# ON IRREDUCIBLE INFINITE CONFORMAL ALGEBRAS

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The associative conformal algebra $\mathrm{Cend}_N$ and the corresponding general Lie conformal algebra $gc_N$ are the most important examples of simple conformal algebras which are not finite (see Sect. 2.10 in [K1]). One of the most important open problems of the theory of conformal algebras is the classification of infinite subalgebras of $\mathrm{Cend}_N$ and of $gc_N$ which act irreducibly on $\mathbb{C}[\partial]^N$. (For a classification of such finite algebras, in the associative case see Theorem 2.6 of the present paper, and in the (more difficult) Lie case see [CK] and [DK].)

The classical Burnside theorem states that any subalgebra of the matrix algebra $\mathrm{Mat}_N\mathbb{C}$ that acts irreducibly on $\mathbb{C}^N$ is the whole algebra $\mathrm{Mat}_N\mathbb{C}$. This is certainly not true for subalgebras of $\mathrm{Cend}_N$ (which is the "conformal" analogue of $\mathrm{Mat}_N\mathbb{C}$). There is a family of infinite subalgebras $\mathrm{Cend}_{N,P}$ of $\mathrm{Cend}_N$, where $P(x) \in \mathrm{Mat}_N\mathbb{C}[x]$, $\det P(x) \neq 0$, that still act irreducibly on $\mathbb{C}[\partial]^N$. One of the conjectures of [K2] states that there are no other infinite irreducible subalgebras of $\mathrm{Cend}_N$. This conjecture was recently proved by Kolesnikov [Ko].

In the Lie conformal case, we have a conjecture on the classification of infinite Lie conformal subalgebras of $\mathrm{gc}_N$ acting irreducibly on $\mathbb{C}[\partial]^N$, see Conjecture 4.4. This conjecture agrees with recent results of E. Zelmanov [Z2] and A. De Sole - V. Kac [DeK].

This is an expanded version of a talk given by the second author at the conference in Guaruja in May, 2004. It is based on a joint work with Victor G. Kac, see [BKL] for details. This is a summary of this work and an updated version with recent results by E. Zelmanov, A. De Sole and V. Kac.

The paper is organized as follows:

- Basic definitions
- Irreducible subalgebras of $\mathrm{Cend}_N$ and finite $\mathrm{Cend}_{N,P}$-modules
- Automorphisms, anti-automorphisms and anti-involutions of $\mathrm{Cend}_{N,P}$
- Irreducible Lie conformal algebras $gc_N$, $oc_{N,P}$ and $spc_{N,P}$

## 1. Basic definitions

An *associative conformal algebra* $R$ is defined as a $\mathbb{C}[\partial]$-module with a $\mathbb{C}$-linear map,

$$R\otimes R \longrightarrow \mathbb{C}[\lambda]\otimes R, \qquad a\otimes b\mapsto a_\lambda b$$

called the $\lambda$-product, and satisfying the axioms ($a$, $b$, $c\in R$),

$$(A1)_\lambda \qquad (\partial a)_\lambda b = -\lambda(a_\lambda b), \qquad a_\lambda(\partial b) = (\lambda+\partial)(a_\lambda b)$$

$$(A2)_\lambda \qquad a_\lambda(b_\mu c) = (a_\lambda b)_{\lambda+\mu} c$$

An associative conformal algebra is called *finite* if it has finite rank as $\mathbb{C}[\partial]$ -module. The notions of homomorphism, ideal and subalgebras of an associative conformal algebra are defined in the usual way (see [K1]).

A *module* over an associative conformal algebra $R$ is a $\mathbb{C}[\partial]$-module $M$ with a $\mathbb{C}$-linear map $R\otimes M \longrightarrow \mathbb{C}[\lambda]\otimes M$, denoted by $a\otimes v\mapsto a_\lambda^M v$, satisfying the properties:

$$(\partial a)_\lambda^M v = [\partial^M, a_\lambda^M]v = -\lambda(a_\lambda^M v), \quad a\in R,\, v\in M,$$
$$a_\lambda^M(b_\mu^M v) = (a_\lambda b)_{\lambda+\mu}^M v, \quad a,b\in R.$$

An $R$-module $M$ is called *trivial* if $a_\lambda v = 0$ for all $a\in R$, $v\in M$ (but it may be non-trivial as a $\mathbb{C}[\partial]$-module).

Given a $\mathbb{C}[\partial]$-module $V$, a *conformal endomorphism* of $V$ is a $\mathbb{C}$-linear map $a: V\to \mathbb{C}[\lambda]\otimes_{\mathbb{C}} V$, denoted by $a_\lambda : V\to V$, such that $[\partial, a_\lambda] = -\lambda a_\lambda$. Denote by Cend$V$ the vector space of all such maps. Cend$V$ has a $\mathbb{C}[\partial]$-module structure:

$$(\partial a)_\lambda := -\lambda a_\lambda .$$

If $V$ is a finite $\mathbb{C}[\partial]$-module, then Cend$V$ has a canonical structure of an associative conformal algebra defined by

$$(a_\lambda b)_\mu v = a_\lambda(b_{\mu-\lambda} v), \qquad a,b\in \text{Cend } V,\ v\in V.$$

*Remark.* Observe that, by definition, a structure of a conformal module over an associative conformal algebra $R$ in a finite $\mathbb{C}[\partial]$-module $V$ is the same as a homomorphism of $R$ to the associative conformal algebra Cend$V$.

We shall use the following notation: $\text{Cend}_N :=\text{Cend}\mathbb{C}[\partial]^N$.

These is a natural isomorphism

$$\text{Cend}_N \simeq \text{Mat}_N \mathbb{C}[\partial, x]$$

and the $\lambda$-product in $\mathrm{Mat}_N\mathbb{C}[\partial, x]$ is

$$A(\partial, x)_\lambda B(\partial, x) = A(-\lambda, x + \lambda + \partial)B(\lambda + \partial, x).$$

We shall work with this presentation of $\mathrm{Cend}_N$. The $\lambda$-action of $\mathrm{Cend}_N$ on $\mathbb{C}[\partial]^N$ is

$$A(\partial, x)\,_\lambda\, v(\partial) = A(-\lambda, \lambda + \partial + \alpha)v(\lambda + \partial), \quad v(\partial) \in \mathbb{C}[\partial]^N.$$

Under the change of basis of $\mathbb{C}[\partial]^N$ by the matrix $C(\partial)$ invertible in $\mathrm{Mat}_N(\mathbb{C}[\partial])$, the symbol $A(\partial, x)$ changes by the formula:

$$A(\partial, x) \longmapsto C(\partial + x)A(\partial, x)C(x)^{-1}. \tag{1.1}$$

Observe that for any $C(x) \in \mathrm{Mat}_N(\mathbb{C}[x])$, with non-zero constant determinant, the map (1.1) gives us an automorphism of $\mathrm{Cend}_N$.

It follows from the formula for $\lambda$-product that

$$\mathrm{Cend}_{P,N} := P(x + \partial)(\mathrm{Cend}_N) \quad \text{and} \quad \mathrm{Cend}_{N,P} := (\mathrm{Cend}_N)P(x),$$

with $P(x) \in \mathrm{Mat}_N(\mathbb{C}[x])$, are right and left ideals, respectively, of $\mathrm{Cend}_N$. In particular, they are subalgebras of $\mathrm{Cend}_N$. Another important subalgebra is

$$Cur_N := Cur\;(\mathrm{Mat}_N\mathbb{C}) = \mathbb{C}[\partial]\,(\mathrm{Mat}_N\mathbb{C}).$$

*Remark.* If $P(x)$ is nondegenerate, i.e., $\det P(x) \neq 0$, then

$$\mathrm{Cend}_{N,P} \simeq \mathrm{Cend}_{N,D},$$

with $D = diag(p_1(x), \cdots, p_N(x))$, where $p_i(x)$ are monic polynomials such that $p_i(x)$ divides $p_{i+1}(x)$. The $p_i(x)$ are called the elementary divisors of $P$. So, up to conjugation, all $\mathrm{Cend}_{N,P}$ are parameterized by the sequence of elementary divisors of $P$.

All left and right ideals of $\mathrm{Cend}_N$ were obtained by B. Bakalov. We extend the classification to $\mathrm{Cend}_{N,P}$.

**Proposition 1.1.** *a) All left ideals in $Cend_{N,P}$, with det $P(x) \neq 0$, are of the form $Cend_{N,QP}$, where $Q(x) \in Mat_N(\mathbb{C}[x])$.*

*b) All right ideals in $Cend_{N,P}$, with det $P(x) \neq 0$, are of the form $Q(\partial + x)Cend_{N,P}$, where $Q(x) \in Mat_N(\mathbb{C}[x])$.*

*c) $Cend_{N,P} \simeq Cend_{P,N}$*

## 2. Irreducible subalgebras of $\mathrm{Cend}_N$ and finite modules over $\mathrm{Cend}_{N,P}$

Given $R$ an associative conformal algebra, we will establish a correspondence between the set of maximal left ideals of $R$ and the set of irreducible $R$-modules. Then we will apply it to the subalgebras $\mathrm{Cend}_{N,P}$.

**Lemma 2.1.** *a) Let $v \in M$ and $\mu \in \mathbb{C}$, then $R_{-\partial-\mu}v$ is an $R$-submodule of $M$.*
*b) Let $M$ be a non-trivial irreducible $R$-module. Then there exists $v \in M$ and $\mu \in \mathbb{C}$ such that $R_{-\partial-\mu}v \neq 0$. In particular, if $M$ is irreducible, then $R_{-\partial-\mu}v = M$.*

By this lemma, given a non-trivial irreducible $R$-module $M$ we can fix $v \in M$ and $\mu \in \mathbb{C}$ such that $R_{-\partial-\mu}v = M$ and consider the following map

$$\phi : R \to M, \qquad r \mapsto r_{-\partial-\mu}v.$$

This is onto and therefore we have that as $R$-modules

$$M \simeq (R/\mathrm{Ker}\ \phi)_\mu. \tag{2.1}$$

where $M_\mu$ is the $\mu$-twisted module of $M$ obtained by replacing $\partial$ by $\partial + \mu$ in the formulas for the action of $R$ on $M$.

On the other hand, it is immediate that given any maximal left ideal I of $R$, we have that $(R/I)_\mu$ is an irreducible $R$-module. Therefore we have

**Theorem 2.2.** *Formula (2.1) defines a surjective map from the set of maximal left ideals of $R$ to the set of equivalence classes of non-trivial irreducible $R$-modules.*

Using this result, we obtain

**Corollary 2.3.** *The $\mathrm{Cend}_{N,P}$-module $\mathbb{C}[\partial]^N$ is irreducible if and only if $\det P(x) \neq 0$. These are all non-trivial irreducible $\mathrm{Cend}_{N,P}$-modules up to equivalence, provided that* $\det P(x) \neq 0$.

This Corollary in the case $P(x) = I$, have been established earlier in [K2], by a completely different method (developed in [KR]). Another proof of this was also given by Retakh in [R].

A subalgebra $S$ of $\mathrm{Cend}_N$ is called *irreducible* if $S$ acts irreducibly in $\mathbb{C}[\partial]^N$.

**Corollary 2.4.** *The following subalgebras of $\mathrm{Cend}_N$ are irreducible: $\mathrm{Cend}_{N,P}$ with $\det P(x) \neq 0$, and $\mathrm{Cur}_N := \mathrm{Mat}_N(\mathbb{C}[\partial])$ or conjugates of it by automorphisms (1.1).*

We have the conformal analog of the Burnside Theorem, originally conjectured in [K2]:

**Conjecture 2.5.** *(proved by Kolesnikov* [Ko], *Feb'2004) Any irreducible subalgebra of* $Cend_N$ *is one of them.*

The (particular case of) classification of finite irreducible subalgebras also follows from the classification in [DK] at the Lie algebra level, see [BKL]:

**Theorem 2.6.** *Any finite irreducible subalgebra of* $Cend_N$ *is a conjugate of* $Cur_N$*.*

Now, we study representation theory of these subalgebras.

*Remark.* It is easy to show that every non-trivial irreducible representation of $Cur_N$ is equivalent to the standard module $\mathbb{C}[\partial]^N$, and that every finite module over $Cur_N$ is completely reducible.

Unfortunately, complete reducibility does not hold for $\mathrm{Cend}_N$. Therefore, we have to study extensions of modules. Here we present the following:

- Classification of all extensions of $\mathrm{Cend}_{N,P}$-modules involving the standard module $\mathbb{C}[\partial]^N$ and finite dimensional trivial modules.
- Classification of all finite modules over $\mathrm{Cend}_N$.

Recall the standard irreducible $\mathrm{Cend}_{N,P}$-module $\mathbb{C}[\partial]^N$ with $\lambda$-action

$$a(\partial, x)P(x)_\lambda v(\partial) = a(-\lambda, \lambda + \partial + \alpha)P(\lambda + \partial)v(\lambda + \partial).$$

Consider the trivial $\mathrm{Cend}_{N,P}$-module over the finite dimensional vector space $V_T$, whose $\mathbb{C}[\partial]$-module structure is given by the linear operator $T$, that is: $\partial \cdot v = T(v)$, $v \in V_T$.

We may assume: $P(x) = diag\{p_1(x), \cdots, p_N(x)\}$ and $\det P \neq 0$.

**Theorem 2.7.** *a) There are no non-trivial extensions of* $Cend_{N,P}$*-modules of the form:*

$$0 \to V_T \to E \to \mathbb{C}[\partial]^N \to 0.$$

*Here and further, all the maps in these sequences are maps of* $Cend_{N,P}$*-modules.*
*b) If there exists a non-trivial extension of* $Cend_{N,P}$*-modules of the form*

$$0 \to \mathbb{C}[\partial]^N \to E \to V_T \to 0,$$

*then* $\det P(\alpha + c) = 0$ *for some eigenvalue* $c$ *of* $T$*. In this case, all torsionless extensions of* $\mathbb{C}[\partial]^N$ *by finite dimensional vector spaces, are parameterized by decompositions* $P(x + \alpha) = R(x)S(x)$ *and can be realized as follows. Consider the* $\lambda$*-action of* $Cend_{N,P}$ *on* $\mathbb{C}[\partial]^N$*:*

$$a(\partial, x)P(x)_\lambda v(\partial) = S(\partial)a(-\lambda, \lambda + \partial + \alpha)R(\lambda + \partial)v(\lambda + \partial).$$

*Then* $S(\partial)\mathbb{C}[\partial]^N$ *is a submodule isomorphic to the standard module, of finite codimension in* $\mathbb{C}[\partial]^N$.
*c) If* $E$ *is a non-trivial extension of* $Cend_{N,P}$*-modules of the form:*

$$0 \to \mathbb{C}[\partial]^N \to E \to \mathbb{C}[\partial]^N \to 0,$$

*then* $E = \mathbb{C}[\partial]^N \otimes \mathbb{C}^2$ *as a* $\mathbb{C}[\partial]$*-module (with trivial action of* $\partial$ *on* $\mathbb{C}^2$*) and* $Cend_{N,P}$ *acts by*

$$a(\partial, x)_\lambda(c(\partial) \otimes u) = a(-\lambda, \lambda + \partial \otimes 1 + 1 \otimes J)c(\lambda + \partial)(1 \otimes u),$$

*where* $J$ *is a* $2 \times 2$ *Jordan block matrix.*

**Corollary 2.8.** *There are no non-trivial extensions of* $Cend_N$*-modules of the form:*

$$0 \to V_T \to E \to \mathbb{C}[\partial]^N \to 0 \quad \text{or} \quad 0 \to \mathbb{C}[\partial]^N \to E \to V_T \to 0$$

**Theorem 2.9.** *Every finite* $Cend_N$*-module is isomorphic to a direct sum of its (finite dimensional) trivial torsion submodule and a free finite* $\mathbb{C}[\partial]$*-module* $\mathbb{C}[\partial]^N \otimes T$ *on which the* $\lambda$*-action is given by*

$$a(\partial, x)_\lambda(c(\partial) \otimes u) = a(-\lambda, \lambda + \partial \otimes 1 + 1 \otimes \alpha)c(\lambda + \partial)(1 \otimes u),$$

*where* $\alpha$ *is an arbitrary operator on* $T$.

## 3. Automorphisms and anti-automorphisms of $Cend_{N,P}$

A $\mathbb{C}[\partial]$-linear map $\sigma : R \to S$ between two associative conformal algebras is called a *homomorphism* (*resp. anti-homomorphism*) if

$$\sigma(a_\lambda b) = \sigma(a)_\lambda \sigma(b) \quad (\text{resp } \sigma(a_\lambda b) = \sigma(b)_{-\lambda-\partial}\, \sigma(a)).$$

An anti-automorphism $\sigma$ is an *anti-involution* if $\sigma^2 = 1$.

**Theorem 3.1.** *Let* $P(x) \in Mat_N \mathbb{C}[x]$ *with det* $P(x) \neq 0$*. Then all automorphisms of* $Cend_{N,P}$ *are those that come from* $Cend_N$ *by restriction. More precisely, any automorphism is of the form:*

$$a(\partial, x)P(x) \longmapsto C(\partial + x)a(\partial, x + \alpha)B(x)P(x),$$

*where* $\alpha \in \mathbb{C}$*, and* $B(x), C(x) \in Mat_N \mathbb{C}[x]$ *are invertible and*

$$P(x + \alpha) = B(x)P(x)C(x).$$

**Theorem 3.2.** *Let $P(x) \in \mathrm{Mat}_N\mathbb{C}[x]$ with det $P(x) \neq 0$. Then we have,*
*a) All non-zero homomorphisms from $\mathrm{Cend}_{N,P}$ to $\mathrm{Cend}_N$ are of the form:*

$$a(\partial, x)P(x) \longmapsto S(\partial + x)a(\partial, x + \alpha)R(x),$$

*where $\alpha \in \mathbb{C}$, and $R(x), S(x) \in \mathrm{Mat}_N\mathbb{C}[x]$ such that*

$$P(x + \alpha) = R(x)S(x).$$

*(b) All non-trivial anti-homomorphisms from $\mathrm{Cend}_{N,P}$ to $\mathrm{Cend}_N$ are of the form:*

$$a(\partial, x)P(x) \longmapsto A(\partial + x)a^t(\partial, -\partial - x + \alpha)B(x),$$

*where $\alpha \in \mathbb{C}$, and $A(x)$ and $B(x)$ are matrices in $\mathrm{Mat}_N\mathbb{C}[x]$ such that*

$$P^t(-x + \alpha) = B(x)A(x).$$

*(c) The conformal algebra $\mathrm{Cend}_{N,P}$ has an anti-automorphism (i.e. it is isomorphic to its opposite conformal algebra) if and only if the matrices $P^t(-x + \alpha)$ and $P(x)$ have the same elementary divisors for some $\alpha \in \mathbb{C}$. In this case, all anti-automorphisms of $\mathrm{Cend}_{N,P}$ are of the form:*

$$a(\partial, x)P(x) \longmapsto Y(\partial + x)a^t(\partial, -\partial - x + \alpha)W(x)P(x),$$

*where $Y(x)$ and $W(x)$ are invertible matrices in $\mathrm{Mat}_N\mathbb{C}[x]$ such that*

$$P^t(-x + \alpha) = W(x)P(x)Y(x).$$

*(d) The conformal algebra $\mathrm{Cend}_{N,P}$ has an anti-involution if and only if there exist an invertible in $\mathrm{Mat}_N\mathbb{C}[x]$ matrix $J(x)$ such that*

$$J^t(-x + \alpha)P^t(-x + \alpha) = \epsilon P(x)J(x) \tag{3.1}$$

*for $\epsilon = 1$ or $-1$. In this case all anti-involutions are given by*

$$\sigma_{P,J,\epsilon,\alpha}(a(\partial, x)P(x)) = \varepsilon J(\partial + x)a^t(\partial, -\partial - x + \alpha)J^t(-x + \alpha)^{-1}P(x)$$

*where $J(x)$ is an invertible in $\mathrm{Mat}_N\mathbb{C}[x]$ matrix satisfying (3.1).*

**Corollary 3.3.** *Let $P(x), Q(x) \in \mathrm{Mat}_N\mathbb{C}[x]$ be two non-degenerate matrices. Then $\mathrm{Cend}_{N,P}$ is isomorphic to $\mathrm{Cend}_{N,Q}$ if and only if there exist $\alpha \in \mathbb{C}$ such that $Q(x)$ and $P(x + \alpha)$ have the same elementary divisors.*

Two anti-involutions $\sigma, \tau$ of an associative conformal algebra $R$ are called *conjugate* if $\sigma = \varphi \circ \tau \circ \varphi^{-1}$ for some automorphism $\varphi$ of $R$.

**Theorem 3.4.** *Any anti-involution of $Cend_N$ is, up to conjugation by an automorphism of $Cend_N$:*

$$a(\partial, x) \mapsto a^*(\partial, -\partial - x),$$

*where $^*$ is the adjoint with respect to a non-degenerate symmetric or skew- symmetric bilinear form over $\mathbb{C}$.*

In [BKL], we also found a characterization of equivalent anti-involutions in $\mathrm{Cend}_{N,P}$ and a relation of anti-involutions for different $P$.

## 4. Lie conformal algebras $gc_N$, $oc_{N,P}$ and $spc_{N,P}$

A *Lie conformal algebra* $R$ is a $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map $R \otimes R \longrightarrow \mathbb{C}[\lambda] \otimes R$, $a \otimes b \mapsto [a_\lambda b]$, called the $\lambda$-bracket, satisfying the following axioms $(a, b, c \in R)$,

$(C1)_\lambda$ $\quad [(\partial a)_\lambda b] = -\lambda [a_\lambda b], \qquad [a_\lambda (\partial b)] = (\lambda + \partial)[a_\lambda b]$

$(C2)_\lambda$ $\quad [a_\lambda b] = -[a_{-\partial-\lambda} b]$

$(C3)_\lambda$ $\quad [a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + [b_\mu [a_\lambda c]].$

A *module* M over a conformal algebra $R$ is a $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map $R \otimes M \longrightarrow \mathbb{C}[\lambda] \otimes M$, $a \otimes v \mapsto a_\lambda v$, satisfying the following axioms $(a,\ b \in R),\ v \in M$,

$(M1)_\lambda$ $\quad (\partial a)^M_\lambda v = [\partial^M, a^M_\lambda] v = -\lambda a^M_\lambda v,$

$(M2)_\lambda$ $\quad [a^M_\lambda, b^M_\mu] v = [a_\lambda b]^M_{\lambda+\mu} v.$

In general, given any associative conformal algebra $R$ with $\lambda$-product $a_\lambda b$, the $\lambda$-bracket defined by

$$[a_\lambda b] := a_\lambda b - b_{-\partial-\lambda} a$$

makes $R$ a Lie conformal algebra.

Let $V$ be a finite $\mathbb{C}[\partial]$-module. The $\lambda$-bracket on $\mathrm{Cend}\, V$, makes it a Lie conformal algebra denoted by $\mathrm{gc}\, V$ and called the *general conformal algebra* (see [DK]).

For any positive integer $N$, we define

$$\mathrm{gc}_N := \mathrm{gc}\, \mathbb{C}[\partial]^N = Mat_N \mathbb{C}[\partial, x],$$

and the $\lambda$-bracket:

$$[A(\partial, x)_\lambda B(\partial, x)] = A(-\lambda, x + \lambda + \partial) B(\lambda + \partial, x) - B(\lambda + \partial, -\lambda + x) A(-\lambda, x).$$

Recall that, any anti-involution in $\mathrm{Cend}_N$ is, up to conjugation

$$\sigma_*(A(\partial, x)) = A^*(\partial, -\partial - x),$$

where $*$ stands for the adjoint with respect to a non-degenerate symmetric or skew-symmetric bilinear form over $\mathbb{C}$. These anti-involutions give us two important subalgebras of $gc_N$: the set of $-\sigma_*$ fixed points is the *orthogonal conformal algebra* $oc_N$ (resp. the *symplectic conformal algebra* $spc_N$), in the symmetric (resp. skew-symmetric) case.

- *Description in terms of conformal bilinear forms:*

The conformal subalgebras $oc_N$ and $spc_N$, as well as the anti-involutions given by Section 3, can be described in terms of conformal bilinear forms. Let $V$ be a $\mathbb{C}[\partial]$-module. A *conformal bilinear form* on $V$ is a $\mathbb{C}$-bilinear map $\langle\ ,\ \rangle_\lambda : V \times V \to \mathbb{C}[\lambda]$ such that

$$\langle \partial v, w\rangle_\lambda = -\lambda \langle v, w\rangle_\lambda = -\langle v, \partial w\rangle_\lambda, \quad \text{for all } v, w \in V.$$

The conformal bilinear form is *non-degenerate* if $\langle v, w\rangle_\lambda = 0$ for all $w \in V$, implies $v = 0$. The conformal bilinear form is *symmetric* (resp. *skew-symmetric*) if $\langle v, w\rangle_\lambda = \epsilon \langle w, v\rangle_{-\lambda}$ for all $v, w \in V$, with $\epsilon = 1$ (resp. $\epsilon = -1$).

Given a conformal bilinear form on a $\mathbb{C}[\partial]$-module $V$, we have a homomorphism of $\mathbb{C}[\partial]$-modules, $L : V \to V^*$, $v \mapsto L_v$, given as usual by

$$(L_v)_\lambda w = \langle v, w\rangle_\lambda, \quad v \in V. \tag{4.1}$$

where $V^*$ is the conformal dual of $V$.

Let $V$ be a free finite rank $\mathbb{C}[\partial]$-module and fix $\beta = \{e_1, \cdots, e_N\}$ a $\mathbb{C}[\partial]$-basis of $V$. Then *the matrix of* $\langle\ ,\ \rangle_\lambda$ *with respect to* $\beta$ is defined as $P_{i,j}(\lambda) = \langle e_i, e_j\rangle_\lambda$. Hence, identifying $V$ with $\mathbb{C}[\partial]^N$, we have

$$\langle v(\partial), w(\partial)\rangle_\lambda = v^t(-\lambda) P(\lambda) w(\lambda). \tag{4.2}$$

Observe that $P^t(-x) = \epsilon P(x)$ with $\epsilon = 1$ (resp. $\epsilon = -1$) if the conformal bilinear form is symmetric (resp. skewsymmetric). We also have that $\mathrm{Im}\, L = P(-\partial)V^*$, where $L$ is defined in (4.1). Indeed, given $v(\partial) \in V$, consider $g_\lambda \in V^*$ defined by $g_\lambda(w(\partial)) = v^t(-\lambda) w(\lambda)$, then by (4.2)

$$(L_{v(\partial)})_\lambda w(\partial) = v^t(-\lambda) P(\lambda) w(\lambda) = g_\lambda(P(\partial) w(\partial)) = (P(-\partial) g)_\lambda (w(\partial)),$$

where in the last equality we are identifying $V^*$ with $\mathbb{C}[\partial]^N$ in the natural way, that is $f \in V^*$ corresponds to $(f_{-\partial} e_1, \cdots, f_{-\partial} e_N) \in \mathbb{C}[\partial]^N$. Therefore, if the conformal bilinear form is non-degenerate, then $L$ gives an isomorphism between $V$ and $P(-\partial)V^*$, with $\det P \neq 0$.

We have the following result:

**Proposition 4.1.** *(a) Let $\langle\ ,\ \rangle_\lambda$ be a non-degenerate symmetric or skew-symmetric conformal bilinear form on $\mathbb{C}[\partial]^N$, and denote by $P(\lambda)$ the matrix of $\langle\ ,\ \rangle_\lambda$ with respect to the standard basis of $\mathbb{C}[\partial]^N$ over $\mathbb{C}[\partial]$. Then the map $aP \mapsto (aP)^*$ from $Cend_{N,P}$ to $Cend_N$ defined by*

$$\langle a_\mu v, w\rangle_\lambda = \langle v, a^*_\mu w\rangle_{\lambda-\mu}.$$

*is the anti-involution of $Cend_{N,P}$ given by*

$$(a(\partial,x)P(x))^* = \epsilon a^t(\partial, -\partial - x)P(x), \tag{4.3}$$

*where $P^t(-x) = \epsilon P(x)$ with $\epsilon = 1$ or $-1$, depending on whether the conformal bilinear form is symmetric or skew-symmetric.*
*(b) Consider the Lie conformal subalgebra of $gc_N$ defined by*

$$\begin{aligned} g_* &= \{a \in Cend_{N,P}\ :\ a^* = -a\,\} \\ &= \{a \in Cend_{N,P}\ :\ \langle a_\mu v, w\rangle_\lambda + \langle v, a_\mu w\rangle_{\lambda-\mu} = 0, \quad \text{for all } v, w \in \mathbb{C}[\partial]^N\}, \end{aligned}$$

*where $^*$ is defined by (4.3). Then under the pairing (4.1) we have $\mathbb{C}[\partial]^N \simeq P(-\partial)(\mathbb{C}[\partial]^N)^*$ as $g_*$-modules.*

Observe that $\mathrm{oc}_N$ (resp. $\mathrm{spc}_N$), can be described as the subalgebra $g_*$ of $gc_N$ in Proposition 4.1(b), with respect to the conformal bilinear form

$$\langle p(\partial)v, q(\partial)w\rangle_\lambda = p(-\lambda)q(\lambda)\,(v,w) \qquad \text{for all } v, w \in \mathbb{C}^N,$$

where $(\cdot,\cdot)$ is a non-degenerate symmetric (resp. skew-symmetric) bilinear form on $\mathbb{C}^N$. For general $P$, see (6.16) in [BKL].

Observe that $gc_{N,P} := gc_N P(x)$ is a conformal subalgebra of $gc_N$, for any $P(x) \in \mathrm{Mat}_N\mathbb{C}[x]$.

A matrix $Q(x) \in \mathrm{Mat}_N\mathbb{C}[x]$ will be called *hermitian* (resp. *skew-hermitian*) if

$$Q^t(-x) = \varepsilon Q(x) \qquad \text{with } \varepsilon = 1 \quad (\text{ resp. } \varepsilon = -1).$$

Up to conjugacy, it suffices to consider the anti-involutions

$$\sigma_{P,\varepsilon}(a(\partial,x)P(x)) = \varepsilon a^t(\partial, -\partial - x)P(x)$$

where $P$ is non-degenerate hermitian or skew-hermitian, depending on whether $\varepsilon = 1$ or $-1$.

Notation ($P$ non-degenerate):

$$\begin{aligned} oc_{N,P} &:= \{a \in \mathrm{Cend}_{N,P}\ :\ \sigma_{P,1}(a) = -a\} \qquad \text{if } P \text{ hermitian} \\ spc_{N,P} &:= \{a \in \mathrm{Cend}_{N,P}\ :\ \sigma_{P,-1}(a) = -a\} \quad \text{if } P \text{ skew-hermitian.} \end{aligned}$$

*Remark.* a) These subalgebras can be obtained in a more invariant form using conformal bilinear forms.
b) In the special case $N = 1$ and $P(x) = x$, the involution $\sigma_{x,-1}$ is the conformal version of the involution used by S. Bloch [B] in connection with certain values of $\zeta$-function.

**Proposition 4.2.** *The subalgebras $gc_{N,P}$, $oc_{N,P}$ and $spc_{N,P}$ with* $\det P(x) \neq 0$ *are simple and act irreducibly on $\mathbb{C}[\partial]^N$.*

Two matrices $a$ and $b$ in $\mathrm{Mat}_N\mathbb{C}[x]$ are called *congruent* if $b = c^*ac$ for some invertible in $\mathrm{Mat}_N\mathbb{C}[x]$ matrix $c$, where $c(x)^* := c(-x)^t$.

**Proposition 4.3.** *(a) The subalgebras $oc_{N,P}$ and $oc_{N,Q}$ (resp. $spc_{N,P}$ and $spc_{N,Q}$) are conjugated by an automorphism of $Cend_N$ if and only if $P$ and $Q$ are congruent hermitian (resp. skew-hermitian) matrices.*
*(b) The subalgebras $oc_{N,P}$ and $spc_{N,Q}$ are not conjugated by any automorphism of $Cend_N$.*

A classification of finite irreducible subalgebras of $gc_N$ was given by D'Andrea-Kac. It is natural to propose:

**Conjecture 4.4.** *Any infinite Lie conformal subalgebra of $gc_N$ acting irreducibly on $\mathbb{C}[\partial]^N$ is conjugate by an automorphism of $Cend_N$ to one of the following subalgebras:*
*(a) $gc_{N,P}$, where* $\det P \neq 0$,
*(b) $oc_{N,P}$, where* $\det P \neq 0$ *and* $P(-x) = P^t(x)$,
*(c) $spc_{N,P}$, where* $\det P \neq 0$ *and* $P(-x) = -P^t(x)$.

This conjecture agrees with the results of the E. Zelamov [Z1]-[Z2] and A. De Sole-V. Kac [DeK]. It is proved in [DeK] that every infinite irreducible Lie conformal subalgebra of $gc_N$ which is $sl_2$-module (with respect to certain Virasoro-like element of $gc_N$) is of type $oc_{N,P}$. On the other hand, E. Zelmanov shows that every simple irreducible Lie conformal subalgebra of $gc_N$ of infinite type that contains $\mathrm{Cur}(sl_2)$, is isomorphic to either $gc_{N,P}$ or $oc_{N,P}$.

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