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Research Article

The Central Extension Defining the Super Matrix Generalization of $W_{1+\infty}$

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We prove that the Lie superalgebra of regular differential operators on the superspace $\mathbb{C}^{M|N}[t,t^{-1}]$ has an essentially unique non-trivial central extension.

1. Introduction

The W infinity algebras naturally arise in various physical systems, such as two-dimensional quantum gravity and the quantum Hall effects (see the review [1, 2] and references there in). The most fundamental one is the $W_{1+\infty}$ which is the central extension of the Lie algebra of regular differential operators on the circle [1–5], and it contains the W_∞ algebra as a subalgebra. Various extensions where constructed: super extension ($W_\infty^{1|1}$) [6, 7], $u(M)$ matrix version of $W_{1+\infty}(W_{1+\infty}^{M})$ [8], and the most general super matrix generalization $W_{1+\infty}^{M|N}$ presented in [1, 2, 9]. It seems difficult to decide where and when the first definition of a (version of) super-W algebra appeared, but a book by Guieu and Roger [10] has a good historical and bibliographic base, including the pioneering papers of Radul where the superanalogues of the Bott-Virasoro cocycles were introduced (see [11]). The original $W_{1+\infty}$ corresponds to $M = 1$, $N = 0$. The general study of representation theory of W infinity algebras started in the remarkable work [4] by Kac and Radul and continued in several works (some of them are [6, 12–14]). Matrix generalizations are deeply related to the main examples of infinite rank conformal algebras (see [15–17]).

The super matrix generalization $W_{1+\infty}^{M|N}$ is defined as *a specific* central extension of the Lie superalgebra of regular differential operators on the superspace $\mathbb{C}^{M|N}[t,t^{-1}]$. Only in the special case of $W_{1+\infty}$ (i.e., $M = 1, N = 0$) was it proved that the 2-cocycle defining this central extension is unique up to coboundary [18]. The main goal of the present work is to extend this result to the super matrix generalization $W_{1+\infty}^{M|N}$. Similar studies of central extensions for q-analogs and other versions can be found in [19, 20].

2. Basic Definitions and Main Result

Let L and $\widehat{L}$ be two Lie superalgebras over $\mathbb{C}$. The Lie superalgebra $\widehat{L}$ is said to be a one-dimensional central extension of L if $\widehat{L}$ is the direct sum of L and $\mathbb{C}C$ as vector spaces and the Lie superbracket in $\widehat{L}$ is given by

$$[a,b]^{\wedge} = [a,b] + \Psi(a,b)C, \quad [a,C]^{\wedge} = 0, \tag{2.1}$$

for all $a,b \in L$, where $[\cdot,\cdot]$ is the Lie bracket in L and $\Psi : L \times L \to \mathbb{C}$ is a *2-cocycle* on L, that is, a bilinear $\mathbb{C}$-valued form satisfying the following conditions for all homogeneous elements $a,b,c \in L$:

$$\begin{aligned} &(1) \quad \Psi(a,b) = -(-1)^{|a||b|}\Psi(b,a), \\ &(2) \quad \Psi([a,b],c) = \Psi(a,[b,c]) - (-1)^{|a||b|}\Psi(b,[a,c]), \end{aligned} \tag{2.2}$$

where $|a|$ denote the parity of a. A central extension is trivial if $\widehat{L}$ is the direct sum of a subalgebra M and $\mathbb{C}C$ as Lie algebras, where M is isomorphic to L. A 2-cocycle corresponding to a trivial central extension is called a *2-coboundary*, and it is given by an $f \in L^*$ as follows:

$$\alpha_f(a,b) = f([a,b]), \tag{2.3}$$

for $a,b \in L$. It is easy to check that α_f is a 2-cocycle. We say that the 2-cocycles Ψ,ϕ are *equivalent* if $\phi - \Psi$ is a 2-coboundary. The second cohomology group of L with coefficients in $\mathbb{C}$ is the set of equivalent classes of 2-cocycles, and it will be denoted by $H^2(L,\mathbb{C})$. If dim $H^2(L,\mathbb{C}) = 1$, we say that L has an essentially unique nontrivial one-dimensional central extension.

Now, we will introduce the Lie superalgebra that will be considered in this work. Let us denote by $\mathrm{Mat}(M \mid N)$ the associative superalgebra of linear transformations on the complex $(M \mid N)$-dimensional superspace $\mathbb{C}^{M|N}$. Namely, we consider the set of all $(M + N) \times (M + N)$ matrices of the form

$$A = \begin{pmatrix} A^0 & A^+ \\ A^- & A^1 \end{pmatrix}, \tag{2.4}$$

where A^0, A^+, A^-, A^1 are $M \times M, M \times N, N \times M, N \times N$ matrices, respectively, with complex entries. The $\mathbb{Z}_2$-gradation is defined by declaring that matrices of the form (2.4) with $A^+ = A^- = 0$ are even, and those with $A^0 = A^1 = 0$ are odd. We denote by $|A|$ the degree of A with respect to this $\mathbb{Z}_2$-gradation. The *supertrace* is defined by

$$\mathrm{Str}(A) = \mathrm{tr}\left(A^0\right) - \mathrm{tr}\left(A^1\right), \tag{2.5}$$

and it satisfies $\mathrm{Str}(AB) = (-1)^{|A||B|}\mathrm{Str}(BA)$.

Let $\mathfrak{D}_{\rm as}$ be the associative algebra of regular differential operators on the circle, that is, the operators on $\mathbb{C}[t, t^{-1}]$ of the form

$$E = e_k(t)\partial_t^k + e_{k-1}(t)\partial_t^{k-1} + \cdots + e_0(t), \quad \text{where } e_i(t) \in \mathbb{C}\left[t, t^{-1}\right]. \tag{2.6}$$

The elements

$$J_k^l = -t^{l+k}(\partial_t)^l \quad (l \in \mathbb{Z}_+, k \in \mathbb{Z}) \tag{2.7}$$

form its basis, where ∂_t denotes d/dt. Another basis of $\mathfrak{D}_{as}$ is

$$L_k^l = -t^k D^l \quad (l \in \mathbb{Z}_+, k \in \mathbb{Z}), \tag{2.8}$$

where $D = t\partial_t$. It is easy to see that

$$J_k^l = -t^k [D]_l. \tag{2.9}$$

Here and further we use the notation

$$[x]_l = x(x-1)\dots(x-l+1). \tag{2.10}$$

Denote by $S\mathfrak{D}_{\rm as}^{M|N}$ the associative superalgebra of $(M+N)\times(M+N)$ (super)matrices with entries in $\mathfrak{D}_{\rm as}$. The $\mathbb{Z}_2$-gradation is the one inherited by the corresponding $\mathbb{Z}_2$-gradation in $\mathrm{Mat}(M \mid N)$. By taking the usual superbracket we make $\mathcal{S}\mathfrak{D}_{\rm as}^{M|N}$ into a Lie superalgebra, which is denoted by $\mathcal{S}\mathfrak{D}^{M|N}$. A set of generators is given by $\{t^s f(D)A \ : \ s \in \mathbb{Z}, \ f \in \mathbb{C}[x], A \in \mathrm{Mat}(M \mid N)\}$.

Let $W_{1+\infty}^{M,N} = \mathcal{S}\mathfrak{D}^{M|N} \oplus \mathbb{C}C$ be the central extension of $\mathcal{S}\mathfrak{D}^{M|N}$ by a one-dimensional vector space with a specified generator C, whose commutation relation for homogeneous elements is given by

$$\begin{aligned}\left[t^r f(D)A, t^s g(D)B\right] = {} & t^{r+s} f(D+s)g(D)AB - (-1)^{|A||B|} t^{r+s} f(D)g(D+r)BA \\ & + \Psi\left(t^r f(D)A, t^s g(D)B\right)C,\end{aligned} \tag{2.11}$$

where the 2-cocycle Ψ is given by

$$\Psi\left(t^r f(D)A, t^s g(D)B\right) = \begin{cases} \left(\displaystyle\sum_{-r\le j\le -1} f(j)g(j+r)\right)\mathrm{Str}(AB) & \text{if } r=-s\ge 0, \\ 0, & \text{if } r+s \neq 0. \end{cases} \tag{2.12}$$

Now, we are in condition to state our main result.

Theorem 2.1. *One has the following: dim* $H^2(\mathcal{S}\mathfrak{D}^{M|N}, \mathbb{C}) = 1$.

3. Proof of Theorem 2.1

We will need the explicit expression of the bracket of basis elements of type (2.9) in $\mathcal{S}\mathfrak{D}^{M|N}$:

$$[t^m[D]_l E_{ij}, t^n[D]_k E_{rs}] = t^{m+n}\Big([D+n]_l[D]_k\delta_{jr}E_{is} - (-1)^{|E_{ij}||E_{rs}|}[D]_l[D+m]_k\delta_{is}E_{rj}\Big). \tag{3.1}$$

In particular, we have

$$\begin{gathered}\left[t^{-1}DE_{ii}, t^m[D]_l E_{ii}\right] = (l+m)t^{m-1}[D]_l E_{ii},\\ \left[t^{-l-1}[D]_l E_{ii}, DE_{ii}\right] = (l+1)t^{-l-1}[D]_l E_{ii},\\ [E_{ii}, t^m[D]_l E_{ij}] = t^m[D]_l E_{ij}, \quad i \neq j.\end{gathered} \tag{3.2}$$

Let β be a 2-cocycle on $\mathcal{S}\mathfrak{D}^{M|N}$. We consider the linear functional in $\mathcal{S}\mathfrak{D}^{M|N}$ defined by

$$\begin{gathered}f_\beta\Big(t^{m-1}[D]_l E_{ii}\Big) = \frac{1}{l+m}\beta\Big(t^{-1}DE_{ii}, t^m[D]_l E_{ii}\Big), \quad l \neq -m,\\ f_\beta\Big(t^{-l-1}[D]_l E_{ii}\Big) = \frac{1}{l+1}\beta\Big(t^{-l-1}[D]_l E_{ii}, DE_{ii}\Big),\\ f_\beta(t^m[D]_l E_{ij}) = \beta(E_{ii}, t^m[D]_l E_{ij}), \quad i \neq j.\end{gathered} \tag{3.3}$$

Then $\beta_1 = \beta - \alpha_{f_\beta}$ is a 2-cocycle on $\mathcal{S}\mathfrak{D}^{M|N}$ that is equivalent to β, and using (3.3), we obtain

$$\begin{gathered}\beta_1\Big(t^{-1}DE_{ii}, t^m[D]_l E_{ii}\Big) = 0, \quad l \neq -m,\\ \beta_1\Big(t^{-l-1}[D]_l E_{ii}, DE_{ii}\Big) = 0,\\ \beta_1(E_{ii}, t^m[D]_l E_{ij}) = 0, \quad i \neq j.\end{gathered} \tag{3.4}$$

In order to complete the proof we need to show that $\Psi = a\beta_1$ for some $a \in \mathbb{C}$. By observing the supertrace that appears in the expression of Ψ in (2.12), we immediately obtain that for any $f, g \in \mathrm{D_{as}}$

$$\Psi(fE_{ij}, gE_{sk}) = 0 \quad \text{if } i \neq k \text{ or } j \neq s. \tag{3.5}$$

In Lemmas 3.1 and 3.2, we will show that β_1 also satisfies (3.5).

Lemma 3.1. *For any* $f, g \in \mathfrak{D}_{\mathrm{as}}$, $\beta_1(fE_{ii}, gE_{sj}) = 0$ *if* $i \neq j$ *or* $i \neq s$.

Proof. *Case* $j = i$ *and* $s \neq i$.

Using that E_{ii} is even, $i \neq s$, and (2.2), we obtain that

$$\begin{aligned}\beta_1(fE_{ii}, gE_{si}) &= \beta_1(fE_{ii}, [E_{ss}, gE_{si}]) = -\beta_1([E_{ss}, gE_{si}], fE_{ii})\\ &= -\beta_1(E_{ss}, [gE_{si}, fE_{ii}]) + \beta_1(gE_{si}, [E_{ss}, fE_{ii}])\\ &= -\beta_1(E_{ss}, (g \circ f)E_{si}) = 0, \quad (\text{using } \ i \neq s \text{ and } (3.4)),\end{aligned} \tag{3.6}$$

where $g \circ f$ is the product in $\mathfrak{D}_{\mathrm{as}}$.

Case $j \neq i$ *and* $s = i$.

In this case we have

$$\begin{aligned}\beta_1(fE_{ii}, gE_{ij}) &= \beta_1(fE_{ii}, [gE_{ij}, E_{jj}]) = -\beta_1([gE_{ij}, E_{jj}], fE_{ii})\\ &= -\beta_1(gE_{ij}, [E_{jj}, fE_{ii}]) + \beta_1(E_{jj}, [gE_{ij}, fE_{ii}]) \quad (\text{by } (2.2))\\ &= \beta_1(E_{jj}, (f \circ g)E_{ij}) = 0 \quad (\text{using } \ i \neq j \text{ and } (3.6)).\end{aligned} \tag{3.7}$$

Case $j \neq i$ *and* $s \neq i$.

By taking the usual bracket, we make the associative algebra $\mathfrak{D}_{\mathrm{as}}$ into a Lie algebra which is denoted by $\mathfrak{D}$. Observe that

$$\mathfrak{D} = \mathcal{S}\mathfrak{D}^{1|0}. \tag{3.8}$$

It is easy to show that $[\mathfrak{D}, \mathfrak{D}] = \mathfrak{D}$; therefore, for any $f \in \mathfrak{D}$, we have

$$f = \sum_l [f_l, h_l], \quad f_l, h_l \in \mathfrak{D}. \tag{3.9}$$

Thus, if $j \neq i$ and $s \neq i$, using (2.2),

$$\begin{aligned}\beta_1(fE_{ii}, gE_{sj}) &= \beta_1\left(\sum_l [f_lE_{ii}, h_lE_{ii}], gE_{sj}\right)\\ &= \sum_l \beta_1(f_lE_{ii}, [h_lE_{ii}, gE_{sj}]) - \sum_l \beta_1(h_lE_{ii}, [f_lE_{ii}, gE_{sj}]) = 0.\end{aligned} \tag{3.10}$$

The proof is finished. □

Lemma 3.2. *For any* $f, g \in \mathfrak{D}_{\mathrm{as}}$ *and* $i \neq j, s \neq k$, $\beta_1(fE_{ij}, gE_{sk}) = 0$ *when* $i \neq k$ *or* $j \neq s$.

Proof. If $i \neq j$ and $k \neq i$, we have

$$\begin{aligned}\beta_1(fE_{ij}, gE_{sk}) &= \beta_1([E_{ii}, fE_{ij}], gE_{sk}) \\ &= \beta_1(E_{ii}, [fE_{ij}, gE_{sk}]) - \beta_1(fE_{ij}, [E_{ii}, gE_{sk}]) \\ &= \delta_{j,s}\beta_1(E_{ii}, (f \circ g)E_{ik}) - \delta_{i,s}\beta_1(fE_{ij}, gE_{ik}) \\ &= -\delta_{i,s}\beta_1(fE_{ij}, gE_{ik}) \quad (\text{using}(3.4)).\end{aligned} \tag{3.11}$$

Hence we have $\beta_1(fE_{ij}, gE_{sk}) = 0$.

Finally, using skew-symmetry and the previous case, if $i \neq j$, $s \neq k$, and $s \neq j$, we have that $\beta_1(fE_{ij}, gE_{sk}) = 0$. □

Now, it remains to consider the expression $\beta_1(fE_{ij}, gE_{ji})$. In order to do it, consider again the Lie algebra $\mathfrak{D} = \mathcal{S}\mathfrak{D}^{1|0}$ (see (3.8)) and denote by $\psi_{\mathfrak{D}}$ the 2-cocycle Ψ defined in (2.12) with $M = 1$ and $N = 0$.

In fact, from the expression of Ψ, we have

$$\Psi(fA, gB) = \psi_{\mathfrak{D}}(f, g)\mathrm{Str}(AB). \tag{3.12}$$

Lemma 3.3. *There exist* $a_i \in \mathbb{C}$ *such that for all* $f, g \in \mathfrak{D}_{\mathrm{as}}$

$$\beta_1(fE_{ii}, gE_{ii}) = a_i\psi_{\mathfrak{D}}(f, g). \tag{3.13}$$

Moreover, the constants a_i *satisfy* $a_i = (-1)^{|E_{ij}|}a_j$ *for all* $i \neq j$.

Proof. Let $\gamma_i : \mathfrak{D} \times \mathfrak{D} \to \mathbb{C}$ be the bilinear map defined by $(i = 1, \ldots, M + N)$

$$\gamma_i(f, g) = \beta_1(fE_{ii}, gE_{ii}). \tag{3.14}$$

Since E_{ii} is even, we have that γ_i is a 2-cocycle in $\mathfrak{D}$.

The following statement was proved in [18] (see Proof of Theorem 2.1 in page 74 and (3.2) and (3.3) in this work): if a 2-cocycle β_1 in $\mathfrak{D}$ satisfies $(l \in \mathbb{Z}_+, m \in \mathbb{Z})$

$$\begin{aligned}\beta_1\left(t^m[D]_l, t^{-1}D\right) &= 0, \\ \beta_1\left(t^{-1-l}[D]_l, D\right) &= 0.\end{aligned} \tag{3.15}$$

Then $\beta_1 = a\psi_{\mathfrak{D}}$ for some $a \in \mathbb{C}$. Now, using (3.4), we have that γ_i satisfies (3.15); thus, we get $\gamma_i = a_i\psi_{\mathfrak{D}}$ for some $a_i \in \mathbb{C}$, proving the first part of this lemma.

In order to prove the second part, consider $i \neq j$. Then

$$\begin{aligned}\beta_1\left(tE_{ii}, t^{-1}E_{ii}\right) &= \beta_1\left(t\left(E_{ii} - (-1)^{|E_{ij}||E_{ji}|}E_{jj}\right), t^{-1}E_{ii}\right) \quad \text{(by Lemma 3.1)} \\ &= \beta_1\left(\left[E_{ij}, tE_{ji}\right], t^{-1}E_{ii}\right) \\ &= \beta_1\left(E_{ij}, \left[tE_{ji}, t^{-1}E_{ii}\right]\right) - (-1)^{|E_{ij}||E_{ji}|}\beta_1\left(tE_{ji}, \left[E_{ij}, t^{-1}E_{ii}\right]\right) \\ &= \beta_1(E_{ij}, E_{ji}) + (-1)^{|E_{ij}||E_{ji}|}\beta_1\left(tE_{ji}, t^{-1}E_{ij}\right).\end{aligned} \tag{3.16}$$

Similarly,

$$\begin{aligned}\beta_1\left(tE_{jj}, t^{-1}E_{jj}\right) &= \beta_1\left(tE_{jj}, t^{-1}\left(E_{jj} - (-1)^{|E_{ij}||E_{ji}|}E_{ii}\right)\right) \quad \text{(by Lemma 3.1)} \\ &= \beta_1\left(tE_{jj}, \left[E_{ji}, t^{-1}E_{ij}\right]\right) \\ &= \beta_1\left(\left[tE_{jj}, E_{ji}\right], t^{-1}E_{ij}\right) + \beta_1\left(E_{ji}, \left[tE_{jj}, t^{-1}E_{ij}\right]\right) \\ &= \beta_1\left(tE_{ji}, t^{-1}E_{ij}\right) - \beta_1(E_{ji}, E_{ij}) \\ &= \beta_1\left(tE_{ji}, t^{-1}E_{ij}\right) + (-1)^{|E_{ij}||E_{ji}|}\beta_1(E_{ij}, E_{ji}).\end{aligned} \tag{3.17}$$

Therefore, $\beta_1(tE_{ii}, t^{-1}E_{ii}) = (-1)^{|E_{ij}||E_{ji}|}\beta_1(tE_{jj}, t^{-1}E_{jj})$, which means that, $a_i = (-1)^{|E_{ij}|}a_j$ for all $i \neq j$, finishing the proof. □

Lemma 3.4. *$\beta_1(E_{ij}, gE_{ji}) = \beta_1(gE_{ij}, E_{ji})$ for $i \neq j$ and $g \in \mathfrak{D}_{\mathrm{as}}$.*

Proof. Since $i \neq j$,

$$\begin{aligned}\beta_1(E_{ij}, gE_{ji}) &= \beta_1(E_{ij}, [E_{ji}, gE_{ii}]) \\ &= \beta_1([E_{ij}, E_{ji}], gE_{ii}) + (-1)^{|E_{ij}||E_{ji}|}\beta_1(E_{ji}, [E_{ij}, gE_{ii}]) \\ &= \beta_1(E_{ii}, gE_{ii}) - (-1)^{|E_{ij}||E_{ji}|}\beta_1(E_{jj}, gE_{ii}) - (-1)^{|E_{ij}||E_{ji}|}\beta_1(E_{ji}, gE_{ij}) \\ &= a_i\psi_{\mathfrak{D}}(1, g) + \beta_1(gE_{ij}, E_{ji}), \quad \text{(by Lemmas 3.3 and 3.1)} \\ &= \beta_1(gE_{ij}, E_{ji}) \quad \text{(by definition of } \psi_{\mathfrak{D}})\end{aligned} \tag{3.18}$$

□

Lemma 3.5. *$\beta_1(fE_{ij}, gE_{ji}) = \beta_1(fE_{ii}, gE_{ii})$ for $i \neq j$ and any $f, g \in \mathfrak{D}_{\text{as}}$.*

Proof. Observe that

$$\begin{aligned} \beta_1(fE_{ii}, gE_{ii}) &= \beta_1([fE_{ij}, E_{ji}], gE_{ii}) \quad \text{(by Lemma 3.1)} \\ &= \beta_1(fE_{ij}, [E_{ji}, gE_{ii}]) - (-1)^{|E_{ij}||E_{ji}|}\beta_1(E_{ji}, [fE_{ij}, gE_{ii}]) \\ &= \beta_1(fE_{ij}, gE_{ji}) + (-1)^{|E_{ij}||E_{ji}|}\beta_1(E_{ji}, (g \circ f)E_{ij}) \\ &= \beta_1(fE_{ij}, gE_{ji}) - \beta_1((g \circ f)E_{ij}, E_{ji}). \end{aligned} \tag{3.19}$$

Similarly,

$$\begin{aligned} \beta_1(fE_{ii}, gE_{ii}) &= (-1)^{|E_{ij}||E_{ji}|}\beta_1(fE_{jj}, gE_{jj}) \quad \text{(by Lemma 3.3)} \\ &= (-1)^{|E_{ij}||E_{ji}|}\beta_1(fE_{jj}, [gE_{ji}, E_{ij}]) \quad \text{(by Lemma 3.1)} \\ &= (-1)^{|E_{ij}||E_{ji}|}\beta_1([fE_{jj}, gE_{ji}], E_{ij}) + (-1)^{|E_{ij}||E_{ji}|}\beta_1(gE_{ji}, [fE_{jj}, E_{ij}]) \\ &= (-1)^{|E_{ij}||E_{ji}|}\beta_1((f \circ g)E_{ji}, E_{ij}) + \beta_1(fE_{ij}, gE_{ji}) \\ &= -\beta_1((f \circ g)E_{ij}, E_{ji}) + \beta_1(fE_{ij}, gE_{ji}) \quad \text{(by Lemma 3.4)}. \end{aligned} \tag{3.20}$$

Hence, from (3.19) and (3.20), we obtain

$$\beta_1([f, g]E_{ij}, E_{ji}) = 0. \tag{3.21}$$

Since $[\mathfrak{D}, \mathfrak{D}] = \mathfrak{D}$, we have $\beta_1(\mathfrak{D}E_{ij}, E_{ji}) = 0$. Therefore, (3.19) becomes the statement of this lemma. □

Proof of Theorem 2.1. From the previous lemmas, one can easily see that $\beta_1 = a_1\Psi$, by observing that the relation between the $a_i's$ in Lemma 3.3 is essentially the supertrace term in expression (2.12) of Ψ. □

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References

[1] H. Awata, M. Fukuma, Y. Matsuo, and S. Odake, "Representation theory of $W_{1+\infty}$ algebra," in *Proceedings of the Workshop Quantum Field Theory, Integrable Models and Beyond*, T. Inami et al., Ed., Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto, Japan, February 1994.

[2] H. Awata, M. Fukuma, Y. Matsuo, and S. Odake, "Representation theory of $W_{1+\infty}$ algebra," *Progress of Theoretical Physics Supplement*, vol. 118, pp. 343–373, 1995.

[3] I. Bakas, B. Khesin, and E. Kiritsis, "The logarithm of the derivative operator and higher spin algebras of W_∞ type," *Communications in Mathematical Physics*, vol. 151, no. 2, pp. 233–243, 1993.

[4] V. Kac and A. Radul, “Quasifinite highest weight modules over the Lie algebra of differential operators on the circle,” *Communications in Mathematical Physics*, vol. 157, no. 3, pp. 429–457, 1993.
[5] C. N. Pope, L. J. Romans, and X. Shen, “Ideals of Kac-Moody algebras and realisations of W_∞,” *Physics Letters B*, vol. 245, no. 1, pp. 72–78, 1990.
[6] H. Awata, M. Fukuma, Y. Matsuo, and S. Odake, “Quasifinite highest weight modules over the super $W_{1+\infty}$ algebra,” *Communications in Mathematical Physics*, vol. 170, no. 1, pp. 151–179, 1995.
[7] E. Bergshoeff, M. Vasiliev, and B. de Wit, “The super-$W_\infty(\lambda)$ algebra,” *Physics Letters B*, vol. 256, no. 2, pp. 199–205, 1991.
[8] S. Odake and T. Sano, “$W_{1+\infty}$ and super-W_∞ algebras with SU(N) symmetry,” *Physics Letters B*, vol. 258, no. 3-4, pp. 369–374, 1991.
[9] S. Odake, “Unitary representations of W infinity algebras,” *International Journal of Modern Physics A*, vol. 7, no. 25, pp. 6339–6355, 1992.
[10] L. Guieu and C. Roger, *L'Algèbre et le Groupe de Virasoro*, Aspects Géométriques et Algébriques, Généralisations, Les Publications CRM, Montreal, Canda, 2007.
[11] A. O. Radul, “Lie algebras of differential operators, their central extensions and W-algebras,” *Functional Analysis and Its Applications*, vol. 25, no. 1, pp. 25–39, 1991.
[12] C. Boyallian, V. G. Kac, J. I. Liberati, and C. H. Yan, “Quasifinite highest weight modules over the Lie algebra of matrix differential operators on the circle,” *Journal of Mathematical Physics*, vol. 39, no. 5, pp. 2910–2928, 1998.
[13] S.-J. Cheng and W. Wang, “Lie subalgebras of differential operators on the super circle,” *Research Institute for Mathematical Sciences Publications*, vol. 39, no. 3, pp. 545–600, 2003.
[14] V. G. Kac and J. I. Liberati, “Unitary quasi-finite representations of W_∞,” *Letters in Mathematical Physics*, vol. 53, no. 1, pp. 11–27, 2000.
[15] C. Boyallian, V. G. Kac, and J. I. Liberati, “On the classification of subalgebras Cend_N and gc_N ,” *Journal of Algebra*, vol. 260, no. 1, pp. 32–63, 2003.
[16] V. Kac, *Vertex Algebras for Beginners*, vol. 10 of *University Lecture Series*, American Mathematical Society, Providence, RI, USA, 2nd edition, 1998.
[17] V. G. Kac, “Formal distribution algebras and conformal algebras,” in *XIIth International Congress of Mathematical Physics (ICMP '97) (Brisbane)*, pp. 80–97, Internat Press, Cambridge, Mass, USA, 1999.
[18] W. L. Li, “2-cocyles on the algebra of differential operators,” *Journal of Algebra*, vol. 122, no. 1, pp. 64–80, 1989.
[19] W. Li and R. L. Wilson, “Central extensions of some Lie algebras,” *Proceedings of the American Mathematical Society*, vol. 126, no. 9, pp. 2569–2577, 1998.
[20] D. Liu and N. Hu, “Derivation algebras and 2-cocycles of the algebras of q-differential operators,” *Communications in Algebra*, vol. 32, no. 11, pp. 4387–4413, 2004.