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Classification of irreducible representations over finite simple Lie conformal superalgebras

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Abstract. This article is a survey on the classification of irreducible representations over finite simple Lie conformal superalgebras.

Contents

1. Introduction

Lie conformal superalgebras encode the singular part of the operator product expansion of chiral fields in two-dimensional quantum field theory [6].

On the other hand, they are closely connected to the notion of formal distribution Lie superalgebra $(\mathfrak{g}, \mathcal{F})$, that is a Lie superalgebra $\mathfrak{g}$ spanned by the coefficients of a family $\mathcal{F}$ of mutually local formal distributions. Namely, to a Lie conformal superalgebra R one can associate a formal distribution Lie superalgebra $(\mathrm{Lie}\, R, R)$ which establishes an equivalence between the category of Lie conformal superalgebras and the category of equivalence classes of formal distribution Lie superalgebras obtained as quotients of Lie R by irregular ideals [**8**].

Finite simple Lie conformal algebras were classified in [**6**] and all their finite irreducible representations were constructed in [**4**]. According to [**6**], any finite simple Lie conformal algebra is isomorphic either to the current Lie conformal

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algebra Cur $\mathfrak{g}$, where $\mathfrak{g}$ is a simple finite-dimensional Lie algebra, or to the Virasoro conformal algebra Vir.

However, the list of finite simple Lie conformal superalgebras is much richer, mainly due to existence of several series of super extensions of the Virasoro conformal algebra.

A complete classification of (linear) finite simple Lie conformal superalgebras was obtained in [**7**], where they proved that any finite simple Lie conformal superalgebras is isomorphic to one of the following:

- Current Lie conformal superalgebras Cur $\mathfrak{g}$, where $\mathfrak{g}$ is a simple finite-dimensional Lie superalgebra,
- $W_n (n \geq 0)$,
- $S_{n,b} (n \geq 2\,,\, b \in \mathbb{C})$,
- $\tilde{S}_n (n \text{ even }, n \geq 2)$,
- $K_n (n \geq 0,\, n \neq 4)$,
- K'_4,
- CK_6.

All finite irreducible representations of the simple conformal superalgebras Cur $\mathfrak{g}$, $K_0 = \mathrm{Vir}$ and K_1 were constructed in [**4**], and those of $S_{2,0}$, $W_1 = K_2$, K_3, and K_4 in [**5**]. More recently, the problem has been solved for all Lie conformal superalgebras from the four series W_n, $S_{n,b}$, $\tilde{S}_n$ and K_n (see [**2**] and [**1**]).

The construction in all these series relies on the observation that the representation theory of a Lie conformal superalgebra R is controlled by the representation theory of the associated (extended) *annihilation algebra* $\mathfrak{g} = (\mathrm{Lie}\, R)_+$ [**4**], thereby reducing the problem to the construction of continuous irreducible modules with discrete topology over the linearly compact superalgebra $\mathfrak{g}$.

The construction of the latter modules consists of two parts. First one constructs a collection of continuous $\mathfrak{g}$-modules $\mathrm{Ind}(F)$, associated to all finite-dimensional irreducible $\mathfrak{g}_0$-modules F, where $\mathfrak{g}_0$ is a certain subalgebra of $\mathfrak{g}$ ($= \mathfrak{gl}(1|n)$ or $\mathfrak{sl}(1|n)$ for the W and S series, and $= \mathfrak{cso}_n$ for the K_n series, for instance).

The irreducible $\mathfrak{g}$-modules $\mathrm{Ind}(F)$ are called *non-degenerate*, and the second part of the problem consists of two parts:

(a) Classify the $\mathfrak{g}_0$-modules F, for which the $\mathfrak{g}$-modules $\mathrm{Ind}(F)$ are non-degenerate, and
(b) construct explicitly the irreducible quotients of $\mathrm{Ind}(F)$, called *degenerate* $\mathfrak{g}$-modules, for reducible $\mathrm{Ind}(F)$.

Both problems have been solved for types W and S in [**2**] and for type K in [**1**]. For types W and S, it turned out, remarkably, that all degenerate modules occur as cokernels of the super de Rham complex, or their duals.

For K_n with $n \geq 4$ (recall that for $0 \leq n \leq 4$ the problem has been solved in [**4**] and [**5**], though in [**5**] the construction for $n = 3$ and 4 is not very explicit), we construct a *contact complex*, which is a certain reduction of the de Rham complex, and show that the cokernels in the contact complex and their duals produce all degenerate $\mathfrak{g}$-modules. As a result, we obtain an explicit construction of all finite irreducible K_n-modules for $n \geq 4$.

We should mention that the construction of our (super) contact complex mimics the beautiful Rumin's construction [**13**] for ordinary (non-super) contact manifolds.

The remaining cases, namely, the representation theory of K_4' (the derived algebra of K_4) and of the exceptional Lie conformal superalgebra CK_6, and the explicit construction of degenerate modules for K_3, will be worked out in a subsequent publication.

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2. Formal distributions, Lie conformal superalgebras and their modules

In this section we introduce the basic definitions and notations in order to have a self-contained work, see [**8, 6, 2, 5**]. Let $\mathfrak{g}$ be a Lie superalgebra. A $\mathfrak{g}$-valued *formal distribution* in one indeterminate z is a formal power series

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \qquad a_n \in \mathfrak{g}.$$

The vector superspace of all formal distributions, $\mathfrak{g}[[z, z^{-1}]]$, has a natural structure of a $\mathbb{C}[\partial_z]$-module. We define

$$\mathrm{Res}_z\, a(z) = a_0.$$

Let $a(z), b(z)$ be two $\mathfrak{g}$-valued formal distributions. They are called *local* if

$$(z-w)^N [a(z), b(w)] = 0 \qquad \text{for} \qquad N >> 0.$$

Let $\mathfrak{g}$ be a Lie superalgebra, a family $\mathcal{F}$ of $\mathfrak{g}$-valued formal distributions is called a *local family* if all pairs of formal distributions from $\mathcal{F}$ are local. Then, the pair $(\mathfrak{g}, \mathcal{F})$ is called a *formal distribution Lie superalgebra* if $\mathcal{F}$ is a local family of $\mathfrak{g}$-valued formal distributions and $\mathfrak{g}$ is spanned by the coefficients of all formal distributions in $\mathcal{F}$. We define the *formal δ-function* by

$$\delta(z-w) = z^{-1} \sum_{n\in\mathbb{Z}} \left(\frac{w}{z}\right)^n.$$

Then it is easy to show ([**8**], Corollary 2.2)), that two local formal distributions are local if and only if the bracket can be represented as a finite sum of the form

$$[a(z), b(w)] = \sum_j [a(z)_{(j)} b(w)]\ \partial_w^j \delta(z-w)/j!,$$

where $[a(z)_{(j)} b(w)] = \mathrm{Res}_z (z-w)^j [a(z), b(w)]$. This is called the *operator product expansion*. Then we obtain a family of operations $_{(n)}$, $n \in \mathbb{Z}_+$, on the space of formal distributions. By taking the generating series of these operations, we define the λ-bracket:

$$[a_\lambda b] = \sum_{n\in\mathbb{Z}_+} \frac{\lambda^n}{n!} [a_{(n)} b].$$

The properties of the λ-bracket motivate the following definition:

Definition 2.1. A *Lie conformal superalgebra* R is a left $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map $R \otimes R \longrightarrow \mathbb{C}[\lambda] \otimes R$, $a \otimes b \mapsto a_\lambda b$, called the λ-bracket, and satisfying the following axioms (a, b, $c \in R$),

Conformal sesquilinearity $\qquad [\partial a_\lambda b] = -\lambda [a_\lambda b], \qquad [a_\lambda \partial b] = (\lambda + \partial)[a_\lambda b],$

Skew-symmetry $\qquad [a_\lambda b] = -(-1)^{p(a)p(b)}[b_{-\lambda-\partial}\ a],$

Jacobi identity $\qquad [a_\lambda[b_\mu c]] = [[a_\lambda b]_{\lambda+\mu}c] + (-1)^{p(a)p(b)}[b_\mu[a_\lambda c]].$

Here and further $p(a) \in \mathbb{Z}/2\mathbb{Z}$ is the parity of a.

A Lie conformal superalgebra is called *finite* if it has finite rank as a $\mathbb{C}[\partial]$-module. The notions of homomorphism, ideal and subalgebras of a Lie conformal superalgebra are defined in the usual way. A Lie conformal superalgebra R is *simple* if $[R_\lambda R] \neq 0$ and contains no ideals except for zero and itself.

Given a formal distribution Lie superalgebra $(\mathfrak{g}, \mathcal{F})$ denote by $\bar{\mathcal{F}}$ the minimal subspace of $\mathfrak{g}[[z, z^{-1}]]$ which contains $\mathcal{F}$ and is closed under all j-th products and invariant under ∂_z. Due to Dong's lemma, we know that $\bar{\mathcal{F}}$ is a local family as well. Then $\mathrm{Conf}(\mathfrak{g}, \mathcal{F}) := \bar{\mathcal{F}}$ is the Lie conformal superalgebra associated to the formal distribution Lie superalgebra $(\mathfrak{g}, \mathcal{F})$.

In order to give the (more or less) reverse functorial construction, we need the following: let $\tilde{R} = R[t, t^{-1}]$ with $\tilde{\partial} = \partial + \partial_t$ and define the bracket [**8**]:

$$[at^n, bt^m] = \sum_{j\in\mathbb{Z}_+} \binom{m}{j} [a_j b] t^{m+n-j}. \tag{2.1}$$

Observe that $\tilde{\partial}\tilde{R}$ is an ideal of $\tilde{R}$ with respect to this bracket. Now, consider $\mathrm{Alg}R = \tilde{R}/\tilde{\partial}\tilde{R}$ with this bracket and let

$$\mathcal{R} = \{\sum_{n\in\mathbb{Z}} (at^n) z^{-n-1} = a\delta(t-z)\ /\ a \in R\}.$$

Then $(\mathrm{Alg}R, \mathcal{R})$ is a formal distribution Lie superalgebra. Note that Alg is a functor from the category of Lie conformal superalgebras to the category of formal distribution Lie superalgebras. On has [**8**]:

$$\mathrm{Conf}(\mathrm{Alg}R) = R, \quad \mathrm{Alg}(\mathrm{Conf}(\mathfrak{g}, \mathcal{F})) = (\mathrm{Alg}\bar{\mathcal{F}}, \bar{\mathcal{F}}).$$

Note also that $(\mathrm{Alg}R, \mathcal{R})$ is the *maximal formal distribution superalgebra* associated to the conformal superalgebra R, in the sense that all formal distribution Lie superalgebras $(\mathfrak{g}, \mathcal{F})$ with $\mathrm{Conf}(\mathfrak{g}, \mathcal{F}) = R$ are quotients of $(\mathrm{Alg}R, \mathcal{R})$ by irregular ideals (that is, an ideal I in $\mathfrak{g}$ with no non-zero $b(z) \in \mathcal{R}$ such that $b_n \in I$). Such formal distribution Lie superalgebras are called *equivalent*.

We thus have an equivalence of categories of Lie conformal superalgebras and equivalence classes of formal distribution Lie superalgebras. So the study of formal distribution Lie superalgebras reduces to the study of Lie conformal superalgebras.

An important tool for the study of Lie conformal superalgebras and their modules is the (extended) annihilation superalgebra. The *annihilation superalgebra* of a Lie conformal superalgebra R is the subalgebra $\mathcal{A}(R)$ (also denoted by $\mathrm{Alg}R_+$) of the Lie superalgebra $\mathrm{Alg}R$ spanned by all elements at^n, where $a \in R, n \in \mathbb{Z}_+$. It is clear from (2.1) that this is a subalgebra, which is invariant with respect to the derivation $\partial = -\partial_t$ of $\mathrm{Alg}R$. The *extended annihilation superalgebra* is defined as

$$\mathcal{A}(R)^e = (\mathrm{Alg}R)^+ := \mathbb{C}\partial \ltimes (\mathrm{Alg}R)_+.$$

Introducing the generating series

$$a_\lambda = \sum_{j\in\mathbb{Z}_+} \frac{\lambda^j}{j!}(at^j),\ a \in R, \tag{2.2}$$

we obtain from (2.1):

$$[a_\lambda, b_\mu] = [a_\lambda b]_{\lambda+\mu}, \quad \partial(a_\lambda) = (\partial a)_\lambda = -\lambda a_\lambda. \tag{2.3}$$

Now let $\mathfrak{g}$ be a Lie superalgebra, and let V be a $\mathfrak{g}$-module. Given a $\mathfrak{g}$-valued formal distribution $a(z)$ and a V-valued formal distribution $v(z)$ we may consider the formal distribution $a(z)v(w)$ and the pair $(a(z), v(z))$ is called *local* if $(z-w)^N(a(z)v(w)) = 0$ for $N >> 0$. As before, we have an expansion of the form:

$$a(z)v(w) = \sum_j \Big(a(z)_{(j)}v(w)\Big)\ \partial_w^j \delta(z-w)/j!,$$

where $a(w)_{(j)}v(w) = \mathrm{Res}_z(z-w)^j a(z)v(w)$ and the sum is finite. By taking the generating series of these operations, we define the λ-*action of* $\mathfrak{g}$ *on* V:

$$a(w)_\lambda v(w) = \sum_{n\in\mathbb{Z}_+} \frac{\lambda^n}{n!}\Big(a(w)_{(n)}v(w)\Big), \qquad \text{(finite sum)}.$$

It has the following properties:

$$\partial_z a(z)_\lambda v(z) = -\lambda a(z)_\lambda v(z), \qquad a(z)_\lambda \partial_z v(z) = (\partial_z + \lambda)(a(z)_\lambda v(z)),$$

and

$$[a(z)_\lambda, b(z)_\mu]v(z) = [a(z)_\lambda b(z)]_{\lambda+\mu} v(z).$$

This motivate the following definition:

Definition 2.2. A *module* M over a Lie conformal superalgebra R is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{C}[\partial]$-module endowed with a $\mathbb{C}$-linear map $R\otimes M \longrightarrow \mathbb{C}[\lambda]\otimes M$, $a\otimes v \mapsto a_\lambda v$, satisfying the following axioms $(a,\, b \in R),\ v \in M$,

$(M1)_\lambda$ $\quad (\partial a)^M_\lambda v = [\partial^M, a^M_\lambda]v = -\lambda a^M_\lambda v,$

$(M2)_\lambda$ $\quad [a^M_\lambda, b^M_\mu]v = [a_\lambda b]^M_{\lambda+\mu} v.$

An R-module M is called *finite* if it is finitely generated over $\mathbb{C}[\partial]$. An R-module M is called *irreducible* if it contains no non-trivial submodule, where the notion of submodule is the usual one.

As before, if $\mathcal{F} \subset \mathfrak{g}[[z,z^{-1}]]$ is a local family and $\mathcal{E} \subset V[[z,z^{-1}]]$ is such that all pairs $(a(z), v(z))$, where $a(z) \in \mathcal{F}$ and $v(z) \in \mathcal{E}$, are local, let $\bar{\mathcal{E}}$ be the minimal subspace of $V[[z,z^{-1}]]$ which contains E and all $a(z)_{(j)}v(z)$ for $a(z) \in \mathcal{F}$ and $v(z) \in \mathcal{E}$, and is ∂_z-invariant. Then it is easy to show that all pairs $(a(z), v(z))$, where $a(z) \in \bar{\mathcal{F}}$ and $v(z) \in \bar{\mathcal{E}}$, are local and $a(z)_{(j)}(\bar{\mathcal{E}}) \subset \bar{\mathcal{E}}$ for all $a(z) \in \bar{\mathcal{F}}$.

Let $\mathcal{F}$ be a local family that spans $\mathfrak{g}$ and let $\mathcal{E} \subset V[[z,z^{-1}]]$ be a family that span V. Then $(V,\mathcal{E})$ is called a *formal distribution module* over the formal distribution Lie superalgebra $(\mathfrak{g},\mathcal{F})$ if all pairs $(a(z), v(z))$, where $a(z) \in \mathcal{F}$ and $v(z) \in \mathcal{E}$, are local. It follows that a formal distribution module $(V,\mathcal{E})$ over a formal distribution Lie superalgebra $(\mathfrak{g},\mathcal{F})$ give rise to a module $\mathrm{Conf}(V,\mathcal{E}) := \bar{\mathcal{E}}$ over the conformal Lie superalgebra $\mathrm{Conf}(\mathfrak{g},\mathcal{F})$.

In the same way as above, we have an equivalence of categories of modules over a Lie conformal superalgebra R and equivalence classes or formal distribution modules over the Lie superalgebra Alg$R$. Namely, given an R-module M, one defines $\tilde{M} = M[t, t^{-1}]$ as a $\tilde{R}$-module with the action similar to (2.1):

$$at^n \cdot vt^m = \sum_{j\in\mathbb{Z}_+} \binom{m}{j} (a_j v) t^{m+n-j}. \tag{2.4}$$

Let $\tilde{\partial} = \partial^M + \partial_t$. Observe that $\tilde{\partial}\tilde{M}$ is invariant with respect to the action of $\tilde{R}$ and $(\tilde{\partial}\tilde{R}) \cdot \tilde{M} = 0$, hence the action of $\tilde{R}$ on $\tilde{M}$ induces a representation of the Lie superalgebra Alg$R = \tilde{R}/\tilde{\partial}\tilde{R}$ on $V(M) := \tilde{M}/\tilde{\partial}\tilde{M}$. Let $\mathcal{M} = \{v\delta(z-t) | v \in M\}$. Then $(V(M), \mathcal{M})$ is a formal distribution module over the formal distribution Lie superalgebra (Alg$R, \mathcal{R}$), which is maximal in the sense that all conformal (Alg$R, \mathcal{R}$) modules $(V, \mathcal{E})$ such that $\bar{\mathcal{E}} \simeq M$ as R-modules are quotients of $(V(M), \mathcal{M})$ by irregular submodules. Such formal distribution modules are called equivalent, and we get an equivalence of categories of R-modules and equivalence classes of formal distribution (Alg$R, \mathcal{R}$)-modules.

Formula (2.3) implies the following important proposition relating modules over a Lie conformal superalgebra R to certain modules over the corresponding extended annihilation superalgebra (Alg$R)^+$.

Proposition 2.3. [**4**] *A module over a Lie conformal superalgebra R is the same as a module over the Lie superalgebra* (Alg$R)^+$ *satisfying the property*

$$a_\lambda m \in \mathbb{C}[\lambda] \otimes M \quad \textit{for any } a \in R, m \in M. \tag{2.5}$$

(One just views the action of the generating series a_λ of (Alg$R)^+$ *as the λ-action of $a \in R$).*

The problem of classifying modules over a Lie conformal superalgebra R is thus reduced to the problem of classifying a class of modules over the Lie superalgebra (Alg$R)^+$.

Let $\mathfrak{g}$ be a Lie superalgebra satisfying the following three conditions (cf. [**5**], p.911):

(L1) $\mathfrak{g}$ is $\mathbb{Z}$-graded of finite depth $d \in \mathbb{N}$, i.e. $g = \oplus_{j\geq -d}\mathfrak{g}_j$ and $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$.

(L2) There exists a semisimple element $z \in \mathfrak{g}_0$ such that its centralizer in $\mathfrak{g}$ is contained in $\mathfrak{g}_0$.

(L3) There exists an element $\partial \in \mathfrak{g}_{-d}$ such that $[\partial, \mathfrak{g}_i] = \mathfrak{g}_{i-d}$, for $i \geq 0$.

Some examples of Lie superalgebras satisfying (L1)-(L3) are provided by annihilation superalgebras of Lie conformal superalgebras.

If $\mathfrak{g}$ is the annihilation superalgebra of a Lie conformal superalgebra, then the modules V over $\mathfrak{g}$ that correspond to finite modules over the corresponding Lie conformal superalgebra satisfy the following conditions:

(1) For all $v \in V$ there exists an integer $j_0 \geq -d$ such that $\mathfrak{g}_j v = 0$, for all $j \geq j_0$.

(2) V is finitely generated over $\mathbb{C}[\partial]$.

Motivated by this, the $\mathfrak{g}$-modules satisfying these two properties are called *finite conformal modules.*

We have a triangular decomposition

$$\mathfrak{g} = \mathfrak{g}_{<0} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{>0}, \qquad \text{with } g_{<0} = \oplus_{j<0}\mathfrak{g}_j,\ g_{>0} = \oplus_{j>0}\mathfrak{g}_j. \tag{2.6}$$

Let $\mathfrak{g}_{\geq 0} = \oplus_{j\geq 0}\mathfrak{g}_j$. Given a $\mathfrak{g}_{\geq 0}$-module F, we may consider the associated induced $\mathfrak{g}$-module

$$\mathrm{Ind}(F) = \mathrm{Ind}_{\mathfrak{g}_{\geq 0}}^{\mathfrak{g}} F = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\geq 0})} F,$$

called the *generalized Verma module* associated to F. We shall identify $\mathrm{Ind}(F)$ with $U(\mathfrak{g}_{<0}) \otimes F$ via the PBW theorem.

Let V be an $\mathfrak{g}$-module. The elements of the subspace

$$\mathrm{Sing}(V) := \{v \in V | \mathfrak{g}_{>0} v = 0\}$$

are called *singular vectors*. For us the most important case is when $V = \mathrm{Ind}(F)$. The $\mathfrak{g}_{\geq 0}$-module F is canonically an $\mathfrak{g}_{\geq 0}$-submodule of $\mathrm{Ind}(F)$, and $\mathrm{Sing}(F)$ is a subspace of $\mathrm{Sing}(\mathrm{Ind}(F))$, called the *subspace of trivial singular vectors*. Observe that $\mathrm{Ind}(F) = F \oplus F_+$, where $F_+ = U_+(\mathfrak{g}_{<0}) \otimes F$ and $U_+(\mathfrak{g}_{<0})$ is the augmentation ideal of the algebra $U(\mathfrak{g}_{<0})$. Then non-zero elements of the space

$$\mathrm{Sing}_+(\mathrm{Ind}(F)) := \mathrm{Sing}(\mathrm{Ind}(F)) \cap F_+$$

are called *non-trivial singular vectors*. The following key result will be used in the classification of irreducible modules.

Theorem 2.4. [**10, 12, 5, 2**] *Let $\mathfrak{g}$ be a Lie superalgebra that satisfies (L1)-(L3).*

(a) If F is an irreducible finite-dimensional $\mathfrak{g}_{\geq 0}$-module, then the subalgebra $\mathfrak{g}_{>0}$ acts trivially on F and $\mathrm{Ind}(F)$ has a unique maximal submodule.

(b) Denote by $\mathrm{Ir}(F)$ the quotient by the unique maximal submodule of $\mathrm{Ind}(F)$. Then the map $F \mapsto \mathrm{Ir}(F)$ defines a bijective correspondence between irreducible finite-dimensional $\mathfrak{g}_0$-modules and irreducible finite conformal $\mathfrak{g}$-modules.

(c) A $\mathfrak{g}$-module $\mathrm{Ind}(F)$ is irreducible if and only if the $\mathfrak{g}_0$-module F is irreducible and $\mathrm{Ind}(F)$ has no non-trivial singular vectors.

This Theorem together with the following results will provide a characterization of all finite irreducible modules over a finite Lie conformal superalgebra in terms of certain (quotients of) induced modules over the extended annihilation algebra.

Denote by $V(M)_+$ the span of elements $\{vt^n | v \in M, n \in \mathbb{Z}_+\}$ in $V(M)$. It is clear from (2.4) that $V(M)^+$ is an $(\mathrm{Alg}R)^+$ submodule, hence an R -module by Proposition 2.3. We denote by $V(M)_+^*$ the restricted dual of $V(M)_+$, i.e. the space of all linear functions on $V(M)_+$ which vanish on all but finite number of subspaces Mt^n, with $n \in \mathbb{Z}_+$. This is an $(\mathrm{Alg}R)^+$-module and hence an R-module as well. The *conformal dual* M^* to an R-module M is defined as

$$M^* = \{f_\lambda : M \to \mathbb{C}[\lambda] \mid f_\lambda(\partial m) = \lambda f_\lambda(m)\},$$

with the structure of $\mathbb{C}[\partial]$-module $(\partial f)_\lambda(m) = -\lambda f_\lambda(m)$, with the following λ-action of R:

$$(a_\lambda f)_\mu(m) = -(-1)^{p(a)(p(f)+1)} f_{\mu-\lambda}(a_\lambda m), \quad a \in R, m \in M.$$

Given a homomorphism of conformal R-modules $T : M \to N$, we define the transpose homomorphism $T^* : N^* \to M^*$ by

$$[T^*(f)]_\lambda(m) = -f_\lambda(T(m))$$

Proposition 2.5. *Let $T : M \to N$ be an injective homomorphism of R-modules such that $N/Im\, T$ is finitely generated torsion-free $\mathbb{C}[\partial]$-module. Then T^* is surjective.*

Proof. Since $N/\mathrm{Im}\ T$ is finitely generated torsion-free, then it is free and therefore a proyective $\mathbb{C}[\partial]$-module. Hence, the short exact sequence $0 \to \mathrm{Im}\ T \to N \to N/\mathrm{Im}\ T \to 0$ is split and $N = \mathrm{Im}\ T \oplus L$ as $\mathbb{C}[\partial]$-module. Now, given $\alpha \in M^*$, we define $\beta \in N^*$ as follows

$$\beta_\lambda(T(m)) = \alpha_\lambda(m), \quad m \in M, \qquad \beta_\lambda(l) = 0, \quad l \in L.$$

Then β is well-defined since T is injective and β belong to N^* since L is a complementary $\mathbb{C}[\partial]$-submodule, finishing the proof. $\square$

Remark 2.6. Observe that the injectivity is not enough (cf. Remark 4.10). Namely, let $R = Vir = \mathbb{C}[\partial]L$ the Virasoro conformal algebra with λ-bracket $[L_\lambda L] = (2\lambda + \partial)L$. Consider the following Vir-modules:

$$\Omega_0 = \mathbb{C}[\partial]m, \ \text{ with } L_\lambda m = (\lambda + \partial)m; \qquad \Omega_1 = \mathbb{C}[\partial]n, \ \text{ with } L_\lambda n = \partial n.$$

Then it is easy to see that the map $d : \Omega_0 \to \Omega_1$ given by $d(m) = \partial n$ is an injective homomorphism of R-modules, but the dual map $d^* : \Omega_1^* \to \Omega_0^*$ given by $d^*(m^*) = \partial n^*$ is not surjective.

The nest result follows by standard arguments using Proposition 2.5.

Proposition 2.7. *Let $T : M \to N$ be a homomorphism of R-modules such that $N/Im\ T$ is finitely generated torsion-free $\mathbb{C}[\partial]$-module. Then the standard map $\psi : N^*/Ker\ T^* \to (M/Ker\ T)^*$, given by $[\psi(\bar{f})]_\lambda(\bar{m}) = f_\lambda(T(m))$ (where by the bar we denote the corresponding class in the quotient) is an isomorphism of R-modules.*

Proposition 2.8. *If M is an R-module finitely generated (over $\mathbb{C}[\partial]$), then $M^{**} \simeq M$.*

Proof. Let $M = \oplus\ \mathbb{C}[\partial]m_i$ (finite sum), with $a_\lambda m_j = \sum_k P_{jk}(\lambda, \partial)m_k$. Then $M^* = \oplus\ \mathbb{C}[\partial]m_i^*$, with $(m_i^*)_\lambda(m_k) = \delta_{i,k}$ and

$$(a_\lambda m_i^*)_\mu(m_j) = -(m_i^*)_{\mu-\lambda}(a_\lambda m_j) = -\sum_k (m_i^*)_{\mu-\lambda}(P_{jk}(\lambda,\partial)m_k) = P_{ji}(\lambda, \mu - \lambda).$$

Therefore,

$$(a_\lambda m_i^*) = -\sum_j P_{ji}(\lambda, -\partial - \lambda)m_j^*,$$

and the last formula shows that by taking the dual again we obtain

$$(a_\lambda m_i^{**}) = \sum_j P_{ij}(\lambda, \partial)m_j^{**}.$$

Hence the map $m_i \mapsto m_i^{**}$ gives us the isomorphism between M and M^{**}. $\square$

Proposition 2.9. *(a) The map $M \longrightarrow V(M)/V(M)_+$ given by $v \mapsto vt^{-1}$ mod $V(M)_+$ is an isomorphism of* $(\mathrm{Alg}R)^+$*- (and R-)modules.*
(b)The map $V(M)_+^* \to M^*$ defined by $f \mapsto f_\lambda$, where

$$f_\lambda(m) = \sum_{j \in \mathbb{Z}_+} \frac{(-\lambda)^j}{j!} f(mt^j),$$

is an isomorphism of $(\mathrm{Alg}R)^+$*- (and R-)modules.*

3. Lie conformal superalgebra K_0, K_1 and $\mathrm{Cur}\,\mathfrak{g}$ and its finite irreducible representations.

For the general definition of the family of conformal Lie superalgebras K_n, see Section 7. In this section we present the conformal Lie superalgebra K_0, that is the Virasoro conformal algebra Vir associated to the (centerless) Virasoro Lie algebra, we define K_1 that corresponds to the (centerless) Neveu-Schwarz algebra and we define the current conformal superalgebras. In all cases, we give the classification of its finite irreducible modules. We will follow the presentation in [**4**], where all the results of this section were obtained.

Consider the (centerless) *Virasoro algebra*, that is the Lie algebra $\mathrm{Vect}\ \mathbb{C}^\times$ of regular vector fields on $\mathbb{C}$, with basis $L_n = -t^{n+1}\dfrac{d}{dt}$ $(n \in \mathbb{Z})$ and commutation relations

$$[L_m, L_n] = (m-n)L_{m+n}.$$

It is a formal distribution Lie algebra spanned by the local formal distribution $L(z) = \sum_{n\in\mathbb{Z}} L_n\, z^{-n-2} = -\delta(t-z)\dfrac{d}{dt}$, since one has:

$$[L(z), L(w)] = \partial_w L(w)\delta(z-w) + 2L(w)\partial_w\delta(z-w).$$

The corresponding conformal algebra is the *Virasoro conformal algebra* $Vir = \mathbb{C}[\partial]\otimes_{\mathbb{C}}\mathbb{C}L = \mathbb{C}[\partial]L$ with λ-bracket (on generator):

$$[L_\lambda L] = (2\lambda + \partial)L.$$

This conformal algebra is also called K_0. In this case the annihilation algebra is $\mathcal{A}(Vir) = \sum_{n\geq -1}\mathbb{C}L_n$, and the extended annihilation algebra is $\mathcal{A}(Vir)^e = \mathbb{C}\partial \ltimes \mathcal{A}(Vir) = \mathbb{C}(\partial - L_{-1}) \oplus \mathcal{A}(Vir)$ a direct sum of ideals, since ∂ acts as $\mathrm{ad}\, L_{-1}$. Hence, in this case the study of conformal modules reduces to the study of conformal modules over $\mathcal{A}(Vir)$. The following result gives the classification of the irreducible modules.

PROPOSITION 3.1. [**4**] *Let $\mathcal{A}(Vir)_{\geq 0} = \sum_{n\geq 0}\mathbb{C}L_n$. Then any non-trivial irreducible conformal module of $\mathcal{A}(Vir)$ is of the form $\mathrm{Ind}_{\mathcal{A}(Vir)_{\geq 0}}^{\mathcal{A}(Vir)}\mathbb{C}_\Delta$, where $\mathbb{C}_\Delta$ is a non-trivial one-dimensional irreducible module over $\mathcal{A}(Vir)_{\geq 0}$ on which L_0 acts as $\Delta \in \mathbb{C}^\times$ and L_j acts as 0 for all $j > 0$.*

Extending these induced modules to the extended annihilation algebra and translating them into the language of conformal modules over Vir, we obtain a 2-parameter family of non-trivial conformal modules given by $M(\Delta, \alpha) = \mathbb{C}[\partial]m$, with λ-action defined (on generators) as

$$L_\lambda m = (\Delta\lambda + \partial + \alpha)m,$$

where $\Delta \in \mathbb{C}^\times$ and $\alpha \in \mathbb{C}$. Using the previous Proposition, we obtain the complete classification:

THEOREM 3.2. [**4**] *The modules $M(\Delta, \alpha)$ with $\Delta \in \mathbb{C}^\times$ and $\alpha \in \mathbb{C}$ are all the finite non-trivial irreducible modules over Vir.*

Translating back to the language of Lie algebras of formal distributions, consider the representations of the Lie algebra $\mathrm{Vect}\ \mathbb{C}^\times$ on the following space of densities:

$$F(\Delta, \alpha) = \mathbb{C}[t, t^{-1}]e^{-\alpha t}(dt)^{1-\Delta},$$

with action defined as follows ($f(t) \in \mathbb{C}[t,t^{-1}], g(t) \in \mathbb{C}[t,t^{-1}]e^{-\alpha t}$):

$$(f(t)\frac{d}{dt}) \cdot (g(t)(dt)^{1-\Delta}) = \big(f(t)\frac{d}{dt}g(t) + (1-\Delta)g(t)\frac{d}{dt}f(t)\big)(dt)^{1-\Delta}.$$

Taking the $F(\Delta,\alpha)$-valued formal distribution

$$m(z) := \sum_{n\in\mathbb{Z}} (t^n e^{-\alpha t}(dt)^{1-\Delta})z^{-n-1} = \delta(t-z)e^{-\alpha t}(dt)^{1-\Delta},$$

and recalling that $L(z) = -\delta(t-z)\frac{d}{dt}$ we have that

$$L(z)m(w) = ((\partial_w + \alpha)m(w))\delta(z-w) + \Delta m(w)\partial_w\delta(z-w).$$

Hence $(F(\Delta,\alpha), \{m(z)\})$ is the formal distribution module over the formal distribution Lie algebra $(\mathrm{Vect}\ \mathbb{C}^\times, \{L(z)\})$, that corresponds to the conformal module $M(\Delta,\alpha)$ over Vir.

The simplest superextension of the Virasoro algebra is the well-known (centerless) *Neveu-Schwarz algebra* $\mathfrak{N}$ which, apart from even basis elements L_n, has odd basis elements G_r, $r \in \frac{1}{2} + \mathbb{Z}$, with commutation relations:

$$[G_r, L_n] = \left(r - \frac{n}{2}\right)G_{r+n}, \qquad [G_r, G_s] = 2L_{r+s}.$$

The Lie conformal superalgebra, associated to $\mathfrak{N}$, is the *Neveu-Schwarz (or N=1) conformal superalgebra*

$$K_1 = R(\mathfrak{N}) := \mathbb{C}[\partial]L + C[\partial]G$$

where the generator L is even, the generator G is odd and the λ-bracket on generators is given by:

$$[L_\lambda L] = (\partial + 2\lambda)L, \quad [L_\lambda G] = (\partial + \frac{3}{2}\lambda)G, \qquad [G_\lambda L] = \frac{1}{2}(\partial + 3\lambda)G, \quad [G_\lambda G] = 2L.$$

The corresponding annihilation algebra in this case is

$$\mathfrak{N}_+ := \mathcal{A}(K_1) = \sum_{n\geq -1}\mathbb{C}L_n + \sum_{r\geq -\frac{1}{2}}\mathbb{C}G_r.$$

and the extended annihilation algebra $\mathcal{A}(K_1)^e = \mathbb{C}\partial \ltimes \mathcal{A}(K_1) = \mathbb{C}(\partial - L_{-1}) \oplus \mathfrak{N}_+$ is a direct sum of ideals, since ∂ acts as $\mathrm{ad}\, L_{-1}$. Hence, in this case the study of conformal modules reduces to the study of conformal modules over $\mathfrak{N}_+$. The following result gives the classification of the irreducible modules.

Proposition 3.3. [**4**] *Let $\mathfrak{N}_{\geq 0} = \sum_{n\geq 0}\mathbb{C}L_n + \sum_{r\geq\frac{1}{2}}\mathbb{C}G_r$. Then any non-trivial irreducible conformal module of $\mathfrak{N}_+$ is of the form $Ind_{\mathfrak{N}_{\geq 0}}^{\mathfrak{N}_+}\mathbb{C}_\Delta$, where $\mathbb{C}_\Delta$ is a non-trivial one-dimensional irreducible module over $\mathfrak{N}_{\geq 0}$ on which L_0 acts as $\Delta \in \mathbb{C}^\times$ and L_j and G_k acts trivially for all $j,k > 0$.*

Extending these induced modules to the extended annihilation algebra and translating them into the language of conformal modules over K_1, we obtain a 2-parameter family of non-trivial conformal modules given by $N(\Delta,\alpha) = \mathbb{C}[\partial]m \oplus \mathbb{C}[\partial]\tilde{m}$, with λ-action defined (on generators) as

$$L_\lambda m = (\Delta\lambda + \partial + \alpha)m, \qquad L_\lambda \tilde{m} = \left((\Delta + \frac{1}{2})\lambda + \partial + \alpha\right)\tilde{m},$$

$$G_\lambda m = \tilde{m}, \qquad\qquad G_\lambda \tilde{m} = (2\Delta\lambda + \partial + \alpha)m,$$

where $\Delta \in \mathbb{C}^\times$ and $\alpha \in \mathbb{C}$. Using the previous proposition, we obtain the complete classification:

Theorem 3.4. [**4**] *The modules $N(\Delta, \alpha)$ with $\Delta \in \mathbb{C}^\times$ and $\alpha \in \mathbb{C}$ are all the finite non-trivial irreducible modules over K_1.*

Let be $\mathfrak{g}$ a finite-dimensional Lie superalgebra. The *current superalgebra* $\tilde{\mathfrak{g}}$ associated to $\mathfrak{g}$ is :

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes C[t, t^{-1}]$$

with bracket given by $[a \otimes t^n, b \otimes t^m] = [a, b] \otimes t^{n+m}$. It is a Lie superalgebra of formal distributions spanned by the following family of pairwise local formal distributions ($a \in \mathfrak{g}$):

$$a(z) = \sum_{n \in \mathbb{Z}} (a \otimes t^n) z^{-n-1}.$$

Indeed, it is immediate to check that

$$[a(z), b(w)] = [a, b](w)\delta(z - w).$$

The corresponding conformal superalgebra is the *current conformal superalgebra* associated to a finite-dimensional Lie superalgebra $\mathfrak{g}$:

$$\text{Cur } \mathfrak{g} := R(\tilde{\mathfrak{g}}) = C[\partial] \otimes_{\mathbb{C}} \mathfrak{g},$$

with the λ-bracket defined on generators by:

$$[a_\lambda b] = [a, b], \qquad \text{for } a,\, b \in \mathfrak{g}.$$

In this case the annihilation algebra is

$$\mathfrak{g}[t] := \mathcal{A}(\text{Cur } \mathfrak{g}) = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t],$$

and the extended annihilation algebra is

$$\tilde{\mathfrak{g}}^+ := \mathcal{A}(\text{Cur } \mathfrak{g})^e = \mathbb{C}\frac{d}{dt} \ltimes \mathfrak{g}[t].$$

Let π be a representation of $\mathfrak{g}[t]$ in a finite-dimensional vector space U, such that $(t^n \otimes \mathfrak{g})U = 0$ for $n >> 0$. This defines on the space $U \otimes \mathbb{C}[t, t^{-1}]$ the structure of a conformal module over $\tilde{\mathfrak{g}}$ by the formula:

$$(a \otimes t^m)(u \otimes t^n) = \sum_{j \in \mathbb{Z}} \binom{m}{j} (\pi(a \otimes t^j)u) \otimes t^{m+n-j}.$$

A special case of this construction is to take a finite-dimensional representation π of the Lie superalgebra $\mathfrak{g}$ in a finite-dimensional vector space U and extend it to $\mathfrak{g}[t]$ by letting $\mathfrak{g} \otimes t\mathbb{C}[t]$ act trivially. Then we have $(a \otimes t^m)(u \otimes t^n) = (\pi(a)u) \otimes t^{m+n}$. In other words, $U \otimes \mathbb{C}[t, t^{-1}]$ is a formal distribution module over the formal distribution Lie algebra $\tilde{\mathfrak{g}}$, since

$$a(z)u(w) = (\pi(a)u)(w)\delta(z - w),$$

where $u(z) = \sum_{n\in\mathbb{Z}}(a\otimes t^n)z^{-n-1}$ for all $u \in U$. Translating back to the language of modules over the conformal algebra Cur $\mathfrak{g}$ we obtain the free $\mathbb{C}[\partial]$-module $M(U) := \mathbb{C}[\partial] \otimes_{\mathbb{C}} U$ with

$$a_\lambda u = \pi(a)u,$$

where $a \in \mathfrak{g}$ and $u \in U$.

Proposition 3.5. [**4**] *Let $\mathfrak{g}$ be a simple finite dimensional Lie superalgebra different from the series $A(m|n)$, for $m \neq n$, $C(n)$ and $W(n)$ (see [**9**]). Let $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ and $\tilde{\mathfrak{g}}^+ = \mathbb{C}\frac{d}{dt} \ltimes \mathfrak{g}[t]$. Then every non-trivial irreducible conformal module over $\tilde{\mathfrak{g}}^+$ is of the form $Ind_{\mathfrak{g}[t]}^{\tilde{\mathfrak{g}}^+} U$, where U is finite dimensional non-trivial irreducible $\mathfrak{g}$-module or else it is the trivial $\mathfrak{g}[t]$-module on which $\dfrac{d}{dt}$ acts as a non-zero scalar.*

Using Proposition 3.5, we obtain the classification:

Theorem 3.6. [**4**] *Let $\mathfrak{g}$ be a simple finite dimensional Lie superalgebra different from the series $A(m|n)$, for $m \neq n$, $C(n)$ and $W(n)$. The modules $M(U)$ where U is finite dimensional non-trivial irreducible $\mathfrak{g}$-module are all the finite non-trivial irreducible modules over Cur $\mathfrak{g}$.*

For the rest of this section, we assume that the simple finite dimensional Lie superalgebra $\mathfrak{g}$ is a member of one of the series $A(m|n)$, for $m \neq n$, $C(n)$ and $W(n)$. In this case, in order to classify irreducible conformal modules over their current algebras, it suffices to consider finite-dimensional irreducible representations of the Lie superalgebra $\mathfrak{g} \otimes \mathbb{C}[t]/t^{n+1}$, where $n \geq 1$ (see [**4**] for details).

Note that it follows from the descriptions of these three series of simple Lie superalgebras in [**9**] that they satisfy the following properties:

(a) $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_l$ is $\mathbb{Z}$-graded such that $\mathfrak{g}_i \subseteq \mathfrak{g}_{\bar{i}}$.

(b) $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathbb{C}c$ is a reductive Lie algebra such that $\mathfrak{a}$ is a semisimple subalgebra and c is a central element.

(c) $\mathfrak{g}_i$ as an $\mathfrak{a}$-module has no trivial summand for $i \neq 0$, and there exists an $\mathfrak{a}$-submodule $\mathfrak{g}_{-1}^* \subseteq \mathfrak{g}_1$ contragradient to $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1 = g_{-1}^* \oplus g_1'$ as $\mathfrak{a}$-modules with $[\mathfrak{g}_1', \mathfrak{g}_{-1}] \subseteq \mathfrak{a}$.

(d) For any non-zero $a \in \mathfrak{g}_{-1}$ and $b \in \mathfrak{g}_{-1}^*$, we have $[a, \mathfrak{g}_{-1}^*] \cap \mathbb{C}c \neq 0$ and $[b, \mathfrak{g}_{-1}] \cap \mathbb{C}c \neq 0$.

Let $\mathfrak{L} = \mathfrak{g} \otimes \mathbb{C}[t]/t^{n+1}$. We want to classify finite-dimensional irreducible $\mathfrak{L}$-modules on which $\mathfrak{g} \otimes t^n$ acts non-trivially. We take $G_0 = \mathfrak{g}_0 + \mathfrak{g}_1' + \sum_{i\geq 2} \mathfrak{g}_i$, and consider the subalgebra $L \subseteq \mathfrak{L}$ defined as follows: If $n = 2k$ $(k \in \mathbb{N})$, we take

$$L = G_0 + G_0 \otimes \mathbb{C}t + \cdots + G_0 \otimes \mathbb{C}t^{k-1} + (G_0 + \mathfrak{g}_1) \otimes \mathbb{C}t^k + \mathfrak{g} \otimes \mathbb{C}t^{k+1} + \cdots + \mathfrak{g} \otimes \mathbb{C}t^{2k}.$$

If $n = 2k + 1$ $(k \in \mathbb{N})$, we take

$$L = G_0 + G_0 \otimes \mathbb{C}t + \cdots + G_0 \otimes \mathbb{C}t^k + \mathfrak{g} \otimes \mathbb{C}t^{k+1} + \cdots + \mathfrak{g} \otimes \mathbb{C}t^{2k+1}.$$

Now, the following result characterizes finite-dimensional irreducible L-modules, and it turns out that every irreducible $\mathfrak{L}$-module on which $\mathfrak{g} \otimes t^n$ acts non-trivially are obtained by inducing from a suitable irreducible L-module.

PROPOSITION 3.7. [**4**] *Every irreducible L-module is an irreducible $\mathfrak{g}_0 \oplus (\mathbb{C}c \otimes \mathbb{C}[t]t)$-module, on which $\mathfrak{g}_1' + \sum_{i\geq 2} \mathfrak{g} + (\mathfrak{g}^c \otimes \mathbb{C}[t]t) \cap L$ acts trivially, where $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{a} + \sum_{i\geq 1} \mathfrak{g}_i$.*

THEOREM 3.8. [**4**] *All finite-dimensional irreducible representations of $\mathfrak{L}$, on which $\mathfrak{g} \otimes t^n$ acts non-trivially, are of the form $\mathrm{Ind}_L^{\mathfrak{L}} V_L$, where V_L is an irreducible representation of L on which $c \otimes t^n$ acts as a non-zero scalar. Furthermore, all such representations are irreducible.*

The corresponding conformal modules produce the classification of all irreducible finite conformal modules over Cur $\mathfrak{g}$, where $\mathfrak{g}$ is in one of the three families.

In the following sections we shall describe most of the remaining Lie conformal superalgebras and their corresponding annihilation superalgebras, and we shall study the induced modules over the corresponding anihilation algebra and its singular vectors in order to apply Theorem 2.4 to get the classification of irreducible finite modules over the Lie conformal superalgebras.

4. Lie conformal superalgebra W_n and finite irreducible representations.

4.1. Definition of W_n. According to [**6**], any finite simple Lie conformal algebra is isomorphic either to the current conformal algebra of a simple finite-dimensional Lie algebra, or to the Virasoro conformal algebra. However, the list of finite simple Lie conformal superalgebras is much richer, mainly due to existence of several series of super extensions of the Virasoro conformal algebra, see [**7**]. The results of this section where obtained in [**2**].

The first series is associated to the Lie superalgebra $W(1,n)$ $(n \geq 1)$. More precisely, let $\Lambda(n)$ be the Grassmann superalgebra in the n odd indeterminates $\xi_1, \xi_2, \dots, \xi_n$. Set $\Lambda(1,n) = \mathbb{C}[t, t^{-1}] \otimes \Lambda(n)$, then

$$W(1,n) = \{a\partial_t + \sum_{i=1}^{n} a_i \partial_i | a, a_i \in \Lambda(1,n)\}, \tag{4.1}$$

where $\partial_i = \frac{\partial}{\partial \xi_i}$ and $\partial_t = \frac{\partial}{\partial t}$ are odd and even derivations respectively. Then $W(1,n)$ is a formal distribution Lie superalgebra with spanning family of (pairwise local) formal distributions:

$$\mathcal{F} = \{\delta(t-z)a \mid a \in W(n)\} \cup \{\delta(t-z)f\partial_t \mid f \in \Lambda(n)\}.$$

where $W(n) = \{\sum_{i=1}^{n} a_i\partial_i | a_i \in \Lambda(n)\}$ is the (finite-dimensional) Lie superalgebra of all derivations of $\Lambda(n)$. The associated Lie conformal superalgebra W_n is defined as

$$W_n = \mathbb{C}[\partial] \otimes (W(n) \oplus \Lambda(n)). \tag{4.2}$$

The λ-bracket is defined as follows $(a, b \in W(n); f, g \in \Lambda(n))$:

$$[a_\lambda b] = [a,b], \quad [a_\lambda f] = a(f) - (-1)^{p(a)p(f)}\lambda f a, \quad [f_\lambda g] = -(\partial + 2\lambda) f g. \tag{4.3}$$

The Lie conformal algebra W_n is simple for $n \geq 0$ and has rank $(n+1)2^n$.

The annihilation subalgebra is

$$\mathcal{A}(W_n) = W(1,n)_+ = \{a\partial_t + \sum_{i=1}^{n} a_i\partial_i | a, a_i \in \Lambda(1,n)_+\}, \tag{4.4}$$

where $\Lambda(1,n)_+ = \mathbb{C}[t] \otimes \Lambda(n)$. The extended annihilation subalgebra is

$$\mathcal{A}(W_n)^e = W(1,n)^+ = \mathbb{C}\partial_t \ltimes W(1,n)_+,$$

and therefore it is isomorphic to the direct sum of $W(1,n)_+$ and a commutative 1-dimensional Lie algebra.

The $\mathbb{Z}$-gradation is obtained by letting

$$\deg\, t = \deg\, \xi_i = 1 = -\deg\, \partial_t = -\deg\, \partial_i.$$

If $\mathfrak{g} = W(1,n)_+$, then $\mathfrak{g}_{-1} =< \partial_t, \partial_1, \ldots, \partial_n >$, where ∂_t is an even element and $\partial_1, \ldots, \partial_n$ are odd elements of a basis in $\mathfrak{g}_{-1}$. Note also that $\mathfrak{g}_0 \simeq gl(1|n)$.

From now on, we shall use the notation $\partial_0 = \partial_t$. Explicitly, we have

$$\mathfrak{g}_0 =< \{t\partial_i, \xi_i\partial_j \ : \ 0 \leq i,j \leq n\} > .$$

In order to write explicitly weights for vectors in $W(1,n)_+$-modules, we would consider the basis

$$t\partial_0; t\partial_0 + \xi_1\partial_1, \ldots, t\partial_0 + \xi_n\partial_n$$

for the Cartan subalgebra H in $W(1,n)_+$, and we write the weight of an eigenvector for the Cartan subalgebra H as a tuple

$$\bar{\mu} = (\mu; \lambda_1, \ldots, \lambda_n)$$

for the corresponding eigenvalues of the basis.

4.2. Modules of Laurent differential forms. 4.2.1 *Restricted dual.* Our algebra $\mathfrak{g} = W(1,n)_+$, and in the next section $S(1,n)_+$, are $\mathbb{Z}$-graded (super)algebras and the modules we intend to study are graded modules, i.e. an $\mathfrak{g}$-module V is a direct sum $V = \oplus_{m\in\mathbb{Z}} V_m$ of finite-dimensional subspaces V_m, and $\mathfrak{g}_k \cdot V_m \subset V_{k+m}$. For a graded module V we define the *restricted dual module* $V^{\#}$ as

$$V^{\#} = \oplus_{m\in\mathbb{Z}} (V_m)^*.$$

hence $V^{\#}$ is a subspace of V^* and it is invariant with respect to the contragradient action, so it defines an $\mathfrak{g}$-module structure. Observe that $(V^{\#})^{\#} = V$.

In our situation, we have $\mathfrak{g}_{-1} = \langle \partial_0, \partial_1, \ldots, \partial_n \rangle$, then any $\mathfrak{g}$-module become a $\mathbb{C}[\partial_0, \partial_1, \ldots, \partial_n]$-module. Hence, a module V is a free $\mathbb{C}[\partial_0, \partial_1, \ldots, \partial_n]$-module if and only if $V^{\#}$ is a cofree module, i.e. it is isomorphic to a direct sum of copies of the standard module $\mathbb{C}[z, \rho_1, \ldots, \rho_n]$, with $\partial_0 \cdot f = \frac{\partial}{\partial z} f$, and $\partial_i \cdot f = \frac{\partial}{\partial \rho_i} f$.

An induced module $\mathrm{Ind}_{\mathfrak{g}_{\geq 0}}^{\mathfrak{g}} F$ is by definition a free $\mathbb{C}[\partial_0, \partial_1, \ldots, \partial_n]$-module, so the co-induced (or produced) module

$$\mathrm{Cnd} F^{\#} = (\mathrm{Ind} F)^{\#}$$

will be cofree.

4.2.2 *Modules of differential forms* . In order to define the differential forms one considers an odd variable dt and even variables $d\xi_1, \ldots, d\xi_n$ and defines the differential forms to be the (super)commutative algebra freely generated by these variables over $\Lambda(1,n)_+ = \mathbb{C}[t] \otimes \Lambda(n)$, or

$$\Omega_+ = \Omega_{n,+} = \Lambda(1,n)_+[d\xi_1, \ldots, d\xi_n] \otimes \Lambda[dt].$$

Generally speaking Ω_+ is just a polynomial (super)algebra over a big set of variables

$$t, \xi_1, \dots, \xi_n, dt, d\xi_1, \dots, d\xi_n,$$

where the parity is

$$p(t) = 0, \quad p(\xi_i) = 1, \quad p(dt) = 1, \quad p(d\xi_i) = 0.$$

These are called *(polynomial) differential forms*, and we define the *Laurent differential forms* to be the same algebra over $\Lambda(1,n) = \mathbb{C}[t, t^{-1}] \otimes \Lambda(n)$:

$$\Omega = \Lambda(1,n)[d\xi_1, \dots, d\xi_n] \otimes \Lambda[dt].$$

We would like to consider a fixed complementary subspace Ω_- to Ω_+ in Ω chosen as follows

$$\Omega_- = t^{-1}\mathbb{C}[t^{-1}] \otimes \Lambda(n) \otimes \mathbb{C}[d\xi_1, \dots, d\xi_n] \otimes \Lambda[dt].$$

For the differential forms we need the usual differential degree that measure only the involvement of the differential variables $dt, d\xi_1, \dots, d\xi_n$, that is

$$\deg\, t = 0, \ \deg\, \xi_i = 0, \ \deg\, dt = 1, \ \deg\, d\xi_i = 1.$$

As a result, the degree of a function is zero an it gives us the *standard $\mathbb{Z}$-gradation* both on Ω and $\Omega_\pm$. As usual, we denote by $\Omega^k, \Omega^k_\pm$ the corresponding graded components.

We denote by Ω^k_c the special subspace of differential forms with constant coefficients in Ω_k.

The operator d is defined on Ω as usual by the rules $d \cdot t = dt, d \cdot \xi_i = d\xi_i, d \cdot d\xi_i = 0$, and the identity

$$d(fg) = (df)g + (-1)^{p(f)} f dg,$$

Observe that d maps both Ω_+ and Ω_- into themselves.

As usual, we extend the natural action of $W(1,n)_+$ on $\Lambda(1,n)$ to the whole Ω by imposing the property

$$D \cdot d = (-1)^{p(D)} d \cdot D, \qquad D \in W(1,n)_+,$$

that is, D (super)commutes with d. It is clear that Ω_+ and all the subspaces Ω^k are invariant. Hence Ω^k_+ and Ω^k are $W(1,n)_+$-modules, which are called the *natural representations* of $W(1,n)_+$ in differential forms.

We define the action of $W(1,n)_+$ on Ω_- via the isomorphism of Ω_- with the factor of Ω by Ω_+. Practically this means that in order to compute $D \cdot f$, where $f \in \Omega_-$, we apply D to f and "disregard terms with non-negative powers of t".

The operator d restricted to $\Omega^k_\pm$ defines an odd morphism between the corresponding representations. Clearly the image and the kernel of such a morphism are submodules in $\Omega^k_\pm$.

Let $\Theta^k_c = (\Omega^k_c)^\#$ and $\Theta^k_+ = (\Omega^k_+)^\#$. In the rest of this subsection, we consider $\mathfrak{g} = W(1,n)_+$.

Proposition 4.1. *For $\mathfrak{g} = W(1,n)_+$ we have:*
(1) The $\mathfrak{g}_0$-module $\Theta^k_c, k \geq 0$ is irreducible with highest weight

$$(0; 0, \dots, 0, -k), \quad k \geq 0.$$

(2) The $\mathfrak{g}$-module Θ^k_+, $k \geq 0$ contains Θ^k_c and this inclusion induces the isomorphism

$$\Theta^k_+ = \text{Ind } \Theta^k_c.$$

(3) The dual maps $d^\# : \Theta^{k+1}_+ \to \Theta^k_+$ are morphisms of $\mathfrak{g}$-modules. The kernel of one of them is equal to the image of the next one and it is a non-trivial proper submodule in Θ^k_+.

Proof. (1) It is well known that Ω^k_c are irreducible and thus Θ^k_+ are also irreducible. Observe that the lowest vector in Ω^k_c is $(d\xi_n)^k$ and it has the weight $(0; 0, \ldots, 0, k)$. Now the sign changes as we go to the dual module and so we get the highest weight of Θ^k_c.

(2) By the definition of the restricted dual, it is the sum of the dual of all the graded components of the initial module. In our case Ω^k_c is the component of the minimal degree in Ω^k_+, so Θ^k_c becomes the component of the maximal degree in Θ^k_+. This implies that $\mathfrak{g}_{>0}$ acts trivially on Θ^k_c, so the morphism Ind $\Theta^k_c \to \Theta^k_+$ is defined. Clearly Ω^k_+ is isomorphic to

$$\Omega^k_c \otimes \mathbb{C}[t, \xi_1, \ldots, \xi_n],$$

so it is a cofree module. Then the module Θ^k_+ is a free $\mathbb{C}[\partial_0, \partial_1, \ldots, \partial_n]$-module and the morphism

$$\text{Ind } \Theta^k_c \to \Theta^k_+$$

is therefore an isomorphism.

(3) Consider the homotopy operator $K : \Omega_{n,+} \to \Omega_{n,+}$ given by

$$K(d\xi_n \, \nu) = \xi_n \nu, \qquad K(\nu) = 0 \quad \text{if } \nu \text{ does not involve } d\xi_n.$$

Let $\varepsilon : \Omega_{n,+} \to \Omega_{n,+}$ be defined by

$$\varepsilon(d\xi_n \, \nu) = \varepsilon(\xi_n \nu) = 0, \qquad \varepsilon(\nu) = \nu \quad \text{if } \nu \text{ does not involve both } d\xi_n \text{ and } \xi_n.$$

One can check that $Kd + dK = \text{Id} - \varepsilon$. Considering the dual maps $K : (\Omega_{n,+})^\# \to (\Omega_{n,+})^\#$ and $\varepsilon : (\Omega_{n,+})^\# \to (\Omega_{n,+})^\#$, we obtain $K^\# d^\# + d^\# K^\# = \text{Id} - \varepsilon^\#$.

Therefore, if $\alpha \in (\Omega_{n,+})^\#$ is a closed form, we get $\alpha = d^\#(K^\# \alpha) + \varepsilon^\#(\alpha)$, and $\varepsilon^\#(\alpha)$ is also a closed form. Observe that $(\varepsilon^\# \alpha)(\nu) = \alpha(\varepsilon(\nu)) = 0$ if ν involve $d\xi_n$ or ξ_n. Hence $\varepsilon^\# \alpha$ is essentially an element in $(\Omega_{n-1,+})^\#$, namely it is equal to an element in $(\Omega_{n-1,+})^\#$ trivially extended in ν's that involve $d\xi_n$ or ξ_n. It follows by induction on n that

$$\alpha = d^\# \alpha_1 + \alpha_0, \tag{4.5}$$

for some $\alpha_0, \alpha_1 \in (\Omega_{n,+})^\#$ and α_0 is a closed form that is a trivial extension of an element $\tilde{\alpha}_0 \in (\Omega_{0,+})^\#$. But $\Omega_{0,+} = \mathbb{C}[t] \otimes \wedge(dt) = \{p(t) + q(t)\, dt \,|\, p, q \in \mathbb{C}[t]\}$ and $\tilde{\alpha}_0 \in (\Omega_{0,+})^\#$ is closed iff $\tilde{\alpha}_0(q(t)dt) = 0$ for all $q \in \mathbb{C}[t]$. In general, it is easy to see that $\gamma \in (\Omega_{0,+})^\#$ is exact iff γ is closed (i.e. $\gamma(q(t)dt) = 0$) and $\gamma(1) = 0$. Therefore, using (4.5), we have $\alpha = d^\# \beta + \alpha_0(1)\, \mathbf{1}^*$, where $\mathbf{1}^*(c\,1) = c$ and zero everywhere else. Since $\mathbf{1}^* \in (\Omega^0_{n,+})^\#$, we get the exactness of the sequence

$$\cdots \xrightarrow{d^\#} (\Omega^2_{n,+})^\# \xrightarrow{d^\#} (\Omega^1_{n,+})^\# \xrightarrow{d^\#} (\Omega^0_{n,+})^\#.$$

$\square$

COROLLARY 4.2. *The $W(1,n)_+$-modules Ω^k_+ of differential forms are isomorphic to the co-induced modules*

$$\Omega^k_+ = \mathrm{Cnd}\ \Omega^k_c.$$

Let us now study the $\mathfrak{g} = W(1,n)_+$-modules Ω^k_-. First, notice that these modules are free as $\mathbb{C}[\partial_0, \partial_1, \ldots, \partial_n]$-modules. Let

$$\xi_* = \xi_1 \cdots \xi_n, \qquad \text{and} \qquad \bar{\Omega}^k_c = t^{-1}\xi_* \Omega^k_c \subset \Omega^k_-. \tag{4.6}$$

PROPOSITION 4.3. *For $\mathfrak{g} = W(1,n)_+$, we have:*

(1) $\bar{\Omega}^k_c$ is an irreducible $\mathfrak{g}_0$-submodule in Ω^k_- with highest weight

$$(-1;0,0,\ldots,0), \quad \textit{for } k=0,$$
$$(0;k,1,\ldots,1), \quad \textit{for } k>0,$$

and $\mathfrak{g}_{>0}$ acts trivially on $\bar{\Omega}^k_c$.

(2) There is an isomorphism $\Omega^k_- = \mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_0} \bar{\Omega}^k_c$.

(3) The differential d gives us $\mathfrak{g}$-module morphisms on Ω^k_- and the kernel and image of d are $\mathfrak{g}$-submodules in Ω^k_-.

(4) The kernel of d and image of d in Ω^k_- for $k \geq 2$ coincide, in Ω^1_- we have Ker $d = \mathbb{C}(t^{-1}dt)+$Im d, *and in Ω^0_-, we have* Ker $d = 0$ *(and the image does not exist).*

PROOF. (1) First of all, $\bar{\Omega}^k_c$ is the maximum total degree component in Ω^k_-, so any element from $\mathfrak{g}_{>0}$ moves it to zero. Also, as $\mathfrak{g}_0$-module $\bar{\Omega}^k_c$ is isomorphic to Ω^k_c multiplied by the 1-dimensional module $\langle t^{-1}\xi_*\rangle$. This permits us to see that its highest vectors are

$$\begin{array}{ll} \langle t^{-1}\xi_*\rangle & \text{for } k=0, \\ \langle t^{-1}\xi_* dt\rangle & \text{for } k=1, \\ \langle t^{-1}\xi_* dt(d\xi_1)^{k-1}\rangle & \text{for } k>1. \end{array}$$

The values of the highest weights are easy to compute.

(2) It is straightforward to see that Ω^0_- is a free rank 1 $\mathbb{C}[\partial_0, \partial_1, \ldots, \partial_n]$-module. Now, the action of $\partial_0, \partial_1, \ldots, \partial_n$ on Ω^k_- is coefficient wise and the fact that Ω^k_- is a free $\mathbb{C}[\partial_0, \partial_1, \ldots, \partial_n]$-module follows. This gives us the isomorphism $\Omega^k_- =\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{g}_0} \bar{\Omega}^k_c$. Parts (3) and (4) are left to the reader. □

The above statement shows us that there are non-trivial submodules in $\Omega^k_\pm$ and Θ^k_+. In fact, these are "almost all" proper submodules and the respective factors are irreducible. These results are discussed in Section 4.4. In order to get this result we need to study singular vectors.

4.3. Singular vectors of W_n-modules. Having in mind the results of Section 2, we introduce the following modules. Given a $gl(1|n)$-module V, we have the associated tensor field $W(1,n)$-module $\mathbb{C}[t,t^{-1}]\otimes\Lambda(n)\otimes V$, which is a formal distribution module spanned by a collection of fields $E = \{\delta(t-z)fv | f \in \Lambda(n), v \in V\}$. The associated conformal W_n-module is

$$\mathrm{Tens}(V) = \mathbb{C}[\partial] \otimes (\Lambda(n) \otimes V)) \tag{4.7}$$

with the following λ-action:

$$a_\lambda(g\otimes v)=a(g)\otimes v+(-1)^{p(a)}\sum_{i,j=1}^n(\partial_i f_j)g\otimes(E_{ij}-\delta_{ij})(v)- \\ -\lambda(-1)^{p(g)}\sum_{j=1}^n f_jg\otimes E_{0j}(v), \tag{4.8}$$

$$f_\lambda(g\otimes v)=(-\partial)(fg\otimes v)+(-1)^{p(fg)}\sum_{i=1}^n(\partial_i f)g\otimes E_{i0}(v)+ \\ +\lambda(fg\otimes E_{00}(v)). \tag{4.9}$$

where $a=\sum_{i=1}^n f_i\partial_i\in W(n), f,g\in\Lambda(n), v\in V$, and $E_{ij}\in gl(1|n)$ are matrix units (they correspond to the level 0 elements $\xi_i\partial_j$ with the notation $\xi_0=t$ and $\partial_0=\partial_t$).

In this case, the modules $M(F)=\ \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}F$ defined in Section 2, correspond to the W_n-module $\mathrm{Tens}(F)$, with F a finite-dimensional (irreducible) $gl(1|n)$-module. When we discuss the highest weight of vectors and singular vectors, we always mean with respect to the upper Borel subalgebra in $\mathfrak{g}=W(1,n)_+$ generated by $\mathfrak{g}_{>0}$ and the elements of $\mathfrak{g}_0$:

$$t\partial_i,\quad \xi_i\partial_j\quad i<j. \tag{4.10}$$

Therefore, in the module $M(V)$, viewed as a module over the annihilation algebra $W(1,n)_+$ (see Proposition 2.3), a vector $m\in M(V)$ is a singular vector if and only if the following conditions are satisfied ($g=\xi_{i_1}\cdots\xi_{i_s}\in\Lambda(n)$, and $\partial_0=\partial_t$)

(s1) $t^n g\partial_i\cdot m=0$ for $n>1$,

(s2) $t^1 g\partial_i\cdot m=0$ except for $g=1$ and $i=0$,

(s3) $t^0 g\partial_j\cdot m=0$ for $s>1$ or $g=\xi_i$ with $i<j$.

(4.11)

We shall frequently use the notation

$$\xi_I=\xi_{i_1}\cdots\xi_{i_s}\in\Lambda(n),\qquad \text{with } I=\{i_1,\dots,i_s\}. \tag{4.12}$$

Therefore, these conditions on a singular vector $m\in\mathrm{Tens}(V)$ translate in terms of the λ-action to (cf. (2.2)):

(S1) $\frac{d^2}{d\lambda^2}(f_\lambda m)=0$ for all $f\in\Lambda(n)$,

(S2) $\frac{d}{d\lambda}(a_\lambda m)=0$ for all $a\in W(n)$,

(S3) $\frac{d}{d\lambda}(f_\lambda m)|_{\lambda=0}=0$ for all $f\in\Lambda(n)$ with $f\neq 1$,

(S4) $(a_\lambda m)|_{\lambda=0}=0$ for all $a=\xi_I\partial_j\in W(n)$ with $|I|>1$ or $a=\xi_i\partial_j$ with $i<j$,

(S5) $(f_\lambda m)|_{\lambda=0}=0$ for all $f=\xi_I\in\Lambda(n)$ with $|I|>1$.

In order to classify the finite irreducible W_n-modules we should solve these equations (S1-5) to obtain the singular vectors. This is done by technical and lengthly reduction lemmas that we shall omit. For details see [**2**].

Recall that we are considering the basis ($\partial_0=\partial_t$)

$$t\partial_0; t\partial_0+\xi_1\partial_1,\dots,t\partial_0+\xi_n\partial_n$$

for the Cartan subalgebra H in $W(1,n)_+$, and we write the weight of an eigenvector for the Cartan subalgebra H as a tuple

$$\bar{\mu} = (\mu; \lambda_1, \dots, \lambda_n) \tag{4.13}$$

for the corresponding eigenvalues of the basis. We can prove the following

PROPOSITION 4.4. [**2**] *Let $n \geq 2$ and m be a non-trivial singular vector in* $\mathrm{Tens}\, V$ *with weight $\bar{\mu}_m$, then we have one of the following:*

(a) $m = \xi^n \otimes v_n$, $\bar{\mu}_m = (0; 0, \dots, 0, -k)$ with $k \geq 0$, v_n is a highest weight vector in V with weight $(0; 0, \dots, 0, -k-1)$, and m is uniquely defined by v_n.

(b) $m = \sum_{l=1}^{n} \xi^l \otimes v_l$, $\bar{\mu}_m = (0; k, 1, \dots, 1)$ with $k \geq 2$, v_1 is a highest weight vector in V with weight $(0; k-1, 1, \dots, 1)$, and m is uniquely defined by v_1.

(c) $m = \partial(\xi_* \otimes w) + \sum_{l=1}^{n} \xi^l \otimes v_l$, $\bar{\mu}_m = (-1; 0, \dots, 0)$, w is a highest weight vector in V with weight $(0; 1, \dots, 1)$, and m is uniquely defined by w.

4.4. Irreducible induced $W(1,n)_+$-modules. In this subsection we consider $\mathfrak{g} = W(1,n)_+$, with $n \geq 2$. Now, we have the following:

THEOREM 4.5. *Let $n \geq 2$ and F be an irreducible $\mathfrak{g}_0$-module with highest weight $\bar{\mu}_*$. Then the $\mathfrak{g}$-modules $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} F$ are irreducible finite continuous modules except for the following cases:*

(a) $\bar{\mu}_* = (0; 0, \dots, 0, -m), m \geq 0$, where $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} F = \Theta_+^m$ and the image $d^\# \Theta_+^{m+1}$ is the only non-trivial proper submodule.

(b) $\bar{\mu}_* = (0; k, 1, \dots, 1), k \geq 1$, where $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} F = \Omega_-^k$. For $k \geq 2$ the image $d\Omega_-^{k-1}$ is the only non-trivial proper submodule. For $k = 1$, both $\mathrm{Im}(d)$ and $\mathrm{Ker}(d)$ are proper submodules. $\mathrm{Ker}(d)$ is a maximal submodule.

REMARK 4.6. Let F be an irreducible $\mathfrak{g}_0$-module with highest weight $\bar{\mu}_* = (-1; 0, \dots, 0)$. Then $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} F = \Omega_-^0$ is an irreducible $\mathfrak{g}$-module. Note that the image of $d : \Omega_-^0 \to \Omega_-^1$ is the submodule in Ω_-^1 generated by the singular vector correponding to the case (c) in Proposition 4.4, but it is not a maximal submodule (see Proposition 4.3 (4)).

PROOF. We know from Theorem 2.4 that in order for $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} F$ to be reducible it has to have non-trivial singular vectors and the possible highest weights of F in this situation are listed in Proposition 4.4 above.

The fact that the induced modules are actually reducible in those cases is known because we have got nice realizations for these induced modules in Propositions 4.1 and 4.3 together with morphisms defined by $d, d^\#$, so kernels and images of these morphisms become submodules.

The subtle thing is to prove that a submodule is really a maximal one. We notice that in each case the factor is isomorphic to a submodule in another induced module so it is enough to show that the submodule is irreducible. This can be proved as follows, a submodule in the induced module is irreducible if it is generated by any highest singular vector that it contains. We see from our list of non-trivial singular vectors that there is at most one such a vector for each case and the images and kernels in question are exactly generated by those vectors, hence they are irreducible. □

COROLLARY 4.7. *The theorem gives us a description of finite continuous irreducible $W(1,n)_+$-modules for $n \geq 2$. Such a module is either $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} F$ for an irreducible finite-dimensional $\mathfrak{g}_0$-module F where the highest weight of F does not belong to the types listed in (a), (b) of the theorem or the factor of an induced module from (a), (b) by its submodule $Ker(d)$.*

4.5. Finite irreducible W_n-modules. In order to give an explicit construction and classification, we need the following notation. Recall that $W(1,n)$ acts by derivations on the algebra of differential forms $\Omega = \Omega(1,n)$, and note that this is a conformal module by taking the family of formal distributions

$$E = \{\delta(z-t)\omega \text{ and } \delta(z-t)\omega \; dt \mid \omega \in \Omega(n)\}$$

Translating this and all other attributes of differential forms, like de Rham differential, etc. into the conformal algebra language, we arrive to the following definitions.

Recall that given an algebra A, the associated current formal distribution algebra is $A[t,t^{-1}]$ with the local family $F = \{a(z) = \sum_{n\in\mathbb{Z}}(at^n)z^{-n-1} = a\delta(z-t)\}_{a\in A}$. The associated conformal algebra is $\mathrm{Cur}A = \mathbb{C}[\partial] \otimes A$ with multiplication defined by $a_\lambda b = ab$ for $a, b \in A$ and extended using sesquilinearity. This is called the *current conformal algebra*.

The conformal algebra of differential forms Ω_n is the current algebra over the commutative associative superalgebra $\Omega(n) + \Omega(n)\; dt$ with the obvious multiplication and parity, subject to the relation $(dt)^2 = 0$:

$$\Omega_n = \mathrm{Cur}(\Omega(n) + \Omega(n)\; dt).$$

The de Rham differential $\tilde{d}$ of Ω_n (we use the tilde in order to distinguish it from the de Rham differential d on $\Omega(n)$) is a derivation of the conformal algebra Ω_n such that:

$$\tilde{d}(\omega_1 + \omega_2 dt) = d\omega_1 + d\omega_2 dt - (-1)^{p(\omega_1)}\partial(\omega_1 dt). \tag{4.14}$$

here and further $\omega_i \in \Omega(n)$.

The standard $\mathbb{Z}_+$-gradation $\Omega(n) = \oplus_{j\in\mathbb{Z}_+}\Omega(n)^j$ of the superalgebra of differential forms by their degree induces a $\mathbb{Z}_+$-gradation

$$\Omega_n = \oplus_{j\in\mathbb{Z}_+}\Omega_n^j, \qquad \text{where } \Omega_n^j = \mathbb{C}[\partial] \otimes (\Omega(n)^j + \Omega(n)^{j-1}\; dt),$$

so that $\tilde{d} : \Omega_n^j \to \Omega_n^{j+1}$.

The contraction ι_D for $D = a + f \in W_n$ is a conformal derivation of Ω_n such that:

$$\begin{aligned} (\tilde{L}_a)_\lambda(\omega_1 + \omega_2 dt) &= L_a\omega_1 + (L_a\omega_2)dt, \\ (\tilde{L}_f)_\lambda\omega &= -(\partial + \lambda)(f\omega), \\ (\tilde{L}_f)_\lambda(\omega dt) &= (-1)^{p(f)+p(\omega)}(df)\omega - \partial(f\omega dt). \end{aligned} \tag{4.15}$$

The properties of $\Omega(1,n)$ imply the corresponding properties of Ω_n given by the following proposition.

PROPOSITION 4.8. (1) $\tilde{d}^2 = 0$.

(2) *The complex $(\Omega_n, \tilde{d}) = \{0 \to \Omega_n^0 \to \cdots \to \Omega_n^j \to \cdots\}$ is exact at all terms Ω_n^j, except for $j = 1$. One has:* $\mathrm{Ker}\; \tilde{d}_{|_{\Omega_n^1}} = \tilde{d}\Omega_n^0 \oplus \mathbb{C}dt$.

(3) $\iota_{D_1}\iota_{D_2} + p(D_1, D_2)\iota_{D_2}\iota_{D_1} = 0$.

(4) $\tilde{L}_D\tilde{d} = (-1)^{p(D)}\tilde{d}\tilde{L}_D$.
(5) $\tilde{L}_D = \tilde{d}\iota_D + (-1)^{p(D)}\iota_D\tilde{d}$.
(6) *The map* $D \mapsto \tilde{L}_D$ *defines a* W_n*-module structures on* Ω_n*, preserving the* $\mathbb{Z}_+$*-gradation and commuting with* $\tilde{d}$*.*

PROOF. Only the proof of (b) requires a comment. Following Proposition 3.2.2 of [**8**], we construct $\mathbb{C}[\partial]$-linear maps $K : \Omega_n \to \Omega_n$ (a homotopy operator) and $\epsilon : \Omega_n \to \Omega_n$ by the formulas ($\omega \in \Omega(n) + \Omega(n)dt$):

$$K(d\xi_n\omega) = \xi_n\omega, \quad K(\omega) = 0 \qquad \text{if } \omega \text{ does not involve } d\xi_n,$$
$$\epsilon(d\xi_n\omega) = \epsilon(\xi_n\omega) = 0, \quad \epsilon(\omega) = \omega \qquad \text{if } \omega \text{ does not involve both } d\xi_n \text{ and } \xi_n.$$

One checks directly that

$$K\tilde{d} + \tilde{d}K = 1 - \epsilon.$$

Therefore, if $\omega \in \Omega_n$ is a closed form, we get $\omega = \tilde{d}(K\omega) + \epsilon(\omega)$. It follows by induction on n that $\omega = \tilde{d}\omega_1 + P(\partial)dt$ for some $\omega_1 \in \Omega_n$ and a polynomial $P(\partial)$. But it is clear from (4.14) that $P(\partial)dt$ is always closed, and it is exact iff $P(\partial)$ is divisible by ∂. □

Since the extended annihilation algebra $W(1,n)^+$ is a direct sum of $W(1,n)_+$ and a 1-dimensional Lie algebra $\mathbb{C}a$, any irreducible $W(1,n)^+$-module is obtained from a $W(1,n)_+$-module M by extending to $W(1,n)^+$, letting $a \mapsto -\alpha$, where $\alpha \in \mathbb{C}$. Translating into the conformal language, we see that all W_n-modules are obtained from conformal $W(1,n)_+$-modules by taking for the action of ∂ the action of $-\partial_t + \alpha I, \alpha \in \mathbb{C}$. We denote by $\mathrm{Tens}_\alpha V$ and $\Omega_{k,\alpha}, \alpha \in \mathbb{C}$, the W_n-modules obtained from $\mathrm{Tens}V$ and Ω_k by replacing in (4.8) and (4.9) respectively ∂ by $\partial + \alpha$.

Now, Theorem 4.5 and Corollary 4.7, along with Theorem 2.4 and Propositions 2.3, 2.9 and 2.7 give us a complete description of finite irreducible W_n-modules.

THEOREM 4.9. *The following is a complete list of non-trivial finite irreducible* W_n*-modules* ($n \geq 2, \alpha \in \mathbb{C}$)*:*

(1) $\mathrm{Tens}_\alpha V$*, where* V *is a finite-dimensional irreducible* $gl(1|n)$*-module different from* $\Lambda^k(\mathbb{C}^{1|n})^*$*,* $k = 1, 2, \ldots$ *and* $\bar{\Omega}^k_c$ *(see (4.6)),* $k = 1, 2, ...,$
(2) $\Omega^*_{k,\alpha}/\mathrm{Ker}\ \tilde{d}^*, k = 1, 2, \ldots$ *, and the same modules with reversed parity,*
(3) W_n*-modules dual to* (2)*, with* $k > 1$*.*

REMARK 4.10. (a) Using Proposition 4.3, we have that the kernel of $\tilde{d}$ and the image of $\tilde{d}$ coincide in Ω_k for $k \geq 2$. Now, since Ω_{k+2} is a free $\mathbb{C}[\partial]$-module of finite rank and $\Omega_{k+1}/\mathrm{Im}\tilde{d} = \Omega_{k+1}/\mathrm{Ker}\tilde{d} \simeq \mathrm{Im}\tilde{d} \subset \Omega_{k+2}$, we obtain that $\Omega_{k+1}/\mathrm{Im}\ \tilde{d}$ is a finitely generated free $\mathbb{C}[\partial]$-module. Therefore, we can apply Proposition 2.7, and we have that

$$\Omega^*_{k+1,\alpha}/\mathrm{Ker}\ \tilde{d}^* \simeq \left(\Omega_{k,\alpha}/\mathrm{Ker}\ \tilde{d}\right)^* \tag{4.16}$$

for $k \geq 1$.

(b) Observe that we can not apply the previous argument for $k = 0$ since, by Proposition 4.3, the image of $\tilde{d}$ has codimension one (over $\mathbb{C}$) in $\mathrm{Ker}\ \tilde{d}$. In fact, (4.16) is not true for $k = 0$. For example, this can be easily seen for $W_0 = Vir$ using the differential map which is explicitly written in Remark 2.6.

(c) Observe that $\Omega_{0,\alpha}$ is an irreducible tensor module (Ker $\tilde{d}=0$, cf. Proposition 4.3), that is why this module is included in case (1) of Theorem 4.9.

(d) Since for a finite rank module M over a Lie conformal superalgebra we have $M^{**}=M$ (see Proposition 2.8), the W_n-modules in case (3) of Theorem 4.9 are isomorphic to $\Omega_{k,\alpha}/ker\ \tilde{d}$, $k=2,3,\ldots$.

(e) Observe that (TensV)* is not isomorphic to TensV*. See Remark 4.13 below for the case of W_1.

Now we will present the case $n=1$ in detail and we shall see that our result agrees with the classification given in [**5**] for $K_2\simeq W_1$. For the general definition of the family K_n, see Section 7. Let us fix some notations. We have

$$W_1=\mathbb{C}[\partial]\otimes(\Lambda(1)\oplus W(1))=\mathbb{C}[\partial]\{1,\xi,\partial_1,\xi\partial_1\}.$$

In [**5**], the conformal Lie superalgebra K_2 is presented as the freely generated module over $\mathbb{C}[\partial]$ by $\{L,J,G^{\pm}\}$. An isomorphism between K_2 and W_1 is explicitly given by

$$L\mapsto -1+\frac{1}{2}\partial\xi\partial_1,\qquad J\mapsto\xi\partial_1,\qquad G^+\mapsto 2\xi,\qquad G^-\mapsto-\partial_1. \tag{4.17}$$

The irreducible modules of W_1 are parameterized by finite-dimensional irreducible representations of $gl(1,1)$ (and the additional twist by alpha that, for simplicity, shall be omitted in the formulas below). The irreducible representations of $gl(1,1)$, denoted by $V_{a,b}$, are parameterized by a and b, the corresponding eigenvalues of e_{11} and e_{22} on the highest weight vector, where e_{ij} denotes the matrix in $gl(1,1)$ with 1 in the ij-place and 0 elsewhere.

If both parameters are equal to zero, the representation is trivial 1-dimensional. Otherwise, either $a+b=0$, the dimension of the $gl(1,1)$-representation is 1, and the corresponding representation of W_1 is one of the tensor modules of rank 2. Or else $a+b$ is non-zero, the dimension of the $gl(1,1)$-representation is 2, and the corresponding tensor module has rank 4.

Explicitly, consider the set of $\mathbb{C}[\partial]$-generators of W_1 $\{1,\,\xi,\,\partial_1,\,\xi\partial_1\}$. Let a and b such that $a+b\neq 0$. Let $V_{a,b}=\mathbb{C}\text{-span}\{v_0,v_1\}$, where v_0 is a highest weight vector. Let $M(a,b)=M(V_{a,b})=\mathbb{C}[\partial]\{v_0,\,v_1,\,w_1=\partial_1 v_0,\,w_0=\partial_1 v_1\}$ be the tensor W_1-module and denote by $L(a,b)$ the irreducible quotient. The action of W_1 in $M(a,b)$ is given explicitly by the following formulas:

$$\begin{aligned}
1_\lambda v_0&=(a\lambda-\partial)v_0, & 1_\lambda v_1&=((a-1)\lambda-\partial)v_1,\\
1_\lambda w_1&=(a\lambda-\partial)w_1, & 1_\lambda w_0&=((a-1)\lambda-\partial)w_0,\\[1ex]
\xi_\lambda v_0&=v_1, & \xi_\lambda v_1&=0,\\
\xi_\lambda w_1&=(a\lambda-\partial)v_0-w_0, & \xi_\lambda w_0&=((a-1)\lambda-\partial)v_1,\\[1ex]
\partial_{1\,\lambda} v_0&=w_1, & \partial_{1\,\lambda} v_1&=(a+b)\lambda v_0+w_0,\\
\partial_{1\,\lambda} w_1&=0, & \partial_{1\,\lambda} w_0&=-(a+b)\lambda w_1\\[1ex]
\xi\partial_{1\,\lambda} v_0&=b\,v_0, & \xi\partial_{1\,\lambda} v_1&=(b+1)\,v_1,\\
\xi\partial_{1\,\lambda} w_1&=(b-1)w_1, & \xi\partial_{1\,\lambda} w_0&=-(a+b)\lambda\,v_0+b\,w_0.
\end{aligned} \tag{4.18}$$

If $a+b\neq 0$ and $a\neq 0$, then $M(a,b)$ is irreducible of rank 4, and the explicit action is given by (4.17). Let $v=p(\partial)v_0+q(\partial)w_0+r(\partial)v_1+s(\partial)w_1$ belong to a submodule of $M(a,b)$. Denote by w the coefficient of the highest power in λ of $\xi_\lambda v$ and by y the coefficient of the highest power in λ of $\xi_\lambda w$.

If $a \neq 1$ then $y = v_1$ (up to a constant factor), therefore v_1 lies in the submodule. If $a = 1$, then by taking the coefficient of the highest power in λ of $\xi\partial_{1\lambda} y$ and using that in this case $b \neq -1$, we also obtain that v_1 lies in the submodule.

Therefore, in any case we have that v_1 lies in any submodule, and by the formulas for the actions on v_1 it is immediate that the other generators also belong to any submodule, proving that $M(a,b)$ is irreducible in this case.

If $a+b \neq 0$ but $a = 0$, it is easy to show as above that $N = \mathbb{C}[\partial]w_1 \oplus \mathbb{C}[\partial](\partial v_0 + w_0)$ is a submodule of $M(0,b)$. Let $L(0,b) = M(0,b)/N = \mathbb{C}[\partial]v_0 \oplus \mathbb{C}[\partial]v_1$, the irreducible quotient of $M(0,b)$, and the action is explicitly given by

$$\begin{aligned} 1_\lambda v_0 &= (-\partial)v_0, & 1_\lambda v_1 &= (-\lambda - \partial)v_1, \\ \xi_\lambda v_0 &= v_1, & \xi_\lambda v_1 &= 0, \\ \partial_{1\lambda} v_0 &= 0, & \partial_{1\lambda} v_1 &= (b\lambda - \partial)v_0, \\ \xi\partial_{1\lambda} v_0 &= b\, v_0, & \xi\partial_{1\lambda} v_1 &= (b+1)\, v_1. \end{aligned} \tag{4.19}$$

If $a+b = 0$, but $a \neq 0$, it is easy to show as above that $M(a,-a) = C[\partial]\{v_0, w_1\}$ is irreducible of rank 2 and the action of W_1 here is given by:

$$\begin{aligned} 1_\lambda v_0 &= (a\lambda - \partial)v_0, & 1_\lambda w_1 &= (a\lambda - \partial)w_1, \\ \xi_\lambda v_0 &= 0, & \xi_\lambda w_1 &= (a\lambda - \partial)v_0, \\ \partial_{1\lambda} v_0 &= w_1, & \partial_{1\lambda} w_1 &= 0, \\ \xi\partial_{1\lambda} v_0 &= -a\, v_0, & \xi\partial_{1\lambda} w_1 &= (-a-1)w_1. \end{aligned} \tag{4.20}$$

Thus we obtain

Corollary 4.11. *The W_1-module $L(a,b)$ as a $\mathbb{C}[\partial]$-module has the following rank: 4 if $a+b \neq 0$ and $a \neq 0$, 2 if $a+b \neq 0$ and $a = 0$, 2 if $a+b = 0$ and $a \neq 0$, 0 if $a = b = 0$. These are all non-trivial finite irreducible W_1-modules.*

Remark 4.12. In [**5**], the irreducible representations of K_2 are classified in terms of parameters Λ and Δ. Using the isomorphism between K_2 and W_1 in (4.17), these parameters are related to ours as follows,

$$a = -\Delta - \frac{\Lambda}{2}\,,\ b = \Lambda.$$

Then it can be easily checked that the above corollary corresponds to Theorem 4.1 in [**5**], and explicit formulas for the λ- action given at the end of section 4 in [**5**], corresponds to ours in each case.

Remark 4.13. It is easy to see that for the case $a+b \neq 0$, $(\mathrm{TensV_{a,b}})^* =$Tens $V_{-a,-b}$, but $(V_{a,b})^* = V_{1-a,-b-1}$. Therefore, in this case we see that Tens V^* is not isomorphic to (Tens $V)^*$.

5. Lie conformal superalgebra S_n and its finite irreducible modules

Recall that the *divergence* of a differential operator $D = \sum_{i=0}^n a_i\partial_i \in W(1,n)$, with $a_i \in \Lambda(1,n)$ and $\partial_0 = \partial_t$ is defined by the formula

$$div\ D = \partial_0 a_0 + \sum_{i=1}^n (-1)^{p(a_i)}\partial_i a_i.$$

The basic property of the divergence is $(D_1, D_2 \in W(1,n))$

$$div\ [D_1, D_2] = D_1(div\ D_2) - (-1)^{p(D_1)p(D_2)} D_2(div\ D_1).$$

It follows that

$$S(1,n) = \{D \in W(1,n)\ :\ div\ D = 0\}$$

is a subalgebra of the Lie superalgebra $W(1,n)$. Similarly,

$$S(1,n)_+ = \{D \in W(1,n)_+\ :\ div\ D = 0\}$$

is a subalgebra of $W(1,n)_+$. We have

$$S(1,n) \text{ (resp. } S(1,n)_+ \text{) } = S(1,n)' \text{ (resp. } S(1,n)'_+ \text{) } \oplus \mathbb{C}\xi_1 \cdots \xi_n \partial_0, \qquad (5.1)$$

where $S(1,n)'$ (resp. $S(1,n)'_+$) denotes the derived subalgebra. It is easy to see that $S(1,n)'$ is a formal distribution Lie superalgebra, see [**7**], Example 3.5.

In order to describe the associated Lie conformal superalgebra, we need to translate the notion of divergence to the "conformal" language as follows. It is a $\mathbb{C}[\partial]$-module map $div : W_n \to$Cur $\Lambda(n)$, given by

$$div\ a = \sum_{i=1} (-1)^{p(f_i)} \partial_i f_i, \qquad div\ f = -\partial \otimes f,$$

where $a = \sum_{i=1}^n f_i \partial_i \in W(n)$ and $f \in \Lambda(n)$. The following identity holds in $\mathbb{C}[\partial] \otimes \Lambda(n)$, where $D_1, D_2 \in W_n$:

$$div\ [D_{1\lambda} D_2] = (D_1)_\lambda (div\ D_2) - (-1)^{p(D_1)p(D_2)} (D_2)_{-\lambda-\partial}(div\ D_1). \qquad (5.2)$$

Therefore,

$$S_n = \{D \in W_n\ :\ div\ D = 0\}$$

is a subalgebra of the Lie conformal superalgebra W_n. It is known that S_n is simple for $n \geq 2$, and finite of rank $n2^n$. Furthermore, it is the Lie conformal superalgebra associated to the formal distribution Lie superalgebra $S(1,n)'$. The annihilation algebra and the extended annihilation algebra is given by

$$\mathcal{A}(S_n) = S(1,n)'_+ \qquad \text{and} \qquad \mathcal{A}(S_n)^e = \mathbb{C}\ ad(\partial_0) \ltimes S(1,n)'_+.$$

Now, we have to study representations of $S(1,n)_+$ and of its derived algebra $S(1,n)'_+$ which has codimension 1. Observe that $S(1,n)_+$ inherits the $\mathbb{Z}$-gradation in $W(1,n)_+$, and denoting by $\mathfrak{g} = S(1,n)_+$ (for the rest of this section), we have that $\mathfrak{g}_{-1} =< \partial_0, \ldots, \partial_n >$ as in $W(1,n)_+$ but the other graded components are strictly smaller than these of $W(1,n)_+$. Observe that $\mathfrak{g}_0 = sl(1|n)$.

In order to consider weights of vectors in $S(1,n)_+$-modules, we take the basis

$$t\partial_0 + \xi_1\partial_1, \ldots, t\partial_0 + \xi_n\partial_n.$$

for the Cartan subalgebra. And the weights are written as $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n)$ for the corresponding eigenvalues.

Propositions 4.1 and 4.3, and Corollary 4.2 holds for $\mathfrak{g} = S(1,n)_+$ with the following minor modification: all highest weights are the same as in the W case, except for the first coordinate that should be removed.

Similarly, if V is an $sl(1|n)$-module, then formulas (4.8) and (4.9) define an S_n-module structure in Tens(V). Indeed, elements $(-1)^{p(f)}\partial(f\partial_i) + \partial_i f$, with $f \in \Lambda(n)$ generate S_n as a $\mathbb{C}[\partial]$-module. It is easy to see that the action of these elements (defined by (4.8) and (4.9)), only involves $E_{ij}(v)$ for $i \neq j$ and $(E_{00} + E_{ii})(v)$ for $i > 0$.

As in the W-case, the classification is reduced to the study of singular vectors in $\mathrm{Tens}(V)$, where V is an $sl(1|n)$-module. Observe that the reduction Lemmas for the singular vector in [**2**] hold in this case, and the proof is basically the same. Therefore, analogous computations give as the following

Proposition 5.1. *Let $n \geq 2$ and V an irreducible finite-dimensional $sl(1|n)$-module. If m is a non-trivial singular vector in the $S(1,n)_+$-module* Tens V *with weight $\bar{\lambda}_m$, then we have one of the following:*

(a) $m = \xi^n \otimes v_n$, $\bar{\lambda}_m = (0,\dots,0,-k)$ with $k \geq 0$, v_n is a highest weight vector in V with weight $(0,\dots,0,-k-1)$, and m is uniquely defined by v_n.

(b) $m = \sum_{l=1}^{n} \xi^l \otimes v_l$, $\bar{\lambda}_m = (k,1,\dots,1)$ with $k \geq 2$, v_1 is a highest weight vector in V with weight $(k-1,1,\dots,1)$, and m is uniquely defined by v_1.

(c) $m = \partial(\xi_* \otimes w) + \sum_{l=1}^{n} \xi^l \otimes v_l$, $\bar{\lambda}_m = (0,\dots,0)$, w is a highest weight vector in V with weight $(1,\dots,1)$, and m is uniquely defined by w.

(d) $m = \partial(\xi^n \otimes w) + \sum_{l=1}^{n-1} \xi_{[l,n]-\{l,n\}} \otimes v_l + \xi^n \otimes v_n$, $\bar{\lambda}_m = (0,\dots,0,-1)$, w is a highest weight vector in V with weight $(1,\dots,1)$, and m is uniquely defined by w.

Using the above proposition, we have

Theorem 5.2. *Let $\mathfrak{g} = S(1,n)_+$ $(n \geq 2)$ and F be an irreducible $\mathfrak{g}_0$-module with highest weight $\bar{\lambda}_*$. Then the $\mathfrak{g}$-modules $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} F$ are irreducible finite continuous modules except for the following cases:*

(a) $\bar{\lambda}_* = (0,\dots,0,-p), p \geq 0$, where $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} F = \Theta_+^p$ and the image $d^{\#}\Theta_+^{p+1}$ is the only non-trivial proper submodule.

(b) $\bar{\lambda}_* = (q,1,\dots,1), q \geq 1$, where $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} F = \Omega_-^q$. For $q \geq 2$ the image $d\Omega_-^{q-1}$ is the only non-trivial proper submodule. For $q = 1$, the proper submodules are $\mathrm{Im}(d)$, $\mathrm{Ker}(d)$ and $\mathrm{Im}(\alpha)$, where α is the composition

$$\alpha : \Theta_+^1 \xrightarrow{d^{\#}} \Theta_+^0 \simeq \Omega_-^0 \xrightarrow{d} \Omega_-^1,$$

and $\mathrm{Ker}(d)$ is the maximal proper submodule.

Proof. Similarly to the case of $W(1,n)_+$, the modules $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} F$ are irreducible except when they have a singular vector and the highest weights of such F, when it could happen, are listed in (a), (b), (c) and (d) of the above Proposition 5.1. The weight $(1,\dots,1)$ is special here because it is relevant to (b), (c) and (d). There are three types of singular vectors possible in this case. The corresponding module $\mathrm{Ind}(F) = \Omega_-^1$ has three different submodules and all three vectors are present. The same argument as for $W(1,n)_+$-modules allows us easily to conclude that the listed submodules are the only ones and the factors are irreducible. □

Corollary 5.3. *The theorem gives us a description of finite continuous irreducible $S(1,n)_+$-modules when $n \geq 2$. Such a module is either $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} F$ for an irreducible finite-dimensional L_0-module F where the highest weight of T does not belong to the types listed in (a), (b) of the theorem or the factor of an induced module from (a), (b) by the submodule* $\mathrm{Ker}(d)$.

COROLLARY 5.4. *The Lie superalgebras $S(1,n)_+$ and $S(1,n)'_+$ have the same finite continuous irreducible modules, and they are described by the previous corollary.*

PROOF. In order to see that Theorem 5.2 also holds for $S(1,n)'_+$, it is basically enough to see that Proposition 5.1 holds in this case. But, if we track the details of the proof, the singular vectors are the same for both Lie superalgebras $S(1,n)'_+$ and $S(1,n)_+$, finishing the proof. $\square$

Now, as in the W_n case, Theorem 5.2 and Corollary 5.4, along with Section 2 give us a complete description of finite irreducible S_n-modules ($n \geq 2$): it is given by Theorem 4.9 in which W_n is replaced by S_n and $gl(1|n)$ is replaced by $sl(1|n)$.

REMARK 5.5. Under the standard isomorphism between S_2 and small N=4 conformal superalgebra it is easy to see that our result agrees with the classification given in [**5**]. Indeed, in [**5**] (Theorem 6.1) the classification of irreducible modules was given in terms of parameters Λ and Δ, and these parameters are related to ours as follows,

$$\lambda_1 = -\Delta + \frac{\Lambda}{2}, \tag{5.3}$$

$$\lambda_2 = -\Delta - \frac{\Lambda}{2}. \tag{5.4}$$

Therefore, the case $2\Delta - \Lambda = 0$ ($\Lambda \in \mathbb{Z}_+$) corresponds to the family $\Omega^*_{\Lambda,\alpha}/\mathrm{Ker}\ \tilde{d}^*$ of rank 4Λ, and the case $2\Delta + \Lambda + 2 = 0$ ($\Lambda \in \mathbb{Z}_+$) corresponds to $\Omega_{\Lambda+1,\alpha}/\mathrm{Ker}\ \tilde{d}$ of rank $4\Lambda + 8$. Therefore, we have one module of rank 4 that corresponds to $\Omega^*_1/Ker\ \tilde{d}^*$, and by Remark 4.10, the dual of this module is Ω_0 (Ker is trivial in this case) and (using Proposition 4.3) Ω_0 is the tensor module $Tens(V)$ where V is the trivial representation, therefore it is reducible with a maximal submodule of codimension 1 (over $\mathbb{C}$).

6. Lie conformal superalgebras $S_{n,b}$ and $\tilde{S}_n$, and their finite irreducible modules

All the results of this section were obtained in [**2**].

- *Case $S_{n,b}$:*

For any $b \in \mathbb{C}$, $b \neq 0$, we take

$$S(1,n,b) = \{D \in W(1,n) | div(e^{bx}D) = 0\}.$$

This is a formal distribution subalgebra of $W(1,n)$. The associated Lie conformal superalgebra is constructed explicitly as follows. Let $D = \sum_{i=1}^n P_i(\partial,\xi)\partial_i + f(\partial,\xi)$ be an element of W_n. We define the deformed divergence as

$$div_b D = div D + bf.$$

It still satisfies equation 5.2, therefore

$$S_{n,b} = \{D \in W_n | div_b D = 0\}$$

is a subalgebra of W_n, which is simple for $n \geq 2$ and has rank $n2^n$. Since $S_{n,0} = S_n$ has been discussed in the previous section, we can (and will) assume that $b \neq 0$.

If $b \neq 0$, the extended annihilation algebra is given by

$$(Alg(S_{n,b}))^+ = \mathbb{C}ad(\partial_0 - b\sum_{i=1}^{n} \xi_i\partial_i) \ltimes S(1,n)_+ \simeq CS(1,n)'_+$$

where $CS(1,n)'_+$ is obtained from $S(1,n)'_+$ by enlarging $sl(1,n)$ to $gl(1,n)$ in the 0th-component.

Therefore, the construction of all finite irreducible modules over $S_{n,b}$ is the same as that for W_n, but without twisting by α. Hence, using Theorem 4.9, we have

THEOREM 6.1. *The following is a complete list of finite irreducible $S_{n,b}$-modules $(n \geq 2, b \in \mathbb{C}, b \neq 0)$:*

(1) Tens$V$, *where V is a finite-dimensional irreducible $gl(1|n)$-module different from $\Lambda^k(\mathbb{C}^{1|n})^*, k = 1, 2, \ldots$ and $\Lambda^k(\mathbb{C}^{1|n})$, $k = 0, 1, 2, \ldots$,*
(2) $\Omega_k^*/\mathrm{Ker}\ \tilde{d}^*, k = 1, 2, \ldots,$ *and the same modules with reversed parity,*
(3) *$S_{n,b}$-modules dual to (b), with $k > 1$.*

- *Case $\tilde{S}_n$:*

Let $n \in \mathbb{Z}_+$ be an even integer. We take

$$\tilde{S}(1,n) = \{D \in W(1,n) | div((1 + \xi_1 \ldots \xi_n)D) = 0\}.$$

This is a formal distribution subalgebra of $W(1,n)$. The associated Lie conformal superalgebra $\tilde{S}_n$ is constructed explicitly as follows:

$$\tilde{S}_n = \{D \in W_n | div((1 + \xi_1 \ldots \xi_n)D) = 0\} = (1 - \xi_1 \ldots \xi_n)S_n.$$

The Lie conformal superalgebra $\tilde{S}_n$ is simple for $n \geq 2$ and has rank $n2^n$.

The extended annihilation algebra is given by

$$(Alg(\tilde{S}_n))^+ = \mathbb{C}ad(\partial_0 - \xi_1 \ldots \xi_n\partial_0) \ltimes S(1,n)'_+ \simeq S(1,n)_+.$$

Therefore, the construction of all finite irreducible modules over $\tilde{S}_n$ is the same as that for S_n, but without the twist by α.

7. Lie conformal superalgebra K_n and its finite irreducible modules

The results of this section where obtained in [**1**].

7.1. Lie conformal superalgebra K_n and annihilation Lie algebra $K(1,n)_+$.

The contact superalgebra $K(1,n)$ is the subalgebra of $W(1,n)$ defined by

$$K(1,n) := \{D \in W(1,n) \mid D\omega = f_D\omega, \text{ for some } f_D \in \Lambda(1,n)\}, \qquad (7.1)$$

where $\omega = dt - \sum_{i=1}^{n} \xi_i d\xi_i$ is the standard contact form, and the action of D on ω is the usual action of vector fields on differential forms.

The space $\Lambda(1,n)$ can be identified with the Lie superalgebra $K(1,n)$ via the map

$$f \mapsto 2f\partial_t + (-1)^{p(f)} \sum_{i=1}^{n} \Big(\xi_i \partial_t f + \partial_i f\Big)\Big(\xi_i \partial_t + \partial_i\Big),$$

the corresponding Lie bracket for elements $f, g \in \Lambda(1,n)$ being

$$[f,g] = \Big(2f - \sum_{i=1}^{n} \xi_i \partial_i f\Big)(\partial_t g) - (\partial_t f)\Big(2g - \sum_{i=1}^{n} \xi_i \partial_i g\Big) + (-1)^{p(f)} \sum_{i=1}^{n} (\partial_i f)(\partial_i g).$$

The Lie superalgebra $K(1,n)$ is a formal distribution Lie superalgebra with the following family of mutually local formal distributions

$$a(z) = \sum_{j\in\mathbb{Z}} (at^j) z^{-j-1}, \text{ for } a = \xi_{i_1} \dots \xi_{i_r} \in \Lambda(n).$$

The associated Lie conformal superalgebra K_n is identified with

$$K_n = \mathbb{C}[\partial] \otimes \Lambda(n), \tag{7.2}$$

the λ-bracket for $f = \xi_{i_1} \dots \xi_{i_r}, g = \xi_{j_1} \dots \xi_{j_s}$ being as follows [**7**]:

$$[f_\lambda g] = \Big((r-2)\partial(fg) + (-1)^r \sum_{i=1}^{n} (\partial_i f)(\partial_i g)\Big) + \lambda(r+s-4)fg. \tag{7.3}$$

The Lie conformal superalgebra K_n has rank 2^n over $\mathbb{C}[\partial]$. It is simple for $n \geq 0, n \neq 4$, and the derived algebra K'_4 is simple and has codimension 1 in K_4.

The annihilation superalgebra is

$$\mathcal{A}(K_n) = K(1,n)_+ = \Lambda(1,n)_+ := \mathbb{C}[t] \otimes \Lambda(n), \tag{7.4}$$

and the extended annihilation superalgebra is

$$\mathcal{A}(K_n)^e = K(1,n)^+ = \mathbb{C}\mathrm{ad}\partial_t \ltimes K(1,n)_+.$$

Note that $\mathcal{A}(K_n)^e$ is isomorphic to the diract sum of $\mathcal{A}(K_n)$ and the trivial 1-dimensional Lie algebra.

The Lie superalgebra $K(1,n)$ is $\mathbb{Z}$-graded by putting

$$\deg(t^m \xi_{i_1} \dots \xi_{i_k}) = 2m + k - 2,$$

and it induces a gradation on $K(1,n)_+$ making it a $\mathbb{Z}$-graded Lie superalgebra of depth 2: $K(1,n)_+ = \oplus_{j\geq -2} (K(1,n)_+)_j$. It is easy to check that $K(1,n)_+$ satisfies conditions (L1)-(L3).

Observe that $K(1,n)_+$ is the subalgebra of

$$W(1,n)_+ = \{a\partial_t + \sum_{i=1}^{n} a_i \partial_i | a, a_i \in \Lambda(1,n)_+\}, \tag{7.5}$$

defined by (cf.(7.1))

$$K(1,n)_+ := \{D \in W(1,n)_+ \mid D\omega = f_D \omega, \text{ for some } f_D \in \Lambda(1,n)_+\}. \tag{7.6}$$

7.2. Induced modules. Using Theorem 2.4, the classification of finite irreducible K_n-modules can be reduced to the study of induced modules for $K(1,n)_+$. Observe that

$$\begin{aligned}(K(1,n)_+)_{-2} &=< \{\mathbf{1}\} >,\\ (K(1,n)_+)_{-1} &=< \{\xi_i \ :\ 1\leq i\leq n\} > \\ (K(1,n)_+)_0 &=< \{t\}\cup\{\xi_i\xi_j \ :\ 1\leq i<j\leq n\} >\end{aligned} \tag{7.7}$$

We shall use the following notation for the basis elements of $(K(1,n)_+)_0$:

$$E_{00}=t, \qquad F_{ij}=-\xi_i\xi_j. \tag{7.8}$$

Observe that $(K(1,n)_+)_0\simeq \mathbb{C}E_{00}\oplus\mathfrak{so}(n)\simeq\mathfrak{cso}(n)$. Take

$$\partial := -\frac{1}{2}\mathbf{1} \tag{7.9}$$

as the element that satisfies (L3) in Section 2.

For the rest of this work, $\mathfrak{g}$ will be $K(1,n)_+$. Let F be a finite-dimensional irreducible $\mathfrak{g}_0$-module, which we extend to a $\mathfrak{g}_{\geq 0}$-module by letting $\mathfrak{g}_j$ with $j>0$ acting trivially. Then we shall identify, as above

$$\operatorname{Ind}(F)\simeq\Lambda(1,n)\otimes F\simeq\mathbb{C}[\partial]\otimes\Lambda(n)\otimes F \tag{7.10}$$

as $\mathbb{C}$-vector spaces. In order to describe the action of $\mathfrak{g}$ in $\operatorname{Ind}(F)$ we introduce the following notation:

$$\begin{aligned}\xi_I &:= \xi_{i_1}\dots\xi_{i_k}, && \text{if}\quad I=\{i_1,\dots,i_k\},\\ \partial_L\,\xi_I &:= \partial_{l_1}\dots\partial_{l_s}\,\xi_I && \text{if}\quad L=\{l_1,\dots,l_s\},\\ \partial_f\,\xi_I &:= \partial_L\,\xi_I && \text{if}\quad f=\xi_L,\\ |f| &:= k && \text{if}\quad f=\xi_{i_1}\dots\xi_{i_k}.\end{aligned} \tag{7.11}$$

In the following theorem, we describe the $\mathfrak{g}$-action on $\operatorname{Ind}(F)$ using the λ-action notation in (2.2), i.e.

$$f_\lambda(g\otimes v)=\sum_{j\geq 0}\frac{\lambda^j}{j!}\,(t^jf)\cdot(g\otimes v)$$

for $f,g\in\Lambda(n)$ and $v\in F$.

THEOREM 7.1. *For any monomials $f,g\in\Lambda(n)$ and $v\in F$, where F is a $\mathfrak{g}_0$-module, we have the following formula for the λ-action of $\mathfrak{g}=K(1,n)_+$ on $\operatorname{Ind}(F)$:*

$$\begin{aligned}&f_\lambda(g\otimes v)=\\ &=(-1)^{p(f)}(|f|-2)\partial(\partial_f g)\otimes v+\sum_{i=1}^{n}\partial_{(\partial_i f)}(\xi_i g)\otimes v+(-1)^{p(f)}\sum_{r<s}\partial_{(\partial_r\partial_s f)}g\otimes F_{rs}v\\ &+\lambda\Big[(-1)^{p(f)}(\partial_f g)\otimes E_{00}v+(-1)^{p(f)+p(g)}\sum_{i=1}^{n}\big(\partial_f(\partial_i g)\big)\xi_i\otimes v+\sum_{i\neq j}\partial_{(\partial_i f)}(\partial_j g)\otimes F_{ij}v\Big]\\ &+\lambda^2(-1)^{p(f)}\sum_{i<j}\partial_f(\partial_i\partial_j g)\otimes F_{ij}v.\end{aligned}$$

The proof of this theorem is ommited (see Appendix A, [**1**]).

We obtained an easier formula for the λ-action in the induced module by taking the Hodge dual of the basis (cf. [**5**], pp. 922). More precisely, for a monomial $\xi_I \in \Lambda(n)$, we let $\overline{\xi_I}$ be its Hodge dual, i.e. the unique monomial in $\Lambda(n)$ such that $\overline{\xi_I}\xi_I = \xi_1 \dots \xi_n$.

The following theorem translates Theorem 7.1 in terms of the Hodge dual basis and gives us the formula that is used to compute the singular vectors.

Theorem 7.2. *Let F be a $\mathfrak{g}_0 = \mathfrak{cso}(n)$-module. Then the λ-action of $K(1,n)_+$ in $\mathrm{Ind}(F) = \mathbb{C}[\partial] \otimes \Lambda(n) \otimes F$, given by Theorem 7.1, is equivalent to the following one:*

$$\begin{aligned}
f_\lambda(g \otimes v) &= (-1)^{\frac{|f|(|f|+1)}{2}+|f||g|} \times \\
&\times \Bigg\{(|f|-2)\partial(fg)\otimes v - (-1)^{p(f)}\sum_{i=1}^{n}(\partial_i f)(\partial_i g)\otimes v - \sum_{r<s}(\partial_r\partial_s f)g\otimes F_{rs}v \\
&+ \lambda\Bigg[fg\otimes E_{00}v - (-1)^{p(f)}\sum_{i=1}^{n}\partial_i\big(f\xi_i g\big)\otimes v + (-1)^{p(f)}\sum_{i\neq j}(\partial_i f)\xi_j g\otimes F_{ij}v\Bigg] \\
&- \lambda^2\sum_{i<j} f\xi_i\xi_j g\otimes F_{ij}v\Bigg\}.
\end{aligned}$$

7.3. Singular vectors. By Theorem 2.4, the classification of irreducible finite modules over the Lie conformal superalgebra K_n reduces to the study of singular vectors in the induced modules $\mathrm{Ind}(F)$, where F is an irreducible finite-dimensional $\mathfrak{cso}(n)$-module. This section will be devoted to the classification of singular vectors.

When we discuss the highest weight of vectors and singular vectors, we always mean with respect to the upper Borel subalgebra in $K(1,n)_+$ generated by $(K(1,n)_+)_{>0}$ and the elements of the Borel subalgebra of $\mathfrak{so}(n)$ in $(K(1,n)_+)_0$. More precisely, recall (7.8), where we defined $F_{ij} = -\xi_i\xi_j \in (K(1,n)_+)_0 \simeq \mathbb{C}E_{00} \oplus \mathfrak{so}(n)$. Observe that F_{ij} corresponds to $E_{ij} - E_{ji} \in \mathfrak{so}(n)$, where E_{ij} are the elements of the standard basis of matrices. Consider the following (standard) notation (cf. [**11**], p.83). Let

$$H_j = i\ F_{2j-1,2j}, \qquad 1 \le j \le m, \tag{7.12}$$

a basis of a Cartan subalgebra of $\mathfrak{g} = \mathfrak{so}(n,\mathbb{C})$, with $m = [\frac{n}{2}]$.

In order to write explicitly weights for vectors in $K(1,n)_+$-modules, we will consider the basis for the Cartan subalgebra $\mathfrak{h}$ in $(K(1,n)_+)_0 \simeq \mathbb{C}E_{00} \oplus \mathfrak{so}(n)$, introduced above:

$$E_{00}; H_1, \dots, H_m, \qquad \text{with } m = \left[\frac{n}{2}\right],$$

and we shall write the weight of an eigenvector for the Cartan subalgebra $\mathfrak{h}$ as an $m+1$-tuple of the corresponding eigenvalues of this basis:

$$\lambda = (\mu; \lambda_1, \dots, \lambda_m). \tag{7.13}$$

Observe that a vector $\vec{m}$ in the $K(1,n)_+$-module $\mathrm{Ind}(F)$ is a singular highest weight vector if and only if the following conditions are satisfied

(S1) $\frac{d^2}{d\lambda^2}\,(f\,{}_\lambda\,\vec{m}) = 0$ for all $f \in \Lambda(n)$,

(S2) $\frac{d}{d\lambda}\,(f\,{}_\lambda\,\vec{m})|_{\lambda=0} = 0$ for all $f = \xi_I$ with $|I| \geq 1$,

(S3) $(f\,{}_\lambda\,\vec{m})|_{\lambda=0} = 0$ for all $f = \xi_I$ with $|I| \geq 3$ or $f \in B_{\mathfrak{so}(n)}$.

In order to classify the finite irreducible K_n-modules we should solve the equations (S1-3) to obtain the singular vectors. The next theorem gives us the complete classification of singular vectors:

Theorem 7.3. *Let F be an irreducible finite-dimensional $\mathfrak{cso}(n)$-module with highest weight λ.*

If $n \geq 4$, then $\vec{m} \in \mathrm{Ind}(F)$ is a non-trivial singular highest weight vector if and only if $\vec{m}$ is one of the following vectors (in the Hodge dual basis):

(a) $\vec{m} = \left(\xi_{\{2\}^c} - i\,\xi_{\{1\}^c}\right) \otimes v_\lambda$, *where v_λ is a highest weight vector of the $\mathfrak{cso}(n)$-module F and $\lambda = (-k; k, 0, \dots, 0)$, with $k \in \mathbb{Z}_{>0}$,*

(b) $$\vec{m} = \sum_{l=1}^{m} \left[\left(\xi_{\{2l\}^c} + i\,\xi_{\{2l-1\}^c}\right) \otimes w_l + \left(\xi_{\{2l\}^c} - i\,\xi_{\{2l-1\}^c}\right) \otimes \overline{w}_l \right] - \\ - \delta_{n,\mathrm{odd}}\; i\,\xi_{\{2m+1\}^c} \otimes w_{m+1},$$

where $w_1 = v_\lambda$ is a highest weight vector of the $\mathfrak{cso}(n)$-module F with highest weight

$$\lambda = (n + k - 2; k, 0, \dots, 0), \text{ for } k \in \mathbb{Z}_{\geq 0},$$

and all $w_l, \overline{w}_l$ are non-zero and uniquely determined by v_λ.

If $n = 3$, then $\vec{m} \in \mathrm{Ind}(F)$ is a non-trivial singular highest weight vector if and only if $\vec{m}$ is one of the following vectors:

(a) $\vec{m} = \left(\xi_{\{2\}^c} - i\,\xi_{\{1\}^c}\right) \otimes v_\lambda$, *where v_λ is a highest weight vector of the $\mathfrak{cso}(3)$-module V and $\lambda = (-k; k)$, with $k \in \frac{1}{2}\mathbb{Z}_{>0}$,*

(b) $\vec{m} = \left(\xi_{\{2\}^c} + i\,\xi_{\{1\}^c}\right) \otimes v_\lambda + \left(\xi_{\{2\}^c} - i\,\xi_{\{1\}^c}\right) \otimes w_1 - i\,\xi_{\{3\}^c} \otimes w_2$,

where v_λ is a highest weight vector of the $\mathfrak{cso}(3)$-module F with highest weight

$$\lambda = (k + 1; k), \text{ for } k \in \frac{1}{2}\mathbb{Z}_{\geq 0} \text{ and } k \neq \frac{1}{2},$$

and all w_l, w_2 are non-zero and uniquely determined by v_λ.

(c) $\vec{m} = \partial\,\left(\xi_* \otimes v_\lambda\right) + \;i\;\xi_{\{1,2\}^c} \otimes v_\lambda - \;2\xi_{\{2,3\}^c} \otimes F_{2,3}v_\lambda + \;2\;\xi_{\{1,3\}^c} \otimes F_{1,3}v_\lambda$, *where v_λ is a highest weight vector of the $\mathfrak{cso}(3)$-module F with highest weight $\lambda = (\frac{3}{2}; \frac{1}{2})$.*

The proof of this theorem was done through several lemmas in appendix B, [**1**]. Since this is quite technical, we will omit it.

REMARK 7.4. (a) The explicit expression of all non-zero vectors $w_l, \bar{w}_l$ in terms of v_λ that appear in the second family of singular vectors for all $n \geq 3$, are given in [**1**].

(b) If n=4, the first family of singular vectors $\vec{m} = \left(\xi_{\{2\}^c} - i\,\xi_{\{1\}^c}\right) \otimes v_\lambda$, where v_λ is a highest weight vector of the $\mathfrak{cso}(4)$-module F and $\lambda = (-k; k, 0)$, with $k \in \mathbb{Z}_{>0}$, corresponds to the family of singular vectors b_2 in Proposition 7.2(i) in [**5**]. Finally, the second family of singular vectors in Theorem 7.3(b), correspond to the family of singular vectors b_5 in Proposition 7.2(ii) in [**5**].

(c) If n=3, the singular vectors in the cases (a), (b) and (c) described in the previous theorem, correspond to the vectors a_2, a_4 and a_6 in Proposition 5.1 in [**5**], respectively. Observe that the families (a) and (b) described for $n \geq 4$ correspond to the families (a) and (b) for $n = 3$, but in the latter case the parameter k is one half a positive integer. Observe that the missing case $(k+1; k)$ with $k = \frac{1}{2}$ in the family (b) is completed by the case (c).

7.4. Modules of differential forms, the contact complex and irreducible induced $K(1,n)_+$-modules. We will use the standard notation introduced in 4.2. Recall that $K(1,n)_+$ is a subalgebra of $W(1,n)_+$, defined by (7.6). Hence Ω_+ and Ω_+^k are $K(1,n)_+$-modules as well.

Observe that the differential of the standard contact form $\omega = dt - \sum_{i=1}^n \xi_i d\xi_i$ is $d\omega = -\sum_{i=1}^n (d\xi_i)^2$, and following Rumin's construction in [**13**], consider for $k \geq 2$

$$I^k = d\omega \wedge \Omega^{k-2} \ + \ \omega \wedge \Omega^{k-1} \ \subset \ \Omega^k, \tag{7.14}$$

$$I_+^k = d\omega \wedge \Omega_+^{k-2} \ + \ \omega \wedge \Omega_+^{k-1} \ \subset \ \Omega_+^k, \tag{7.15}$$

and $I^1 = \omega \wedge \Omega^0$, $I_+^1 = \omega \wedge \Omega_+^0$, $I^0 = 0 = I_+^0$. It is clear that $d(I^k) \subseteq I^{k+1}$ and $d(I_+^k) \subseteq I_+^{k+1}$, and using (7.6) it is easy to prove that I^k and I_+^k are $K(1,n)_+$-submodules of Ω^k and Ω_+^k, respectively. Therefore we have the following *contact complex* of $K(1,n)_+$-modules (we also denote by d the induced maps in the quotients):

$$0 \longrightarrow \mathbb{C} \xrightarrow{d} \Omega_+^0 \xrightarrow{d} \Omega_+^1/I_+^1 \xrightarrow{d} \Omega_+^2/I_+^2 \xrightarrow{d} \cdots \tag{7.16}$$

Let $\mathbb{C}[d\xi_i]^l \subseteq \Omega_+^l$ be the subspace of homogeneous polynomials in $d\xi_1, \ldots, d\xi_n$ of degree l. Using that the action of $\mathfrak{cso}(n) = \mathbb{C}\ E_{00} \oplus \mathfrak{so}(n) = (K(1,n)_+)_0$ in Ω_+^l is given by

$$E_{00} \longmapsto 2\,t\,\partial_t + \sum_{i=1}^n \xi_i\,\partial_i, \ \ F_{ij} \longmapsto \xi_i \partial_j - \xi_j \partial_i, \tag{7.17}$$

it follows that $\mathbb{C}[d\xi_i]^l$ is a $\mathfrak{cso}(n)$-invariant subspace. Now, consider $\Gamma^l = \pi(\mathbb{C}[d\xi_i]^l)$, where $\pi : \Omega_+^l \longrightarrow \Omega_+^l/I_+^l$, and take $\Theta^l = (\Gamma^l)^{\#}$. Here and further, we denote by $\#$ the restricted dual, that is the sum of the dual of all the graded components of the initial module, as in 4.2.1. Then, we have

PROPOSITION 7.5. *(1) The $\mathfrak{cso}(n)$-module $\Theta^l, l \geq 0$, is irreducible with highest weight $(-l; l, 0, \ldots, 0)$.*

(2) The $K(1,n)_+$-module $(\Omega_+^l/I_+^l)^\#, l \geq 0$, contains Θ^l and this inclusion induces the isomorphism

$$(\Omega_+^l/I_+^l)^\# = \mathrm{Ind}(\Theta^l).$$

(3) The dual maps $d^\# : (\Omega_+^{l+1}/I_+^{l+1})^\# \to (\Omega_+^l/I_+^l)^\#$ are morphisms of $K(1,n)_+$-modules. The kernel of one of them is equal to the image of the next one and it is a non-trivial proper submodule in $(\Omega_+^l/I_+^l)^\#$.

COROLLARY 7.6. *The following $K(1,n)_+$-modules are isomorphic*

$$\Omega_+^k/I_+^k = (\mathrm{Ind}(\Gamma^k))^*.$$

Let us now study the $K(1,n)_+$-modules Ω_-^k. Recall that we identified (via isomorphism) Ω_-^k with Ω^k/Ω_+^k. Let $\widetilde{\pi} : \Omega^k \to \Omega^k/\Omega_+^k = \Omega_-^k$. Observe that $I_-^k = \widetilde{\pi}(I^k)$ is a $K(1,n)_+$-submodule of Ω_-^k, and $d(I_-^k) \subseteq I_-^{k+1}$. Let

$$\xi_* = \xi_1 \cdots \xi_n, \qquad \text{and} \qquad \Gamma_-^k = t^{-1}\xi_*\Omega_c^k \subset \Omega_-^k.$$

PROPOSITION 7.7. *For $\mathfrak{g} = K(1,n)_+$, we have:*

(1) The $\mathfrak{cso}(n)$-module Γ_-^k is an irreducible submodule of Ω_-^k with highest weight

$$(n+k-2; k, 0, \ldots, 0), \quad \textit{for } k \geq 0,$$

and $\mathfrak{g}_{>0}$ acts trivially on Γ_-^k.

(2) There is a $\mathfrak{g}$-module isomorphism $\Omega_-^k/I_-^k = \mathrm{Ind}(\Gamma_-^k)$.

(3) The differential d gives us $\mathfrak{g}$-module morphisms on Ω_-^k/I_-^k, and the kernel and image of d are $\mathfrak{g}$-submodules in Ω_-^k/I_-^k.

(4) The kernel of d and image of d in Ω_-^k/I_-^k for $k \geq 2$ coincide, in Ω_-^1/I_-^1 we have $\mathrm{Ker}\ d = \mathbb{C}(\overline{t^{-1}dt}) + \mathrm{Im}\ d$*, and in Ω_-^0, we have* $\mathrm{Ker}\ d = 0$.

PROOF. (1) First, a simple computation shows that $\mathfrak{g}_{>0}$ maps Γ_-^k to zero. Also, as a $\mathfrak{g}_0$-module, Γ_-^k is isomorphic to the space of harmonic polinomials in $d\xi_1, \ldots, d\xi_n$ of degree k multiplied by the 1-dimensional module $\langle t^{-1}\xi_*\rangle$. This permits us to see that its highest weight vectors are

$$\begin{array}{ll} \langle t^{-1}\xi_*\rangle & \text{for } k = 0, \\ \langle t^{-1}\xi_*(d\xi_1 - id\xi_2)^k\rangle & \text{for } k \geq 1. \end{array}$$

The values of the highest weights are easy to compute using (7.17).

(2) It is straightforward to see that Ω_-^0 is a free rank 1 $\mathbb{C}[\partial_0, \partial_1, \ldots, \partial_n]$-module. Now, the action of $\partial_0, \partial_1, \ldots, \partial_n$ on Ω_-^k/I_-^k is coefficientwise, hence the fact that Ω_-^k/I_-^k is a free $\mathbb{C}[\partial_0, \partial_1, \ldots, \partial_n]$-module follows. This gives us the isomorphism $\Omega_-^k/I_-^k = \mathrm{Ind}(\Gamma_-^k)$.

(3) It follows immediately from the fact that d commutes with the action of vector fields.

(4) Let $\alpha \in \Omega_-^k$ be such that $d\alpha \in I_-^{k+1}$. Then $d\alpha = \omega\wedge\beta + d\omega\wedge\gamma$, with $\beta \in \Omega_-^k$ and $\gamma \in \Omega_-^{k-1}$. Observe that $d(\alpha - \omega\wedge\gamma) = \omega\wedge(\beta - d\gamma)$, hence, by replacing α by another representative, we may assume that $\gamma = 0$. Since $0 = d^2\alpha = d(\omega\wedge\beta) = d\omega\wedge\beta - \omega\wedge d\beta$, then $d\omega\wedge d\alpha = d\omega\wedge(\omega\wedge\beta) = \omega\wedge\omega\wedge d\beta = 0$. Therefore, $d\alpha \in \mathrm{Ker}(d\omega\wedge\ \cdot\ ) = 0$. But the differential complex $(\Omega_-^\bullet, d)$ is exact except for $k = 1$ (see Proposition 4.3), proving the statement. □

In the last part of this subsection, we classify the irreducible induced $K(1,n)_+$-modules. Let $\mathfrak{g} = K(1,n)_+$. Now, we have the following:

THEOREM 7.8. *Let F_λ be an irreducible $\mathfrak{g}_0$-module with highest weight λ.*
If $n \geq 4$, then the $\mathfrak{g}$-module $\mathrm{Ind}(F_\lambda)$ is an irreducible (finite conformal) module except for the following cases:

(a) $\lambda = (-l; l, 0, \ldots, 0), l \geq 0$, $\mathrm{Ind}(F_\lambda) = (\Omega_+^l/I_+^l)^\#$, and $d^\#(\Omega_+^{l+1}/I_+^{l+1})^\#$ is the only non-trivial proper submodule.

(b) $\lambda = (n+k-2; k, 0, \ldots, 0), k \geq 1$, and $\mathrm{Ind}(F_\lambda) = \Omega_-^k/I_-^k$. For $k \geq 2$ the image $d\Omega_-^{k-1}/I_-^{k-1}$ is the only non-trivial proper submodule. For $k = 1$, both $\mathrm{Im}(d)$ and $\mathrm{Ker}(d)$ are proper submodules, and $\mathrm{Ker}(d)$ is a maximal submodule.

PROOF. We know from Theorem 2.4 that in order for the $\mathfrak{g}$-module $\mathrm{Ind}(F)$ to be reducible it has to have non-trivial singular vectors and the possible highest weights of F in this situation are listed in Theorem 7.3 above.

The fact that the induced modules are actually reducible in those cases is known because we have got nice realizations for these induced modules in Propositions 7.5 and 7.7 together with morphisms defined by $d, d^\#$, so kernels and images of these morphisms become submodules.

The subtle thing is to prove that a submodule is really a maximal one. We notice that in each case the factor is isomorphic to a submodule in another induced module so it is enough to show that the submodule is irreducible. This can be proved as follows, a submodule in the induced module is irreducible if it is generated by any highest singular vector that it contains. We see from our list of non-trivial singular vectors that there is at most one such a vector for each case and the images and kernels in question are exactly generated by those vectors, hence they are irreducible. □

COROLLARY 7.9. *The theorem gives us a description of finite conformal irreducible $K(1,n)_+$-modules for $n \geq 4$. Such a module is either $\mathrm{Ind}(F)$ for an irreducible finite-dimensional $\mathfrak{g}_0$-module F, where the highest weight of F does not belong to the types listed in (a), (b) of the theorem, or the factor of an induced module from (a), (b) by its submodule $Ker(d)$.*

7.5. Finite irreducible K_n-modules. Recall (see Section 4.5) that the conformal algebra of differential forms Ω_n is the current algebra over the commutative associative superalgebra $\Omega(n) + \Omega(n)\ dt$ with the obvious multiplication and parity, subject to the relation $(dt)^2 = 0$:

$$\Omega_n = \mathrm{Cur}(\Omega(n) + \Omega(n)\ dt).$$

The de Rham differential $\tilde{d}$ of Ω_n (we use the tilde in order to distinguish it from the de Rham differential d on $\Omega(n)$) is a derivation of the conformal algebra Ω_n such that:

$$\tilde{d}(\omega_1 + \omega_2 dt) = d\omega_1 + d\omega_2 dt - (-1)^{p(\omega_1)}\partial(\omega_1 dt). \tag{7.18}$$

here and further $\omega_i \in \Omega(n)$.

The standard $\mathbb{Z}_+$-gradation $\Omega(n) = \oplus_{j\in\mathbb{Z}_+}\Omega(n)^j$ of the superalgebra of differential forms by their degree induces a $\mathbb{Z}_+$-gradation

$$\Omega_n = \oplus_{j\in\mathbb{Z}_+}\Omega_n^j, \qquad \text{where } \Omega_n^j = \mathbb{C}[\partial]\otimes(\Omega(n)^j + \Omega(n)^{j-1}\ dt),$$

so that $\tilde{d} : \Omega_n^j \to \Omega_n^{j+1}$.

Let $\omega = dt - \sum_{i=1}^n \xi_i d\xi_i \in \Omega_n^1$. Observe that $\tilde{d}\omega = -\sum_{i=1}^n (d\xi_i)^2$. Now, we define, for $j \geq 2$,

$$\begin{aligned} I_n^j &= \mathbb{C}[\partial]\otimes\left(\omega\wedge\Omega(n)^{j-1}\ +\ d\omega\wedge\Omega(n)^{j-2}\ dt\right)\ \subset\ \Omega_n^j, \\ I_n^1 &= \mathbb{C}[\partial]\otimes(\omega\wedge\Omega(n)^0), \qquad I^0 = 0. \end{aligned} \tag{7.19}$$

It is clear that $\tilde{d}(I_n^j) \subseteq I_n^{j+1}$, and it is easy to prove that I_n^j is K_n-submodules of Ω_n^j. Therefore, we get a Rumin conformal complex $(\Omega_n^j/I_n^j, \tilde{d})$, where we also denote by $\tilde{d}$ the differential in the quotient.

Let V be a finite dimensional irreducible $\mathfrak{cso}(n)$-module, using the results of Section 2 and recalling that the annihilation algebra of K_n is $K(1,n)_+$, we have that the $K(1,n)_+$-modules $\mathrm{Ind}\,(V)$ studied in the previous section are K_n-modules with the λ-action given by Theorem 7.2. We denote by $\mathrm{Tens}\,(V)$ the corresponding K_n-module.

Since the extended annihilation algebra $K(1,n)^+$ is a direct sum of $K(1,n)_+$ and a 1-dimensional Lie algebra $\mathbb{C}a$, any irreducible $K(1,n)^+$-module is obtained from a $K(1,n)_+$-module M by extending to $K(1,n)^+$, letting $a \mapsto -\alpha$, where $\alpha \in \mathbb{C}$. Translating into the conformal language (see Proposition 2.3), we see that all K_n-modules are obtained from conformal $K(1,n)_+$-modules by taking for the action of ∂ the action of $-\partial_t + \alpha I, \alpha \in \mathbb{C}$. We denote by $\mathrm{Tens}_\alpha V$ and $\Omega_{k,\alpha}, \alpha \in \mathbb{C}$, the K_n-modules obtained from $\mathrm{Tens} V$ and Ω_k by replacing ∂ by $\partial + \alpha$ in the corresponding actions.

As in [**2**], we see that Theorem 7.8 and Corollary 7.9, along with Theorem 2.4 and Propositions 2.3, 2.7 and 2.8, give us a complete description of finite irreducible K_n-modules, namely we obtain the following theorem.

THEOREM 7.10. *The following is a complete list of non-trivial finite irreducible K_n-modules ($n \geq 4, \alpha \in \mathbb{C}$):*

(1) $\mathrm{Tens}_\alpha V$*, where V is a finite-dimensional irreducible $\mathfrak{cso}(n)$-module with highest weight different from $(-k; k, 0, \ldots, 0)$ and $(n+k-2; k, 0, \ldots, 0)$ for $k = 1, 2, \ldots$,*

(2) $\left(\Omega_n^k/I_n^k\right)^*_\alpha \Big/ \mathrm{Ker}\ \tilde{d}^*, k = 1, 2, \ldots$ *, and the same modules with reversed parity,*

(3) *K_n-modules dual to (2), with $k > 1$.*

REMARK 7.11. (a) Using Proposition 7.7, we have that the kernel of $\tilde{d}$ and the image of $\tilde{d}$ coincide in Ω_n^k/I_n^k for $k \geq 2$. Now, since Ω_n^{k+2}/I_n^{k+2} is a free $\mathbb{C}[\partial]$-module of finite rank and $(\Omega_n^{k+1}/I_n^{k+1})/\mathrm{Im}\,\tilde{d} = (\Omega_n^{k+1}/I_n^{k+1})/\mathrm{Ker}\,\tilde{d} \simeq \mathrm{Im}\,\tilde{d} \subset \Omega_n^{k+2}/I_n^{k+2}$, we obtain that $(\Omega_n^{k+1}/I_n^{k+1})/\mathrm{Im}\,\tilde{d}$ is a finitely generated free $\mathbb{C}[\partial]$-module. Therefore, we can apply Proposition 2.6 in [**2**], and we have that

$$\left(\Omega_n^{k+1}/I_n^{k+1}\right)^*/\mathrm{Ker}\ \tilde{d}^* \simeq \left((\Omega_n^k/I_n^k)/\mathrm{Ker}\ \tilde{d}\right)^* \tag{7.20}$$

for $k \geq 1$.

(b) Since for a free finite rank module M over a Lie conformal superalgebra we have $M^{**} = M$, using (7.20), the K_n-modules in case (3) of Theorem 7.10 are isomorphic to $(\Omega_n^k/I_n^k)_\alpha/\mathrm{Ker}\ \tilde{d}$, $k = 1, 2, \ldots$.

(c) Let V be a finite-dimensional (one dimesional in fact) irreducible $\mathfrak{cso}(n)$-module with highest weight $(0; 0, \ldots, 0)$. Observe that the module Tens V has a maximal submodule of codimension 1 over $\mathbb{C}$. Hence, the irreducible quotient is the one dimensional (over $\mathbb{C}$) trivial K_n -module. Therefore, we excluded the case $k = 0$ in Theorem 7.10(2).

(d) Let V be a finite-dimensional irreducible $\mathfrak{cso}(n)$-module with highest weight $(n-2; 0, 0, \ldots, 0)$. Observe that in case (3) in Theorem 7.10, we excluded $k = 1$, because in this case the dual corresponds to the module $\mathrm{Tens}_\alpha\ V$, which is isomorphic to $\Omega_{0,\alpha}$ and it is an irreducible tensor module, therefore this module is included in case (1) of Theorem 7.10.

(e) The case $K_2 \simeq W_1$ was studied in full detail at the end of Section 4.5.

8. Work in progress

In order to complete the classification of finite simple modules of all conformal superalgebras, the remaining cases are:

- K_3, in this case the classification is known, but it still remains to give a nice realization of the irreducible finite modules.
- K_4', the Lie conformal superalgebra which is the derived subalgebra of $K_4 = K_4' \oplus \mathbb{C}\xi_1 \ldots \xi_4$.
- CK_6. This is a simple rank 32 subalgebra of K_6, whose even part is $W_0 \ltimes \mathrm{Cur}\ \mathfrak{so}_6$ and whose odd part is spanned by six primary fields of conformal weight 3/2 and ten primary fields of conformal weight 1/2. For the explicit form of the commutation relations, as well as for more detailed information on CK_6, see [**3**].

These cases are part of a work in progress and will be worked out in subsequent publications. At this moment we have some reduction lemmas that produce the classification of singular vectors of CK_6.

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