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# On the classification of subalgebras of $\mathrm{Cend}_N$ and $\mathrm{gc}_N$

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**Abstract**

The problem of classification of infinite subalgebras of $\mathrm{Cend}_N$ and of $\mathrm{gc}_N$ that acts irreducibly on $\mathbb{C}[\partial]^N$ is discussed in this paper.


## 0. Introduction

Since the pioneering papers [2,4], there has been a great deal of work towards understanding of the algebraic structure underlying the notion of the operator product expansion (OPE) of chiral fields of a conformal field theory. The singular part of the OPE encodes the commutation relations of fields, which leads to the notion of a Lie conformal algebra [11,12].

In the past few years a structure theory [7], representation theory [5,6] and cohomology theory [1] of finite Lie conformal algebras has been developed.

The associative conformal algebra $\mathrm{Cend}_N$ and the corresponding general Lie conformal algebra $\mathrm{gc}_N$ are the most important examples of simple conformal algebras which are not finite (see [11, Section 2.10]). One of the most urgent open problems of the theory of conformal algebras is the classification of infinite subalgebras of $\mathrm{Cend}_N$ and of $\mathrm{gc}_N$ which

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act irreducibly on $\mathbb{C}[\partial]^N$. (For a classification of such finite algebras, in the associative case see Theorem 5.2 of the present paper, and in the (more difficult) Lie case see [5] and [7].)

The classical Burnside theorem states that any subalgebra of the matrix algebra $\mathrm{Mat}_N\,\mathbb{C}$ that acts irreducibly on $\mathbb{C}^N$ is the whole algebra $\mathrm{Mat}_N\,\mathbb{C}$. This is certainly not true for subalgebras of $\mathrm{Cend}_N$ (which is the "conformal" analogue of $\mathrm{Mat}_N\,\mathbb{C}$). There is a family of infinite subalgebras $\mathrm{Cend}_{N,P}$ of $\mathrm{Cend}_N$, where $P(x) \in \mathrm{Mat}_N\,\mathbb{C}[x]$, $\det P(x) \neq 0$, that still act irreducibly on $\mathbb{C}[\partial]^N$. One of the conjectures of [12] states that there are no other infinite irreducible subalgebras of $\mathrm{Cend}_N$.

One of the results of the present paper is the classification of all subalgebras of $\mathrm{Cend}_1$ and determination of the ones that act irreducibly on $\mathbb{C}[\partial]$ (Theorem 2.1). This result proves the above-mentioned conjecture in the case $N = 1$. For general $N$ we can prove this conjecture only under the assumption that the subalgebra in question is unital (see Theorem 5.3). This result is closely related to a difficult theorem of A. Retakh [16] (but we avoid using it).

Next, we describe all finite irreducible modules over $\mathrm{Cend}_{N,P}$ (see Corollary 3.5). This is done by using the description of left ideals of the algebras $\mathrm{Cend}_{N,P}$ (see Proposition 1.3(a)). Further, we describe all extensions between non-trivial finite irreducible $\mathrm{Cend}_{N,P}$-modules and between non-trivial finite irreducible and trivial finite-dimensional modules (Theorem 3.8). This leads us to a complete description of finite $\mathrm{Cend}_N$-modules (Theorem 3.10).

Next we describe all automorphisms of $\mathrm{Cend}_{N,P}$ (Theorems 4.1 and 4.2). We also classify all homomorphisms and anti-homomorphisms of $\mathrm{Cend}_{N,P}$ to $\mathrm{Cend}_N$ (Theorem 4.3). This gives, in particular, a classification of anti-involutions of $\mathrm{Cend}_{N,P}$. One case of such an anti-involution ($N = 1$, $P = x$) was studied by S. Bloch [3] on the level of the Lie algebra of differential operators on the circle to link representations of the corresponding subalgebra to the values of $\zeta$-function. Representation theory of the subalgebra corresponding to the anti-involution of $\mathrm{Cend}_1$ was developed in [14].

The subspace of anti-fixed points of an anti-involution of $\mathrm{Cend}_{N,P}$ is a Lie conformal subalgebra that still acts irreducibly on $\mathbb{C}[\partial]^N$. This leads us to Conjecture 6.8 on classification of infinite Lie conformal subalgebras of $\mathrm{gc}_N$ acting irreducibly on $\mathbb{C}[\partial]^N$. This conjecture agrees with the results of the papers [8,18].

## 1. Left and right ideals of $\mathrm{Cend}_{N,P}$

First we introduce the basic definitions and notations, see [11]. An *associative conformal algebra* $R$ is defined as a $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map,

$$R \otimes R \to \mathbb{C}[\lambda] \otimes R, \quad a \otimes b \mapsto a_\lambda b$$

called the $\lambda$-product, and satisfying the following axioms ($a, b, c \in R$),

$(\mathrm{A1})_\lambda$ $(\partial a)_\lambda b = -\lambda(a_\lambda b)$, $a_\lambda(\partial b) = (\lambda + \partial)(a_\lambda b)$,
$(\mathrm{A2})_\lambda$ $a_\lambda(b_\mu c) = (a_\lambda b)_{\lambda+\mu} c$.

An associative conformal algebra is called *finite* if it has finite rank as a $\mathbb{C}[\partial]$-module. The notions of homomorphisms, ideals, and subalgebras of an associative conformal algebra are defined in the usual way.

A *module* over an associative conformal algebra $R$ is a $\mathbb{C}[\partial]$-module $M$ endowed with a $\mathbb{C}$-linear map $R \otimes M \to \mathbb{C}[\lambda] \otimes M$, denoted by $a \otimes v \mapsto a_\lambda^M v$, satisfying the properties:

$$(\partial a)_\lambda^M v = \left[\partial^M, a_\lambda^M\right] v = -\lambda\left(a_\lambda^M v\right), \quad a \in R,\ v \in M,$$
$$a_\lambda^M\left(b_\mu^M v\right) = (a_\lambda b)_{\lambda+\mu}^M v, \quad a, b \in R.$$

An $R$-module $M$ is called *trivial* if $a_\lambda v = 0$ for all $a \in R$, $v \in M$ (but it may be non-trivial as a $\mathbb{C}[\partial]$-module).

Given two $\mathbb{C}[\partial]$-modules $U$ and $V$, a *conformal linear map* from $U$ to $V$ is a $\mathbb{C}$-linear map $a : U \to \mathbb{C}[\lambda] \otimes_{\mathbb{C}} V$, denoted by $a_\lambda : U \to V$, such that $[\partial, a_\lambda] = -\lambda a_\lambda$, that is $\partial^V a_\lambda - a_\lambda \partial^U = -\lambda a_\lambda$. The vector space of all such maps, denoted by $\operatorname{Chom}(U, V)$, is a $\mathbb{C}[\partial]$-module with

$$(\partial a)_\lambda := -\lambda a_\lambda.$$

Now, we define $\operatorname{Cend} V := \operatorname{Chom}(V, V)$ and, provided that $V$ is a finite $\mathbb{C}[\partial]$-module, $\operatorname{Cend} V$ has a canonical structure of an associative conformal algebra defined by

$$(a_\lambda b)_\mu v = a_\lambda(b_{\mu-\lambda} v), \quad a, b \in \operatorname{Cend} V,\ v \in V.$$

**Remark 1.1.** Observe that, by definition, a structure of a conformal module over an associative conformal algebra $R$ in a finite $\mathbb{C}[\partial]$-module $V$ is the same as a homomorphism of $R$ to the associative conformal algebra $\operatorname{Cend} V$.

For a positive integer $N$, let $\operatorname{Cend}_N = \operatorname{Cend} \mathbb{C}[\partial]^N$. It can also be viewed as the associative conformal algebra associated to the associative algebra $\operatorname{Diff}^N \mathbb{C}^\times$ of all $N \times N$ matrix valued regular differential operators on $\mathbb{C}^\times$, that is (see [11, Section 2.10] for more details)

$$\operatorname{Conf}\left(\operatorname{Diff}^N \mathbb{C}^\times\right) = \bigoplus_{n \in \mathbb{Z}_+} \mathbb{C}[\partial] J^n \otimes \operatorname{Mat}_N \mathbb{C}$$

with $\lambda$-product given by ($J_A^k = J^k \otimes A$)

$$J_A^k{}_\lambda J_B^l = \sum_{j=0}^{k} \binom{k}{j} (\lambda + \partial)^j J_{AB}^{k+l-j}.$$

Given $\alpha \in \mathbb{C}$, the natural representation of $\operatorname{Diff}^N \mathbb{C}^\times$ on $\mathrm{e}^{-\alpha t}\mathbb{C}^N[t, t^{-1}]$ gives rise a conformal module structure on $\mathbb{C}[\partial]^N$ over $\operatorname{Conf}(\operatorname{Diff}^N \mathbb{C}^\times)$, with $\lambda$-action

$$J_A^m{}_\lambda v = (\lambda + \partial + \alpha)^m A v, \quad m \in \mathbb{Z}_+,\ v \in \mathbb{C}^N.$$

Now, using Remark 1.1, we obtain a natural homomorphism of conformal associative algebras from $\mathrm{Conf}(\mathrm{Diff}^N\,\mathbb{C}^\times)$ to $\mathrm{Cend}_N$, which turns out to be an isomorphism (see [7] and [11, Proposition 2.10]).

In order to simplify the notation, we will introduce the following bijective map, called the *symbol*,

$$\mathrm{Symb}: \mathrm{Cend}_N \to \mathrm{Mat}_N\,\mathbb{C}[\partial, x],$$
$$\sum_k A_k(\partial)J^k \mapsto \sum_k A_k(\partial)x^k,$$

where $A_k(\partial) \in \mathrm{Mat}_N(\mathbb{C}[\partial])$. The transferred $\lambda$-product is

$$A(\partial, x)_\lambda B(\partial, x) = A(-\lambda, x + \lambda + \partial)B(\lambda + \partial, x). \tag{1.1}$$

The above $\lambda$-action of $\mathrm{Cend}_N$ on $\mathbb{C}[\partial]^N$ is given by the following formula:

$$A(\partial, x)_\lambda v(\partial) = A(-\lambda, \lambda + \partial + \alpha)v(\lambda + \partial), \quad v(\partial) \in \mathbb{C}[\partial]^N. \tag{1.2}$$

Note also that under the change of basis of $\mathbb{C}[\partial]^N$ by the matrix $C(\partial)$ invertible in $\mathrm{Mat}_N(\mathbb{C}[\partial])$, the symbol $A(\partial, x)$ changes by the formula:

$$A(\partial, x) \mapsto C(\partial + x)A(\partial, x)C(x)^{-1}. \tag{1.3}$$

Observe that for any $C(x) \in \mathrm{Mat}_N(\mathbb{C}[x])$, with non-zero constant determinant, the map (1.3) gives us an automorphism of $\mathrm{Cend}_N$.

It follows immediately from the formula for $\lambda$-product that

$$\mathrm{Cend}_{P,N} := P(x + \partial)(\mathrm{Cend}_N) \quad \text{and} \quad \mathrm{Cend}_{N,P} := (\mathrm{Cend}_N)P(x),$$

with $P(x) \in \mathrm{Mat}_N(\mathbb{C}[x])$, are right and left ideals, respectively, of $\mathrm{Cend}_N$. Another important subalgebra is

$$\mathrm{Cur}_N := \mathrm{Cur}(\mathrm{Mat}_N\,\mathbb{C}) = \mathbb{C}[\partial](\mathrm{Mat}_N\,\mathbb{C}). \tag{1.4}$$

**Remark 1.2.** If $P(x)$ is non-degenerate, i.e., $\det P(x) \neq 0$, then by elementary transformations over the rows (left multiplications) we can make $P(x)$ upper triangular without changing $\mathrm{Cend}_{N,P}$. After that, applying to $\mathrm{Cend}_{N,P}$ an automorphism of $\mathrm{Cend}_N$ of the form (1.3), with $\det C(x) = 1$ (in order to multiply $P$ on the right, which are elementary transformations over the columns), we get $\mathrm{Cend}_{N,P} \simeq \mathrm{Cend}_{N,D}$, with $D = \mathrm{diag}(p_1(x), \ldots, p_N(x))$, where $p_i(x)$ are monic polynomials such that $p_i(x)$ divides $p_{i+1}(x)$. The $p_i(x)$ are called the elementary divisors of $P$. So, up to conjugation, all $\mathrm{Cend}_{N,P}$ are parameterized by the sequence of elementary divisors of $P$.

All left and right ideals of $\mathrm{Cend}_N$ were obtained by B. Bakalov. Now, we extend the classification to $\mathrm{Cend}_{N,P}$.

**Proposition 1.3.** (a) *All left ideals in* $\mathrm{Cend}_{N,P}$, *with* $\det P(x) \neq 0$, *are of the form* $\mathrm{Cend}_{N,QP}$, *where* $Q(x) \in \mathrm{Mat}_N(\mathbb{C}[x])$.

(b) *All right ideals in* $\mathrm{Cend}_{N,P}$, *with* $\det P(x) \neq 0$, *are of the form* $Q(\partial + x)\,\mathrm{Cend}_{N,P}$, *where* $Q(x) \in \mathrm{Mat}_N(\mathbb{C}[x])$.

**Proof.** (a) By Remark 1.2, we may suppose that $P$ is diagonal with $\det P(x) \neq 0$. Denote by $p_1(x), \ldots, p_N(x)$ the diagonal coefficients.

Let $J \subseteq \mathrm{Cend}_N$ be a left ideal. First, let us see that $J$ is generated over $\mathbb{C}[\partial]$ by $I := J \cap \mathrm{Mat}_N(\mathbb{C}[x])$. If $a(\partial, x) = \sum_{i=0}^{m} \partial^i a_i(x) \in J$, then

$$\begin{aligned} E_{k,k} P(x)_\lambda a(\partial, x) &= p_k(\lambda + \partial + x) E_{k,k} a(\lambda + \partial, x) \\ &= p_k(\lambda + \partial + x) E_{k,k} \left( \sum_i (\lambda + \partial)^i a_i(x) \right) \in \mathbb{C}[\lambda] \otimes J, \end{aligned} \tag{1.5}$$

using that $\det P(x) \neq 0$ and considering the coefficient of the maximal power of $\lambda$ in (1.5), we get $E_{k,k} a_m(x) \in J$ for all $k$. Hence $a_m(x) \in J$. Applying the same argument to $a(\partial, x) - \partial^m a_m(x) \in J$, and so on, we get $a_i(x) \in J$ for all $i$. Therefore, $J$ is generated over $\mathbb{C}[\partial]$ by $I := J \cap \mathrm{Mat}_N(\mathbb{C}[x])$.

If $a(x) \in I$, then

$$\begin{aligned} E_{i,j} P(x)_\lambda a(x) &= p_j(\lambda + \partial + x) E_{i,j} a(x) \\ &= \lambda^{\max} E_{i,j} a(x) + \text{lower terms} \in \mathbb{C}[\lambda] \otimes J. \end{aligned} \tag{1.6}$$

Therefore, $\mathrm{Mat}_N(\mathbb{C}) \cdot I \subseteq I$.

Now, considering the next coefficient in $\lambda$ in (1.6) if $p_j$ is non-constant, or the constant term in $\lambda$ of $x E_{i,j} P(x)_\lambda a(x)$ if $p_j$ is constant, we get that $x a(x) \in I$. It follows that $I$ is a left ideal of $\mathrm{Mat}_N(\mathbb{C}[x])$. But all left ideals of $\mathrm{Mat}_N(\mathbb{C}[x])$ are principal, i.e., of the form $\mathrm{Mat}_N(\mathbb{C}[x]) R(x)$, since $\mathrm{Mat}_N(\mathbb{C}[x])$ and $\mathbb{C}[x]$ are Morita equivalent. This completes the proof of (a).

In a similar way, but using the expression $a(\partial, x) = \sum_i \partial^i \tilde{a}_i(\partial + x)$, we get (b). □

**Proposition 1.4.** $\mathrm{Cend}_{N,P} \simeq B(\partial + x)(\mathrm{Cend}_N) A(x)$ *if* $P(x) = A(x)B(x)$. *In particular,* $\mathrm{Cend}_{N,P} \simeq \mathrm{Cend}_{P,N}$.

**Proof.** It is easy to see that the map $a(\partial, x)P(x) \to B(\partial + x)a(\partial, x)A(x)$ is an isomorphism provided that $P(x) = A(x)B(x)$. □

## 2. Classification of subalgebras of $\mathrm{Cend}_1$

We can identify $\mathrm{Cend}_1$ with $\mathbb{C}[\partial, x]$, then the $\lambda$-product is

$$r(\partial, x)_\lambda s(\partial, x) = r(-\lambda, \lambda + \partial + x) s(\lambda + \partial, x), \tag{2.1}$$

where $r(\partial, x), s(\partial, x) \in \mathbb{C}[\partial, x]$.

The main result of this section is

**Theorem 2.1.** (a) *Any subalgebra of* $\mathrm{Cend}_1$ *is one of the following*:

(1) $\mathbb{C}[\partial]$;
(2) $\mathbb{C}[\partial, x]\ p(x)$, *with* $p(x) \in \mathbb{C}[x]$;
(3) $\mathbb{C}[\partial, x]q(\partial + x)$, *with* $q(x) \in \mathbb{C}[x]$;
(4) $\mathbb{C}[\partial, x]p(x)q(\partial + x) = \mathbb{C}[\partial, x]p(x) \cap \mathbb{C}[\partial, x]q(\partial + x)$, *with* $p(x), q(x) \in \mathbb{C}[x]$.

(b) *The subalgebras* $\mathbb{C}[\partial, x]\ p(x)$ *with* $p(x) \neq 0$, *and* $\mathbb{C}[\partial]$ *are all the subalgebras of* $\mathrm{Cend}_1$ *that act irreducibly on* $\mathbb{C}[\partial]$.

In order to prove Theorem 2.1, we first need some lemmas and the following important notation. Given $r(\partial, x) \in \mathbb{C}[\partial, x]$, we denote by $r_i$ and $\tilde{r}_j$ the coefficients uniquely determined by

$$r(\partial, x) = \sum_{i=0}^{n} r_i(x)\partial^i = \sum_{j=0}^{m} \tilde{r}_j(\partial + x)\partial^j \tag{2.2}$$

with $r_n(x) \neq 0$ and $\tilde{r}_m(\partial + x) \neq 0$.

**Lemma 2.2.** *Let* $S$ *be a subalgebra of* $\mathrm{Cend}_1$ *and let* $t(\partial) \in \mathbb{C}[\partial]$ *be a non-zero polynomial.*

(a) *If* $t(\partial) \in S$, *then* $\mathbb{C}[\partial] \subseteq S$.
(b) *If* $t(\partial), r(\partial, x) \in S$ *and* $r(\partial, x)$ *depends non-trivially on* $x$, *then* $S = \mathrm{Cend}_1$. *In particular, if* $1 \in S$, *then either* $S = \mathbb{C}[\partial]$ *or* $S = \mathrm{Cend}_1$.

**Proof.** (a) If $t(\partial) \in S$, we deduce from the maximal coefficient in $\lambda$ of $t(\partial)_\lambda t(\partial) = t(-\lambda)t(\lambda + \partial)$ that $1 \in S$, proving (a).

(b) From (a), we have that $1 \in S$. Then the coefficients of $\lambda$ in $r(\partial, x)_\lambda 1 = r(-\lambda, \lambda + \partial + x)$ are in $S$. Therefore, using notation (2.2), we obtain that $\tilde{r}_j(\partial + x) \in S$ for all $j$. Since $r(\partial, x)$ depends non-trivially on $x$, there exist $j_0$ such that $\tilde{r}_{j_0}$ is non-constant, that is $\tilde{r}_{j_0}(z) = \sum_{i=0}^{l} a_i z^i$ with $a_l \neq 0$ and $l > 0$. Now, using that $\mathbb{C}[\partial] \subseteq S$ and

$$1_\lambda \tilde{r}_{j_0}(\partial + x) = \tilde{r}_{j_0}(\lambda + \partial + x) = \lambda^l + \big(la_l(\partial + x) + a_{l-1}\big)\lambda^{l-1} + \text{lower powers in } \lambda$$

we obtain that $x \in S$. Then by induction and taking $\lambda$-products of type $x_\lambda x^k$ we see that $x^{k+1} \in S$ for all $k \geqslant 1$, proving (b). □

**Lemma 2.3.** *Let* $S$ *be a subalgebra of* $\mathrm{Cend}_1$, *let* $p(x)$ *and* $q(x)$ *be two non-constant polynomials.*

(a) *If* $p(x) \in S$, *then* $\mathbb{C}[\partial, x]p(x) \subseteq S$.

(b) *If* $q(\partial+x)\in S$, *then* $\mathbb{C}[\partial,x]q(\partial+x)\subseteq S$.
(c) *If* $p(x)q(\partial+x)\in S$, *then* $\mathbb{C}[\partial,x]p(x)q(\partial+x)\subseteq S$.

**Proof.** Part (a) and (b) follows from the proof of (c).

(c) Assume that $q(x+\partial)p(x)\in S$. Then, we compute $q(x+\partial)p(x)_\lambda q(x+\partial)p(x) = q(x+\partial)p(\lambda+\partial+x)q(\lambda+x+\partial)p(x)$, and looking at the monomial of highest degree minus one, we get that $(x+\partial)q(x+\partial)p(x)\in S$, and since by definition $S$ is a $\mathbb{C}[\partial]$-module, we deduce that $q(x+\partial)\tilde{p}(x) := xq(x+\partial)p(x)\in S$. Applying this argument to $q(x+\partial)\tilde{p}(x)$ we deduce that $x^k q(x+\partial)p(x)\in S$ for any $k\in\mathbb{Z}_+$, and therefore $q(x+\partial)p(x)\mathbb{C}[\partial,x]\subseteq S$. □

**Lemma 2.4.** *Let $S$ be a subalgebra of* $\mathrm{Cend}_1$ *which does not contain* 1.

(a) *Let $p(x)$ be of minimal degree such that $p(x)\in S$. Then $\mathbb{C}[\partial,x]p(x)=S$.*
(b) *Let $q(\partial+x)$ be of minimal degree such that $q(\partial+x)\in S$. Then $S=\mathbb{C}[\partial,x]q(\partial+x)$.*
(c) *Let $q(\partial+x)p(x)$ be of minimal degree (in $x$) such that $q(\partial+x)p(x)\in S$. Then $S=p(x)q(\partial+x)\mathbb{C}[\partial,x]$.*

**Proof.** (a) From Lemma 2.3(a), we have that $p(x)\mathbb{C}[\partial,x]\subseteq S$ (by our assumption, $p(x)$ is non-constant). Now, suppose that there exists $q(\partial,x)\in S$ with $q(\partial,x)\notin p(x)\mathbb{C}[\partial,x]$ and $p$ as above. Then, by applying the division algorithm to each coefficient of $q(\partial,x)=\sum_{k=0}^{l} q_k(x)\partial^k$, we may write $q(\partial,x)=t(\partial,x)p(x)+r(\partial,x)$ with $r(\partial,x)=\sum_{k=0}^{n} r_k(x)\partial^k=\sum_{j=0}^{m}\tilde{r}_j(\partial+x)\partial^k$ and $\deg r_k<\deg p$ (cf. notation (2.2)). Using that $p(x)\mathbb{C}[\partial,x]\subseteq S$, we obtain that $r(\partial,x)\in S$. Now, since

$$r(\partial,x)_\lambda r(\partial,x)=r(-\lambda,\lambda+\partial+x)r(\lambda+\partial,x), \tag{2.3}$$

looking at the coefficient of maximum degree in $\lambda$ in (2.3), we get: $r_n(x)\tilde{r}_m(x+\partial)\in S$. By our assumption, one of the polynomials in this product is non-constant. If $\tilde{r}_m(x+\partial)$ is constant, then $r_n(x)\in S$, but $\deg r_n<\deg p$ which is a contradiction. If $r_n(x)$ is constant, then $\tilde{r}_m(x+\partial)\in S$. Then, looking at the leading coefficient of the following polynomial in $\lambda$: $p(x)_\lambda\tilde{r}_m(x+\partial)=p(\lambda+\partial+x)\tilde{r}_m(x+\lambda+\partial)$ we have that $1\in S$, which contradicts our assumption.

If neither $\tilde{r}_m(x+\partial)$ nor $r_n(x)$ are constants, we look at $p(x)_\lambda\tilde{r}_m(x+\partial)r_n(x) = p(\lambda+\partial+x)\tilde{r}_m(\lambda+x+\partial)r_n(x)\in S$ and looking at the coefficient of maximum degree in $\lambda$ we get that $r_n(x)\in S$, which contradicts the minimality of $p(x)$.

(b) The proof is the same as that of (a).

(c) We may assume that $p$ and $q$ are non-constant polynomials, otherwise we are in the cases (a) or (b). By Lemma 2.3(c), we have $p(x)q(x+\partial)\mathbb{C}[\partial,x]\subseteq S$. Let $t(\partial,x)\in S$, but $t(\partial,x)\notin\mathbb{C}[\partial,x]p(x)q(x+\partial)$. Then we may have three cases:

(1) $t(\partial,x)\in p(x)\mathbb{C}[\partial,x]$ or

(2) $t(\partial,x)\in q(\partial+x)\mathbb{C}[\partial,x]$ or

(3) $t(\partial,x)\notin p(x)\mathbb{C}[\partial,x]$ and $t(\partial,x)\notin q(\partial+x)\mathbb{C}[\partial,x]$.

Note that these cases are mutually exclusive. Suppose we are in Case (1), so that $t(\partial,x)=p(x)r(\partial,x)$ with $r(\partial,x)\notin q(\partial+x)\mathbb{C}[\partial,x]$. Then we get $r(\partial,x)=q(\partial+$

$x)\tilde{r}(\partial, x) + s(\partial, x)$, with $s(\partial, x) \neq 0$, and (using notation (2.2)) deg $\tilde{s}_k <$ deg $q$ for all $k = 0, \ldots, m$. Therefore, we have that $t(\partial, x) = p(x)r(\partial, x) = p(x)q(\partial + x)\tilde{r}(\partial, x) + p(x)s(\partial, x)$ and then $p(x)s(\partial, x) \in S$. Now, we can compute:

$$p(x)s(\partial, x)_\lambda p(x)q(\partial + x) = p(\lambda + \partial + x)s(-\lambda, \lambda + \partial + x)p(x)q(\lambda + \partial + x)$$

and looking at the coefficient of maximum degree in $\lambda$, we have (using notation (2.2)) that $p(x)\tilde{s}_m(\partial + x) \in S$ which is a contradiction.

Similarly, Case (2) also leads to a contradiction.

In the remaining Case (3) we may assume that deg $p \leqslant$ deg $q$ since the case of the opposite inequality is completely analogous. We have $t(\partial, x) \in S$, but $\notin \mathbb{C}[\partial, x]p(x)$. Then

$$t(\partial, x) = p(x)h(\partial, x) + r(\partial, x) \tag{2.4}$$

with $0 \neq r(\partial, x) = \sum_{k=0}^{n} r_k(x)\partial^k = \sum_{j=0}^{m} \tilde{r}_j(\partial + x)\partial^k$ where deg $r_k <$ deg $p$ and deg $\tilde{r}_j <$ deg $p$.

If $h(\partial, x) \in \mathbb{C}[\partial, x]q(\partial + x)$, then $r(\partial, x) \in S$, but the leading coefficient of

$$p(x)q(\partial + x)_\lambda r(\partial, x) = p(\lambda + \partial + x)q(\partial + x)r(\lambda + \partial, x)$$

is in $S$ which is $q(\partial + x)r_n(x)$, and this contradicts the assumption of minimality of $p(x)q(\partial + x)$.

So, suppose that $h(\partial, x) \notin \mathbb{C}[\partial, x]q(\partial + x)$. Then $h(\partial, x) = \tilde{h}(\partial, x)q(\partial + x) + s(\partial, x)$ with $0 \neq s(\partial, x) = \sum_{k=0}^{l} s_k(x)\partial^k = \sum_{j=0}^{m} \tilde{s}_j(\partial + x)\partial^k$ and deg $\tilde{s}_j <$ deg $q$. By (2.4) we have $p(x)s(\partial, x) + r(\partial, x) \in S$. Now, we compute:

$$\begin{aligned} &\big(p(x)s(\partial, x) + r(\partial, x)\big)_\lambda p(x)q(\partial + x) \\ &\quad = \big(p(\lambda + \partial + x)s(-\lambda, \lambda + \partial + x) + r(-\lambda, \lambda + \partial + x)\big)p(x)q(\lambda + \partial + x). \end{aligned}$$

Then the leading coefficient in $\lambda$ is either $p(x)\tilde{s}_m(\partial + x) \in S$, which is impossible since deg $\tilde{s}_m <$ deg $q$, or $p(x)\tilde{r}_m(\partial + x) \in S$. But in the latter case, deg $\tilde{r}_m \geqslant$ deg $q$, but by construction deg $\tilde{r}_m <$ deg $p$, and this contradicts the assumption deg $p \leqslant$ deg $q$. □

**Proof of Theorem 2.1.** (a) Let $S$ be a non-zero subalgebra of $\mathrm{Cend}_1$. If $S \subseteq \mathbb{C}[\partial]$ then by Lemma 2.2(a) we have that $S = \mathbb{C}[\partial]$. Therefore we may assume that there is $r(\partial, x) \in S$ which depends non-trivially on $x$. Recall that we can write $r(\partial, x) = \sum_{i=0}^{m} p_i(x)\partial^i = \sum_{j=0}^{n} q_j(\partial + x)\partial^j$. We have

$$\begin{aligned} r(\partial, x)_\lambda r(\partial, x) &= r(-\lambda, \lambda + \partial + x)r(\lambda + \partial, x) \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n} q_j(\partial + x)p_i(x)(-\lambda)^j(\lambda + \partial)^i. \end{aligned}$$

Then, considering the leading coefficient of this $\lambda$-polynomial, we have $p_m(x)q_n(\partial + x) \in S$. Therefore, we may have one of the following situations:

(1) $p_m(x)$ and $q_n(\partial + x)$ are constant,
(2) $q_n(\partial + x)$ is constant and $p_m(x)$ is non-constant,
(3) $p_m(x)$ is constant and $q_n(\partial + x)$ is non-constant, or
(4) both polynomials non-constant.

Let us see what happens in each case.

(1) By Lemma 2.2(b), we have that $S = \mathrm{Cend}_1$.

(2) In this case, we may take $p(x) \in S$ of minimal degree, then using Lemma 2.4(a) we have $S = \mathbb{C}[\partial, x]p(x)$.

(3) It is completely analogous to (2).

(4) Here, we have that $p(x)q(x + \partial) \in S$ and, again we may assume that it has minimal degree. Now, by Lemma 2.4(c), we finish the proof of (a).

The proof of (b) is straightforward. □

## 3. Finite modules over $\mathrm{Cend}_{N,P}$

Given an associative conformal algebra $R$ (not necessarily finite), we will establish a correspondence between the set of maximal left ideals of $R$ and the set of irreducible $R$-modules. Then we will apply it to the subalgebras $\mathrm{Cend}_{N,P}$.

First recall that the following property holds in an $R$-module $M$ (cf. [7, Remark 3.3]):

$$a_\lambda(b_{-\partial-\mu} v) = (a_\lambda b)_{-\partial-\mu} v, \quad a, b \in R,\ v \in M. \tag{3.1}$$

**Remark 3.1.** (a) Let $v \in M$ and fix $\mu \in \mathbb{C}$, then due to (3.1) we have that $R_{-\partial-\mu} v$ is an $R$-submodule of $M$.

(b) $\mathrm{Tor}\, M$ is a trivial $R$-submodule of $M$ [7, Lemma 8.2].

(c) If $M$ is irreducible and $M = \mathrm{Tor}\, M$, then $M \simeq \mathbb{C}$.

(d) If $M$ is a non-trivial finite irreducible $R$-module, then $M$ is free as a $\mathbb{C}[\partial]$-module.

**Lemma 3.2.** *Let $M$ be a non-trivial irreducible $R$-module. Then there exists $v \in M$ and $\mu \in \mathbb{C}$ such that $R_{-\partial-\mu} v \neq 0$. In particular, $R_{-\partial-\mu} v = M$ if $M$ is irreducible.*

**Proof.** Suppose that $R_{-\partial-\mu} v = 0$ for all $v \in M$ and $\mu \in \mathbb{C}$, then we have that $r_{-\partial-\mu} v = 0$ in $\mathbb{C}[\mu] \otimes M$ for all $r \in R$ and $v \in M$. Thus writing down $r_{-\partial-\mu} v$ as a polynomial in $\mu$ and looking at the $n$-products that are going to appear in this expansion, we conclude that $r_\lambda v = 0$ for all $v \in M$ and $r \in R$. Hence $M$ is a trivial $R$-module, a contradiction. □

By Lemma 3.2, given a non-trivial irreducible $R$-module $M$ we can fix $v \in M$ and $\mu \in \mathbb{C}$ such that $R_{-\partial-\mu} v = M$ and consider the following map:

$$\phi : R \to M, \quad r \mapsto r_{-\partial-\mu} v.$$

Observe that $\phi(\partial r) = (\partial + \mu)\phi(r)$ and using (3.1) we also have $\phi(r_\lambda s) = r_\lambda \phi(s)$. Therefore, the map $\phi$ is a homomorphism of $R$-modules into $M_{-\mu}$, where $M_\mu$ is the

$\mu$-twisted module of $M$ obtained by replacing $\partial$ by $\partial + \mu$ in the formulas for the action of $R$ on $M$, and $\mathrm{Ker}(\phi)$ is a maximal left ideal of $R$. Clearly this map is onto $M_{-\mu}$.

Therefore we have that $M_{-\mu} \simeq (R/\operatorname{Ker}\phi)$ as $R$-modules, or equivalently,

$$M \simeq (R/\operatorname{Ker}\phi)_{\mu}. \tag{3.2}$$

On the other hand, it is immediate that given any maximal left ideal $I$ of $R$, we have that $(R/I)_{\mu}$ is an irreducible $R$-module. Therefore we have proved the following

**Theorem 3.3.** *Formula* (3.2) *defines a surjective map from the set of maximal left ideals of $R$ to the set of equivalence classes of non-trivial irreducible $R$-modules.*

**Remark 3.4.** (a) Observe that given an $R$-module $M$ and $v \in M$, the set $I = \{a \in R \mid a_\lambda v = 0\}$ is a left ideal, but not necessarily $M \simeq R/I$. For example, consider $\mathbb{C}[\partial]$ as a $\mathrm{Cend}_1$-module, then the kernel of $a \mapsto a_\lambda v$ is $\{0\}$.

(b) If we fix $\mu \in \mathbb{C}$, there are examples of irreducible modules where $R_{-\partial-\mu} v = 0$ for all $v \in M$ (cf. Lemma 3.2). Indeed, consider $\mathbb{C}[\partial]$ as a $\mathrm{Cend}_{1,(x+\mu)}$-module.

Using Remark 3.1, Proposition 1.3 and Theorem 3.3, we have

**Corollary 3.5.** *The* $\mathrm{Cend}_{N,P}$*-module* $\mathbb{C}[\partial]^N$ *defined by* (1.2) *is irreducible if and only if* $\det P(x) \neq 0$. *These are all non-trivial irreducible* $\mathrm{Cend}_{N,P}$*-modules up to equivalence, provided that* $\det P(x) \neq 0$.

Note that Corollary 3.5 in the case $P(x) = I$, have been established earlier in [12], by a completely different method (developed in [13]). Another proof of this was also given in [17].

A subalgebra $S$ of $\mathrm{Cend}_N$ is called *irreducible* if $S$ acts irreducibly in $\mathbb{C}[\partial]^N$.

**Corollary 3.6.** *The following subalgebras of* $\mathrm{Cend}_N$ *are irreducible*: $\mathrm{Cend}_{N,P}$ *with* $\det P(x) \neq 0$, *and* $\mathrm{Cur}_N := \mathrm{Mat}_N(\mathbb{C}[\partial])$ *or conjugates of it by automorphisms* (1.3).

**Remark 3.7.** It is easy to show that every non-trivial irreducible representation of $\mathrm{Cur}_N$ is equivalent to the standard module $\mathbb{C}[\partial]^N$, and that every finite module over $\mathrm{Cur}_N$ is completely reducible.

We will finish this section with the classification of all extensions of $\mathrm{Cend}_{N,P}$-modules involving the standard module $\mathbb{C}[\partial]^N$ and finite-dimensional trivial modules, and the classification of all finite modules over $\mathrm{Cend}_N$.

We shall work with the standard irreducible $\mathrm{Cend}_{N,P}$-module $\mathbb{C}[\partial]^N$ with $\lambda$-action (see (1.2))

$$a(\partial, x)P(x)_\lambda v(\partial) = a(-\lambda, \lambda + \partial + \alpha)P(\lambda + \partial)v(\lambda + \partial).$$

Consider the trivial $\mathrm{Cend}_{N,P}$-module over the finite-dimensional vector space $V_T$, whose $\mathbb{C}[\partial]$-module structure is given by the linear operator $T$, that is: $\partial \cdot v = T(v)$, $v \in V_T$. As usual, we may assume that $P(x) = \mathrm{diag}\{p_1(x), \ldots, p_N(x)\}$. We shall assume that $\det P \neq 0$.

**Theorem 3.8.** (a) *There are no non-trivial extensions of* $\mathrm{Cend}_{N,P}$*-modules of the form*:

$$0 \to V_T \to E \to \mathbb{C}[\partial]^N \to 0.$$

*Here and further, all the maps in these sequences are maps of* $\mathrm{Cend}_{N,P}$*-modules.*

(b) *If there exists a non-trivial extension of* $\mathrm{Cend}_{N,P}$*-modules of the form*

$$0 \to \mathbb{C}[\partial]^N \to E \to V_T \to 0, \tag{3.3}$$

*then* $\det P(\alpha + c) = 0$ *for some eigenvalue* $c$ *of* $T$*. In this case, all torsionless extensions of* $\mathbb{C}[\partial]^N$ *by finite-dimensional vector spaces, are parameterized by decompositions* $P(x+\alpha) = R(x)S(x)$ *and can be realized as follows. Consider the following isomorphism of conformal algebras*:

$$\mathrm{Cend}_{N,P} \to S(\partial + x)\,\mathrm{Cend}_N\, R(x), \quad a(\partial, x)P(x) \mapsto S(\partial + x)a(\partial, x)R(x),$$

*where* $P(x+\alpha) = R(x)S(x)$ (*this is the isomorphism between* $\mathrm{Cend}_{N,S}$ *and* $\mathrm{Cend}_{S,N}$ (*Proposition* 1.4), *restricted to* $\mathrm{Cend}_{N,R}\, S(x)$). *Using this isomorphism, we get an action of* $\mathrm{Cend}_{N,P}$ *on* $\mathbb{C}[\partial]^N$:

$$a(\partial, x)P(x)_\lambda v(\partial) = S(\partial)a(-\lambda, \lambda + \partial + \alpha)R(\lambda + \partial)v(\lambda + \partial).$$

*Then* $S(\partial)\mathbb{C}[\partial]^N$ *is a submodule isomorphic to the standard module, of finite codimension in* $\mathbb{C}[\partial]^N$.

(c) *If* $E$ *is a non-trivial extension of* $\mathrm{Cend}_{N,P}$*-modules of the form*:

$$0 \to \mathbb{C}[\partial]^N \to E \to \mathbb{C}[\partial]^N \to 0,$$

*then* $E = \mathbb{C}[\partial]^N \otimes \mathbb{C}^2$ *as a* $\mathbb{C}[\partial]$*-module* (*with trivial action of* $\partial$ *on* $\mathbb{C}^2$) *and* $\mathrm{Cend}_{N,P}$ *acts by*

$$a(\partial, x)_\lambda\big(c(\partial) \otimes u\big) = a(-\lambda, \lambda + \partial \otimes 1 + 1 \otimes J)c(\lambda + \partial)(1 \otimes u), \tag{3.4}$$

*where* $J$ *is a* $2 \times 2$ *Jordan block matrix.*

**Proof.** (a) Consider a short exact sequence of $R = \mathrm{Cend}_{N,P}$-modules

$$0 \to T \to E \to V \to 0, \tag{3.5}$$

where $V$ is irreducible finite, and $T$ is trivial (finite-dimensional vector space). Take $v \in E$ with $v \notin T$, and let $\mu \in \mathbb{C}$ be such that $A := R_{-\partial-\mu} v \neq 0$. Then we have three possibilities.

(1) The image of $A$ in $V$ is 0, then $A = T$, which is impossible since $A$ corresponds to a left ideal of $\mathrm{Cend}_{N,P}$.

(2) The image of $A$ in $V$ is $V$ and $A \cap T = 0$, then $A$ is isomorphic to $V$, hence the exact sequence splits.

(3) The image of $A$ in $V$ is $V$ and $T' = A \cap T \neq 0$. Now, if $T' = T$ then $A = E$ and $E$ is a cyclic module, which is impossible since it has torsion. If $T' \neq T$, we consider the exact sequence $0 \to T' \to A \to V \to 0$, by an inductive argument on the dimension of the trivial module, the last sequence splits, i.e., $A = T' \oplus V' \subset E$ with $V' \simeq V$, hence $E = T \oplus V'$ as $\mathrm{Cend}_{N,P}$-modules, proving (a).

(b) We may assume without loss of generality that $\alpha = 0$. Consider an extension of $\mathrm{Cend}_{N,P}$-modules of the form (3.3). As a vector space $E = \mathbb{C}[\partial]^N \oplus V_T$. We have, for $v \in V_T$:

$$\begin{aligned} \partial v &= T(v) + g_v(\partial), \quad \text{where } g_v(\partial) \in \mathbb{C}[\partial]^N, \\ x^l BP(x)_\lambda v &= f_l^{v,B}(\lambda,\partial), \qquad \text{where } f_l^{v,B}(\lambda,\partial) \in \big(\mathbb{C}[\partial]^N\big)[\lambda],\ B \in \mathrm{Mat}_N \mathbb{C}. \end{aligned} \tag{3.6}$$

Let $P(x) = \sum_{i=0}^m Q_i x^i$. Since

$$\begin{aligned} \big(x^k AP(x)_\lambda x^l BP(x)\big)_{\lambda+\mu} v &= (\lambda+\partial+x)^k AP(\lambda+\partial+x) x^l BP(x)_{\lambda+\mu} v \\ &= \sum_{i=0}^m \sum_{j=0}^{i+k} \binom{i+k}{j} (\lambda+\partial)^{i+k-j} x^{j+l} AQ_i BP(x)_{\lambda+\mu} v \\ &= \sum_{i=0}^m \sum_{j=0}^{i+k} \binom{i+k}{j} (-\mu)^{i+k-j} f_{j+l}^{v,AQ_iB}(\lambda+\mu,\partial) \end{aligned}$$

and

$$\begin{aligned} x^k AP(x)_\lambda \big(x^l BP(x)_\mu v\big) &= x^k AP(x)_\lambda \big(f_l^{v,B}(\mu,\partial)\big) \\ &= (\lambda+\partial)^k AP(\lambda+\partial) f_l^{v,B}(\mu,\lambda+\partial) \end{aligned}$$

must be equal by $(\mathrm{A2})_\lambda$, we have the functional equation

$$\begin{aligned} &(\lambda+\partial)^k AP(\lambda+\partial) f_l^{v,B}(\mu,\lambda+\partial) \\ &\quad = \sum_{i=0}^m \sum_{j=0}^{i+k} \binom{i+k}{j} (-\mu)^{i+k-j} f_{j+l}^{v,AQ_iB}(\lambda+\mu,\partial). \end{aligned} \tag{3.7}$$

If we put $\mu = 0$ in (3.7), we get

$$(\lambda+\partial)^k AP(\lambda+\partial) f_l^{v,B}(0,\lambda+\partial)=\sum_{i=0}^{m} f_{i+k+l}^{v,AQ_iB}(\lambda,\partial). \tag{3.8}$$

Since the right-hand side of (3.8) is symmetric in $k$ and $l$, so is the left-hand side, hence, in particular, we have

$$(\lambda+\partial)^k AP(\lambda+\partial) f_0^{v,B}(0,\lambda+\partial)=AP(\lambda+\partial) f_k^{v,B}(0,\lambda+\partial).$$

Taking $A=I$ and using that $\det P\neq 0$, we get

$$f_k^{v,B}(0,\lambda+\partial)=(\lambda+\partial)^k f_0^{v,B}(0,\lambda+\partial). \tag{3.9}$$

Furthermore, by $(\mathrm{A1})_\lambda$, we have $[\partial, x^k AP(x)_\lambda]v=-\lambda x^k AP(x)_\lambda v$, which gives us the next condition:

$$(\lambda+\partial) f_k^{v,A}(\lambda,\partial)=f_k^{T(v),A}(\lambda,\partial)+(\lambda+\partial)^k AP(\lambda+\partial) g_v(\lambda+\partial). \tag{3.10}$$

We shall prove that if $c$ is an eigenvalue of $T$ and $p_j(c)\neq 0$ for all $1\leqslant j\leqslant N$, then (after a change of complement) the generalized eigenspace of $T$ corresponding to the eigenvalue $c$ is a trivial submodule of $E$ (hence is a non-zero torsion submodule). Indeed, let $\{v_1,\dots,v_s\}$ be vectors corresponding to one Jordan block of $T$ associated to $c$, that is $T(v_1)=cv_1$ and $T(v_{i+1})=cv_{i+1}+v_i$ for $i\geqslant 1$. Then (3.10) with $v=v_1$ becomes

$$(\lambda+\partial-c) f_k^{v_1,A}(\lambda,\partial)=(\lambda+\partial)^k AP(\lambda+\partial) g_{v_1}(\lambda+\partial). \tag{3.11}$$

Observe that the right-hand side of (3.11) depends on $\lambda+\partial$, so $f_k^{v_1,A}(\lambda,\partial)=f_k^{v_1,A}(0,\lambda+\partial)$. Then using (3.9), we have

$$\begin{aligned} f_k^{v_1,A}(\lambda,\partial) &= f_k^{v_1,A}(0,\lambda+\partial)\\ &= (\lambda+\partial)^k f_0^{v_1,A}(0,\lambda+\partial)=(\lambda+\partial)^k f_0^{v_1,A}(\lambda,\partial). \end{aligned} \tag{3.12}$$

Similarly, considering (3.10) with $v=v_{i+1}$ ($i\geqslant 1$), we get

$$\begin{aligned} (\lambda+\partial-c) f_k^{v_{i+1},A}(\lambda,\partial) &= f_k^{v_i,A}(\lambda,\partial)+(\lambda+\partial)^k AP(\lambda+\partial) g_{v_{i+1}}(\lambda+\partial)\\ &= (\lambda+\partial)^k\big[f_0^{v_i,A}(0,\lambda+\partial)+AP(\lambda+\partial) g_{v_{i+1}}(\lambda+\partial)\big]. \end{aligned} \tag{3.13}$$

Again, since the right-hand side of (3.13) depends only on $\lambda+\partial$, we have that (3.12) also holds for any $v_i$.

Using that $p_j(c)\neq 0$ ($j=1,\dots,N$) (recall that $P$ is diagonal), and taking $A=E_{i,j}$, we obtain from (3.11) with $k=0$ that

$$f_0^{v_1,A}(\lambda,\partial)=AP(\lambda+\partial) h_{v_1}(\lambda+\partial), \tag{3.14a}$$

where $g_{v_1}(\partial) = (\partial - c)h_{v_1}(\partial)$. Now, (3.13) with $k = 0$ and $i = 1$ becomes (by (3.14a))

$$\begin{aligned}(\lambda + \partial - c) f_0^{v_2,A}(\lambda, \partial) &= f_0^{v_1,A}(\lambda, \partial) + AP(\lambda + \partial) g_{v_2}(\lambda + \partial) \\ &= AP(\lambda + \partial)\big(h_{v_1}(\lambda + \partial) + g_{v_2}(\lambda + \partial)\big).\end{aligned}$$

As in (3.14a), we get

$$f_0^{v_2,A}(\lambda, \partial) = AP(\lambda + \partial) h_{v_2}(\lambda + \partial),$$

where $g_{v_2}(\partial) + h_{v_1}(\partial) = (\partial - c)h_{v_2}(\partial)$. Similarly, we obtain for all $i \geqslant 1$,

$$f_0^{v_{i+1},A}(\lambda, \partial) = AP(\lambda + \partial) h_{v_{i+1}}(\lambda + \partial), \tag{3.14b}$$

where $g_{v_{i+1}}(\partial) + h_{v_i}(\partial) = (\partial - c)h_{v_{i+1}}(\partial)$. Changing the basis to $v_i' = v_i - h_{v_i}(\partial)$, we have from (3.12) and (3.14) that $x^k AP(x)_\lambda v_i' = 0$ and

$$\begin{aligned}\partial v_1' &= T(v_1) + g_{v_1}(\partial) - \partial h_{v_1}(\partial) \\ &= cv_1 + (\partial - c)h_{v_1}(\partial) - \partial h_{v_1}(\partial) = cv_1', \\ \partial v_{i+1}' &= T(v_{i+1}) + g_{v_{i+1}}(\partial) - \partial h_{v_{i+1}}(\partial) \\ &= cv_{i+1} + v_i + (\partial - c)h_{v_{i+1}}(\partial) - \partial h_{v_{i+1}}(\partial) - h_{v_i}(\partial) \\ &= cv_{i+1}' + v_i'.\end{aligned} \tag{3.15}$$

Hence, the $T$-invariant subspace spanned by $\{v_i'\}$ is a trivial submodule of $E$. Therefore, if $p_j(c) \neq 0$ for all $j$ and all eigenvalues $c$ of $T$, then $E$ is a trivial extension. This proves the first part of (b).

Now suppose that the extension $E$ of $\mathbb{C}[\partial]^N$ by a finite-dimensional vector space have no non-zero trivial submodule (equivalently, $E$ is torsionless). By Remark 3.1(b), $E$ must be a free $\mathbb{C}[\partial]$-module of rank $N$.

Then, the problem reduces to the study of a $\mathrm{Cend}_{N,P}$-module structure on $E = \mathbb{C}[\partial]^N$, but using Remark 1.1, this is the same as a non-zero homomorphism from $\mathrm{Cend}_{N,P}$ to $\mathrm{Cend}_N$. So, the end of this proof also gives us the classification of all these homomorphisms.

Denote by $\phi : \mathrm{Cend}_{N,P} \to \mathrm{Cend}_N$ the (non-zero) homomorphism associated to $E$. It is an embedding (due to irreducibility) of free $\mathbb{C}[\partial]$-modules $\mathbb{C}[\partial]^N \to \mathbb{C}[\partial]^N$, hence it is given by a non-degenerate matrix $S(\partial) \in \mathrm{Mat}_N\, \mathbb{C}[\partial]$. Hence the action on $E$ of $\mathrm{Cend}_{N,P}$ is given by the formula:

$$\phi\big(a(\partial, x)P(x)\big)_\lambda\big(S(\partial)v\big) = S(\partial)a(-\lambda, \lambda + \partial + \alpha)P(\lambda + \partial + \alpha)v \quad \text{for all } v \in \mathbb{C}^N.$$

Furthermore, we have:

$$\begin{aligned}\big(\phi\big(a(\partial,x)P(x)\big)S(x)\big)_\lambda v &= \phi\big(a(\partial,x)P(x)\big)_\lambda\big(S(\partial)v\big)\\ &= \big(S(\partial+x)a(\partial,x+\alpha)P(x+\alpha)\big)_\lambda v \quad \text{for all } v\in\mathbb{C}^N.\end{aligned}$$

Hence $\phi(a(\partial,x)P(x)) = S(\partial+x)a(\partial,x+\alpha)P(x+\alpha)S^{-1}(x)$, and this is in $\mathrm{Cend}_N$ if and only if $R(x) := P(x+\alpha)S^{-1}(x) \in \mathrm{Mat}_N\,\mathbb{C}[x]$, proving (b).

(c) Consider a short exact sequence of $R = \mathrm{Cend}_{N,P}$-modules

$$0 \to V \to E \to V' \to 0, \tag{3.16}$$

where $V$ and $V'$ are irreducible finite. Take $v \in E$ with $v \notin V$, and let $\mu \in \mathbb{C}$ be such that $A := R_{-\partial-\mu}v \neq 0$. Then we have three possibilities.

(1) The image of $A$ in $V'$ is 0, then $A = V$, which is impossible because $v \notin V$.

(2) The image of $A$ in $V'$ is $V'$ and $A \cap V = 0$, then $A$ is isomorphic to $V'$, hence the exact sequence splits.

(3) The image of $A$ in $V'$ is $V'$ and $A \cap V = V$, hence $A = E$ and $E$ is a cyclic module, hence corresponds to a left ideal which is contained in a unique maximal ideal (otherwise the sequence splits). It is easy to see then that $E$ is the indecomposable module given in (3.4), where $J$ is the $2\times 2$ Jordan block. □

**Corollary 3.9.** *There are no non-trivial extensions of* $\mathrm{Cend}_N$*-modules of the form*:

$$0 \to V_T \to E \to \mathbb{C}[\partial]^N \to 0 \quad \textit{or} \quad 0 \to \mathbb{C}[\partial]^N \to E \to V_T \to 0.$$

**Theorem 3.10.** *Every finite* $\mathrm{Cend}_N$*-module is isomorphic to a direct sum of its* (*finite-dimensional*) *trivial torsion submodule and a free finite* $\mathbb{C}[\partial]$*-module* $\mathbb{C}[\partial]^N \otimes T$ *on which the* $\lambda$*-action is given by*

$$a(\partial,x)_\lambda\big(c(\partial)\otimes u\big) = a(-\lambda, \lambda+\partial\otimes 1 + 1\otimes\alpha)c(\lambda+\partial)(1\otimes u), \tag{3.17}$$

*where* $\alpha$ *is an arbitrary operator on* $T$.

**Proof.** Consider a short exact sequence of $R = \mathrm{Cend}_N$-modules

$$0 \to V \to E \to V' \to 0,$$

where $V$ and $V'$ are irreducible finite. By Theorem 3.8(c), the exact sequence split or $E$ is the indecomposable module that corresponds to a $2\times 2$ Jordan block $J$, i.e., $E = \mathbb{C}[\partial]^N \otimes \mathbb{C}^2$, and $R$ acts via (3.17), where $\alpha = J$.

Next, using Corollary 3.9, the short exact sequences of $R$-modules $0 \to V \to E \to C \to 0$ and $0 \to C \to E \to V \to 0$, where $C$ is a trivial 1-dimensional $R$-module, and $V$ is a standard $R$-module (1.2), split.

Recall [11] that an $R$-module is the same as a module over the associated extended annihilation algebra $(\operatorname{Alg} R)^- = \mathbb{C}\partial \ltimes (\operatorname{Alg} R)_-$, where $(\operatorname{Alg} R)_-$ is the annihilation algebra. For $R = \mathrm{Cend}_N$ one has:

$$(\operatorname{Alg} R)_- = \left(\mathrm{Diff}^N \mathbb{C}\right), \qquad (\operatorname{Alg} R)^- = \mathbb{C}\partial \ltimes (\operatorname{Alg} R)_-,$$

where $\partial$ acts on $(\operatorname{Alg} R)_-$ via $-ad\partial_t$. Furthermore, viewed as an $(\operatorname{Alg} R)_-$-module, all modules (1.2) are equivalent to the module $F = \mathbb{C}[t, t^{-1}]^N/\mathbb{C}[t]^N$, and the modules (1.2) are obtained by letting $\partial$ act as $-\partial_t + \alpha$.

Let $M$ be a finite $R$-module. Then it has finite length and, by Corollary 3.5, all its irreducible subquotients are either trivial 1-dimensional or are isomorphic to a standard $R$-module (1.2). Since the exact sequence splits when restricted to $(\operatorname{Alg} R)_-$, we conclude that, viewed as an $(\operatorname{Alg} R)_-$-module, $M$ is a finite direct sum of modules equivalent to $F$ or trivial 1-dimensional. Thus, viewed as an $(\operatorname{Alg} R)_-$-module, $M = S \oplus (F \otimes T)$, where $S$ and $T$ are trivial $(\operatorname{Alg} R)_-$-modules. The only way to extend this $M$ to an $(\operatorname{Alg} R)^-$-module is to let $\partial$ act as operators $\alpha$ and $\beta$ on $T$ and $S$, respectively, and as $-\partial_t$ on $F$, which gives (3.17). $\square$

**Remark 3.11.** Theorem 3.10 was stated in [12], and another proof of it was given in [17].

## 4. Automorphisms and anti-automorphisms of $\mathrm{Cend}_{N,P}$

A $\mathbb{C}[\partial]$-linear map $\sigma : R \to S$ between two associative conformal algebras is called a *homomorphism* (*respectively anti-homomorphism*) if

$$\sigma(a_\lambda b) = \sigma(a)_\lambda \sigma(b) \quad (\text{respectively } \sigma(a_\lambda b) = \sigma(b)_{-\lambda-\partial}\sigma(a)).$$

An anti-automorphism $\sigma$ is an *anti-involution* if $\sigma^2 = 1$.

An important example of an anti-involution of $\mathrm{Cend}_N$ is:

$$\sigma\big(a(\partial, x)\big) = a^t(\partial, -x - \partial), \tag{4.1}$$

where the superscript $t$ stands for the transpose of a matrix.

By Corollary 3.5 we know that all irreducible finite $\mathrm{Cend}_N$-modules are of the form ($\alpha \in \mathbb{C}$):

$$a(\partial, x)_\lambda v(\partial) = a(-\lambda, \lambda + \partial + \alpha) v(\lambda + \partial).$$

Hence, twisting one of these modules by an automorphism of $\mathrm{Cend}_N$ gives again one of these modules, and we get the following

**Theorem 4.1.** *All automorphisms of* $\mathrm{Cend}_N$ *are of the form*:

$$a(\partial, x) \mapsto C(\partial + x) a(\partial, x + \alpha) C(x)^{-1},$$

*where* $\alpha \in \mathbb{C}$ *and* $C(x)$ *is a matrix with a non-zero constant determinant.*

This result can be generalized as follows.

**Theorem 4.2.** *Let* $P(x) \in \operatorname{Mat}_N \mathbb{C}[x]$ *with* $\det P(x) \neq 0$. *Then all automorphisms of* $\operatorname{Cend}_{N,P}$ *are those that come from* $\operatorname{Cend}_N$ *by restriction. More precisely, any automorphism is of the form*:

$$a(\partial, x)P(x) \mapsto C(\partial + x)a(\partial, x + \alpha)B(x)P(x), \tag{4.2}$$

*where* $\alpha \in \mathbb{C}$, *and* $B(x)$ *and* $C(x)$ *are invertible matrices in* $\operatorname{Mat}_N \mathbb{C}[x]$ *such that*

$$P(x + \alpha) = B(x)P(x)C(x). \tag{4.3}$$

**Proof.** Let $\pi'(a) = \pi(s(a))$, where $\pi$ is the standard representation and $s$ is an automorphism of $\operatorname{Cend}_{N,P}$. Since it is equivalent to the standard representation due to Corollary 3.5, we deduce that $s(a(\partial, x)) = C(\partial + x)a(\partial, x + \alpha)C(x)^{-1}$ for some invertible (in $\operatorname{Mat}_N \mathbb{C}[x]$) matrix $C(x)$. But $C(\partial + x) \operatorname{Cend}_{N,P} C(x)^{-1} = \operatorname{Cend}_{N,P}$ if and only if (4.3) holds. Indeed, we have: $C(\partial + x)\ P(x + \alpha)C(x)^{-1} = A(\partial, x)P(x)$ for some $A(\partial, x) \in \operatorname{Cend}_N$. Taking determinants of both sides of this equality, we see that $\det A(\partial, x)$ is a non-zero constant. Hence $B(x) := P(x + \alpha)C(x)^{-1}P(x)^{-1}$ is invertible in $\operatorname{Mat}_N \mathbb{C}[x]$, finishing the proof. □

**Theorem 4.3.** *Let* $P(x) \in \operatorname{Mat}_N \mathbb{C}[x]$ *with* $\det P(x) \neq 0$. *Then we have*:

(a) *All non-zero homomorphisms from* $\operatorname{Cend}_{N,P}$ *to* $\operatorname{Cend}_N$ *are of the form*

$$a(\partial, x)P(x) \mapsto S(\partial + x)a(\partial, x + \alpha)R(x), \tag{4.4}$$

*where* $\alpha \in \mathbb{C}$, *and* $R(x)$ *and* $S(x)$ *are matrices in* $\operatorname{Mat}_N \mathbb{C}[x]$ *such that*

$$P(x + \alpha) = R(x)S(x). \tag{4.5}$$

(b) *All non-trivial anti-homomorphisms from* $\operatorname{Cend}_{N,P}$ *to* $\operatorname{Cend}_N$ *are of the form*

$$a(\partial, x)P(x) \mapsto A(\partial + x)a^t(\partial, -\partial - x + \alpha)B(x), \tag{4.6}$$

*where* $\alpha \in \mathbb{C}$, *and* $A(x)$ *and* $B(x)$ *are matrices in* $\operatorname{Mat}_N \mathbb{C}[x]$ *such that*

$$P^t(-x + \alpha) = B(x)A(x). \tag{4.7}$$

(c) *The conformal algebra* $\operatorname{Cend}_{N,P}$ *has an anti-automorphism* (*i.e., it is isomorphic to its opposite conformal algebra*) *if and only if the matrices* $P^t(-x + \alpha)$ *and* $P(x)$ *have the same elementary divisors for some* $\alpha \in \mathbb{C}$. *In this case, all anti-automorphisms of* $\operatorname{Cend}_{N,P}$ *are of the form*

$$a(\partial, x)P(x) \mapsto Y(\partial + x)a^t(\partial, -\partial - x + \alpha)W(x)P(x), \tag{4.8}$$

*where $Y(x)$ and $W(x)$ are invertible matrices in* $\mathrm{Mat}_N\,\mathbb{C}[x]$ *such that*

$$P^t(-x+\alpha) = W(x)P(x)Y(x). \tag{4.9}$$

(d) *The conformal algebra* $\mathrm{Cend}_{N,P}$ *has an anti-involution if and only if there exist an invertible in* $\mathrm{Mat}_N\,\mathbb{C}[x]$ *matrix $Y(x)$ such that*

$$Y^t(-x+\alpha)P^t(-x+\alpha) = \epsilon P(x)Y(x) \tag{4.10}$$

*for $\epsilon = 1$ or $-1$. In this case all anti-involutions are given by*

$$\sigma_{P,Y,\epsilon,\alpha}\big(a(\partial,x)P(x)\big) = \varepsilon Y(\partial+x)a^t(\partial,-\partial-x+\alpha)Y^t(-x+\alpha)^{-1}P(x), \tag{4.11}$$

*where $Y(x)$ is an invertible in* $\mathrm{Mat}_N\,\mathbb{C}[x]$ *matrix satisfying* (4.10).

**Proof.** (a) Follows by the end of proof of Theorem 3.8(b).

(b) Since composition of two anti-homomorphisms is a homomorphism, using the anti-involution (4.1) we see that any anti-homomorphism must be of the form

$$a(\partial,x)P(x) \to R^t(-\partial-x)a^t(\partial,-\partial-x+\alpha)S^t(-x) \tag{4.12}$$

with $P(x+\alpha) = R(x)S(x)$. Then, (4.6) and (4.7) follows by taking $A(x) = S^t(-x)$ and $B(x) = R^t(-\partial-x)$.

(c) Let $\phi$ be an anti-automorphism of $\mathrm{Cend}_{N,P}$. In particular, it is an anti-homomorphism as in part (b), whose image is $\mathrm{Cend}_{N,P}$. Then, for all $a(\partial,x)P(x) \in \mathrm{Cend}_{N,P}$, we have that $\phi(a(\partial,x)P(x)) = A(\partial+x)a^t(\partial,-\partial-x+\alpha)B(x) \in \mathrm{Cend}_{N,P}$. Then taking $a(\partial,x)$ to be the identity matrix we have that

$$A(\partial+x)B(x) = b(\partial,x)P(x), \quad \text{for some } b(\partial,x) \in \mathrm{Cend}_{N,P}\,. \tag{4.13}$$

Since $P^t(-x+\alpha) = B(x)A(x)$, taking determinant of both sides of (4.13), and comparing its highest degrees in $x$, we deduce that $\det b(\partial,x)$ is a (non-zero) constant. Therefore $\det A(x)$ is also a (non-zero) constant. Now, from (4.13), we see that $A^{-1}(\partial+x)b(\partial,x)$ does not depend on $\partial$. Then we have $B(x) = W(x)P(x)$, where $W(x) = A^{-1}(\partial+x)b(\partial,x)$ is an invertible matrix. Therefore,

$$\phi\big(a(\partial,x)P(x)\big) = A(\partial+x)a^t(\partial,-\partial-x+\alpha)W(x)P(x), \tag{4.14}$$

with $A$, $W$ invertible matrices such that

$$W(x)P(x)A(x) = P^t(-x+\alpha). \tag{4.15}$$

(d) Now suppose that $\phi$ is an anti-involution. Then it is as in (4.8), and it also satisfies $\phi^2 = \mathrm{Id}$. This condition implies that

$$a(\partial, x)P(x) = Y(\partial + x)W^t(-\partial - x + \alpha)a(\partial, x)Y^t(-x + \alpha)W(x)P(x) \tag{4.16}$$

for all $a(\partial, x) \in \mathrm{Cend}_{N,P}$. Denote $Z(x) = Y^t(-x+\alpha)W(x)$. Taking $a(\partial, x) = \mathrm{Id}$ in (4.16) and using that $\det P(x) \neq 0$, we have $Y(\partial + x)W^t(-\partial - x + \alpha) = Z^{-1}(x)$. Now, (4.16) becomes $a(\partial, x)P(x) = Z^{-1}(x)a(\partial, x)Z(x)P(x)$. Hence, we obtain $Z(x) = \varepsilon\,\mathrm{Id}$, where $\varepsilon$ is a constant. Thus, $Y^{-1}(x) = \varepsilon W^t(-x + \alpha)$. From (4.9) we deduce that

$$P(x)Y(x) = \varepsilon\big(P(-x + \alpha)Y(-x + \alpha)\big)^t. \tag{4.17}$$

This condition is also sufficient. There exists an anti-involution if (4.17) holds for some invertible matrix $Y$, and it is given by

$$\phi\big(a(\partial, x)P(x)\big) = \varepsilon Y(\partial + x)a^t(\partial, -\partial - x + \alpha)Y^t(-x + \alpha)^{-1}P(x),$$

with $\varepsilon = 1$ or $-1$. □

Two anti-involutions $\sigma, \tau$ of an associative conformal algebra $R$ are called *conjugate* if $\sigma = \varphi \circ \tau \circ \varphi^{-1}$ for some automorphism $\varphi$ of $R$. Recall that two matrices $a$ and $b$ in $\mathrm{Mat}_N\,\mathbb{C}[x]$ are called *$\alpha$-congruent* if $b = c^* a\, c$ for some invertible in $\mathrm{Mat}_N\,\mathbb{C}[x]$ matrix $c$, where $c(x)^* := c(-x + \alpha)^t$. We shall simply call them *congruent* if $\alpha = 0$. The following proposition gives us a characterization of equivalent anti-involutions $\sigma_{P,Y,\epsilon,\alpha}$ in $\mathrm{Cend}_{N,P}$ (defined in (4.11)) and relates anti-involutions for different $P$.

**Proposition 4.4.** (a) *The anti-involutions $\sigma_{P,Y_1,\epsilon_1,\alpha}$ and $\sigma_{P,Y_2,\epsilon_2,\gamma}$ of* $\mathrm{Cend}_{N,P}$ *are conjugate if and only if $\epsilon_1 = \epsilon_2$ and $P(x + (\gamma - \alpha)/2)Y_2(x + (\gamma - \alpha)/2)$ is $\alpha$-congruent to $P(x)Y_1(x)$.*

(b) *Let $\varphi_Y$ be the automorphism of* $\mathrm{Cend}_N$ *given by*

$$\varphi_Y\big(a(\partial, x)\big) = Y(\partial + x)^{-1}a(\partial, x)Y(x),$$

*where $Y$ is an invertible matrix in* $\mathrm{Mat}_N\,\mathbb{C}[x]$, *and let $P$ and $Y$ satisfying* (4.10). *Then*

$$\sigma_{P,Y,\epsilon,\alpha} = \varphi_Y^{-1} \circ \sigma_{PY,I,\epsilon,\alpha} \circ \varphi_Y. \tag{4.18}$$

(c) *Let $c_\alpha$ be the automorphism of* $\mathrm{Cend}_N$ *given by $c_\alpha(a(\partial, x)) = a(\partial, x + \alpha)$, where $\alpha \in \mathbb{C}$. Suppose that $P^t(-x + \alpha) = \epsilon P(x)$, for $\epsilon = 1$ or $-1$, then $Q(x) := P(x + \alpha/2)$ satisfies $Q^t(-x) = \epsilon Q(x)$ and*

$$\sigma_{P,I,\epsilon,\alpha} = c_{\alpha/2}^{-1} \circ \sigma_{Q,I,\epsilon,0} \circ c_{\alpha/2}. \tag{4.19}$$

**Proof.** (a) Let $\varphi_{B,C,\alpha}$ be the automorphism of $\mathrm{Cend}_{N,P}$ given by (4.2) and (4.3). A straightforward computation shows that $\varphi^{-1}_{B,C,\beta} \circ \sigma_{P,Y,\epsilon,\alpha} \circ \varphi_{B,C,\beta} = \sigma_{P,\bar{Y},\epsilon,2\beta+\alpha}$, where $\bar{Y}(x) = C^{-1}(x-\beta)Y(x-\beta)B^t(-x+\alpha+\beta)$ and $P(x+\beta) = B(x)P(x)C(x)$. Hence, if $\sigma_{P,Y_1,\epsilon_1,\alpha}$ and $\sigma_{P,Y_2,\epsilon_2,\gamma}$ are conjugate, then $\epsilon_1 = \epsilon_2$ and $Y_2(x) = C^{-1}(x-\beta)Y(x-\beta)B^t(-x+\alpha+\beta)$, with $\beta = \gamma - \alpha/2$. Therefore, $P(x+\beta)Y_2(x+\beta) = B(x)P(x)Y_1(x)B^t(-x+\alpha)$, that is $P(x+(\gamma-\alpha)/2)Y_2(x+(\gamma-\alpha)/2)$ is $\alpha$-congruent to $P(x)Y_1(x)$.

Conversely, suppose that $P(x+(\gamma-\alpha)/2)Y_2(x+(\gamma-\alpha)/2) = B(x)P(x)Y_1(x)B^t \times (-x+\alpha)$ for some $B(x)$ invertible matrix in $\mathrm{Mat}_N\,\mathbb{C}[x]$. Recall that $Y_1$ and $Y_2$ are invertible. Then $C(x) := Y_1(x)B^t(-x+\alpha)Y_2(x+(\gamma-\alpha)/2)^{-1}$ is an invertible matrix in $\mathrm{Mat}_N\,\mathbb{C}[x]$, satisfies $P(x+(\gamma-\alpha)/2) = B(x)P(x)C(x)$, and it is easy to check that the anti-involutions are conjugated by the automorphism $\varphi_{B,C,(\gamma-\alpha)/2}$, proving (a).

Parts (b) and (c) are straightforward computations. □

**Theorem 4.5.** *Any anti-involution of* $\mathrm{Cend}_N$ *is, up to conjugation by an automorphism of* $\mathrm{Cend}_N$:

$$a(\partial, x) \mapsto a^*(\partial, -\partial - x),$$

*where* $*$ *is the adjoint with respect to a non-degenerate symmetric or skew-symmetric bilinear form over* $\mathbb{C}$.

**Proof.** Using Theorem 4.3(d), we have that any anti-involution of $\mathrm{Cend}_N$ has the form $\sigma(a(\partial,x)) = C(\partial+x)a^t(\partial,-\partial-x+\alpha)C(x)^{-1}$, where $C(x)$ is an invertible matrix such that $C^t(x) = \varepsilon C(-x+\alpha)$, with $\varepsilon = 1$ or $-1$. By Proposition 4.4(c), we may suppose that $\alpha = 0$. Now, the proof follows because $C(x)$ is congruent to a constant symmetric or skew-symmetric matrix, by the following general theorem of Djokovic. □

**Theorem 4.6** (Djokovic [9,10]). *If $A$ is invertible in* $\mathrm{Mat}_N(\mathbb{C}[x])$ *and* $A^* = A$ (*respectively* $A^* = -A$) *where* $A(x)^* = A^t(-x)$, *then $A$ is congruent to a symmetric* (*respectively skew-symmetric*) *matrix over* $\mathbb{C}$.

**Proof.** The symmetric case follows by Proposition 5 in [9]. The skew-symmetric case was communicated to us by D. Djokovic and we will give the details here. Suppose $A^* = -A$. By [15, Theorem 2.2.1, Chapter 7] it follows that $A$ has to be isotropic, i.e., there exists a non-zero vector $v$ in $\mathbb{C}[x]^N$ such that $v^*Av = 0$. We can assume that $v$ is primitive (i.e., the greatest common divisor of its coordinates is 1). But then $\mathbb{C}[x]v$ is a direct summand: $\mathbb{C}[x]^N = \mathbb{C}[x]v \oplus M$, for some $\mathbb{C}[x]$-submodule $M$ of $\mathbb{C}[x]^N$. Then we have $\mathbb{C}[x]^N = (\mathbb{C}[x]v)^\perp \oplus M^\perp$ and $M^\perp$ is a free rank one $\mathbb{C}[x]$-module, that is $M^\perp = \mathbb{C}[x]w$ for some $w \in \mathbb{C}[x]^N$. Since $\mathbb{C}[x]v \subseteq (\mathbb{C}[x]v)^\perp$, the submodule $P = \mathbb{C}[x]v + \mathbb{C}[x]w$ is free of rank two. If $Q = M \cap (\mathbb{C}[x]v)^\perp$, then since $\mathbb{C}[x]v \subseteq (\mathbb{C}[x]v)^\perp$ we have $(\mathbb{C}[x]v)^\perp = \mathbb{C}[x]v \oplus Q$ and

$$\mathbb{C}[x]^N = \left(\mathbb{C}[x]v\right)^\perp \oplus \mathbb{C}[x]w = P \oplus Q,$$

with $Q = P^{\perp}$. Choose $w' \in P$ such that $v^* A w' = 1$. Then $v, w'$ must be a free basis of $P$ and the corresponding $2 \times 2$ block is of the form

$$\begin{pmatrix} 0 & 1 \\ -1 & f \end{pmatrix}$$

for some skew element $f = g - g^*$ (cf. [9, Proposition 5]). One can now replace $f$ by 0, by taking the basis $v, w' - gv$, and use induction to finish the proof. □

**Remark 4.7.** We do not know any counter-examples to the following generalization of Djokovic's theorem: If $A \in \mathrm{Mat}_N(\mathbb{C}[x])$ and $A^* = A$ (respectively $A^* = -A$) where $A(x)^* = A^t(-x)$, then $A$ is congruent to a direct sum of $1 \times 1$ matrices of the form $(p(x))$ where $p$ is an even (respectively odd) polynomial and $2 \times 2$ matrices of the form

$$\begin{pmatrix} 0 & q(x) \\ \varepsilon q(-x) & 0 \end{pmatrix},$$

where $q(x)$ is a polynomial, and $\varepsilon = 1$ (respectively $\varepsilon = -1$).[1]

As a consequence of Theorem 4.3, we have the following result.

**Theorem 4.8.** *Let $P(x), Q(x) \in \mathrm{Mat}_N\, \mathbb{C}[x]$ be two non-degenerate matrices. Then $\mathrm{Cend}_{N,P}$ is isomorphic to $\mathrm{Cend}_{N,Q}$ if and only if there exist $\alpha \in \mathbb{C}$ such that $Q(x)$ and $P(x+\alpha)$ have the same elementary divisors.*

**Proof.** We may assume that $P$ is diagonal. Let $\phi : \mathrm{Cend}_{N,P} \to \mathrm{Cend}_{N,Q}$ be an isomorphism. In particular it is a homomorphism from $\mathrm{Cend}_{N,P}$ to $\mathrm{Cend}_N$ whose image is $\mathrm{Cend}_{N,Q}$. Then, by Theorem 4.3(a), we have that $\phi(a(\partial, X)P(X)) = A(\partial + x)a(\partial, x + \alpha)B(x)$, with $P(x+\alpha) = B(x)A(x)$. In particular

$$A(\partial + x)a(\partial, x+\alpha)B(x) = Q(x) \tag{4.20}$$

for some $a(\partial, x)P(x) \in \mathrm{Cend}_{N,P}$.

Taking determinant in both sides of (4.20), and comparing its highest degrees in $\partial$, we can deduce that $\det A(x)$ is constant. Now, define the isomorphism $\phi_2 = \chi_A \circ \phi : \mathrm{Cend}_{N,P} \to \mathrm{Cend}_{N,QA}$, where $\chi_A(a(\partial, x)) = A^{-1}(\partial + x)a(\partial, x)A(x)$. Hence $\phi_2(a(\partial, x)P(x)) = a(\partial, x+\alpha)B(x)A(x)$. Since $\phi_2$ is an isomorphism, we have that

$$B(x)A(x) = D(x)Q(x)A(x) \quad \text{and} \quad C(x)B(x)A(x) = Q(x)A(x)$$

[1] This conjecture has been proved recently by D. Djokovic and F. Szechtman, "Solution of the congruence problem for arbitrary hermitian and skew-hermitian matrices over polynomials rings", and independently by L. Vaserstein.

for some $C(x)$ and $D(x)$ (obviously $C$ and $D$ do not depend on $\partial$). Comparing these two formulas, we have that $C(x)D(x) = \mathrm{Id}$. Then both are invertible matrices, and $Q(x)A(x) = C(x)B(x)A(x) = C(x)P(x+\alpha)$ for some invertible matrices $A$ and $C$. □

## 5. On irreducible subalgebras of $\mathrm{Cend}_N$

In this section we study the conformal analog of the Burnside Theorem. Recall that a subalgebra of $\mathrm{Cend}_N$ is called irreducible if it acts irreducibly on $\mathbb{C}[\partial]^N$. The following is the conjecture from [12] on the classification of such subalgebras:

**Conjecture 5.1.** *Any irreducible subalgebra of* $\mathrm{Cend}_N$ *is either* $\mathrm{Cend}_{N,P}$ *with* $\det P(x) \neq 0$ *or* $C(x+\partial)\,\mathrm{Cur}_N\, C(x)^{-1}$ *(i.e., is a conjugate of* $\mathrm{Cur}_N$*), where* $\det C(x) = 1$*. As before,* $\mathrm{Cur}_N = \mathrm{Mat}_N(\mathbb{C}[\partial])$.

The classification of finite irreducible subalgebras follows from the classification in [7] at the Lie algebra level:

**Theorem 5.2.** *Any finite irreducible subalgebra of* $\mathrm{Cend}_N$ *is a conjugate of* $\mathrm{Cur}_N$.

**Proof.** Let $R$ be a finite irreducible subalgebra of $\mathrm{Cend}_N$. Then the Lie conformal algebra $R_-$ (with the bracket $[a_\lambda b] = a_\lambda b - b_{-\partial-\lambda} a$), of course, still acts irreducibly on $\mathbb{C}[\partial]^N$. By the conformal analogue of the Cartan–Jacobson theorem [7] applied to $R_-$, a conjugate $R_1$ of $R$ either contains the element $xI$, or is contained in $\mathrm{Mat}_N\,\mathbb{C}[\partial]$. The first case is ruled out since then $R_1$ is infinite. In the second case, by the same theorem, $R_1$ contains $\mathrm{Cur}\,\mathfrak{g}$, where $\mathfrak{g} \subset \mathrm{Mat}_N\,\mathbb{C}$ is a simple Lie algebra acting irreducibly on $\mathbb{C}^N$, provided that $N > 1$.

By the classical Burnside theorem, we conclude that $R_1 = \mathrm{Mat}_N\,\mathbb{C}[\partial]$ in the case $N > 1$. It is immediate to see that the same is true if $N = 1$ (or we may apply Theorem 2.1). □

**Theorem 5.3.** *If* $S \subseteq \mathrm{Cend}_N$ *is an irreducible subalgebra such that* $S$ *contains the identity matrix* $\mathrm{Id}$*, then* $S = \mathrm{Cur}_N$ *or* $S = \mathrm{Cend}_N$.

**Proof.** Since $\mathrm{Id} \in S$, and using the idea of (1.5), we have that $S = \mathbb{C}[\partial]A$, where $A = S \cap \mathrm{Mat}_N\,\mathbb{C}[x]$. Observe that $A$ is a subalgebra of $\mathrm{Mat}_N\,\mathbb{C}[x]$. Indeed,

$$P(x)Q(x) = P(x)_\lambda Q(x)|_{\lambda=-\partial} \in S \quad \text{for all } P, Q \in A.$$

In order to finish the proof, we should show that $A = \mathrm{Mat}_N\,\mathbb{C}$ or $A = \mathrm{Mat}_N\,\mathbb{C}[x]$. Observe that $A$ is invariant with respect to $\mathrm{d}/\mathrm{d}x$, using that $P(x)_\lambda(\mathrm{Id}) = P(\lambda+\partial+x) \in \mathbb{C}[\lambda] \otimes S$ and Taylor's expansion.

Let $A_0 \subset \mathrm{Mat}_N\,\mathbb{C}$ be the set of leading coefficients of matrices from $A$. This is obviously a subalgebra of $\mathrm{Mat}_N\,\mathbb{C}$ that acts irreducibly on $\mathbb{C}^N$. Otherwise we would have a non-trivial $A_0$-invariant subspace $u \subset \mathbb{C}^N$. Let $U$ denote the space of vectors in $\mathbb{C}[\partial]^N$ whose leading coefficients lie in $u$; this is a $\mathbb{C}[\partial]$-submodule. But we have:

$$a(x)_\lambda u(\partial) = a\ (\lambda+\partial)u(\lambda+\partial) = \sum_{j\geqslant 0} \frac{\lambda^j}{j!}\big(a(\lambda+\partial)u(\lambda+\partial)\big)^{(j)}\big|_{\lambda=0},$$

where $(j)$ stands for $j$th derivative with respect to $\lambda$. Since both $A$ and $U$ are invariant with respect to the derivative by the indeterminate, we conclude that $U$ is invariant with respect to $A$, hence with respect to $S = \mathbb{C}[\partial]A$.

Thus, $A_0 = \mathrm{Mat}_N\,\mathbb{C}$. Therefore $A$ is a subalgebra of $\mathrm{Mat}_N\,\mathbb{C}[x]$ that contains $\mathrm{Mat}_N\,\mathbb{C}$ and is $\mathrm{d}/\mathrm{d}x$-invariant. If $A$ is larger than $\mathrm{Mat}_N\,\mathbb{C}$, applying $\mathrm{d}/\mathrm{d}x$ a suitable number of times, we get that $A$ contains a matrix of the form $xa$, where $a$ is a non-zero constant matrix (we can always subtract the constant term). Hence $A \supset x(\mathrm{Mat}_N\,\mathbb{C})a(\mathrm{Mat}_N\,\mathbb{C}) = x\,\mathrm{Mat}_N\,\mathbb{C}$, hence $A$ contains $x^k\,\mathrm{Mat}_N(\mathbb{C})$ for all $k \in \mathbb{Z}_+$. □

## 6. Lie conformal algebras $\mathbf{gc}_N$, $\mathbf{oc}_{N,P}$ and $\mathbf{spc}_{N,P}$

A *Lie conformal algebra* $R$ is a $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map $R\otimes R \to \mathbb{C}[\lambda]\otimes R$, $a\otimes b \mapsto [a_\lambda b]$, called the $\lambda$-bracket, satisfying the following axioms $(a, b, c \in R)$,

(C1)$_\lambda$ $[(\partial a)_\lambda b] = -\lambda[a_\lambda b]$, $[a_\lambda(\partial b)] = (\lambda+\partial)[a_\lambda b]$,
(C2)$_\lambda$ $[a_\lambda b] = -[a_{-\partial-\lambda} b]$,
(C3)$_\lambda$ $[a_\lambda[b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + [b_\mu[a_\lambda c]]$.

A *module* $M$ over a Lie conformal algebra $R$ is a $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map $R\otimes M \to \mathbb{C}[\lambda]\otimes M$, $a\otimes v \mapsto a_\lambda v$, satisfying the following axioms $(a, b \in R,\ v \in M)$,

(M1)$_\lambda$ $(\partial a)^M_\lambda v = [\partial^M, a^M_\lambda]v = -\lambda a^M_\lambda v$,
(M2)$_\lambda$ $[a^M_\lambda, b^M_\mu]v = [a_\lambda b]^M_{\lambda+\mu} v$.

Let $U$ and $V$ be modules over a Lie conformal algebra $R$. Then, the $\mathbb{C}[\partial]$-module $N := \mathrm{Chom}(U, V)$ has an $R$-module structure defined by

$$\big(a^N_\lambda \varphi\big)_\mu u = a^V_\lambda(\varphi_{\mu-\lambda} u) - \varphi_{\mu-\lambda}\big(a^U_\lambda u\big), \tag{6.1}$$

where $a \in R$, $\varphi \in N$ and $u \in U$. Therefore, one can define the contragradient $R$-module $U^* = \mathrm{Chom}(U, \mathbb{C})$, where $\mathbb{C}$ is viewed as the trivial $R$-module and $\mathbb{C}[\partial]$-module. We also define the tensor product $U\otimes V$ of $R$-modules as the ordinary tensor product with $\mathbb{C}[\partial]$-module structure $(u \in U, v \in V)$:

$$\partial(u\otimes v) = \partial u\otimes v + u\otimes \partial v$$

and $\lambda$-action defined by $(r \in R)$:

$$r_\lambda(u\otimes v) = r_\lambda u\otimes v + u\otimes r_\lambda v.$$

**Proposition 6.1.** *Let $U$ and $V$ be two $R$-modules. Suppose that $U$ has finite rank as a $\mathbb{C}[\partial]$-module. Then $U^*\otimes V \simeq \mathrm{Chom}(U,V)$ as $R$-modules, with the identification $(f\otimes v)_\lambda(u) = f_{\lambda+\partial^V}(u)v$, $f\in U^*$, $u\in U$ and $v\in V$.*

**Proof.** Define $\varphi: U^*\otimes V \to \mathrm{Chom}(U,V)$ by $\varphi(f\otimes v)_\lambda(u) = f_{\lambda+\partial^V}(u)\,v$. Observe that $\varphi$ is $\mathbb{C}[\partial]$-linear, since

$$\begin{aligned}\varphi\big(\partial(f\otimes v)\big)_\lambda(u) &= \varphi(\partial f\otimes v + f\otimes \partial v)_\lambda(u) = (\partial f)_{\lambda+\partial^V}(u)v + f_{\lambda+\partial^V}(u)\partial v\\ &= -\big(\lambda+\partial^V\big)f_{\lambda+\partial^V}(u)v + f_{\lambda+\partial^V}(u)\partial v = -\lambda f_{\lambda+\partial^V}(u)v\\ &= -\lambda\varphi(f\otimes v)_\lambda(u) = \partial\big(\varphi(f\otimes v)\big)_\lambda(u)\end{aligned}$$

and $\varphi$ is a homomorphism, since

$$\begin{aligned}\varphi\big(r_\lambda(f\otimes v)\big)_\mu(u) &= \varphi(r_\lambda f\otimes v + f\otimes r_\lambda v)_\mu(u)\\ &= (r_\lambda f)_{\mu+\partial^V}(u)v + f_{\mu+\partial^V}(u)(r_\lambda v)\\ &= -f_{\mu-\lambda+\partial^V}(r_\lambda u)v + f_{\mu+\partial^V}(u)(r_\lambda v)\end{aligned}$$

and

$$\begin{aligned}\big(r_\lambda\big(\varphi(f\otimes v)\big)\big)_\mu(u) &= r_\lambda\big(\varphi(f\otimes v)_{\mu-\lambda}(u)\big) - \varphi(f\otimes v)_{\mu-\lambda}(r_\lambda u)\\ &= r_\lambda\big(f_{\mu-\lambda+\partial^V}(u)v\big) - f_{\mu-\lambda+\partial^V}(r_\lambda u)v\\ &= f_{\mu+\partial^V}(u)(r_\lambda v) - f_{\mu-\lambda+\partial^V}(r_\lambda u)v.\end{aligned}$$

The homomorphism $\varphi$ is always injective. Indeed, if $\varphi(f\otimes v)=0$, then $f_{\mu+\partial^V}(u)v=0$ for all $u\in U$. Suppose that $v\neq 0$, then $f_{\lambda+\partial^V}=0$, that is $f=0$.

It remains to prove that $\varphi$ is surjective provided that $U$ has finite rank as a $\mathbb{C}[\partial]$-module. Let $g\in\mathrm{Chom}(U,V)$, and $U=\mathbb{C}[\partial]\{u_1,\dots,u_n\}$. Then, there exist $v_{ik}\in V$ such that

$$g_\lambda(u_i) = \sum_{k=0}^{m_i}\big(\lambda+\partial^V\big)^k v_{ik} = \sum_{k=0}^{m_i}\varphi(f_{ik}\otimes v_{ik})_\lambda(u_i),$$

where $f_{ik}\in U^*$ is defined (on generators) by $f_{ik}(u_j)=\delta_{i,j}\lambda^k$. Therefore, $g = \varphi(\sum_{i=0}^n\sum_{k=0}^{m_i} f_{ik}\otimes v_{ik})$, finishing the proof. □

In general, given any associative conformal algebra $R$ with $\lambda$-product $a_\lambda b$, the $\lambda$-bracket defined by

$$[a_\lambda b] := a_\lambda b - b_{-\partial-\lambda}a \tag{6.2}$$

makes $R$ a Lie conformal algebra.

Let $V$ be a finite $\mathbb{C}[\partial]$-module. The $\lambda$-bracket (6.2) on Cend $V$, makes it a Lie conformal algebra denoted by gc $V$ and called the *general conformal algebra* (see [7,11] and [12]). For any positive integer $N$, we define $\mathrm{gc}_N := \mathrm{gc}\,\mathbb{C}[\partial]^N = \mathrm{Mat}_N\,\mathbb{C}[\partial, x]$, and the $\lambda$-bracket (6.2) is by (1.1):

$$\big[A(\partial,x)_\lambda B(\partial,x)\big] = A(-\lambda, x+\lambda+\partial)B(\lambda+\partial, x) - B(\lambda+\partial, -\lambda+x)A(-\lambda, x).$$

Recall that, by Theorem 4.5, any anti-involution in $\mathrm{Cend}_N$ is, up to conjugation

$$\sigma_*\big(A(\partial,x)\big) = A^*(\partial, -\partial - x), \tag{6.3}$$

where $*$ stands for the adjoint with respect to a non-degenerate symmetric or skew-symmetric bilinear form over $\mathbb{C}$. These anti-involutions give us two important subalgebras of $\mathrm{gc}_N$: the set of $-\sigma_*$ fixed points is the *orthogonal conformal algebra* $\mathrm{oc}_N$ (respectively the *symplectic conformal algebra* $\mathrm{spc}_N$), in the symmetric (respectively skew-symmetric) case.

**Proposition 6.2.** *The subalgebras* $\mathrm{oc}_N$ *and* $\mathrm{spc}_N$ *are simple.*

**Proof.** We will prove that $\mathrm{oc}_N$ is simple. The proof for $\mathrm{spc}_N$ is similar. Let $I$ be a non-zero ideal of $\mathrm{oc}_N$. Let $0 \neq A(\partial,x) \in I$, then $A(\partial,x) = \sum_{i=0}^m \partial^i a_i(x) = \sum_{j=0}^n \partial^j \tilde{a}_j(\partial+x)$, with $a_i(x), \tilde{a}_j(x) \in \mathrm{Mat}_N\,\mathbb{C}[x]$. Now, using that $A(\partial,x) = -A^t(\partial, -\partial-x)$, we obtain that $n = m$ and $a_i(x) = -\tilde{a}_i^t(-x)$. Computing the $\lambda$-bracket

$$\big[xE_{ij} - (-\partial - x)E_{ji\lambda}A(\partial,x)\big] = \lambda^{m+1}\big(E_{ij}a_m(x) - a_m^t(-\partial-x)E_{ji}\big) + \lambda^m\cdots$$

we deduce that $E_{ij}a_m(x) - a_m^t(-\partial-x)E_{ji} \in I$, with $a_m \neq 0$. By taking appropriate $i$ and $j$, we have that there exist polynomials $b_k(x)$ such that $\sum_{k=1}^N (b_k(x)E_{ik} - b_k(-\partial - x)E_{ki}) \in I$, with $b_k \neq 0$ for some $k \neq i$. Now by computing $[(2x+\partial)E_{rr\,\lambda} \sum_{k=1}^N (b_k(x)E_{ik} - b_k(-\partial-x)E_{ki})]$ and looking at its leading coefficient in $\lambda$, we show that $E_{ri} - E_{ir} \in I$, with $r \neq i$. Taking brackets with elements in $o_N$, we have $E_{jl} - E_{lj} \in I$ for all $j \neq l$. Now, we can see from the $\lambda$-brackets $[xE_{ri} - (-\partial-x)E_{ir\,\lambda}E_{ir} - E_{ri}] = (2x+\partial)(E_{ii} - E_{rr})$ and $[(2x+\partial)E_{ii\,\lambda}(2x+\partial)(E_{ii} - E_{rr})] = \lambda(2x+\partial)E_{ii}$, that $(2x+\partial)E_{ii} \in I$ for all $i$. The other generators are obtained by $(k \neq i, j)$

$$\big[(-x)^k E_{ik} - (\partial+x)^k E_{ki\,\lambda}E_{jk} - E_{kj}\big]\big|_{\lambda=0} = x^k E_{ij} - (-\partial-x)^k E_{ji}.$$

Similarly, we can see that $(x^k - (-\partial-x)^k)E_{ii} \in I$, finishing the proof. □

The conformal subalgebras $\mathrm{oc}_N$ and $\mathrm{spc}_N$, as well as the anti-involutions given by (6.3), and their generalizations can be described in terms of conformal bilinear forms. Let $V$ be a $\mathbb{C}[\partial]$-module. A *conformal bilinear form* on $V$ is a $\mathbb{C}$-bilinear map $\langle\,,\,\rangle_\lambda : V \times V \to \mathbb{C}[\lambda]$

such that

$$\langle \partial v, w\rangle_\lambda = -\lambda\langle v, w\rangle_\lambda = -\langle v, \partial w\rangle_\lambda, \quad \text{for all } v, w \in V.$$

The conformal bilinear form is *non-degenerate* if $\langle v, w\rangle_\lambda = 0$ for all $w \in V$, implies $v = 0$. The conformal bilinear form is *symmetric* (respectively *skew-symmetric*) if $\langle v, w\rangle_\lambda = \epsilon\langle w, v\rangle_{-\lambda}$ for all $v, w \in V$, with $\epsilon = 1$ (respectively $\epsilon = -1$).

Given a conformal bilinear form on a $\mathbb{C}[\partial]$-module $V$, we have a homomorphism of $\mathbb{C}[\partial]$-modules, $L : V \to V^*$, $v \mapsto L_v$, given as usual by

$$(L_v)_\lambda w = \langle v, w\rangle_\lambda, \quad v \in V. \tag{6.4}$$

Let $V$ be a free finite rank $\mathbb{C}[\partial]$-module and fix $\beta = \{e_1, \dots, e_N\}$ a $\mathbb{C}[\partial]$-basis of $V$. Then *the matrix of* $\langle\, ,\, \rangle_\lambda$ *with respect to* $\beta$ is defined as $P_{i,j}(\lambda) = \langle e_i, e_j\rangle_\lambda$. Hence, identifying $V$ with $\mathbb{C}[\partial]^N$, we have

$$\big\langle v(\partial), w(\partial)\big\rangle_\lambda = v^t(-\lambda)P(\lambda)w(\lambda). \tag{6.5}$$

Observe that $P^t(-x) = \epsilon P(x)$ with $\epsilon = 1$ (respectively $\epsilon = -1$) if the conformal bilinear form is symmetric (respectively skew-symmetric). We also have that $\operatorname{Im} L = P(-\partial)V^*$, where $L$ is defined in (6.4). Indeed, given $v(\partial) \in V$, consider $g_\lambda \in V^*$ defined by $g_\lambda(w(\partial)) = v^t(-\lambda)w(\lambda)$, then by (6.5)

$$(L_{v(\partial)})_\lambda w(\partial) = v^t(-\lambda)P(\lambda)w(\lambda) = g_\lambda\big(P(\partial)w(\partial)\big) = \big(P(-\partial)g\big)_\lambda\big(w(\partial)\big),$$

where in the last equality we are identifying $V^*$ with $\mathbb{C}[\partial]^N$ in the natural way, that is $f \in V^*$ corresponds to $(f_{-\partial}e_1, \dots, f_{-\partial}e_N) \in \mathbb{C}[\partial]^N$. Therefore, if the conformal bilinear form is non-degenerate, then $L$ gives an isomorphism between $V$ and $P(-\partial)V^*$, with $\det P \neq 0$.

Suppose that we have a non-degenerate conformal bilinear form on $V = \mathbb{C}[\partial]^N$ which is also symmetric or skew-symmetric. Denote by $P(\lambda)$ the matrix of this bilinear form with respect to the standard basis of $\mathbb{C}[\partial]^N$. Then for each $a \in \operatorname{Cend}_N$ and $w \in V$, the map ${f^{a,w}}_\lambda(v) := \langle w, a_\mu v\rangle_{\lambda-\mu}$ is in $\mathbb{C}[\mu] \otimes V^*$, that is ${f^{a,w}}_\lambda$ is a $\mathbb{C}$-linear map, ${f^{a,w}}_\lambda(\partial v) = \lambda {f^{a,w}}_\lambda(v)$ and depends polynomially on $\mu$, because $\deg_\mu {f^{a,w}}_\lambda(v) \leqslant \max\{\deg_\mu {f^{a,w}}_\lambda(e_i)\colon\ i = 1, \dots, N\}$. Observe that if we restrict to $\operatorname{Cend}_{N,P}$, then ${f^{aP,w}}_\lambda = (P(-\partial)f^{a,w})_\lambda \in \operatorname{Im} L$. Therefore, since $\langle\, ,\, \rangle_\lambda$ is non-degenerate, there exists a unique $(aP)^*_\mu w \in \mathbb{C}[\mu] \otimes V$ such that ${f^{aP,w}}_\lambda(v) = \langle w, aP_\mu v\rangle_{\lambda-\mu} = \langle (aP)^*_\mu w, v\rangle_\lambda$. Thus, we have attached to each $aP \in \operatorname{Cend}_{N,P}$ a map $(aP)^* : V \to \mathbb{C}[\mu] \otimes V$, $w \mapsto (aP)^*_\mu w$, where the vector $(aP)^*_\mu w$ is determined by the identity

$$\langle aP_\mu v, w\rangle_\lambda = \big\langle v, (aP)^*_\mu w\big\rangle_{\lambda-\mu}.$$

Observe that $(aP)^*_\mu(\partial w) = (\partial + \mu)(aP)^*_\mu w$, that is $(aP)^* \in \operatorname{Cend} V$. Indeed,

$$\begin{aligned}\big\langle v, (aP)^*_\mu(\partial w)\big\rangle_{\lambda-\mu} &= \langle aP_\mu v, \partial w\rangle_\lambda = \lambda\langle aP_\mu v, w\rangle_\lambda \\ &= -\big\langle \partial(aP_\mu v), w\big\rangle_\lambda = \langle \mu aP_\mu v, w\rangle_\lambda - \langle aP_\mu \partial v, w\rangle_\lambda \\ &= \mu\big\langle v, (aP)^*_\mu w\big\rangle_{\lambda-\mu} - \big\langle \partial v, (aP)^*_\mu w\big\rangle_{\lambda-\mu} \\ &= \big\langle v, (\mu+\partial)(aP)^*_\mu w\big\rangle_{\lambda-\mu}.\end{aligned}$$

Moreover we have the following result:

**Proposition 6.3.** (a) *Let $\langle\,,\,\rangle_\lambda$ be a non-degenerate symmetric or skew-symmetric conformal bilinear form on $\mathbb{C}[\partial]^N$, and denote by $P(\lambda)$ the matrix of $\langle\,,\,\rangle_\lambda$ with respect to the standard basis of $\mathbb{C}[\partial]^N$ over $\mathbb{C}[\partial]$. Then the map $aP \mapsto (aP)^*$ from $\mathrm{Cend}_{N,P}$ to $\mathrm{Cend}_N$ defined by*

$$\langle a_\mu v, w\rangle_\lambda = \big\langle v, a^*_\mu w\big\rangle_{\lambda-\mu} \tag{6.6}$$

*is the anti-involution of $\mathrm{Cend}_{N,P}$ given by*

$$\big(a(\partial, x)P(x)\big)^* = \epsilon a^t(\partial, -\partial - x)P(x), \tag{6.7}$$

*where $P^t(-x) = \epsilon P(x)$ with $\epsilon = 1$ or $-1$, depending on whether the conformal bilinear form is symmetric or skew-symmetric.*

(b) *Consider the Lie conformal subalgebra of $\mathrm{gc}_N$ defined by*

$$\begin{aligned} g_* &= \{a \in \mathrm{Cend}_{N,P}\colon a^* = -a\} \\ &= \big\{a \in \mathrm{Cend}_{N,P}\colon \langle a_\mu v, w\rangle_\lambda + \langle v, a_\mu w\rangle_{\lambda-\mu} = 0, \text{ for all } v, w \in \mathbb{C}[\partial]^N\big\},\end{aligned}$$

*where $^*$ is defined by (6.7). Then under the pairing (6.4) we have $\mathbb{C}[\partial]^N \simeq P(-\partial)(\mathbb{C}[\partial]^N)^*$ as $g_*$-modules.*

**Proof.** (a) First let us check that $\varphi(aP) = (aP)^*$ defines an anti-homomorphism from $\mathrm{Cend}_{N,P}$ to $\mathrm{Cend}_N$. Since $(a, b \in \mathrm{Cend}_{N,P})$

$$\begin{aligned}\big\langle v, (a_\mu b)^*_\gamma w\big\rangle_{\lambda-\gamma} &= \big\langle (a_\mu b)_\gamma v, w\big\rangle_\lambda = \big\langle a_\mu(b_{\gamma-\mu} v), w\big\rangle_\lambda \\ &= \big\langle b_{\gamma-\mu} v, a^*_\mu w\big\rangle_{\lambda-\mu} = \big\langle v, b^*_{\gamma-\mu}\big(a^*_\mu w\big)\big\rangle_{\lambda-\gamma} \\ &= \big\langle v, \big(b^*_{\gamma-\mu} a^*\big)_\gamma w\big\rangle_{\lambda-\gamma},\end{aligned}$$

we have that $\varphi(a_\mu b)_\gamma = (\varphi(b)_{\gamma-\mu}\varphi(a))_\gamma = (\varphi(b)_{-\partial-\mu}\varphi(a))_\gamma$ (the last equality is an obvious identity in $\mathrm{Cend}_N$).

Now, using Theorem 4.3(b), we have that

$$\varphi\big(a(\partial, x)P(x)\big) = A(\partial + x)a^t(\partial, -\partial - x + \alpha)B(x),$$

with $\alpha \in \mathbb{C}$ and $P^t(-x+\alpha) = B(x)A(x)$. Replacing $\varphi(aP)$ in (6.6) and using (6.5), we obtain

$$P(\lambda-\mu)a^t(-\mu,\mu-\lambda)P(\lambda) = P(\lambda-\mu)A(\lambda-\mu)a^t(-\mu,\mu-\lambda+\alpha)B(\lambda), \quad \text{for all } a(\partial,x). \tag{6.8}$$

Taking $a(\partial,x) = I$ and using that $\det P \neq 0$, we have $P(\lambda) = A(\lambda-\mu)B(\lambda)$. Since the left-hand side does not depend on $\mu$, we get $A = A(x) \in \mathrm{Mat}_N\,\mathbb{C}$, with $\det A \neq 0$. Using that $\epsilon P(x-\alpha) = P^t(-x+\alpha) = B(x)A$, then (6.8) become

$$a^t(-\mu,\mu-\lambda)\epsilon B(\lambda+\alpha)A = Aa^t(-\mu,\mu-\lambda+\alpha)B(\lambda), \quad \text{for all } a(\partial,x).$$

In particular, we have $\epsilon B(\lambda+\alpha)A = AB(\lambda)$. Hence $a^t(-\mu,\mu-\lambda)A = Aa^t(-\mu,\mu-\lambda+\alpha)$ for all $a(\partial,x)$, getting $\alpha = 0$ and $A = cI$. Therefore,

$$\varphi\big(a(\partial,x)P(x)\big) = \epsilon a^t(\partial,-\partial-x)P(x),$$

with $P^t(-x) = \epsilon P(x)$ with $\epsilon = 1$ or $-1$, depending on whether the conformal bilinear form is symmetric or skew-symmetric, getting (a).

(b) Using (6.4), we obtain for all $a \in g_*$ and $v, w \in \mathbb{C}[\partial]^N$ that

$$(L_{a_\mu v})_\lambda(w) = \langle a_\mu v, w\rangle_\lambda = -\langle v, a_\mu w\rangle_{\lambda-\mu} = -(L_v)_{\lambda-\mu}(a_\mu w) = \big(a_\mu(L_v)\big)_\lambda(w)$$

finishing the proof. □

Observe that $\mathrm{oc}_N$ (respectively $\mathrm{spc}_N$), can be described as the subalgebra $g_*$ of $\mathrm{gc}_N$ in Proposition 6.3(b), with respect to the conformal bilinear form

$$\big\langle p(\partial)v, q(\partial)w\big\rangle_\lambda = p(-\lambda)q(\lambda)(v,w) \quad \text{for all } v, w \in \mathbb{C}^N,$$

where $(\cdot,\cdot)$ is a non-degenerate symmetric (respectively skew-symmetric) bilinear form on $\mathbb{C}^N$. For general $P$, see (6.12) below.

Then, $\mathrm{oc}_N$ (respectively $\mathrm{spc}_N$) is the $\mathbb{C}[\partial]$-span of $\{y_A^n := x^nA - (-\partial-x)^nA^*\colon\ A \in \mathrm{Mat}_N\,\mathbb{C}\}$, where $^*$ stands for the adjoint with respect to a non-degenerate symmetric (respectively skew-symmetric) bilinear form over $\mathbb{C}$. Therefore we have that $\mathrm{gc}_N = \mathrm{oc}_N \oplus M_N$ (respectively $\mathrm{gc}_N = \mathrm{spc}_N \oplus M_N$), where $M_N$ is the set of $\sigma_*$-fixed points, i.e.

$$M_N = \mathbb{C}[\partial]\text{-span of } \big\{w_A^n := x^nA + (-\partial-x)^nA^*\colon\ A \in \mathrm{Mat}_N\,\mathbb{C}\big\}. \tag{6.9}$$

We are using the same notation $M_N$ in the symmetric and skew-symmetric case. Observe that $M_N$ is an $\mathrm{oc}_N$-module (respectively $\mathrm{spc}_N$-module) with the action given by

$$\begin{aligned} {y_A^n}_\lambda w_B^m \;=\; &(\lambda+\partial+w_{AB})^n w_{AB}^m - (-\partial-w_{A^*B})^n w_{A^*B}^m \\ &+ (-1)^n(-\lambda-\partial-w_{AB^*})^{m+n} - (-\lambda+w_{BA})^m w_{BA}^n. \end{aligned} \tag{6.10}$$

Let us give a more conceptual understanding of the module $M_N$. Let $V = \mathbb{C}[\partial]^N$. By definition, $V^* = \mathrm{Chom}(V, \mathbb{C}) = \{\alpha : \mathbb{C}[\partial]^N \to \mathbb{C}[\lambda]\colon \alpha_\lambda \partial = \lambda \alpha_\lambda\}$ and given $\alpha \in V^*$ it is completely determined by the values in the canonical basis $\{e_i\}$ of $\mathbb{C}^N$, this is $p_\alpha(\lambda) := (\alpha_\lambda e_1, \dots, \alpha_\lambda e_N) \in \mathbb{C}[\lambda]^N$. Thus, we may identify $V^* \simeq \mathbb{C}[\lambda]^N$ and $\mathbb{C}[\partial]$-module structure is given by $(\partial p)(\lambda) = -\lambda p(\lambda)$.

We have that $\mathrm{gc}_N$ acts on $V$ by the $\lambda$-action

$$A(\partial, x)_\lambda v(\partial) = A(-\lambda, \lambda + \partial) v(\lambda + \partial), \quad v(\partial) \in \mathbb{C}[\partial]^N,$$

and on $V^*$ by the contragradient action, given by

$$A(\partial, x)_\lambda v(\partial) = -{}^t A(-\lambda, -\partial) v(\lambda + \partial), \quad v(\partial) \in \mathbb{C}[\partial]^N.$$

It is easy to check that $(V^*)^* \simeq V$ as $\mathrm{gc}_N$-modules. Observe that by Proposition 6.3(b), $V \simeq V^*$ as $\mathrm{oc}_N$-modules and $\mathrm{spc}_N$-modules.

We define the 2*nd exterior power* $\Lambda^2(V)$ and the 2*nd symmetric power* $S^2(V)$ in the usual way with the induced $\mathbb{C}[\partial]$-module and $\mathrm{gc}_N$-module structures.

**Proposition 6.4.** (a) *$V \otimes V = S^2(V) \oplus \Lambda^2(V)$ is the decomposition of $V \otimes V$ into a direct sum of irreducible $\mathrm{gc}_N$-modules. $V^* \otimes V$ is isomorphic to the adjoint representation of $\mathrm{gc}_N$.*

(b) *$\mathrm{gc}_N \simeq V \otimes V = S^2(V) \oplus \Lambda^2(V)$ is the decomposition of $\mathrm{gc}_N$ into a direct sum of irreducible $\mathrm{oc}_N$-modules, where $\Lambda^2(V)$ is isomorphic to the adjoint representation of $\mathrm{oc}_N$, and $M_N \simeq S^2(V)$ as $\mathrm{oc}_N$-modules.*

(c) *$\mathrm{gc}_N \simeq V \otimes V = S^2(V) \oplus \Lambda^2(V)$ is the decomposition of $\mathrm{gc}_N$ into a direct sum of irreducible $\mathrm{spc}_N$-modules, where $S^2(V)$ is isomorphic to the adjoint representation of $\mathrm{spc}_N$, and $M_N \simeq \Lambda^2(V)$ as $\mathrm{spc}_N$-modules.*

**Proof.** (a) Follows from Proposition 6.1 and part (b).

(b) Define $\varphi : V \otimes V \to \mathrm{gc}_N$ by

$$\varphi\big(p(\partial)e_i \otimes q(\partial)e_j\big) = p(-x)q(x + \partial)E_{ji}.$$

It is easy to check that this is an $\mathrm{oc}_N$-module isomorphism. Note that $\sigma_*$ defined in (6.3) corresponds via $\varphi$ to $\sigma(p(\partial)e_i \otimes q(\partial)e_j) = q(\partial)e_j \otimes p(\partial)e_i$. Therefore it is immediate that $M_N \simeq S^2(V)$ and $\Lambda^2(V) \simeq \mathrm{oc}_N$. It remains to see that $M_N$ is an irreducible $\mathrm{oc}_N$-module. Let $W \neq 0$ be a $\mathrm{oc}_N$-submodule of $M_N$ and $0 \neq w(\partial, x) = \sum_{i,j} q_{ij}(\partial, x)E_{ij} \in W$. We may suppose that $q_{11} \neq 0$. Computing $[y^1_{E_{11}\lambda} w(\partial, x)]$ and looking at the highest degree of $\lambda$ that appears in the component $E_{11}$, we deduce that there exists in $W$ an element of the form $w' = \sum_i (p_i(\partial, x)E_{1i} + q_i(\partial, x)E_{i1})$, with $p_1 = q_1 = 1$. Now, computing $[y^1_{E_{12}\lambda} w'(\partial, x)]$ we have that $w'' = r(\partial, x)E_{11} + w^1_{E_{12}}$ + terms out of the first column and row $\in W$. And from $[y^1_{E_{11}\lambda} w''(\partial, x)]$ and looking at the highest degree in $\lambda$, we have that if $r(\partial, x)$ is non-constant, $w^0_{E_{11}} \in W$, and if $r(\partial, x)$ is constant, $w^0_{E_{11}} + w^1_{E_{12}} \in W$. In both cases, by (6.10) we have that $w^0_I \in W$. Now, looking at ($n \gg 0$ and $A$ arbitrary)

$$y^n_{A\lambda} w^0_I = \lambda^n 2w^0_A + \lambda^{n-1} 2n\big(\partial w^0_A + w^1_A\big) + \lambda^{n-2} 2\binom{n}{2}\big(\partial^2 w^0_A + 2\partial w^1_A + w^2_A\big) + \cdots$$

we get $W = M_N$, finishing part (b).

(c) The proof is similar to (b), with $\varphi : V \otimes V \to \mathrm{gc}_N$ defined by $\varphi(p(\partial)e_i \otimes q(\partial)e_j) = p(-x)q(x+\partial)E^\dagger_{ij}$, where $E^\dagger_{ij} = -E_{j,N/2+i}$, $E^\dagger_{N/2+i,N/2+j} = E_{N/2+j,i}$, $E^\dagger_{i,N/2+j} = -E_{N/2+j,N/2+i}$ and $E^\dagger_{N/2+i,j} = -E_{j,i}$, for all $1 \leqslant i, j \leqslant N/2$. □

Observe that $\mathrm{gc}_{N,P} := \mathrm{gc}_N\, P(x)$ is a Lie conformal subalgebra of $\mathrm{gc}_N$, for any $P(x) \in \mathrm{Mat}_N\, \mathbb{C}[x]$.

A matrix $Q(x) \in \mathrm{Mat}_N\, \mathbb{C}[x]$ will be called *hermitian* (respectively *skew-hermitian*) if

$$Q^t(-x) = \varepsilon Q(x) \quad \text{with } \varepsilon = 1 \text{ (respectively } \varepsilon = -1).$$

Denote by $o_{P,Y,\varepsilon,\alpha}$ the subalgebra of $\mathrm{gc}_{N,P}$ of $-\sigma_{P,Y,\varepsilon,\alpha}$-fixed points. By Proposition 4.4(b), (c), we have the following isomorphisms, obtained by conjugating by automorphisms of $\mathrm{Cend}_N$

$$o_{P,Y,\varepsilon,\alpha} \simeq o_{PY,I,\varepsilon,\alpha} \simeq o_{Q,I,\varepsilon,0}, \tag{6.11}$$

where $Q(x) = (PY)(x + \alpha/2)$ is hermitian or skew-hermitian, depending on whether $\varepsilon = 1$ or $-1$. Therefore, up to conjugacy, we may restrict our attention to the family of subalgebras (6.11), that is it suffices to consider the anti-involutions

$$\sigma_{P,I,\varepsilon,0}\big(a(\partial,x)P(x)\big) = \varepsilon a^t(\partial, -\partial - x)P(x),$$

where $P$ is non-degenerate hermitian or skew-hermitian, depending on whether $\varepsilon = 1$ or $-1$. From now on we shall use the following notation

$$\begin{aligned} \mathrm{oc}_{N,P} &:= o_{P,I,1,0} \quad &&\text{if } P \text{ is hermitian,} \\ \mathrm{spc}_{N,P} &:= o_{P,I,-1,0} \quad &&\text{if } P \text{ is skew-hermitian.} \end{aligned} \tag{6.12}$$

These subalgebras are those obtained in Proposition 6.3(b) in a more invariant form. In the special case $N = 1$ and $P(x) = x$, the involution $\sigma_{x,I,-1,0}$ is the conformal version of the involution given by Bloch in [3].

Note that $\mathrm{gc}_{N,P} \simeq \mathrm{oc}_N \cdot P(x) \oplus M_N \cdot P(x)$. If $P$ is hermitian, then $\mathrm{oc}_{N,P} = \mathrm{oc}_N \cdot P(x)$ and $M_N \cdot P(x)$ is an $\mathrm{oc}_{N,P}$-module. If $P$ is skew-hermitian, then $\mathrm{spc}_{N,P} = M_N \cdot P(x)$, and $\mathrm{oc}_N \cdot P(x)$ is a $\mathrm{spc}_{N,P}$-module.

**Remark 6.5.** (a) The subalgebras $\mathrm{gc}_N$, $\mathrm{gc}_{N,xI}$, $\mathrm{oc}_N$ and $\mathrm{spc}_{N,xI}$ contain the conformal Virasoro subalgebra $\mathbb{C}[\partial](x + \alpha\partial)I$, for $\alpha$ arbitrary, $\alpha = 0$, $\alpha = \frac{1}{2}$ and $\alpha = 0$, respectively.

(b) Let $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, then by (6.11) we obtain

$$\mathrm{spc}_N = o_{I,J,-1,0} \simeq o_{J,I,-1,0} = \mathrm{spc}_{N,J}\,.$$

(c) The proof of Proposition 6.2 still works for $\mathrm{oc}_{N,P}$ and $\mathrm{spc}_{N,P}$ with $\det P(x) \neq 0$ if $P(x)$ satisfies the property that for each $i$ there exists $j$ such that $\deg P_{ij}(x) > \deg P_{ik}(x)$ for all $k \neq j$. Hence, by Remark 4.7 and the footnote to it, all Lie conformal algebras $\mathrm{oc}_{N,P}$ and $\mathrm{spc}_{N,P}$ with $\det P(x) \neq 0$ are simple.

**Proposition 6.6.** *The subalgebras* $\mathrm{oc}_{N,P}$ *and* $\mathrm{spc}_{N,P}$, *with* $\det P(x) \neq 0$, *act irreducibly on* $\mathbb{C}[\partial]^N$.

**Proof.** Let $M$ be a non-zero $\mathrm{oc}_{N,P}$-submodule of $\mathbb{C}[\partial]^N$ and take $0 \neq v(\partial) \in M$. Since $\det P(x) \neq 0$, there exists $i$ such that $P(y)v(y)$ has non-zero $i$th coordinate that we shall denote by $b(y)$. Recall that $\{(x^k A - (-\partial - x)^k A^t)P(x) \mid A \in \mathrm{Mat}_N \mathbb{C}\}$ generates $\mathrm{oc}_{N,P}$. Now, looking at the highest degree in $\lambda$ in

$$(2x + \partial)E_{ii} P(x)_\lambda v(\partial) = (\lambda + 2\partial) b(\partial + \lambda) e_i$$

we deduce that $e_i \in M$. Now, since the $i$th column of $P = (P_{r,j})$ is non-zero, we can take $k$ such that $P_{k,i}(x) \neq 0$ has maximal degree in $x$, in the $i$th column. Then, considering the $\lambda$ action of $(xE_{jk} - (-\partial - x)E_{kj})P(x)$ on $e_i$, for $j = 1, \ldots, N$, and looking at the highest degree in $\lambda$, we have that $e_j \in M$ for all $j = 1, \ldots, N$. Therefore $M = \mathbb{C}[\partial]^N$. A similar argument also works for $\mathrm{spc}_{N,P}$. □

**Proposition 6.7.** (a) *The subalgebras* $\mathrm{oc}_{N,P}$ *and* $\mathrm{oc}_{N,Q}$ (*respectively* $\mathrm{spc}_{N,P}$ *and* $\mathrm{spc}_{N,Q}$) *are conjugated by an automorphism of* $\mathrm{Cend}_N$ *if and only if* $P$ *and* $Q$ *are congruent hermitian* (*respectively skew-hermitian*) *matrices.*

(b) *The subalgebras* $\mathrm{oc}_{N,P}$ *and* $\mathrm{spc}_{N,Q}$ *are not conjugated by any automorphism of* $\mathrm{Cend}_N$.

**Proof.** By Theorem 4.1, any automorphism of $\mathrm{Cend}_N$ has the form $\varphi_A(a(\partial, x)) = A(\partial + x)a(\partial, x + \alpha)A(x)^{-1}$, with $A(x)$ an invertible matrix in $\mathrm{Mat}_N \mathbb{C}[x]$. Suppose that the restriction of $\varphi_A$ to $\mathrm{oc}_{N,P}$ gives us an isomorphism between $\mathrm{oc}_{N,P}$ and $\mathrm{oc}_{N,Q}$. Then $\varphi_A(a(\partial, x)P(x)) = A(\partial + x)a(\partial, x + \alpha)D(x)Q(x)$ for all $a(\partial, x) \in \mathrm{oc}_N$, where $D$ is an invertible matrix in $\mathrm{Mat}_N \mathbb{C}[x]$ and $P(x + \alpha) = D(x)Q(x)A(x)$. But the image is in $\mathrm{oc}_{N,Q}$ if and only if (applying $\sigma_{Q,I,1,0}$)

$$a(\partial, x - \alpha)R(x) = R^t(-\partial - x)a(\partial, x + \alpha) \quad \text{for all } a(\partial, x) \in \mathrm{oc}_N,$$

where $R(x) = A^t(-x)D(x)^{-1}$. Therefore, we must have $\alpha = 0$ and $R = c\,\mathrm{Id}$ ($c \in \mathbb{C}$), that is $D(x) = cA^t(-x)$. Hence $P(x) = cA^t(-x)Q(x)A(x)$, proving (a). Part (b) follows by similar arguments. □

A classification of finite irreducible subalgebras of $\mathrm{gc}_N$ was given in [7]. In view of the discussion of this section, it is natural to propose the following conjecture.

**Conjecture 6.8.** *Any infinite Lie conformal subalgebra of* $\mathrm{gc}_N$ *acting irreducibly on* $\mathbb{C}[\partial]^N$ *is conjugate by an automorphism of* $\mathrm{Cend}_N$ *to one of the following subalgebras*:

(a) $\mathrm{gc}_{N,P}$, *where* $\det P \neq 0$,
(b) $\mathrm{oc}_{N,P}$, *where* $\det P \neq 0$ *and* $P(-x) = P^t(x)$,
(c) $\mathrm{spc}_{N,P}$, *where* $\det P \neq 0$ *and* $P(-x) = -P^t(x)$.

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