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Matrix-valued bispectral operators and quasideterminants

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Abstract
We consider a matrix-valued version of the bispectral problem, that is, find differential operators $L\left(x, \frac{\mathrm{d}}{\mathrm{d}x}\right)$ and $B\left(z, \frac{\mathrm{d}}{\mathrm{d}z}\right)$ with matrix coefficients such that there exists a family of matrix-valued common eigenfunctions $\psi(x, z)$:

$$L\left(x, \frac{\mathrm{d}}{\mathrm{d}x}\right)\psi(x,z) = f(z)\psi(x,z), \qquad \psi(x,z)B\left(z, \frac{\mathrm{d}}{\mathrm{d}z}\right) = \Theta(x)\psi(x,z),$$

where f and Θ are matrix-valued functions. Using quasideterminants, we prove that the operators L obtained by non-degenerated rational matrix Darboux transformations from $g\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)D$ are bispectral operators, where $g(y) \in \mathbb{C}[y]$ and D is a diagonal matrix. We also give a procedure to find an explicit formula for the operator B extending previous results in the scalar case.

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1. Introduction

In [1], Duistermaat and Grünbaum started the study of bispectral operators. From the beginning, this problem showed its connection with the KdV and KP hierarchies, as well as the Calogero–Moser system (see [2–4]). This problem was expanded in several directions, for a general discussion of the bispectral problem see [5].

The first few attempts to construct bispectral operators in the matrix-valued case were done by Zubelli. In [6, 7], he established the bispectral property of certain AKNS–ZS operators, and in [8] he constructed bispectral operators by using matrix Darboux transformations. For the discrete–continuous matrix version of this problem see [9] and references therein about the recent and interesting developments on matrix-valued orthogonal polynomials having the bispectral property. As was pointed out in [9], the non-commutative version of this problem is very rich and subtle.

In the present work, we give a general construction of matrix-valued bispectral operators using matrix Darboux transformations and the theory of quasideterminants developed by Gelfand and Retakh [10]. In this way, we extend the well-known results in the scalar-valued case developed in [2, 4, 11]. The main ingredient in our construction is a matrix-valued extension of Reach's lemma [12] by using quasideterminants (cf [4, 11, 12]). Although it might seem that the results of this work are very particular, since D is a diagonal matrix, considering differential operators with arbitrary matrix functions coefficients is really a very tough task. Observe that the AKNS–ZS bispectral operators studied in [6] are examples of our general results.

This work is organized as follows: in section 2 we recall some results on matrix Darboux transformations. In section 3, we give the properties of quasideterminants that are needed, the Reach's lemma is proved and the main result is presented. In section 4, we explicitly show the bispectrality of the examples exposed at the end of [13], and some concluding remarks are presented in section 5.

2. Matrix Darboux transformations

We shall consider *matrix differential operators* of the form

$$L = a_m(x)\frac{\mathrm{d}}{\mathrm{d}x}^m + a_{m-1}(x)\frac{\mathrm{d}}{\mathrm{d}x}^{m-1} + \cdots + a_0(x)$$

where the coefficients $a_i(x)$ are $d \times d$ matrix-valued functions. It is called *monic* if $a_m(x) = I$ is the $d \times d$ identity matrix. If $a_m(x) \neq 0$, then m is called the order of L.

Definition 2.1 (cf [13])**.** *We will say that the monic matrix differential operator L is obtained from another operator L_0 of the same type, by matrix Darboux transformations (MDT) if there exists a monic matrix differential operator A which intertwines L and L_0, that is*

$$LA = AL_0.$$

The order of A is called the order of the MDT.

The classical Darboux transformation (see [1] and references therein) corresponds to the scalar case $d = 1$, with order 1. Moreover, in this case it is possible to prove that $A = \frac{\mathrm{d}}{\mathrm{d}x} - (\log\psi)'$, where ψ is some eigenfunction of L_0, namely

$$L_0\psi = \lambda\psi.$$

Hence, we have the factorization $L_0 - \lambda I = BA$ for some operator B of order $m - 1$, where m is the order of L_0. The Darboux transformation of L_0 (associated with ψ) is

$$L = AB + \lambda I.$$

But, in the matrix case, with $d > 1$, a MDT of order 1 is in general not related to any factorization of $L_0 - \lambda I$ in contrast to the scalar case.

The following result characterizes the operators related by the MDT.

Theorem 2.2 [13]**.** *If the operator L is obtained by a MDT from the operator L_0 with intertwining operator A, i.e. $LA = AL_0$, then*

$$L_0(\ker A) \subseteq \ker A.$$

Conversely, for any nd-dimensional L_0-invariant subspace V of d-vector functions there exist operators A and L such that $\ker A = V$ *and* $LA = AL_0$.

Proof. See [13] and references therein. Some results of [14] are needed. □

From now on, we shall work in the special case $L_0 = g\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)D$, where $g \in \mathbb{C}[y]$ and $D = \mathrm{diag}(l_1, \ldots, l_d)$ is the $d \times d$ diagonal matrix with entries l_i's.

In this case, any L_0-invariant space V is generated by the columns of a $d \times nd$ matrix Φ satisfying the condition

$$g\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) D\Phi = \Phi C, \tag{2.1}$$

where C is a constant $nd \times nd$ matrix which can be assumed, by a suitable change of base in V, to be in its Jordan form. Suppose for a while that C is a single Jordan block of the form

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{pmatrix}$$

for some constant λ. Denote by v_i the ith column of Φ. By (2.1), we have that

$$g\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) l_i (v_1)_i = \lambda (v_1)_i,$$

where $(v_1)_i$ is the ith coordinate of v_1. Thus, we have that each $(v_1)_i$ is a solution of a homogeneous constant coefficient differential equation, therefore a quasipolynomial, i.e. it has the form $\sum_{i=1}^{m} p_i(x)\,\mathrm{e}^{\mu_i x}$, with $p_i(x) \in \mathbb{C}[x]$ and $\mu_i \in \mathbb{C}$.

Now, again by (2.1), we have that $g\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) l_i (v_2)_i = (v_1)_i + \lambda (v_2)_i$, where $(v_2)_i$ is the ith coordinate of v_2. Thus each $(v_2)_i$, with $i = 1, \ldots, d$, is a solution of a (non-homogeneous) constant coefficient differential equation, namely,

$$\left(g\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) l_i - \lambda \right) (v_2)_i = (v_1)_i.$$

Since we already know that each $(v_1)_i$ is a quasipolynomial, it is a well-known result, using the Green functions (see, e.g., [15], p 110–117), that $(v_2)_i$, with $i = 1, \ldots, d$, is also a quasipolynomial.

Recursively, one can show that each $(v_r)_i$ is a quasipolynomial for $r \geqslant 1$ and $i = 1, \ldots, d$.

Similar arguments apply for C with an arbitrary number of Jordan blocks since they are independent.

Thus, in our case, V is formed by vectors with quasipolynomial coordinates: $v = (v_1, \ldots, v_d)^t \in V$ and $v_i = \sum_j p_{ij}(x)\,\mathrm{e}^{\lambda_j x}$, with p_{ij} polynomials in x.

Now, for $L_0 = g\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) D$, consider a basis of an L_0-invariant space V, it corresponds to the columns of a $d \times nd$ matrix Φ satisfying (2.1), and Φ can be thought as $nd \times d$ matrices $f_1, \ldots, f_n$, i.e. $\Phi = (f_1, \ldots, f_n)$. These matrices $f_1, \ldots, f_n$ generate the kernel of the intertwining operator A associated with the MDT obtained from the L_0-invariant space V. Therefore, we have proved that $f_i = \sum_l R_{il}(x)\,\mathrm{e}^{\lambda_{il} x}$, with $\lambda_{il} \in \mathbb{C}$ and $R_{il}(x) \in \mathrm{Mat}_{n\times n}(\mathbb{C}[x])$. We shall consider the following special case (cf [11]):

Definition 2.3. *A MDT is called rational if it comes from an L_0-invariant space $V = \langle f_1, \ldots, f_n \rangle$ where $f_i = Q_i(x)\,\mathrm{e}^{\lambda_i x}$ with Q_i being a matrix polynomial.*

In the scalar case, this notion was defined and studied in [1, 4, 11]. Another motivation for this name is that the intertwining operator A in this case will have matrix rational coefficients, this can be proved by using the explicit formula for A in (3.5) and property

(3.4) for quasideterminants (see section 3). Observe that MDT were previously studied in [13], where they prove that all matrix Schrödinger operators

$$L = -D^2 + U(z), \qquad D = \frac{\mathrm{d}}{\mathrm{d}z},$$

with a potential $U(z)$ being a $d \times d$ rational matrix-valued function, have trivial monodromy if they are obtained by MDT from $L_0 = -D^2$. See [16] for the relation of MDT with the matrix KdV equation.

3. Quasideterminants and bispectral operators

In the first part of this section, we will review the definition of quasideterminant and some of its properties. For details, we refer to [10].

Let $A = (a_{ij})$, $1 \leqslant i, j \leqslant n$, be a matrix with formal non-commuting entries a_{ij}. We denote $A^{\alpha\beta}$, $1 \leqslant \alpha, \beta \leqslant n$, as the $n-1$ order matrix obtained from A by removing the αth row and the βth column.

Definition 3.1. *For a matrix A over a ring with unit the quasideterminant $|A|_{pq}$ is defined if the matrix A^{pq} is invertible. In this case,*

$$|A|_{pq} = a_{pq} - \sum_{\substack{i\neq p\\ j\neq q}} a_{pj} b_{ji} a_{iq}, \tag{3.1}$$

where b_{ji} are the entries of the matrix $(A^{pq})^{-1}$.

If the entries of the matrix A commute with each other, it is easy to see that

$$|A|_{pq} = (-1)^{p+q} \frac{\det A}{\det A^{pq}}.$$

Therefore, quasideterminants correspond to a generalization of a fraction of determinants. In the following theorem, we summarize some of the properties of quasideterminants that will be used in this work (for a complete study see [10] and references therein).

Theorem 3.2 [10]**.** *Let $A = (a_{ij})$ be an $n \times n$ matrix over a ring R with unit.*

(1) For a square matrix $A = (a_{ij})$ with formal entries

$$HI(A) = (|A|_{ij}), \tag{3.2}$$

where I denotes the involution $I(A) = A^{-1}$ and $H(A) = \left(a_{ji}^{-1}\right)$ is the Hadamard inverse of A. Thus, we have

$$|A|_{ij} \cdot b_{ji} = 1, \tag{3.3}$$

where $B = A^{-1} = (b_{rs})$.

(2) Multiplications of columns: let C be the matrix obtained from A by multiplying its j-th column by a scalar μ from the right. Then,

$$|C|_{il} = \begin{cases} |A|_{ij}\mu, & \text{if} \quad l = j; \\ |A|_{il}, & \text{if} \quad l \neq j \text{ and } \quad \mu \text{ is invertible.} \end{cases} \tag{3.4}$$

(3) Addition of columns: let C be the matrix constructed by adding to some column of A its l-th column multiplied by a scalar λ from the right. Then,

$$|A|_{ij} = |C|_{ij}, \qquad i = 1, \ldots, n, \qquad j = 1, \ldots, l-1, l+1, \ldots, n.$$

(4) If $|A|_{ij}$ is defined, the following statements are equivalent:

(i) $|A|_{ij} = 0$;

(ii) the i-th row of A is a left linear combination of the other rows of this matrix;

(iii) the j-th column of A is a right linear combination of the other columns of this matrix.

(5) For any $k \neq p$ *and any* $l \neq q$

$$|A|_{pq} = a_{pq} - \sum_{j \neq q} a_{pj} (|A^{pq}|_{kj})^{-1} |A^{pj}|_{kq}$$

if all terms in these expressions are defined.

Consider $L_0 = g\left(\frac{d}{dx}\right) D$ as in section 2. Let V be an nd-dimensional L_0-invariant subspace of d-vector functions. Let us combine the vectors of V as columns of $nd \times d$ matrices $\Phi_1, \ldots, \Phi_n$, namely $V = \langle \Phi_1, \ldots, \Phi_n \rangle$. The intertwining operator A given in theorem 2.2 can be written in terms of $\Phi_1, \ldots, \Phi_n$ as follows:

$$A(\Phi) = |W(\Phi_1, \ldots, \Phi_n, \Phi)|_{n+1,n+1} \tag{3.5}$$

where

$$W(\Phi_1, \ldots, \Phi_n, \Phi) = \begin{pmatrix} \Phi_1 & \cdots & \Phi_n & \Phi \\ \vdots & \ddots & \vdots & \vdots \\ \Phi_1^{(n-1)} & \cdots & \Phi_n^{(n-1)} & \Phi^{(n-1)} \\ \Phi_1^{(n)} & \cdots & \Phi_n^{(n)} & \Phi^{(n)} \end{pmatrix}$$

is the *Wronski matrix* of $\Phi_1, \ldots, \Phi_n, \Phi$ (cf theorem 1.1 in [14]).

Now, we can state the analog of Reach's lemma that will be useful to prove the main result of this paper (see [12] for the original scalar version and [4, 11] for other applications).

Lemma 3.3. *Let* $f_1, \ldots, f_{n+1}$ *and* p *be arbitrary smooth matrix-valued functions such that* $W(f_1, \ldots, f_n)$, $|W(f_1, \ldots, f_n)|_{nj}$ *and* $W(f_1, \ldots, \widehat{f_j}, \ldots, f_n)$ *are invertible for all* $j = 1, \ldots, n$. *Define*

$$F(x) = f_{n+1}(x) \left(\int p(x)\, dx \right) - \sum_{j=1}^{n} f_j(x)$$

$$\times \int (|W(f_1, \ldots, f_n)|_{nj}(x))^{-1} |W(f_1, \ldots, \hat{f}_j, \ldots, f_{n+1})|_{n,n+1}(x) p(x)\, dx. \tag{3.6}$$

Then,

$$|W(f_1, \ldots, f_n, F)|_{n+1,n+1}(x) = |W(f_1, \ldots, f_{n+1})|_{n+1,n+1}(x) \left(\int p(x)\, dx \right). \tag{3.7}$$

Proof. By theorem 3.2.4, the following identity

$$0 = \left| \begin{pmatrix} f_1(x) & f_2(x) & \cdots & f_{n+1}(x) \\ f_1'(x) & f_2'(x) & \cdots & f_{n+1}'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_{n+1}^{(n-1)}(x) \\ f_1^{(i)}(x) & f_2^{(i)}(x) & \cdots & f_{n+1}^{(i)}(x) \end{pmatrix} \right|_{n+1,n+1}$$

holds for $i = 0, \ldots, n-1$, and expanding it with respect to the last row (see theorem 3.2.5) we obtain

$$0 = f_{n+1}^{(i)}(x) - \sum_{j=1}^{n} f_j^{(i)}(x)(|W(f_1,\ldots,f_n)|_{nj}(x))^{-1}|W(f_1,\ldots,\hat{f}_j,\ldots,f_{n+1})|_{n,n+1}(x) \quad (3.8)$$

for all $i = 0, \ldots, n-1$. Observe that, by hypothesis, all quasideterminants used before are well defined. Now, in order to prove (3.7), we need to compute the derivatives of F defined in (3.6). We have

$$\begin{aligned} F'(x) &= f'_{n+1}(x)\left(\int p(x)\,\mathrm{d}x\right) \\ &\quad - \sum_{j=1}^{n} f'_j(x)\int (|W(f_1,\ldots,f_n)|_{nj}(x))^{-1}|W(f_1,\ldots,\hat{f}_j,\ldots,f_{n+1})|_{n,n+1}(x)p(x)\,\mathrm{d}x \\ &\quad + \left(f_{n+1}(x) - \sum_{j=1}^{n} f_j(x)(|W(f_1,\ldots,f_n)|_{nj}(x))^{-1}\right. \\ &\quad \left. \times |W(f_1,\ldots,\hat{f}_j,\ldots,f_{n+1})|_{n,n+1}(x)\right)p(x) \end{aligned}$$

and the last term is zero by (3.8) with $i = 0$. Similarly, and using (3.8) with different values of i, it is easy to see that

$$\begin{aligned} F^{(i)}(x) &= f_{n+1}^{(i)}(x)\left(\int p(x)\,\mathrm{d}x\right) - \sum_{j=1}^{n} f_j^{(i)}(x)\int (|W(f_1,\ldots,f_n)|_{nj}(x))^{-1} \\ &\quad \times |W(f_1,\ldots,\hat{f}_j,\ldots,f_{n+1})|_{n,n+1}(x)p(x)\,\mathrm{d}x \end{aligned} \quad (3.9)$$

for all $i = 0, \ldots, n$.

Inserting (3.9) into $|W(f_1,\ldots,f_n,F)|_{n+1,n+1}$, most of the terms in the last column disappear by subtracting multiples of the first n columns by scalars from the right (see column elimination for quasideterminants in theorem 3.2.3), and all that remains is

$$|W(f_1,\ldots,f_n,F)|_{n+1,n+1}(x) = \left|\begin{pmatrix} f_1(x) & f_2(x) & \cdots & f_{n+1}(x)\left(\int p(x)\,\mathrm{d}x\right) \\ f'_1(x) & f'_2(x) & \cdots & f'_{n+1}(x)\left(\int p(x)\,\mathrm{d}x\right) \\ \vdots & \vdots & & \vdots \\ f_1^{(n)}(x) & f_2^{(n)}(x) & \cdots & f_{n+1}^{(n)}(x)\left(\int p(x)\,\mathrm{d}x\right) \end{pmatrix}\right|_{n+1,n+1}$$

from which the lemma follows by theorem 3.2.2. □

Remark 3.4. Observe that $L_0 = g\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)D$, with D being a constant diagonal matrix, is trivially a bispectral operator since $\psi_0(x,z) = \mathrm{e}^{xz}Q$, with Q being a constant non-singular matrix, satisfies

$$L_0\psi_0(x,z) = \psi_0(x,z)f(z),$$

where $f(z) = g(z)Q^{-1}DQ$, and $\psi_0(x,z)B_0 = h(x)\psi_0(x,z)$, where $B_0 = \tilde{h}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)\tilde{D}$, $h(x) = \tilde{h}(x)Q\tilde{D}Q^{-1}$ and $\tilde{h}(y) \in \mathbb{C}[y]$. Observe that if $LA = AL_0$, then $\psi(x,z) = A\psi_0(x,z)$ satisfies

$$L\psi(x,z) = \psi(x,z)f(z). \quad (3.10)$$

Motivated by the previous lemma, we consider the following definition.

Definition 3.5. *A MDT is called non-degenerated if it comes from an L_0-invariant space $V = \langle f_1, \dots, f_n\rangle$ such that $K = W(f_1, \dots, f_n)$, $W(f_1, \dots, \widehat{f_j}, \dots, f_n)$ and the elements $(K^{-1})_{jn}$ are invertible for all $j = 1, \dots, n$.*

Now we will prove the main result of this paper.

Theorem 3.6. *Let $L_0 = g\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)D$, with D being a constant diagonal matrix and $g \in \mathbb{C}[y]$. Then any matrix differential operator L obtained by a non-degenerated rational matrix Darboux transformation from the operator L_0 is bispectral.*

Proof. Suppose L is obtained by a non-degenerated rational matrix Darboux transformation of $L_0 = g\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)D$. Then, the rational MDT comes from an L_0-invariant space $V = \langle f_1, \dots, f_n\rangle$ where $f_i(x) = P_i(x)\,\mathrm{e}^{\lambda_i x}$ and $P_i(x)$ is a matrix polynomial for $i = 1, \dots, n$.

Set $f_{n+1}(x) = \mathrm{e}^{xz}Q$ with Q being a constant non-singular matrix. Recall that f_i's, $i = 1, \dots, n$, span the kernel of the intertwining operator

$$A(f) = |W(f_1, \dots, f_n, f)|_{n+1,n+1}$$

in (3.5). Take $\psi(x, z) = A(f_{n+1}(x))$, by remark 3.4, it is enough to show that

$$\psi(x, z)B\left(z, \frac{\mathrm{d}}{\mathrm{d}z}\right) = \psi(x, z)\Theta(x)$$

for some matrix differential operator B in z and $\Theta(x) \in \mathbb{C}[x]$. We write $\Theta(x) = \int f_0(x)\,\mathrm{d}x$ for some polynomial $f_0(x)$.

Since $f_i(x) = P_i(x)\,\mathrm{e}^{\lambda_i x}$ for $i = 1, \dots, n$, the exponentials $\mathrm{e}^{\lambda_i x}$ appear in each column of $W(f_1, \dots, f_n)$. Hence, using property (2) of quasideterminats in theorem 3.2, we obtain

$$|W(f_1, \dots, f_n)|_{nj} = |R(x)|_{nj} \cdot \mathrm{e}^{\lambda_j x},$$

with $R(x)$ being a matrix polynomial. Observe that for $i \neq j$, the (invertible) factor $\mathrm{e}^{\lambda_i x}$ can be removed because the quasideterminant of type $|\cdot|_{nj}$ does not change by (3.4).

Observe that the non-degeneracy of the Darboux transformation and property (3.3) allows us to apply the previous lemma and quarantines the existence of all quasideterminants involved in the following computations. By (3.3), we have that

$$|R(x)|_{nj}((R(x))^{-1})_{jn} = 1.$$

Similarly,

$$|W(f_1, \dots, \hat{f}_j, \dots, f_{n+1})|_{n,n+1} = |S_j(x, z)|_{n,n} \cdot \mathrm{e}^{xz}Q,$$

where $S_j(x, z)$ is a matrix whose entries depend polynomially on x and z; in fact, the variable z only appears in the last column. Moreover, using (3.1), it is easy to see that the quasideterminant $|S_j(x, z)|_{n,n}$ depends polynomially on z, and as a rational function in x. Take $r(x)$ the monic scalar polynomial of minimal degree such that the following expression depend polynomially on z and x for all $j = 1, \dots, n$:

$$\begin{aligned}\mathrm{e}^{(\lambda_j - z)x} \cdot |W(f_1, \dots, f_n)|_{nj}^{-1} \cdot |W(f_1, \dots, \hat{f}_j, \dots, f_{n+1})|_{n,n+1} \cdot r(x) \\ = (|R(x)|_{nj})^{-1} \cdot |S_j(x, z)|_{n,n} \cdot Q \cdot r(x). \qquad (3.11)\end{aligned}$$

Fix $p(x) = r(x)q(x)$, for some arbitrary polynomial q. Observe that in this case we apply the previous lemma to a scalar-valued function p, even though in the previous lemma p is a matrix-valued function. Then, we have that

$$\begin{aligned} F(x) = {} & f_{n+1}(x)\left(\int p(x)\,\mathrm{d}x\right) - \sum_{j=1}^{n} f_j(x) \\ & \times \int (|W(f_1,\ldots,f_n)|_{nj}(x))^{-1}|W(f_1,\ldots,\hat{f}_j,\ldots,f_{n+1})|_{n,n+1}(x)p(x)\,\mathrm{d}x \\ = {} & f_{n+1}(x)\left(\int p(x)\,\mathrm{d}x\right) - \sum_{j=1}^{n} P_j(x)\,\mathrm{e}^{\lambda_j x} \\ & \times \int (|W(f_1,\ldots,f_n)|_{nj}(x))^{-1}|W(f_1,\ldots,\hat{f}_j,\ldots,f_{n+1})|_{n,n+1}(x)p(x)\,\mathrm{d}x \\ = {} & f_{n+1}(x)\left(\int p(x)\,\mathrm{d}x\right) - \sum_{j=1}^{n} P_j(x)\,\mathrm{e}^{\lambda_j x} \\ & \times \int (|R(x)|_{nj})^{-1}|S(x,z))|_{n,n+1}\,\mathrm{e}^{x(z-\lambda_j)}Qp(x)\,\mathrm{d}x. \end{aligned}$$

After integrating by parts in the second term of the RHS in the last equation and using (3.11), we have

$$F(x) = \left(P_{n+1}(x)\left(\int p(x)\,\mathrm{d}x\right) - \sum_{j=1}^{n} P_j(x)T(z,x)\right)\mathrm{e}^{xz}, \tag{3.12}$$

where $T(z,x)$ is a matrix polynomial in x whose coefficients are rational funcions in z. Hence, $F(x)$ depends polynomially on x and we have that

$$F(x) = \mathrm{e}^{xz}QB\left(z,\frac{\mathrm{d}}{\mathrm{d}z}\right), \tag{3.13}$$

for some differential operator $B\left(z,\frac{\mathrm{d}}{\mathrm{d}z}\right)$ with matrix coefficients whose entries are rational functions in z.

Thus, using lemma (3.3) and (3.4), we conclude that

$$\begin{aligned} \psi(x,z)B\left(z,\frac{\mathrm{d}}{\mathrm{d}z}\right) &= |W(f_1,\ldots,f_n,\mathrm{e}^{xz}Q)|_{n+1,n+1}B\left(z,\frac{\mathrm{d}}{\mathrm{d}z}\right) \\ &= \left|W\left(f_1,\ldots,f_n,\mathrm{e}^{xz}Q\cdot B\left(z,\frac{\mathrm{d}}{\mathrm{d}z}\right)\right)\right|_{n+1,n+1} \\ &= |W(f_1,\ldots,f_n,F)|_{n+1,n+1} = |W(f_1,\ldots,\mathrm{e}^{xz}Q)|_{n+1,n+1}\left(\int p(x)\,\mathrm{d}x\right) \\ &= \psi(x,z)\left(\int p(x)\,\mathrm{d}x\right), \end{aligned}$$

finishing the proof. □

Remark 3.7. (a) From the proof we can deduce that given L as in theorem 3.6, for any $\Theta(x)$ such that $\Theta'(x)$ is divisible by $r(x)$ where $r(x)$ is as in (3.11), there exists a differential operator B in z such that $\Theta(x)$ is its eigenvalue.

(b) Observe that the proof of theorem 3.6 gives a procedure to obtain the explicit formula for the operator B for each $\Theta(x)$ as in the previous remark. Namely, let $f_1,\ldots,f_n$ be a basis of the L_0-invariant space that describes the non-degenerated rational matrix Darboux transformation that produces the new operator L. Then, by using quasideterminants, one can compute explicitly $F(x)$ as in (3.12), and then, using (3.13), we obtain the explicit formula for the operator B.

4. Examples

In this section, we shall explicitly show the bispectrality of the examples of Schroedinger operators given at the end of [13]. The computation of the explicit formula for the operator B was done by using the construction explained in remark 3.7(b), for just one particular choice of Θ in each example.

Example 4.1. Consider $L_0 = -\frac{d^2}{dx^2} I$. Let V be the L_0-invariant vector space spanned by the columns of

$$f_1 = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}.$$

The intertwining operator whose kernel is V is given by

$$A = \frac{d}{dx} I - \begin{pmatrix} \frac{1}{x} & -\frac{1}{x^2} \\ 0 & \frac{1}{x} \end{pmatrix}.$$

In this case,

$$L = \frac{d^2}{dx^2} I - \begin{pmatrix} -\frac{1}{x^2} & \frac{2}{x^3} \\ 0 & -\frac{1}{x^2} \end{pmatrix},$$

and $\psi(x, z) = A(e^{xz})$. The operator

$$B = \frac{d^3}{dz^3}\left(\frac{I}{3}\right) - \frac{d^2}{dz^2}\left(\frac{I}{z}\right) + \frac{d}{dz}\left(\frac{I}{z^2}\right) + \begin{pmatrix} 0 & \frac{1}{z^2} \\ 0 & 0 \end{pmatrix}$$

satisfies $\psi(x, z)B = \Theta(x)\psi(x, z)$, with $\Theta(x) = \frac{x^3}{3}$. And in general, for any Θ such that $\Theta' = x^2 q(x)$ with $q(x) = \mathbb{C}[x]$, it is possible to find the corresponding B.

Example 4.2. Let $L = -I\frac{d^2}{dx^2} + U(x)$ be the matrix Schroedinger operator whose potential $U(x)$ with three second-order poles is given by

$$U(x) = \frac{P_u}{(x-u)^2} + \frac{P_v}{(x-v)^2} + \frac{P_w}{(x-w)^2}$$

where the projectors P_u, P_v, P_w are defined as follows:

$$P_u = \frac{2}{(u-v)(u-w)} \begin{pmatrix} vw - u^2 & u(-vw + u^2) \\ w - 2u + v & u(-w + 2u - v) \end{pmatrix}$$

$$P_v = \frac{2}{(v-u)(v-w)} \begin{pmatrix} uw - v^2 & -v(uw - v^2) \\ w - 2v + u & -v(w - 2v + u) \end{pmatrix}$$

and

$$P_w = \frac{2}{(w-u)(w-v)} \begin{pmatrix} uv - w^2 & -w(uv - w^2) \\ u - 2w + v & -w(u - 2w + v) \end{pmatrix}.$$

In this case, L is obtained by MDT from $L_0 = -I\frac{d^2}{dx^2}$ with respect to the L_0-invariant vector space V generated by the column vectors of the following matrices:

$$f_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad f_2 = \begin{pmatrix} x & x^2 \\ 0 & -2x \end{pmatrix}, \qquad f_2 = \begin{pmatrix} 0 & a(x) \\ x^2 & b(x) \end{pmatrix}$$

where

$$a(x) = \tfrac{1}{3}x^4(u + v + w) - \tfrac{4}{3}(uv + uw + vw)x^3 + 4uvwx^2$$
$$b(x) = x^4 - \tfrac{4}{3}(u + v + w)x^3.$$

The intertwining operator is given by

$$A = I\frac{\mathrm{d}^3}{\mathrm{d}x^3} - \frac{1}{2}\left(\frac{P_u}{(x-u)} + \frac{P_v}{(x-v)} + \frac{P_w}{(x-w)}\right)\left(I\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\frac{\mathrm{d}}{\mathrm{d}x}\right)$$

and $\psi(x,z) = A(\mathrm{e}^{xz})$ satisfies $L\psi(x,z) = -z^2\psi(x,z)$. After some computations, it is possible to see that $r(x) = (x-u)(x-v)(x-w)$ satisfies (3.11). Hence, for any Θ such that $\Theta'(x) = r(x)q(x)$, with $q(x) \in \mathbb{C}[x]$, there exists a matrix differential operator B in z satisfying $\psi(x,z)B = \Theta(x)\psi(x,z)$. In the special case $\Theta' = r$, the operator B is given by

$$B = \frac{\mathrm{d}^4}{\mathrm{d}z^4}b_4 + \frac{\mathrm{d}^3}{\mathrm{d}z^3}b_3 + \frac{\mathrm{d}^2}{\mathrm{d}z^2}b_2 + \frac{\mathrm{d}^1}{\mathrm{d}z^1}b_1 + b_0$$

where $b_4 = \frac{1}{4}I$, $b_3 = -\left(\frac{1}{3}(v+u+w) + \frac{3}{x}\right)I$ and

$$b_2 = \begin{pmatrix} \frac{1}{2}(uv+uw+vw) + \frac{3(w+v+u)}{x} + \frac{18}{x^2} & -\frac{(v+u+w)}{x^2} \\ 0 & \frac{1}{2}(uv+uw+vw) + \frac{3(w+v+u)}{x} + \frac{15}{x^2} \end{pmatrix}$$

$$b_1 = \begin{pmatrix} -uvw - \frac{3(uw+vw+uv)}{x} - \frac{11(v+u+w)}{x^2} - \frac{60}{x^3} & \frac{8(w+v+u)}{x^3} + \frac{uv+uw+vw}{x^2} \\ \frac{3}{x^2} & -uvw - \frac{3(uw+vw+uv)}{x} - \frac{11(v+u+w)}{x^2} - \frac{36}{x^3} \end{pmatrix}$$

$$b_0 = \begin{pmatrix} \frac{3uvw}{x} + \frac{5(uv+uw+vw)}{x^2} + \frac{17(w+v+u)}{x^3} + \frac{90}{x^4} & -\frac{18(u+w+v)}{x^4} - \frac{5(uv+uw+vw)}{x^3} \\ -\frac{v+u+w}{x^2} - \frac{9}{x^3} & \frac{3(uvw)}{x} + \frac{6(uw+vw+uv)}{x^2} + \frac{15(v+u+w)}{x^3} + \frac{36}{x^4} \end{pmatrix}$$

5. Concluding remarks

There is no doubt that the theory of quasideterminants will play an important role in the study of matrix bispectral operators and matrix orthogonal polynomials (a discrete–continuous instance of it). For example, orthogonal polynomials as quasideterminants of moments matrices were introduced in [17]. This general definition suggests that quasideterminant language is obviously the right language to understand [18].

An interesting open problem is the possible relation between matrix bispectral operators and the matrix version of the Calogero–Moser system, as well as the matrix KdV (or KP) equation (cf [3, 4]).

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