L_A^2}\,\|\sigma_n - v_0\|_X^2 \leq (A^{-1}(v_0 - f), v_0 - \sigma_n)_X \leq \|A^{-1}(v_0 - f)\|_X\|\sigma_n - v_0\|_X,$$

which proves that the sequence $\{\sigma_n - v_0\}$ is bounded in X. This implies that the sequence $\{\sigma_n\}$ is bounded in X and, using (56) we deduce that $\{u_n\}$ is a bounded sequence in X.

(Step iii) We prove the inequality

$$(Au_n + f - v, u_n)_X \leq 0 \qquad \forall\, v \in K,\ n \in \mathbb{N}. \tag{61}$$

Let $n \in \mathbb{N}$ and $v \in K$. We use (55)–(57) and equality $v = P_K v$ to see that

$$\begin{aligned}&(Au_n + f - v, u_n)_X = (\sigma_n - v, A^{-1}(\sigma_n - f))_X \\ &\quad = -\frac{1}{\lambda_n}(\sigma_n - v, \sigma_n - P_K\sigma_n)_X = -\frac{1}{\lambda_n}(\sigma_n - v, (\sigma_n - P_K\sigma_n) - (v - P_K v))_X\end{aligned}$$

which shows that

$$(Au_n + f - v, u_n)_X = -\frac{1}{\lambda_n}\Big[\|\sigma_n - v\|_X^2 - (P_K\sigma_n - P_K v, \sigma_n - v)_X\Big]. \tag{62}$$

Recall that $\lambda_n > 0$ and, moreover, (10) implies that

$$(P_K\sigma_n - P_K v, \sigma_n - v)_X \leq \|P_K\sigma_n - P_K v\|_X\|\sigma_n - v\|_X \leq \|\sigma_n - v\|_X^2.$$

Therefore, using (62), we deduce that (61) holds.

(Step iv) We prove that there exists $M > 0$, such that

$$d(Au_n + f, K)_X \le M\lambda_n \qquad \forall\, n \in \mathbb{N}. \tag{63}$$

Let $n \in \mathbb{N}$. We use (55) and (57) to see that

$$d(Au_n + f, K) = d(\sigma_n, K) = \|\sigma_n - P_K\sigma_n\|_X = \lambda_n \|A^{-1}(\sigma_n - f)\|_X. \tag{64}$$

On the other hand, it follows from the proof of Step (ii) that the sequence $\{\sigma_n\}$ is bounded in X. Therefore, using the properties of the operator A^{-1}, we deduce that there exists $M > 0$, which does not depend on n. such that

$$\|A^{-1}(\sigma_n - f)\|_X \le M \qquad \forall\, n \in \mathbb{N}. \tag{65}$$

Inequality (63) is now a consequence of relations (64) and (65).

(Step v) End of proof. We now combine inequalities (61) and (63) with assumptions (53) to see that condition (19) is satisfied with $\varepsilon_n = M\lambda_n$. Finally, we use Theorem 2 to conclude that the convergence $u_n \to u$ in X holds. □

5. An Example in Contact Mechanics

In this section, we apply the abstract results in Sections 3 and 4 in the variational analysis of a mathematical model that describes the bilateral contact between an elastic body and a foundation. The classical formulation of the problem is the following.

Problem 2 ($\mathcal{M}$)**.** *Find a displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$, such that*

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}) \qquad \text{in} \quad \Omega, \tag{66}$$

$$\mathrm{Div}\, \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \qquad \text{in} \quad \Omega, \tag{67}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on} \quad \Gamma_1, \tag{68}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_2 \qquad \text{on} \quad \Gamma_2, \tag{69}$$

$$u_\nu = 0 \qquad \text{on} \quad \Gamma_3, \tag{70}$$

$$\|\boldsymbol{\sigma}_\tau\| \le g, \quad \boldsymbol{\sigma}_\tau = -g\,\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \ \text{ if } \ \boldsymbol{u}_\tau \ne \boldsymbol{0} \qquad \text{on} \quad \Gamma_3. \tag{71}$$

Here, $\Omega \subset \mathbb{R}^d$ ($d \in \{2, 3\}$) is a domain with smooth boundary Γ divided into three measurable disjoint parts, Γ_1, Γ_2, and Γ_3, such that *meas* $(\Gamma_1) > 0$. It represents the reference configuration of the elastic body. Moreover, $\boldsymbol{\nu}$ is the unit outward normal to Γ, $\mathbb{S}^d$ denotes the space of second order symmetric tensors on $\mathbb{R}^d$. and, below, we use the notation "$\cdot$", $\|\cdot\|$, and $\boldsymbol{0}$ for the inner product, the norm, and the zero element of the spaces $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively. We use notation $\boldsymbol{x} = (x_i)$ to represent a generic point in $\Omega \cup \Gamma$.

We now provide a short description of Problem $\mathcal{M}$ and send the reader to [1,2,17–20] for more details and comments. First, Equation (66) represents the constitutive law of the material, in which $\mathcal{A}$ is the elasticity operator and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ denotes the linearized strain tensor. Equation (67) is the equilibrium equation, in which $\boldsymbol{f}_0$ denotes the density of body forces acting on the body. The boundary condition (68) is the displacement condition, which we use because we assume that the body is held fixed on the part Γ_1 on its boundary. Condition (69) is the traction boundary condition. It models the fact that a traction of density $\boldsymbol{f}_2$ is acting on the part Γ_2 of the surface of the body. The boundary conditions (70) and (71) are the interface laws on Γ_3, where the body is assumed to be in contact with an obstacle, the so-called foundation. Here, u_ν and $\boldsymbol{u}_\tau$ denote the normal and tangential displacement, respectively, and $\boldsymbol{\sigma}_\tau$ is the tangential part of the stress vector $\boldsymbol{\sigma\nu}$. Condition (70) is the bilateral contact condition and condition (71) represents the Tresca friction law, in which g denotes the friction bound.

In the analysis of Problem $\mathcal{M}$, we use the standard notation for Sobolev and Lebesgue spaces associated with Ω and Γ. Moreover, for an element $\boldsymbol{v} \in H^1(\Omega)^d$, we still write $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ to Γ and its normal and tangential components are denoted by v_ν and $\boldsymbol{v}_\tau$ on Γ, provided by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$. In addition, recall that $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$ with $\sigma_\nu = \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{\nu}$.

Next, for the displacement field, we need the space V, and for the stress and strain fields, we need the space Q, defined as follows:

$$V = \{\, \boldsymbol{v} \in H^1(\Omega)^d : \ \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \ \Gamma_1,\ v_\nu = 0 \ \text{ on } \ \Gamma_3 \,\},$$

$$Q = \{\, \boldsymbol{\sigma} = (\sigma_{ij}) : \ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \quad \forall\, i,\, j = 1, \ldots, d \,\}.$$

We use the notation $\boldsymbol{\varepsilon}$ for the deformation operator, that is,

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\,(u_{i,j} + u_{j,i}) \quad \forall\, u \in V,$$

where an index that follows a comma denotes the partial derivative with respect to the corresponding component of $\boldsymbol{x}$, e.g., $u_{i,j} = \frac{\partial u_i}{\partial x_j}$. It is well known that the spaces V and Q are real Hilbert spaces endowed with the inner products

$$(\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, dx. \tag{72}$$

The associated norms on these spaces are denoted by $\|\cdot\|_V$ and $\|\cdot\|_Q$, respectively. Note that, from the definition of the inner product in the spaces V and Q, we have

$$\|\boldsymbol{v}\|_V = \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_Q \quad \forall\, \boldsymbol{v} \in V. \tag{73}$$

In the study of Problem $\mathcal{M}$, we assume that the operator $\mathcal{A}$ satisfies the following condition.

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\
\text{(b) There exists } L_{\mathcal{A}} > 0 \text{ such that} \\
\qquad \|\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)\| \le L_{\mathcal{A}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\
\qquad\quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(c) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\
\qquad (\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \ge m_{\mathcal{A}}\, \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\
\qquad\quad \forall\, \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\
\text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega, \\
\qquad \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\
\text{(e) The mapping } \boldsymbol{x} \mapsto \mathcal{A}(\boldsymbol{x}, \boldsymbol{0}) \text{ belongs to } Q.
\end{array}\right. \tag{74}$$

Moreover, the density of body forces and the friction bound are such that

$$\boldsymbol{f}_0 \in L^2(\Omega)^d, \qquad \boldsymbol{f}_2 \in L^2(\Gamma_2)^d. \tag{75}$$

$$g > 0. \tag{76}$$

Assume now that $(\boldsymbol{u}, \boldsymbol{\sigma})$ represents a couple of regular functions that satisfy (66)–(71). Then, using standard arguments, it follows that

$$u \in V, \quad \int_\Omega \sigma \cdot (\varepsilon(v) - \varepsilon(u))\,dx + g\int_{\Gamma_3} \|v_\tau\|\,da - g\int_{\Gamma_3} \|u_\tau\|\,da \tag{77}$$
$$\geq \int_\Omega f_0 \cdot (v - u)\,dx + \int_{\Gamma_2} f_2 \cdot (v - u)\,da \quad \forall\, v \in V.$$

We now introduce the operator $A : Q \to Q$, the functional $j : V \to \mathbb{R}$ the element $f \in V$, and the set K defined by

$$(A\sigma, \tau)_Q = \int_\Omega \mathcal{A}\sigma \cdot \tau\,dx \qquad \forall\, \sigma,\ \tau \in Q, \tag{78}$$

$$j(v) = \int_{\Gamma_3} \|v_\tau\|\,da, \qquad \forall\, v \in V, \tag{79}$$

$$(f, v)_V = \int_\Omega f_0 \cdot v\,dx + \int_{\Gamma_2} f_2 \cdot v\,da \qquad \forall\, v \in V, \tag{80}$$

$$K = \Big\{\, \tau \in Q \,:\, (\tau, \varepsilon(v))_Q + gj(v) \geq (f, v)_V \ \ \forall\, v \in V \,\Big\}. \tag{81}$$

Then, using (77) and notation (79), (80) we obtain that

$$(\sigma, \varepsilon(v) - \varepsilon(u))_Q + gj(v) - gj(u) \geq (f, v - u)_V. \tag{82}$$

We now use (82) with $v = 2u$ and $v = \mathbf{0}_V$ to find that

$$(\sigma, \varepsilon(u))_Q + gj(u) = (f, u)_V. \tag{83}$$

Therefore, by (82) and (83), we see that

$$(\sigma, \varepsilon(v))_Q + gj(v) \geq (f, v)_V.$$

This inequality and the definition (81) imply that

$$\sigma \in K. \tag{84}$$

Next, we use (81) and (83) to deduce that

$$(\tau - \sigma, \varepsilon(u))_Q \geq 0 \qquad \forall\, \tau \in K$$

and, with notation $\omega = \varepsilon(u)$ for the strain field, we see that

$$(\tau - \sigma, \omega)_Q \geq 0 \qquad \forall\, \tau \in K. \tag{85}$$

On the other hand, the constitutive law (66), definition (78), and equality $\omega = \varepsilon(u)$ show that

$$(\sigma, \tau)_Q = (A\omega, \tau)_Q \qquad \forall\, \tau \in Q$$

and, therefore,

$$\sigma = A\omega. \tag{86}$$

We now combine (84)–(86) to deduce that

$$A\omega \in K, \qquad (A\omega - \tau, \omega)_Q \leq 0 \qquad \forall\, \tau \in K. \tag{87}$$

Finally, inequality (87) and (6) lead to the following variational formulation of Problem $\mathcal{M}$.

Problem 3. $\mathcal{M}^V$. *Find a strain field* $\omega \in Q$ *such that*

$$-\omega \in N_K(A\omega). \tag{88}$$

We now consider the sequences $\{\boldsymbol{f}_{0n}\}$, $\{\boldsymbol{f}_{2n}\}$, $\{g_n\}$ such that, for each $n \in \mathbb{N}$, the following hold.

$$\boldsymbol{f}_{0n} \in L^2(\Omega)^d, \qquad \boldsymbol{f}_{2n} \in L^2(\Gamma_2)^d. \tag{89}$$

$$g_n \geq g. \tag{90}$$

$$\boldsymbol{f}_{0n} \to \boldsymbol{f} \quad \text{in } L^2(\Omega)^d, \qquad \boldsymbol{f}_{2n} \to \boldsymbol{f}_2 \quad \text{in } L^2(\Gamma_2)^d. \tag{91}$$

$$g_n \to g. \tag{92}$$

Then, for each $n \in \mathbb{N}$, we consider the element $\boldsymbol{f}_n \in V$ and the set K_n provided by

$$(\boldsymbol{f}_n, \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_{0n} \cdot \boldsymbol{v}\, dx + \int_{\Gamma_2} \boldsymbol{f}_{2n} \cdot \boldsymbol{v}\, da \qquad \forall\, \boldsymbol{v} \in V, \tag{93}$$

$$K_n = \Big\{ \boldsymbol{\tau} \in Q \,:\, (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + g_n j(\boldsymbol{v}) \geq (\boldsymbol{f}_n, \boldsymbol{v})_V \ \ \forall\, \boldsymbol{v} \in V \Big\}, \tag{94}$$

together with the following problem.

Problem 4. $\mathcal{M}_n^V$. *Find a strain field $\boldsymbol{\omega}_n \in Q$, such that*

$$-\boldsymbol{\omega}_n \in N_{K_n}(\mathcal{A}\boldsymbol{\omega}_n). \tag{95}$$

Our main result in this section is the following.

Theorem 6. *Assume (74)–(76), (89) and (90). Then, Problem $\mathcal{M}^V$ has a unique solution $\boldsymbol{\omega}$, and, for each $n \in \mathbb{N}$, Problem $\mathcal{M}_n^V$ has a unique solution $\boldsymbol{\omega}_n$. Moreover, if (91) and (92) hold, then $\boldsymbol{\omega}_n \to \boldsymbol{\omega}$ in Q.*

Proof. For the existence part, we use Theorem 1 on space $X = Q$. First, we note that

$$(\boldsymbol{w}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{w}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q \qquad \forall\, \boldsymbol{w},\ \boldsymbol{v} \in V \tag{96}$$

and, as $gj(\boldsymbol{v}) \geq 0$ for each $\boldsymbol{v} \in V$, using definition (81), we deduce that $\boldsymbol{\varepsilon}(\boldsymbol{f}) \in K$ and, therefore, K is nonempty. On the other hand, it is easy to see that K is a convex subset of Q. We conclude from here that condition (3) is satisfied. In addition, using assumption (74), we see that

$$(\mathcal{A}\boldsymbol{\sigma} - \mathcal{A}\boldsymbol{\tau}, \boldsymbol{\sigma} - \boldsymbol{\tau})_Q \geq m_{\mathcal{A}} \|\boldsymbol{\sigma} - \boldsymbol{\tau}\|_Q^2, \qquad \|\mathcal{A}\boldsymbol{\sigma} - \mathcal{A}\boldsymbol{\tau}\|_Q \leq L_{\mathcal{A}}\, \|\boldsymbol{\sigma} - \boldsymbol{\tau}\|_Q$$

for all $\boldsymbol{\sigma}, \boldsymbol{\tau} \in Q$. Therefore, condition (4) holds with $m_A = m_{\mathcal{A}}$ and $L_A = L_{\mathcal{A}}$. We are now in a position to use Theorem 1 with $f = \boldsymbol{0}_Q$ to deduce the unique solvability of the inclusion (88). The unique solvability of the inclusion and (95) follows from the same argument.

Assume now that convergences (91) and (92) hold. Then, using definitions (93) and (80), it is easy to see that $\boldsymbol{f}_n \to \boldsymbol{f}$ in V and, therefore, (73) implies that

$$\boldsymbol{\varepsilon}(\boldsymbol{f}_n) \to \boldsymbol{\varepsilon}(\boldsymbol{f}) \quad \text{in } Q. \tag{97}$$

On the other hand, using definitions (81) and (94) of sets K and K_n, together with equality (96), it is easy to set that the following equivalence holds, for each $n \in \mathbb{N}$:

$$\boldsymbol{\sigma} \in K \quad \Longleftrightarrow \quad \frac{g_n}{g}\, \boldsymbol{\sigma} - \frac{g_n}{g}\, \boldsymbol{\varepsilon}(\boldsymbol{f}) + \boldsymbol{\varepsilon}(\boldsymbol{f}_n) \in K_n.$$

We deduce from here that

$$K_n = a_n K + b_n \quad \text{with} \quad a_n = \frac{g_n}{g} \quad \text{and} \quad b_n = \boldsymbol{\varepsilon}(\boldsymbol{f}_n) - \frac{g_n}{g}\,\boldsymbol{\varepsilon}(\boldsymbol{f}).$$

It follows from here that condition (41) is satisfied. Moreover, the convergences (92) and (97) guarantee that the sequences $\{a_n\}$ and $\{b_n\}$ defined above satisfy conditions (40). The convergence result in Theorem 6 is now a direct consequence of Theorem 4. □

Theorem 6 is important from a mechanical point of view as it shows that the weak solution of the contact problems $\mathcal{M}$ also continuously depends on the density of body forces, the density of the traction forces, and the friction bound.

6. An Application in Solid Mechanics

In this section, we provide an example of inclusion in solid mechanics for which the results in Theorem 5 work. More precisely, we introduce and analyze two nonlinear constitutive laws for elastic materials. To this end, again, we use notation $\mathbb{S}^d$ for the space of second order symmetric tensors on $\mathbb{R}^d$ with $d \in \{1,2,3\}$, and recall that the indices i, j, k, and l run between 1 and d. Our construction below is based on rheological arguments, which can be found in [21].

The first constitutive law is obtained by connecting an elastic rheological element in parallel with a rigid–elastic element with constraints. Therefore, we have an additive decomposition of the total stress $\boldsymbol{\sigma} \in \mathbb{S}^d$, i.e.,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^E + \boldsymbol{\sigma}^{RC}. \tag{98}$$

Here, $\boldsymbol{\sigma}^E$ is the stress in the elastic element and $\boldsymbol{\sigma}^{RP}$ is the stress in the rigid–elastic element with constraints. We denote the strain tensor by $\boldsymbol{\varepsilon} \in \mathbb{S}^d$ and we recall that, as the connection is in parallel, this tensor is the same in the two rheological components we considered. We also assume that the constitutive law of the elastic element is provided by

$$\boldsymbol{\sigma}^E = A\boldsymbol{\varepsilon} \tag{99}$$

in which $A = (A_{ijkl}) : \mathbb{S}^d \to \mathbb{S}^d$ is a fourth order tensor. Moreover, we assume that the constitutive law of the rigid-elastic element is provided by

$$\boldsymbol{\varepsilon} \in N_K(\boldsymbol{\sigma}^{RC}) \tag{100}$$

where $K \subset \mathbb{S}^d$ represents the set of constraints and, as usual, N_K represents the outward normal cone to K. The interior of K in the topology of $\mathbb{S}^d$ is denoted by $int\,K$. Then, for stress fields $\boldsymbol{\sigma}^{RC}$, such that $\boldsymbol{\sigma}^{RC} \in int\,K$ we have $N_K(\boldsymbol{\sigma}^{RC}) = \mathbf{0}$ and, therefore, Equation (100) implies that $\boldsymbol{\varepsilon} = \mathbf{0}$. We conclude that this equation describes a rigid behavior. For stress fields $\boldsymbol{\sigma}^{RC}$ such that $\boldsymbol{\sigma}^{RC} \in K - int\,K$ we can have $\boldsymbol{\varepsilon} \neq \mathbf{0}$ and therefore, (100) describes a nonlinear elastic behaviour. An example of set of constraints is the von Mises convex used in [18,22], for instance. It is given by

$$K = \{\, \boldsymbol{\tau} \in \mathbb{S}^d \ : \|\boldsymbol{\tau}_D\| \le k \,\} \tag{101}$$

where $\boldsymbol{\tau}_D$ represents the deviatoric part of the tensor $\boldsymbol{\tau}$ and k is a given yield limit.

We now use relations (98)–(100) to write

$$\boldsymbol{\varepsilon} \in N_K(\boldsymbol{\sigma}^{RC}) = N_K(\boldsymbol{\sigma} - \boldsymbol{\sigma}^E) = N_K(\boldsymbol{\sigma} - A\boldsymbol{\varepsilon})$$

and, using notation $\boldsymbol{\omega} = -\boldsymbol{\varepsilon}$ we obtain the following constitutive law:

$$-\boldsymbol{\omega} = N_K(A\boldsymbol{\omega} + \boldsymbol{\sigma}) \tag{102}$$

The second consitutive law is obtained by connecting a linearly elastic rheological element in parallel with a rigid–elastic rheological element without constraints. Therefore, we keep the notation $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}^E$, and $\boldsymbol{\varepsilon}$ introduced above and we denote the stress in the rigid–elastic element by $\boldsymbol{\sigma}^{RE}$. We have

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^E + \boldsymbol{\sigma}^{RE}. \tag{103}$$

and we assume now that the constitutive law of the rigid–elastic element is provided by

$$\boldsymbol{\varepsilon} = \frac{1}{\lambda}(\boldsymbol{\sigma}^{RE} - P_K\boldsymbol{\sigma}^{RE}). \tag{104}$$

Here, again K represents the domain of rigidity of the material, assumed to be a nonempty closed convex subset of $\mathbb{S}^d$ and, in addition, P_K denotes the projection operator on K and $\lambda > 0$ is a provided elastic coefficient. Note that for stress fields $\boldsymbol{\sigma}^{RE}$, such that $\boldsymbol{\sigma}^{RE} \in K$, we have $\boldsymbol{\sigma}^{RE} = P_K\boldsymbol{\sigma}^{RE}$ and, therefore, (104) implies that $\boldsymbol{\varepsilon} = \mathbf{0}$, which shows that this equation describes a rigid behavior. For stress fields $\boldsymbol{\sigma}^{RE}$, such that $\boldsymbol{\sigma}^{RE} \notin K$, we have $\boldsymbol{\varepsilon} \neq \mathbf{0}$.

We now use relations (104), (103) and (99) to write

$$\begin{aligned}\boldsymbol{\varepsilon} &= \frac{1}{\lambda}(\boldsymbol{\sigma}^{RE} - P_K\boldsymbol{\sigma}^{RE}) = \frac{1}{\lambda}\big(\boldsymbol{\sigma} - \boldsymbol{\sigma}^E - P_K(\boldsymbol{\sigma} - \boldsymbol{\sigma}^E)\big)\\ &= \frac{1}{\lambda}\big(\boldsymbol{\sigma} - A\boldsymbol{\varepsilon} - P_K(\boldsymbol{\sigma} - A\boldsymbol{\varepsilon})\big)\end{aligned}$$

and, using notation $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_\lambda = -\boldsymbol{\omega}_\lambda$ in order to underline the dependence of the strain field on the coefficient λ, we obtain the following constitutive law:

$$\boldsymbol{\omega}_\lambda + \frac{1}{\lambda}\big(A\boldsymbol{\omega}_\lambda + \boldsymbol{\sigma} - P_K(A\boldsymbol{\omega}_\lambda + \boldsymbol{\sigma})\big) = 0 \tag{105}$$

A brief comparaison between the constitutive laws (105) and (102) reveals the fact that (102) is expressed in terms of inclusions and involves unilateral constraints. In contrast, the law (105) is in the form of an equality and does not involve unilateral constraints. For these reasons, we say that (105) is more regular that the constitutive law (102). Consider now the following assumptions.

$$A : \mathbb{S}^d \to \mathbb{S}^d \text{ is a positively symmetric fourth order tensor.} \tag{106}$$

$$K \subset \mathbb{S}^d \text{ is an nonempty closed subset.} \tag{107}$$

Our main result in this section is the following.

Theorem 7. *Assume (106) and (107). Then, for every stress tensor $\boldsymbol{\sigma} \in \mathbb{S}^d$ there exists a unique solution $\boldsymbol{\omega} \in \mathbb{S}^d$ to include (102) and, for every $\boldsymbol{\sigma} \in \mathbb{S}^d$ and $\lambda > 0$, there exists a unique solution $\boldsymbol{\omega}_\lambda \in \mathbb{S}^d$ to Equation (105). Moreover, $\boldsymbol{\omega}_\lambda \to \boldsymbol{\omega}$ in $\mathbb{S}^d$ is denoted as $\lambda \to 0$.*

Theorem 7 is a direct consequence of Theorem 5. In addition to the mathematical interest in this theorem, it is important from a mechanical point of view as it shows that:

- a stress field $\boldsymbol{\sigma} \in \mathbb{S}^d$ results in a unique strain field $\boldsymbol{\varepsilon} \in \mathbb{S}^d$ associated with the constitutive law (102);
- a stress field $\boldsymbol{\sigma} \in \mathbb{S}^d$ results in a unique strain field $\boldsymbol{\varepsilon}_\lambda \in \mathbb{S}^d$ associated with the constitutive law (105);
- the constitutive law (102) can be approached by the more regular constitutive law (105) for a small elasticity coefficient λ.

7. Conclusions

In this paper, we considered a stationary inclusion in a Hilbert space X, for which we provided a convergence criterion, Theorem 2. This criterion provided necessary and sufficient conditions on the sequence $\{u_n\}$, which guaranteed its convergence to the solution u of the inclusion problem.We used this criterion to deduce the continuous dependence results of the solution with respect to the data, as well as a convergence result in the study of an associated penalty problem. Besides the novelty of Theorem 2, we illustrated its use in contact and solid mechanics. This represents a new evidence of the cross fertilization between the models and applications, on the one hand, and the nonlinear functional analysis, on the other hand.

Our results in this work should be extended in several directions.First, it would be interesting to relax the assumption (4) concerning the operator A. Second, it would be interesting to use the abstract result in Theorem 2 in order to obtain the continuous dependence of the solution with respect to all of the data (K, A, f). The use of Theorem 2 in the study of the convergence of the solution of a discrete version of the inclusion (1) as the discretization parameter converges also represents a problem that deserves to be studied in the future. Computer simulations of these theoretical convergence results would be welcome, too. Extensions to evolutionary inclusions (similar to those used in [5], for instance) could also be investigated. Any result in this last direction would have applications in the study of constitutive laws with viscoelastic or viscoplastic materials and the associated contact problems.

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