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Comment

Comment on Maia, A.V.D.M.; Bakke, K. Topological Effects of a Spiral Dislocation on Quantum Revivals. *Universe* 2022, *8*, 168

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1. Introduction

In a series of papers, Maia and Bakke [1–4] studied several quantum-mechanical models in which a particle moves in an elastic medium with a spiral dislocation. They considered a variety of interactions: a harmonic potential [1,4] a nonuniform radial electric field [2], a uniform axial magnetic field [2], and the effect of rotation [3]. Maia and Bakke also took into account the effect of a hard wall in the radial part of the Schrödinger equation [1,2]. The Schrödinger equation for all those models is separable in cylindrical coordinates, and they arrived at somewhat similar eigenvalue equations in most of those papers that can be solved in terms of a confluent hypergeometric function. In order to obtain exact analytical eigenvalues, Maia and Bakke [1–3] resorted to some approximations, but they did not discuss their effect on the accuracy of the results. Maia and Bakke commonly set $\hbar = 1$ and $c = 1$ [1,2,4] (in some cases they did not even mention this fact [3]), and as a result their analytical expressions exhibit an obvious inconsistence in their units as discussed elsewhere [5].

The purpose of this paper is the analysis of the results for the harmonic oscillator in an elastic medium with a spiral dislocation that appeared in this journal [4]. In Section 2, we outline the approximate analytical results obtained by Maia and Bakke [4]; in Section 3, we derive exact analytical results in order to test the approximate ones just mentioned; and in Section 4, we summarize the main results and draw conclusions.

2. Outline of the Model

The model resembles a harmonic oscillator in three dimensions

$$H = -\frac{\hbar^2}{2m}\nabla^2 + \frac{m\omega^2}{2}r^2, \tag{1}$$

where m is the mass of the particle, ω the oscillator frequency, and $r^2 = x^2 + y^2$. The spiral dislocation is embodied in the line element

$$ds^2 = dr^2 + 2\beta dr d\phi + \left(\beta^2 + r^2\right)d\phi^2 + dz^2, \tag{2}$$

where $0 \leq r < \infty$, $0 \leq \phi \leq 2\pi$, $-\infty < z < \infty$, which determines the form of the Laplace–Beltrami operator ∇^2. It is clear that if s, r, and z have unit of length, then β should also have this unit.

Upon writing the solution to $H\psi = E\psi$ as $\psi(r,\theta,\phi) = e^{il\phi}e^{ikz}f(r)$, Maia and Bakke [4] derived the radial equation

$$\left(1 + \frac{\beta^2}{r^2}\right)f''(r) + \left(\frac{1}{r} - \frac{\beta^2}{r^3}\right)f'(r) - \frac{l^2}{r^2+\beta^2}f(r) - m^2\omega^2 r^2 f(r) + \left(2mE - k^2\right)f(r) = 0, \tag{3}$$

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where $-\infty < k < \infty$ and $l = 0, \pm 1, \pm 2, \ldots$ is the rotational quantum number. The units in this equation appear to be inconsistent, but the authors stated that they chose "the units where $\hbar = 1$ and $c = 1$" [4] (although c is not expected to appear anywhere in the present model equations). Unfortunately, Maia and Bakke never specified the actual units of length, energy, etc. as one should reasonably do [5] (we have recently carried out a pedagogical discussion of this issue with some detail [6]). We will discuss this point in more detail in Section 3.

Maia and Bakke [4] chose an ansatz of the form

$$f(r) = g(x) = x^{|l|/2} e^{-x/2} u(x), \tag{4}$$

where $x = m\omega(r^2 + \beta^2)$. Obviously, $x_0 \leq x < \infty$ and $g(x_0) = 0$ at $x_0 = m\omega\beta^2$. From the properties of the confluent hypergeometric function, Maia and Bakke [4] derived the approximate analytical spectrum

$$E_n^{MB\pm} = -\omega\left(2n + \frac{1}{2} + \frac{m\omega\beta^2}{2}\right) + \frac{4m\omega^2\beta^2}{\pi^2}\left[1 \pm \sqrt{1 - \frac{\pi^2(4n+1)}{4m\omega\beta^2}}\right] + \frac{k^2}{2m}, \tag{5}$$

where $n = 0, 1, \ldots$ is the radial quantum number. This expression suggests that $m\omega\beta^2$ is dimensionless; consequently, ω and k should also be dimensionless. There are two intriguing features: first, the fact that the authors did not indulge in explaining the meaning of the $\pm$ sign, and second, the fact that E_n does not depend on l. However, the most striking fact about this expression is that the eigenvalues are complex unless $m\omega\beta^2 \geq \pi^2(4n+1)/4$, which is inconsistent with the condition $0 < \beta < 1$ considered in earlier papers by the same authors who stated that "Since $0 < \beta < 1$, we can consider $\beta^2 \ll 1$" [2] or "Note that, since $0 < \beta < 1$, then, we can assume that $\beta^2 \ll 1$" [3]. Here, Maia and Bakke [4] stated that "In the present work, by contrast, we do not assume that β is very small". Notice that they did not explicitly indicate that small values of β are not allowed. However, if we take the limit $\beta \to 0$, then the energies $E_n^{MB\pm}$ yield a negative spectrum instead of the expected harmonic-oscillator one. It is therefore clear that the spectrum given by Equation (5) cannot be correct. Maia and Bakke [3] simply stated that "In addition, the radial quantum number possesses an upper limit given by"

$$n_{\max} < \frac{m\omega\beta^2}{\pi^2} - \frac{1}{4}. \tag{6}$$

3. Exact Analytical Solutions

Before solving the eigenvalue equation for the model outlined above, it seems necessary to discuss the units in a reasonable way. In order to obtain a dimensionless equation, we first choose a unit of length, say L, and define the dimensionless quantities $(\tilde{s}, \tilde{r}, \tilde{z}, \tilde{\beta}) = (s/L, r/L, z/L, \beta/L)$ that lead to the dimensionless Laplace–Beltrami operator $\tilde{\nabla}^2 = L^2\nabla^2$. In principle, L is arbitrary. In the present case, it is convenient to choose $L = (\hbar/m\omega)^{1/2}$ because the Hamiltonian operator takes the simple form

$$\tilde{H} = \frac{mL^2}{\hbar^2} H = -\frac{1}{2}\tilde{\nabla}^2 + \frac{1}{2}\tilde{r}^2, \tag{7}$$

where $\hbar^2/(mL^2) = \hbar\omega$ is the unit of energy (see [6] for details). Notice that this choice of units is equivalent to setting $\hbar = 1$, $m = 1$, and $\omega = 1$ [5]. It is also important to take into account that $kz = \tilde{k}\tilde{z}$, where $\tilde{k} = kL$. When we solve the eigenvalue equation $H\psi = E\psi$, we obtain energies of the form $E = E(\hbar, m, \omega, \beta, k)$, and when we solve $\tilde{H}\tilde{\psi} = \tilde{E}\tilde{\psi}$ they are $\tilde{E} = \tilde{E}(\tilde{\beta}, \tilde{k})$. The connection is given by $E(\hbar, m, \omega, \beta, k) = \hbar\omega E(1, 1, 1, \tilde{\beta}, \tilde{k}) = \hbar\omega\tilde{E}(\tilde{\beta}, \tilde{k})$. Since we do not know which are the units of length and energy used by Maia and Bakke [4], then we simply set $m = 1$ and $\omega = 1$ in their expressions and remove the tilde on $\tilde{E}$, $\tilde{\beta}$, and $\tilde{k}$ in order to compare results.

For example, the differential equation for $g(x)$ is

$$g''(x) + \frac{1}{x}g'(x) + \left[\frac{2E - k^2 + \beta^2}{4x} - \frac{l^2}{4x^2} - \frac{1}{4}\right]g(x) = 0, \tag{8}$$

where E, k, and β are assumed to be dimensionless.

It is convenient to consider the following cases:

Case I: $0 \leq x < \infty$, $g(x \to \infty) = 0 \Rightarrow E_{n,l}, g_{n,l}(x)$

Case II: $x_0 \leq x < \infty$, $g(x_0) = 0$, $g(x \to \infty) = 0 \Rightarrow E_{n,l}^{II}, g_{n,l}^{II}(x)$

Case III: $0 \leq x \leq x_0$, $g(x_0) = 0 \Rightarrow E_{n,l}^{III}, g_{n,l}^{III}(x)$

In order to obtain exact analytical results for Case I, we apply the Frobenius (power-series) method based on the expansion

$$u(x) = \sum_{j=0}^{\infty} c_j x^j, \tag{9}$$

that leads to the recurrence relation

$$c_{j+1} = A_j c_j,\ A_j = -\frac{\beta^2 + 2E - k^2 - 2(2n + |l| + 1)}{4(n+1)(n + |l| + 1)},\ j = 0, 1, \ldots. \tag{10}$$

The truncation condition $c_n \neq 0$, $c_{n+1} = 0$, $n = 0, 1, \ldots$, leads to $c_j = c_{j,n,l} = 0$ for all $j > n$ (see [7] for further analysis) and to the exact eigenvalues

$$E_{n,l} = 2n + |l| + 1 - \frac{\beta^2}{2} + \frac{k^2}{2}, \tag{11}$$

that agree with those obtained earlier by Maia and Bakke [1]. When $E = E_{n,l}$, A_j takes the simpler form

$$A_j = \frac{j - n}{(j+1)(j + |l| + 1)}. \tag{12}$$

It is convenient to define

$$W_{n,l} = E_{n,l} + \frac{\beta^2}{2} - \frac{k^2}{2} = 2n + |l| + 1. \tag{13}$$

The exact solutions are thus given by

$$g_{n,l}(x) = x^{|l|/2} e^{-x/2} \sum_{j=0}^{n} c_{j,n,l} x^j. \tag{14}$$

The interesting fact is that the solutions $g_{n,l}(x)$ for Case I enable us to obtain some particular solutions for Case II and Case III. In what follows, we illustrate the strategy by means of some simple examples. For simplicity, we arbitrarily set $c_{0,n,l} = 1$. From

$$g_{1,0}(x) = e^{-x/2}(1 - x), \tag{15}$$

we realize that $g_{0,0}^{II}(x) = g_{1,0}(x)$ for $x_0 = 1$ and $W_{0,0}^{II} = 3$. Exactly in the same way, we have

$$g_{0,1}^{II}(x) = g_{1,1}(x) = \frac{\sqrt{x}}{2} e^{-x/2}(2 - x),\ x_0 = 2,\ W_{0,1}^{II} = 4, \tag{16}$$

or

$$\begin{aligned} g_{0,0}^{II} &= g_{2,0}(x) = e^{-x/2}\left(\frac{x^2}{2} - 2x + 1\right),\ x_0 = 2 + \sqrt{2},\ W_{0,0}^{II} = 5 \\ g_{1,0}^{II} &= g_{2,0}(x),\ x_0 = 2 - \sqrt{2},\ W_{1,0}^{II} = 5. \end{aligned} \tag{17}$$

It is clear that, by straightforward generalization of this obvious procedure, we can obtain other solutions $g^{II}_{n',l}(x)$ and $W^{II}_{n',l}$ for particular values of x_0 given by the zeros of $g_{n,l}(x)$. Note that we can also derive the solutions $g^{III}_{n',l}(x)$ and $W^{III}_{n',l}$ for Case III (but we will not discuss them in this comment).

In order to compare the present exact results with the approximate ones derived by Maia and Bakke [4], we define $W^{MB\pm}_{n,l} = E^{MB\pm}_{n,l} + \beta^2/2 - k^2/2$ for $m = 1$ and $\omega = 1$ in Equation (5). As argued above, the analytical expression of those authors is only valid for $x_0 = \beta^2 \geq \pi^2/4 \approx 2.47$ when $n = 0$. Figure 1 shows that $W^{MB-}_{0,0}$ is absurdly small, while, on the other hand, $W^{MB+}_{0,0}$ appears to be somewhat better but cannot be considered to be a reasonable approximation. Although it increases with x_0 as the exact result, the slope is wrong. It is worth pointing out that Maia and Bakke [4] appeared to believe that both solutions are suitable because they did not choose one of them.

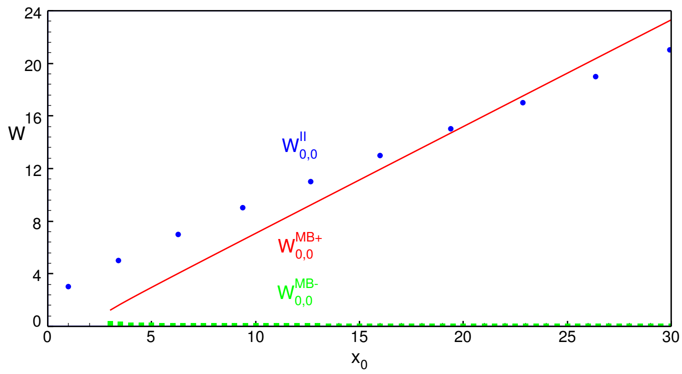

Figure 1. Exact eigenvalues $W^{II}_{0,0}$ and those calculated by means of Equation (5).

It is not possible to determine the difference $W^{II}_{n,l} - W^{II}_{n,l'}$ directly from our exact eigenvalues because we have them for different values of x_0. However, a reasonable strategy is to fit each $W^{II}_{n,l}$ by means of a suitable curve and compare them. We have found that a quadratic polynomial yields reasonable results in the interval $0 < x_0 < 30$, as shown in Figure 2 for $n = 0$ and $l = 0$. Proceeding exactly in this way, we have

$$\begin{aligned} W^{II}_{0,0} &= 2.500106874 + 0.7124767969x_0 - 0.003261278934x_0^2, \\ W^{II}_{0,1} &= 2.77275271 + 0.6804213543x_0 - 0.002389996785x_0^2, \\ W^{II}_{0,2} &= 3.133268997 + 0.6494487589x_0 - 0.001678639142x_0^2. \end{aligned} \tag{18}$$

It is clear that the exact eigenvalues $W^{II}_{n,l}$ appear to be almost independent of the rotational quantum number only for sufficiently large values of $x_0 = \beta^2$. However, since $0 < \beta < 1$ [1–3], the independence of $E^{MB\pm}_n$ with respect to l appears to be just an artifact of the approximation proposed by Maia and Bakke [4].

In closing, we want to stress the point that the ansatz (4) is unsuitable for obtaining the solutions of Case II for arbitrary values of x_0. To understand it, simply note that the only solutions to the differential Equation (8) with the behavior at origin and infinity given by Equation (4) are just the functions (14) with the eigenvalues (13) as shown for another quantum-mechanical model [7]. Consequently, the ansatz (4) will give solutions for Case II only for the zeros of the solutions for Case I. In other words, the Expression (5) is utterly wrong.

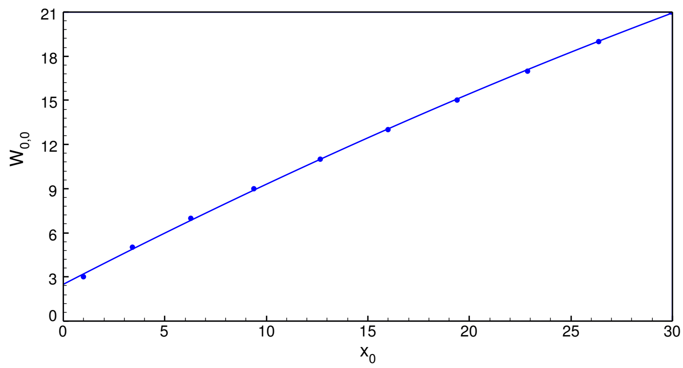

Figure 2. Exact eigenvalues $W_{0,0}^{II}$ (points) and polynomial fit (continuous line).

4. Conclusions

In this comment, we have shown that the analytical eigenvalues derived by Maia and Bakke [4] are unsuitable for any physical application because they exhibit an obvious inconsistency in their units. The spectrum is real only for sufficiently large values of β^2; this is inconsistent with previous statements that this parameter should be small [1–3]. Apparently, Maia and Bakke [4] forgot this fact in the paper analyzed here. We may reasonably assume that their arguments about quantum revivals, based on the wrong eigenvalues $E_n^{MB\pm}$, are far from correct.

Conflicts of Interest: The author declares no conflict of interest.

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