ON THE SPECTRAL DATA OF PERTURBED STARK OPERATORS IN THE HALF-LINE WITH MIXED BOUNDARY CONDITIONS

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Abstract: We discuss some asymptotic formulas for the spectral data of perturbed Stark operators associated with the differential expression

$$-\frac{d^2}{dx^2}+x+q(x),\quad x\in[0,\infty),\quad q\in L^1(0,\infty),$$

and boundary condition $\varphi'(0)-b\varphi(0)=0$, $b\in\mathbb{R}$.

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1 INTRODUCTION

This paper is concerned with the spectral analysis of self-adjoint operators associated with a differential expression of the form

$$\tau_q=-\frac{d^2}{dx^2}+x+q(x),\quad x\in[0,\infty),$$

where q is a real-valued function that lies in $L^1(\mathbb{R}_+)$.

By standard theory (see e.g. [7, Ch. 6]), τ_q is in the limit-circle case at 0 and in the limit-point case at ∞. Hence (the closure of) the minimal operator H_q' defined by τ_q is symmetric and has deficiency indices $(1,1)$. The self-adjoint extensions of H_q' are defined by imposing the usual boundary condition at $x=0$. Namely, given $b\in\mathbb{R}\cup\{\infty\}$,

$$\mathcal{D}(H_{q,b})=\begin{Bmatrix}\varphi\in L^2(\mathbb{R}_+):\varphi,\varphi'\in\mathrm{AC_{loc}}([0,\infty)),\ \tau_q\varphi\in L^2(\mathbb{R}_+),\\ \varphi'(0)-b\varphi(0)=0 \text{ if } b\in\mathbb{R},\ \varphi(0)=0 \text{ if } b=\infty\end{Bmatrix},\quad H_{q,b}\varphi=\tau_q\varphi. \tag{1}$$

Moreover, $H_{q,b}$ has only simple, discrete spectrum, with a finite number of negative eigenvalues (if any) because it is semi-bounded from below. Let $\psi(q,z,x)$ be the unique (up to a constant multiple) square-integrable solution to the eigenvalue problem $\tau_q\varphi=z\varphi$, $z\in\mathbb{C}$. According to the Borg–Marchenko uniqueness theorem [3], $H_{q,b}$ is uniquely determined by the spectral data consisting of the set of eigenvalues

$$\{\lambda_n(q,b)\}_{n=1}^{\infty}=\{\lambda\in\mathbb{R}:w(q,b,\lambda)=0\},\quad w(q,b,z):=\begin{cases}\psi'(q,z,x)-b\psi(q,z,x), & b\in\mathbb{R},\\ \psi(q,z,x), & b=\infty,\end{cases}$$

along with the set of (logarithmic) norming constants[1]

$$\{\kappa_n(q,b)\}_{n=1}^{\infty},\quad \kappa_n(q,b)=\begin{cases}\log\left(\dfrac{\psi(q,\lambda_n(q),0)}{\dot{w}(q,b,\lambda_n(q))}\right), & b\in\mathbb{R},\\[2ex] \log\left(-\dfrac{\psi'(q,\lambda_n(q),0)}{\dot{\psi}(q,\lambda_n(q),0)}\right), & b=\infty.\end{cases} \tag{2}$$

[1] The norming constants are given by (minus) the residues of the Weyl function $m(z)$ —which in this case is a meromorphic Herglotz function— at the eigenvalues. That is,

$$e^{\kappa_n(q,b)}:=-\lim_{\epsilon\to0}i\epsilon\, m(q,b,\lambda_n(q,b)+i\epsilon)=\|\phi(q,b,\lambda_n(q),\cdot)\|^2,$$

where $\phi(q,b,z,x)$ is the solution to $\tau_q\varphi=z\varphi$ that obeys either $\phi(q,b,z,0)=1$ and $\phi'(q,b,z,0)=b$ if $b\in\mathbb{R}$ or else $\phi(q,\infty,z,0)=0$. The expression (2) follows after applying the identity $\partial_x(\psi\,\dot{\psi}'-\psi'\dot{\psi})=-\psi^2$. The notation $\varphi':=\partial_x\varphi$ and $\dot{\varphi}:=\partial_z\varphi$ is used throughout this work.

A main task of the (inverse) spectral analysis of this kind of operators is to obtain a precise, i.e. sharp, characterization of the spectral data. This usually can be accomplished when q lies in certain subspaces that are themselves Hilbert spaces (for instance, see [1, 5]). For the problem in hand we consider real-valued perturbations q that belong to the Hilbert space

$$\mathfrak{A}_r := \left\{q \in \mathcal{A}_r \cap \mathrm{AC}[0,\infty) : q' \in \mathcal{A}_r\right\}, \quad \|q\|^2_{\mathfrak{A}_r} := \|q\|^2_{\mathcal{A}_r} + \left\|q'\right\|^2_{\mathcal{A}_r},$$

where

$$\mathcal{A}_r := L^2_{\mathbb{R}}(\mathbb{R}_+, (1+x)^r dx), \quad \|q\|_{\mathcal{A}_r} := \|q\|_{L^2(\mathbb{R}_+,(1+x)^r dx)},$$

and $r > 1$ is arbitrary but fixed. It is easy to verify that $\mathcal{A}_r \subset L^1(\mathbb{R}_+)$ (as long as $r > 1$). The following characterization of the spectral data for the Dirichlet problem is shown in [6]; here $\lambda_n(q) := \lambda_n(q,\infty)$ and $\kappa_n(q) := \kappa_n(q,\infty)$:

Theorem 1 *For every $n \in \mathbb{N}$, $\lambda_n : \mathcal{A}_r \to \mathbb{R}$ and $\kappa_n : \mathcal{A}_r \to \mathbb{R}$ are real analytic maps.*[2] *Moreover, in terms of*

$$\omega_r(n) := \begin{cases} n^{-1/3}\log^{1/2} n & \text{if } r \in (1,2), \\ n^{-1/3} & \text{if } r \in [2,\infty), \end{cases} \tag{3}$$

one has the following asymptotics:[3]

$$\lambda_n(q) = -a_n + \pi\frac{\int_0^\infty \mathrm{Ai}^2(x+a_n)q(x)dx}{(-a_n)^{1/2}} + O\big(n^{-1/3}\omega_r^2(n)\big)$$

and

$$\kappa_n(q) = -2\pi\frac{\int_0^\infty \mathrm{Ai}(x+a_n)\,\mathrm{Ai}'(x+a_n)q(x)dx}{(-a_n)^{1/2}} + O\big(\omega_r^3(n)\big),$$

uniformly on bounded subsets of $\mathfrak{A}_r$.

In this paper we discuss some preliminary steps toward generalizing Theorem 1 to the problem with mixed (including Neumann) boundary condition corresponding to $b \in \mathbb{R}$. We will make use of the (minimal) assumption $q \in L^1(\mathbb{R}_+)$ exclusively.

2 THE UNPERTURBED PROBLEM

The eigenvalue problem $\tau_0\varphi = z\varphi$ has the square-integrable solution

$$\psi_0(z,x) = \sqrt{\pi}\,\mathrm{Ai}(x-z),$$

where Ai denotes the Airy function of the first kind;[4] a summary of their relevant properties can be found in [4, Ch. 9]. Thus, the set of eigenvalues and norming constants of $H_b := H_{0,b}$ can be written as

$$\left\{\lambda_n(b) \in \mathbb{R} : \mathrm{Ai}'(-\lambda_n(b)) - b\,\mathrm{Ai}(-\lambda_n(b)) = 0,\ \kappa_n(b) = \log\left(\frac{\mathrm{Ai}(-\lambda_n(b))}{\lambda_n(b)\,\mathrm{Ai}(-\lambda_n(b)) + b\,\mathrm{Ai}'(-\lambda_n(b))}\right)\right\}_{n=1}^{\infty}.$$

[2]Let $\mathcal{B}$ be a Hilbert space over $\mathbb{K}$, and let $\mathcal{U} \subset \mathcal{B}$ be open. A map $f : \mathcal{U} \to \mathbb{K}$ is (Fréchet) differentiable at $q \in \mathcal{U}$ if there exists a linear functional $d_q f : \mathcal{B} \to \mathbb{K}$ such that

$$\lim_{v\to 0} \frac{|f(q+v) - f(q) - d_q f(v)|}{\|v\|_{\mathcal{B}}} = 0.$$

The map f is continuously differentiable on $\mathcal{U}$ if it is differentiable at every point in $\mathcal{U}$ and the resulting map $df : \mathcal{U} \to L(\mathcal{B},\mathbb{K})$ is continuous. If $\mathcal{B}$ is a complex Hilbert space, then f is analytic on an open subset $\mathcal{U}$ of $\mathcal{B}$ if it is continuously differentiable there. Finally, let $\mathcal{B}^{\mathbb{C}}$ be the complexification of a real Hilbert space $\mathcal{B}$ and let $\mathcal{U} \subset \mathcal{B}$ be open. Then $f : \mathcal{U} \to \mathbb{R}$ is real analytic on $\mathcal{U}$ if for every $q \in \mathcal{U}$ there exists $\mathcal{V}_q \subset \mathcal{B}^{\mathbb{C}}$ open and an analytic map $h_q : \mathcal{V}_q \to \mathbb{C}$ such that $f(v) = h_q(v)$ for all $v \in \mathcal{U} \cap \mathcal{V}_q$ (assumed non-empty).

[3]In this paper an order relation of the form

$$f(n) = O(g(n))$$

implicitly assume $n \in \mathbb{N}$ and (of course) $n \to \infty$.

[4]The factor $\sqrt{\pi}$ does not play any role in this work but we keep it to facilitate comparison with [6].

Proposition 1 *Given $b \in \mathbb{R}$, one has*

$$\lambda_n(b) = -a_n' - \frac{b}{a_n'} + O(n^{-4/3}) \quad \textit{and} \quad \kappa_n(b) = -\log(-a_n') + \frac{b^2}{a_n'} + O(n^{-4/3}),$$

where a_n' denotes the n-th zero of the derivative of the function Ai (see [4, Ch. 9]).

Proof. The first assertion can be shown by using the method of successive resubstitutions like in [2]. A computation yields

$$\begin{aligned}\kappa_n(b) &= -\log\lambda_n(b) - \log\left(1 + b\frac{\mathrm{Ai}'(-\lambda_n(b))}{\lambda_n(b)\,\mathrm{Ai}(-\lambda_n(b))}\right)\\ &= -\log\lambda_n(b) - b\frac{\mathrm{Ai}'(-\lambda_n(b))}{\lambda_n(b)\,\mathrm{Ai}(-\lambda_n(b))} + O\left(\left(\frac{\mathrm{Ai}'(-\lambda_n(b))}{\lambda_n(b)\,\mathrm{Ai}(-\lambda_n(b))}\right)^2\right),\end{aligned}$$

from which the second assertion follows. □

3 Main results (in a relative sense)

Let us define

$$\omega(q,z) := \int_0^\infty \frac{|q(x)|}{\sqrt{1+|x-z|}}\,dx.$$

Clearly $\omega(q,z)$ is well defined if we only assume $q \in L^1(\mathbb{R}_+, (1+x)^{-1/2}dx)$, although in such a case we cannot control its behavior when $z \to \infty$. Some decaying property is warranted if q belongs to $L^1(\mathbb{R}_+)$; this is shown next.

Lemma 1 *Assume $q \in L^1(\mathbb{R}_+)$. Then $\omega(q,z) \to 0$ as $z \to \infty$.*

Proof. Given $\varepsilon > 0$, choose $x_* > 0$ and $\mu_* > x_*$ such that

$$\int_{x_*}^\infty |q(x)|\,dx < \frac{\varepsilon}{2} \quad \text{and} \quad \frac{1}{\sqrt{\mu_* - x_*}} < \frac{\varepsilon}{2\,\|q\|_1}.$$

Suppose $|\mathrm{Im}(z)| > \mu_*$. Then $|x - z| > \mu_*$ for any $x > 0$. Hence,

$$\frac{1}{\sqrt{1+|x-z|}} \le \frac{1}{\sqrt{\mu_*}} < \frac{\epsilon}{2\,\|q\|_1}$$

for all $x \in \mathbb{R}_*$, which in turn implies $\omega(q,z) < \epsilon$. A similar reasoning applies when $|\mathrm{Im}(z)| \le \mu_*$ and $\mathrm{Re}(z) < -\mu_*$. Finally, suppose that $|\mathrm{Im}(z)| \le \mu_*$ and $\mathrm{Re}(z) > \mu_*$. Since $\omega(q,z) \le \omega(q, \mathrm{Re}(z))$, it suffices to consider $z = \mu \in \mathbb{R}$ with $\mu > \mu_*$. Then,

$$\omega(q,\mu) < \frac{1}{\sqrt{1+|x_* - \mu_*|}}\int_0^{x_*}|q(x)|\,dx + \int_{x_*}^\infty |q(x)|\,dx < \varepsilon.$$

Thus, we have shown that $\omega(q,z) < \varepsilon$ whenever $|\mathrm{Re}(z)| + |\mathrm{Im}(z)| > \mu_*$. □

Let us define the contours

$$\mathcal{F}^m := \left\{z \in \mathbb{C} : |\zeta| = \left(m - \tfrac{5}{4}\right)\pi\right\}, \quad \mathcal{F}_n := \left\{z \in \mathbb{C} : \left|\zeta - \left(n - \tfrac{3}{4}\right)\pi\right| = \tfrac{\pi}{2}\right\}, \quad m,n \in \mathbb{N}, \quad m \ge 2.$$

Clearly $\mathcal{F}_n$ encloses exactly one zero of $\mathrm{Ai}'(-z)$, namely $-a_n'$, for sufficiently large values of n.

Lemma 2 *There exists $m_0, n_0 \in \mathbb{N}$ such that, for every $m \ge m_0$ and $n \ge n_0$, the following statement holds true:*

$$\sigma(z) g_A(-z) < 16\sqrt{\pi}\left|\mathrm{Ai}'(-z)\right|,$$

whenever $z \in \mathcal{F}^m$ or $z \in \mathcal{F}_n$.

Proof. This inequality follows from argument like in the proof of Lemma A.2 of [6]. □

Proposition 2 *Suppose* $(q,b) \in L^1(\mathbb{R}_+) \times \mathbb{R}$. *Then the eigenvalues and norming constants of* $H_{q,b}$ *satisfy*

$$\lambda_n(q,b) = -a_n' + O\big(n^{-1/3}\big) \quad \textit{and} \quad \kappa_n(q,b) = -\log(-a_n') + o(1).$$

Proof. Let us abbreviate

$$w(z) = w(q,b,z), \quad \omega(z) = \omega(q,z), \quad \psi_0(z) = \psi_0(z,0), \quad \psi(z) = \psi(q,z,0), \quad \text{et cetera.}$$

Resorting to Lemma 3.1 of [6], we obtain

$$\big|w(z) - \psi_0'(z)\big| \leq \big|\psi'(z) - \psi_0'(z)\big| + |b|\,|\psi(z)| \leq Ce^{C\omega(z)}\left[\omega(z) + \frac{|b|}{(\sigma(z))^2}\right]\sigma(z)g_A(-z),$$

where $\sigma(w) := 1 + |w|^{1/4}$ and $g_A(w) := \exp(-\frac{2}{3}\operatorname{Re} w^{3/2})$. According to Lemma 1, there exists $n_1 \in \mathbb{N}$ such that

$$e^{C\omega(z)} \leq 2 \quad \text{and} \quad \omega(z) + \frac{|b|}{(\sigma(z))^2} \leq \frac{1}{32C},$$

whenever $|z| \geq (\frac{3}{2}\pi(n_1 - \frac{5}{4}))^{2/3}$. Then, by Lemma 2, there exists $n_2 \geq n_1$ such that

$$\big|w(z) - \psi_0'(z)\big| < \big|\psi_0'(z)\big| \tag{4}$$

for all $z \in \mathcal{F}^{n_2}$. Moreover, by increasing n_2 if necessary, we can assume that $\mathcal{F}^{n_2}$ encloses the (finitely many) negative zeros of $w(z)$. Finally, increasing n_2 one more time if necessary, we can ensure that (4) holds true for z on every contour $\mathcal{F}_n$ whenever $n \geq n_2$. Now, Rouché's theorem implies that $w(z)$ has as many zeros as $\psi_0'(z)$ within $\mathcal{F}^{n_2}$ and exactly one zero $\lambda_n = \lambda_n(q,b)$ nearby $-a_n'$ within each $\mathcal{F}_n$ with $n \geq n_2$. That is, for sufficiently large n,

$$\left|\tfrac{2}{3}\lambda_n^{3/2} - \tfrac{2}{3}(-a_n')^{3/2}\right| \leq \pi$$

whence the asymptotics for the eigenvalues follows. Finally, a computation like in the proof of Proposition 1 yields

$$\kappa_n(q,b) = -\log\lambda_n(q,b) + O(\omega(q,\lambda_n(q,b))),$$

which in turn implies the asymptotics for the norming constants. □

Clearly these asymptotic expansions are rather coarse. They cannot be improved because we have insufficient control on the decay of the function $\omega(q,z)$ if we just assume $q \in L^1(\mathbb{R}_+)$. However, sharper results akin to Theorem 1 are expected if $q \in \mathfrak{A}_r$; this will be the subject of a subsequent paper.

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