

Numerical Solution and Validation Concerning a Descriptive Model of a Simultaneous Heat and Mass Transfer Process

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Abstract – This paper deals with the numerical solution of an initial moving boundary problem (IMBVP) arising from a simplified version of a free boundary problem (IFBVP) which was formulated in another paper, as a mathematical model of heat and moisture transfer during the bubbling stage of an immersion frying process of stick shaped natural potato. The IMBVP was solved numerically using an explicit finite difference method, immobilizing the moving boundary through the use of the well known Landau transformation. The simulation output provides the temperature and moisture profiles and the amount of free moisture lost by vaporization. The model was validated by comparing the predicted results with experimental data. A good agreement was obtained. **Copyright © 2011 Praise Worthy Prize S.r.l. - All rights reserved.**

Keywords: Potato Immersion Frying, Free-Moving Boundary Models, Moisture Desorption Front, Numerical Solution

Nomenclature

A	Total lateral surface area (m^2)
a	Parameter defined by eqn. (20)
C	Volumetric desorbed moisture concentration (kg/m^3)
C_s	Specific heat of potato ($\text{J}/\text{kg } ^\circ\text{C}$)
C_0	Initial volumetric moisture concentration (kg/m^3)
D	Diffusivity (m^2/s)
$e(t)$	Moving boundary position at time t (m)
F	Initial free moisture content
h	Convective heat transfer parameter ($\text{W}/\text{m}^2 \text{ } ^\circ\text{C}$)
ΔH	Water vaporization heat (J/kg)
k_s	Effective thermal conductivity ($\text{W}/\text{m K}$)
K_x	Rate constant of moisture loss (min^{-1})
$Q(x)$	Initial temperature profile of potato sample ($^\circ\text{C}$)
R	Half – thickness of potato sample (m)
$S(t)$	Free boundary position at time t (m)
T	Temperature profile inside the potato ($^\circ\text{C}$)
T_b	Bulk temperature of oil bath ($^\circ\text{C}$)
X_0	Initial moisture content ($\text{kg}/\text{kg db}$)
X_e	Equilibrium moisture content ($\text{kg}/\text{kg db}$)
x, t	Spatial (m) and time (s) coordinate respectively
$Z(t)$	Amount of vaporized moisture up to time t (kg)

I. Introduction

As it can be seen in specific reported literature [1]-[9] on the subject, immersion frying is an important historical method to prepare fast food.

This accounts for the economic relevance of the industrial processes involved.

In another paper [10] the so called “Initial Period” of the immersion frying process during which the preheating and bubble stages take place, was interpreted and modeled taking into account experimental observations and other arguments. Such process was viewed as a classical free boundary problem [11] to provide a mathematical descriptive model. In such model, the one – dimensional spatial domain $0 \leq x \leq R$, which coincides with the thickness of the slab potato sample, is characterized during the “Initial Period” by the presence of a free boundary which is localized at each time t by the function $S=S(t)$.

It is noticed that such free boundary represents the instantaneous position of a moisture desorption front which moves from the external interface potato – oil ($x=R$), penetrating the potato sample and reaching the center of it at a time τ such that $S(\tau)=0$.

Such free boundary divides the overall one dimensional spatial domain (OSD) $0 \leq x \leq R$ of the solid body in two regions:

- **Core region (CR)** : $0 \leq x \leq S(t)$
- **Peripheral region (PR)**: $S(t) \leq x \leq R$,

The last region, initially non – existent, increase its thickness along the frying period under analysis.

The corresponding equations to mass and heat transfer during the “Initial Period” of immersion frying process are summarized in Eqs. (1) to (13):

Core region (CR)

$$C(x, t_1) = C_0 \quad (1)$$

Peripheral region (PR)

$$S(t_1) = R \quad (12)$$

$$\frac{\partial C}{\partial t} = D \frac{\partial^2 C}{\partial x^2} \quad (2)$$

$$K_x = 0,78 \left(\frac{T_b}{170} \right)^{1,61} \quad (13)$$

Overall region (OSD)

$$\rho_s C_s \frac{\partial T}{\partial t} = k_s \frac{\partial^2 T}{\partial x^2} \quad (3)$$

The numerical coefficients used in (11) and (13) were adopted from those reported in [13] concerning the study of a similar immersion frying processes under similar conditions.

Boundary conditions

$$C(S(t), t) = C_0 \quad t > t_1 \quad (4)$$

$$-4AD \frac{\partial C}{\partial x}(R, t) = \omega_v \quad t > t_1 \quad (5)$$

where ω_v denotes the vaporization rate,

$$\frac{\partial T}{\partial x}(0, t) = 0 \quad t > t_1 \quad (6)$$

$$k_s \frac{\partial T}{\partial x}(R, t) - D \cdot \Delta H \frac{\partial C}{\partial x}(R, t) = h [T_b - T(R, t)] \quad t > t_1 \quad (7)$$

Initial conditions

$$C(x, t_1) = C_0 \quad (8)$$

$$T(x, t_1) = Q(x) \quad (9)$$

$$Q(x) = 1,5 \times 10^8 x^3 - 9,13 \times 10^5 x^2 + 2,05 \times 10^3 x + 20 \quad (10)$$

where the expression of the function $Q=Q(x)$ in (10) denotes a correlation function of a set of couples of values temperature vs. position obtained from an analytical expression in term of the transcendental function: *erf*, *erfc*, *exp*, for the temperature profile along the thickness of the potato sample evaluated at the ends of the preheating step ($t=t_1$) in the immersion frying process [12],

Interfacial mass balance

$$\frac{dS}{dt} = \frac{-\frac{K_x}{60} R^2 (\rho_s - C_0) (X_0 - X_e) \exp(-K_x t)}{C_0 (R + S(t)) - \int_{S(t)}^R C(x, t) dx} \quad (11)$$

where the interface mass balance provides by (11) is the equation of motion of the moving front,

II. Mathematical Model

Taking into account the considerations outlined in another paper, [10] a mixed empirical–predictive model consisting of an initial – free boundary value problem (IFBVP) for a set of two partial parabolic equations, one for the desorbed free moisture on (PR) and the other for the temperature profile on (OSD), together with an initial value problem for a first order ordinary differential equation descriptive of the dynamical behavior of the free boundary $S=S(t)$ was developed. Such initial value problem springs from the respective mass conservation for the free moisture on the free boundary $S=S(t)$.

It is important to notice the nonlinear nature of the mathematical descriptive model used. Such characteristic is due to the presence of the moving boundary which divides the **core** and **peripheral** regions and implies the coupling of the respective heat and mass transfer parabolic equations.

Free boundary problems as descriptive mathematical models of a wide class of technological and industrial processes, may be found in areas such as heat and mass transfer, diffusion – reaction, and others [14]–[21]. The research developed by other authors [22]–[27], are the pioneer contributions that provides predictive models to describe an immersion frying process applied to the cooking of solid substances (natural potato as an important particular case).

Eqs. (1) to (13) resume the resulting mathematical model developed in another paper [10] describing the “Initial Period” of the immersion frying process.

It is well known [28]–[30] the wellposedness of the mathematical free boundary model summarized in Eqs. (1) to (13), concerning the local, global (in time) existence and uniqueness of the solution, and its continuous dependence upon data for it.

However, it is known that only for few special cases of simple geometries and boundary data, an analytical (exact) solution can be found for free boundary problems [31].

Hence, a numerical solution must be searched. In such area, some numerical techniques have been developed and reported in literature [32]–[35].

At this point, it is important to notice that in the case of free boundary problems, even to attain a numerical solution, the free boundary (unknown moving boundary)

becomes a difficulty, and it is common the occurrence of some stability and convergence problems.

In this sense, initial moving boundary problems are not so difficult to handle.

In view of the previous considerations, the main objectives of this research were:

1. To deduce an adequate simplified mathematical model (that is, an initial moving boundary value problem (IMBVP), to play the role of the IFBVP which is given in Eqs. (1) to (13).
So, the original IFBVP under study was approximated by an associated Initial Moving Boundary Value Problem: IMBVP
2. To solve numerically such IMBVP
3. To validate the corresponding predicted data from the numerical solution.
Such numerical solution provides temperature profiles, moisture profiles, and the amount of vaporized desorbed moisture during the period of the process analyzed.

III. The Approximating Initial Moving Boundary Model

If we consider the dynamic behavior of the free boundary position $S=S(t)$ along the bubble stage of the immersion frying process, given by the Initial Value Problem (IVP) consisting in Eqs. (11) and (12) and we neglect the coupling term:

$$\int_{S(t)}^R C(x,t) dx \quad (14)$$

on the right hand side of Eq. (11), we obtain the following first order IVP for a real function which is denoted as $e=e(t)$:

$$\frac{de}{dt} = \frac{-\frac{K_x}{60}(\rho_s - C_0)(X_0 - X_e) \exp(-K_x t)}{C_0 \cdot (R + e(t))} \quad (15)$$

$$e(t_1) = R \quad (16)$$

The preceding submodel (15) – (16) is supported in view of the following bounds for $C_0(R+S(t))$ and for the coupling term given by Eq. (14):

$$C_0 R \leq C_0 (R + S(t)) \leq 2C_0 R \quad (17)$$

$$0 \leq \int_{S(t)}^R C(x,t) dx \leq C_0 R \quad (18)$$

The explicit solution for the IVP given by Eqs. (15) and (16), is:

$$e(t) = 2 \left[\frac{a}{K_x} (\exp(K_x t_1) - \exp(K_x t)) \right]^{1/2} \quad (19)$$

with the parameter a given as:

$$a = \frac{K_x (\rho_s - C_0)}{60 C_0} (X_0 - X_e) \quad (20)$$

Typical values for parameters and physical properties in the IVP (15) – (16) are summarized in Table I.

TABLE I
PARAMETERS AND PHYSICAL PROPERTIES DATA

Property	Value	Source
ρ_s	1087 kg/m ³	Experimental
C_0	217 kg free moisture/m ³ sample	Experimental
K_x	0.78 min ⁻¹	Experimental
X_0	4 kg/kg db	[12]
X_e	0.54 kg/kg db	[12]

The Initial Moving Boundary Value Problem (IMBVP) which approximates the original Initial Free Boundary Value Problem (IFBVP) given by Eqs. (1) to (13), is obtained replacing the Eqs. (11) and (12) by Eqs. (15) and (16), respectively.

In Fig. 1, the graphic representation of the function $e=e(t)$ given by Eq. (19), and the experimental values for the free boundary position $S=S(t)$ as resulting from observations outlined in [10], are illustrated. The corresponding linear regression curve denoted by $\sigma(t)$ (given by Eq. (21)) of such experimental values $S(t)$ is also illustrated in such figure:

$$\sigma(t) = -9 \times 10^{-5} t u_{(t-10)} + 0.0098 \quad (\text{m}) \quad (21)$$

The corresponding result of a linear regression analysis of the predicted values and the experimentally determined values for the frying process in oil bath temperature of 180 °C, show that:

$$r^2=0.9683 \quad m=-9 \times 10^{-5} \quad n=0.0098$$

where r^2 , m and n denote the complement to one of the residual sum of squares, slope and the intercept respectively.

Hence, it can be seen a good agreement between predicted and experimental values for the moisture desorption front position as a function of time during the “Initial Period”

Taking into account the preceding results, it is clear that, at first glance, by looking for information regarding the solution of the IFBVP given by Eq. (1) to (13), it is reasonable to solve an associate Initial Moving Boundary Value Problem (IMBVP), obtained from the already mentioned IFBVP, replacing the sub model given by Eqs. (11) and (12) by Eqs. (15) and (16) respectively.

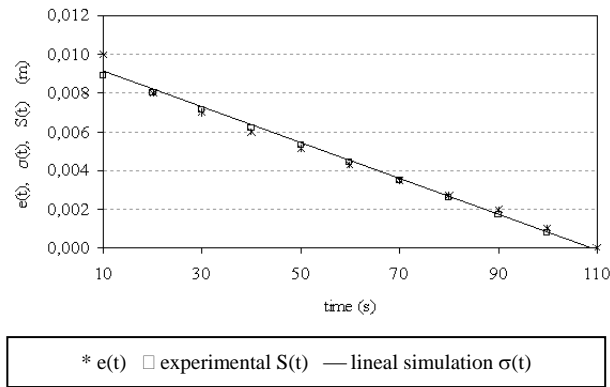


Fig. 1. The function $e(t)$, $S(t)$ and $\sigma(t)$

From the mathematical point of view this is a very advantageous fact, since the pre cited IMBVP is less complicated to handle than the original IFBVP.

IV. Model Solution

Now, we shall solve the IMBVP consisting in Eqs. (1) to (10), (13), (15) and (16).

An analytical solution is not possible for such IMBVP. So, it is only possible a numerical solution. For that purpose, we can use the explicit solution given by Eq. (19) for the moving boundary $e=e(t)$, of the sub-model (15)-(16), in order to solve numerically the IMBVP. So that, in the system given by Eqs. (1) to (10), we insert the expression given by Eq. (19) for the function $e=e(t)$ instead of $S=S(t)$. Then the resulting system is solved numerically.

To obtain the values of temperature $T=T(x,t)$, an explicit finite - difference method was used, while the values of the concentration $C=C(x,t)$ were computed “immobilizing” the moving boundary using a Landau’s coordinate transformation [25] and solving the resulting model by an implicit difference method.

The temperature distribution for $t=100s$ along the thickness of the potato sample is shown in Fig. 2. It is important to take into account that the function $Q(x)$ represented in Fig. 2, is the temperature distribution along the thickness of the potato sample evaluated at $t=100s$, when the bubble stage begin.

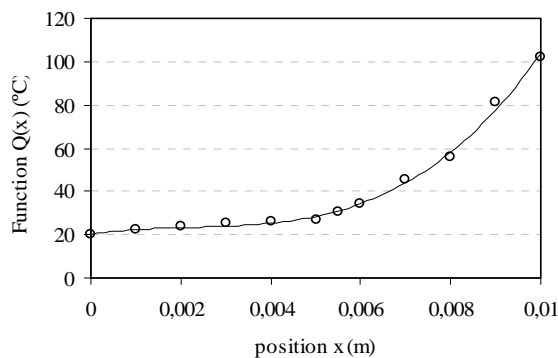


Fig. 2. Temperature along the position in the potato sample, at $t=100s$

The moving boundary position $\sigma(t)$ vs. time is shown in the upper part of Fig. 3, the correspondig desorbed moisture concentration at each time in the spatial domain $[\sigma(t),0.01]$ is illustrated in the lower part of the same figure.

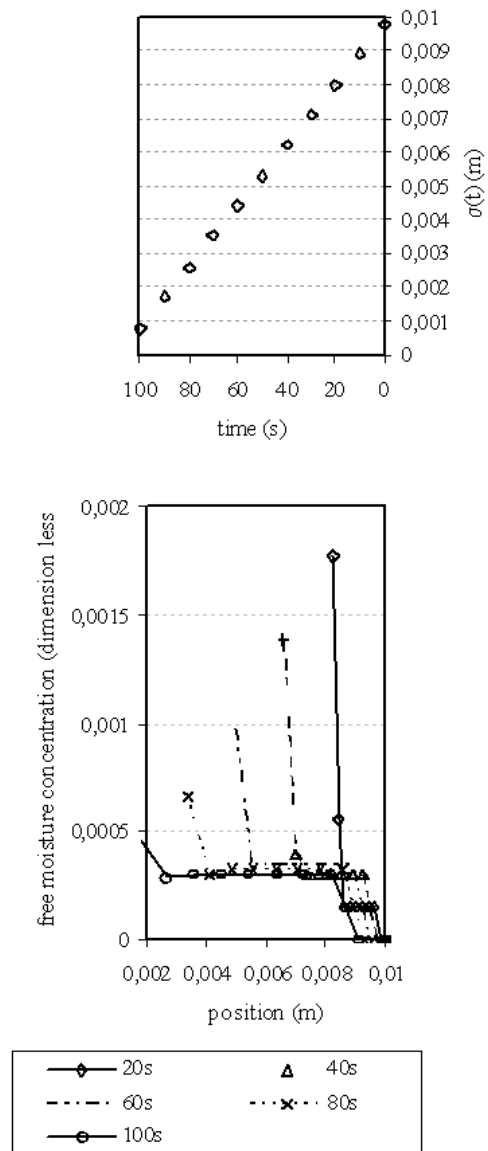


Fig. 3. Moving boundary position $\sigma(t)$ and desorbed free moisture concentration profiles

V. Model Validation

Taking into account [10] regarding the corresponding mass balance for the moisture lost by vaporization, the following equations can be written in order to obtain the amount $Z=Z(t)$ of vaporized desorbed free moisture on the external interface oil – potato, up to time t along the immersion frying:

$$Z(t) = -A \int_{t_1}^t D \frac{\partial C}{\partial x}(R, \tau) d\tau \quad (22)$$

$$D \frac{\partial C}{\partial x}(R, t) = \frac{1}{4A} \frac{dM}{dt} \quad (23)$$

$$\frac{dM}{dt} = 4LC_0(R + \sigma(t)) \frac{d\sigma}{dt} \quad (24)$$

where, $M(t)$ denotes the instantaneous desorbed moisture contained inside the **peripheral region** ($\sigma(t) \leq x \leq R$) of the potato sample, and A is the total lateral surface area on which the vaporization takes place.

Hence, from (22) to (24) we find:

$$Z(t) = -LC_0 \frac{d\sigma}{dt} [R + \sigma(t)] d\tau \quad (25)$$

Typical values for geometrical and physical properties, R , L and C_0 respectively, in Eq. (25) are: $R=0.01\text{m}$, $L=0.1\text{m}$; $C_0=217 \text{ kg/m}^3$, $t_1=6$ to 10s , $t=90$ to 100s .

Hence, in order to validate the model, we obtain $Z(t)$ for $t=100\text{s}$ from Eq. (25) and the resulting value is compared with the experimental estimation for the amount of moisture loss by vaporization during the first 100s (“Initial Period”) of the immersion frying process.

Such amount F_0 was observed [10] to be of the order of 20% of the total initial moisture contained in the potato sample.

From Eq. (25) we find:

$$Z(t) = 2,97 \times 10^{-3} \text{kg} \quad (26)$$

And, in view of the $3.3 \cdot 10^{-3} \text{kg}$ value of F_0 [10] it is observed that $Z(t)$ is in the order of 90% of F_0 , which can be considered an acceptable agreement.

Therefore, the IMBVP shown in previous sections is confirmed to be a good alternative in order to provide a descriptive mathematical model of the “Initial Period” during the immersion frying process discussed.

Furthermore, as a consequence, the initial free boundary value problem represents a valid alternative as mathematical model to describe immersion frying process.

VI. Conclusion

Preliminary results coming out of the confrontation between experimental data regarding the temporal localization of the moisture desorption front, and the corresponding predicted values for the dynamic position of such front, as resulting from the simulation of a simplified IVP, respect to the original one for the free boundary or desorption front position $S=S(t)$, were detected and taken into account.

Then, the difficult task implied by the mathematical solution of the predictive semi analytical model consisting of an initial free boundary value problem

(IFBVP), involving heat and mass transfer, such as it was formulated [10] was accomplished.

In such sense, the original Initial Free Boundary Value Problem (IFBVP) was replaced by an associate Initial Moving Boundary Value Problem (IMBVP) whose solution was obtained numerically.

An acceptable agreement between experimental data and predicted values by the IMBVP model was obtained. An excellent explicit analytical approximate expression which provides the dynamical behavior of the free moisture desorption front during the earlier stage of a natural potato immersion frying process, has been revealed and obtained.

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References

- [1] J. Aguilera, Fritura de Alimentos. *Temas en Tecnología de Alimentos*. CYTED. Instituto Politécnico Nacional. México. (Vol. 1, Cap.5, 1996)
- [2] G. Varela, A. Bender, I. Morton, Frying of Food. Principles, Changes, New Approaches. *Ellis Horwood series in Food Science and Technology*. (VCH Publishers. 1988).
- [3] M. Blumenthal, R. Stier, Optimization of Deep – Fat frying Operations. *Trends. Food Sci. Techno* 1.1 (1991) 144 – 148.
- [4] M. Gamble, P. Rice, J. Selman, Relationship Between Oil Uptake and Moisture Loss During Frying of Potato Slices From c.v. Record U.K. Tubers. *Int. J. Food Sci. Tech.*, 22 (1987) 233.
- [5] Ch. Keller, F. Escher, Heat and Mass Transfer During Deep Fat Frying of Potato Products. *Int Congress on Engineering and Food* 5. Cologne Germany. May 25 –June 1 (1989).
- [6] L. Levine, Understanding Frying Operations. Part II. *Cereal Foods World*, (1990) 359 - 514.
- [7] N. Mittelman, S. Mizrahi, Z. Berk, Heat and Mass Transfer in Frying. *Engineering and Food*. McKenna, B.M. (Elsevier Applied Science Publishers, 1984, Chap. 12)
- [8] G. Reddy, H. Das, H. Kinetics of Deep-Fat Frying of Potato and Optimization of Process Variables. *Journal of Food Science*, 30(2) (1993) 105 - 108.
- [9] P. Rice, M. Gamble, Technical Note: Modeling Moisture Loss During Potato Slice Frying. *International Journal of Food Science and Technology*, 24, (1989) pp. 183-187.
- [10] L. Villa, J. Gottifredi, A. Bouciguez, Some Considerations on a Simultaneous Heat and Mass Transfer Food Process. Model Formulation. *International Review of Chemical Engineering* Vol. 3, N° 2, (2011) pp. 265-271.
- [11] J. Crank, The Mathematics of Diffusion. *Clarendon Press*, Oxford, 1956.
- [12] Villa, L.T. Freído por inmersión profunda: Modelado y Análisis Matemático de la Dinámica Inicial. *FACENA*, Vol. 20 (2004) 47-65.
- [13] M. Krokida, V. Oreopoulou, Z. Maroulis, Water loss and oil uptake as a function of frying time. *Journal of Food Engineering* 44 (2000) 39 - 46
- [14] M. Ozisik, *Heat conduction*, (John Wiley and Sons, New York, 1980, Chapter 10)
- [15] M. Primicerio, Problemi di Diffusione a Frontiera Libera. *Bolletino Unione Matematica Italiana*, (5) 18-A, (1981) pp.11-68.
- [16] D. Tarzia, A Bibliography on Moving-Free Boundary Problems for the Heat Diffusion Equation. The Stefan Problem, *MAT – Serie A*, Rosario, # 2 (2000), with 5869 Titles on the Subject, 300 pages. www.austral.edu.ar/MAT-serieA/2 (2000)/

- [17] L. Villa, O. Quiroga, A moving boundary model for chemical dissolution in solid-liquid systems. Pseudo steady-state approximation ACH-Models in Chemistry 135 (1-2), (1998) pp 109-117
- [18] L. Villa, Some Remarks on a Moving Boundary Problem in a Fluid Reaction-Diffusion System. *Journal of Mathematical Analysis and Applications*. Vol 142, N° 2. (1989) pp 431 - 440
- [19] L. Villa, (1990). Some Remarks on Heat conduction Process with Change of Phase (Stefan Problem). *Journal Mathematical Analysis and Applications*. 151 (1990) pp 455 - 460
- [20] D. Tarzia, L. Villa, On the Free Boundary Problem in the Wen – Langmuir Shrinking Core Model for Non – Catalytic Gas – Solid Reactions. *Mechanic (Journal of the Italian Association of theoretical and applied mechanics)*. 24 (1989) 86 - 92
- [21] A. Fasano, L. Villa, Some Remarks. On the curvature of the Free Boundary in a Stefan Problem with Appearance of a Phase. *Bollettino della Unione Matematica Italiana* (6), 1-8, (1982) pp.743-752
- [22] B. Farkas, Modeling Immersion Frying as a Moving Boundary Problem. PhD Dissertation, University of California, Davis. 1994
- [23] B. Farkas, R. Singh, T. Rumsey, Mathematical Modeling of Immersion Frying: A Novel use of Drying Theory. *Proceeding of the 9th International Drying Symposium*, Gold Coast. Australia. 1995 pp 1 - 4
- [24] B. Farkas, R. Singh, T. Rumsey, Modeling Heat and Mass Transfer in Immersion Frying. **I.** Model Development. *J. Food Engng.*, 29, (1996) pp.211-226.
- [25] B. Farkas, R. Singh, T. Rumsey, Modeling Heat and Mass Transfer in Immersion Frying. **II.** Model Solution and Verification. *J. Food Engng.*, 29, (1996) 227-248.
- [26] M. Farid, X. Chen, The analysis of heat and mass transfer during frying of food using a moving boundary solution procedure. *Heat and Mass Transfer* **34** (1998), pp.69-77.
- [27] M. Farid, A unified approach to the heat and mass transfer in melting, solidification, frying and different drying process. *Chemical Engineering Sciences* **56** (2001), 5419-5427.
- [28] A. Friedman, Remarks On Stefan – Type Free Boundary Problems for Parabolic Equations. *Journal of Mathematics and Mechanics*, Vol. 19, N° 6, (1960) pp 885 - 903.
- [29] J. Cannon, J. Dougals, The stability of the boundary in a Stefan problem. *Ann. Scuola Norm. Sup Pisa. Ser III*, 21, (1967) pp83 – 91
- [30] B. Sherman, Free Boundary Problems for the Heat Equation in which the Moving Interface Coincides Initially with the Fixed Face. *Journal of Mathematical Analysis and Applications* 33 (1971) pp 449-466
- [31] J. Crank, *Free and Moving Boundary Problems*. (Clarendon Press, Oxford, 1984)
- [32] H. Landau, Heat conduction in a melting solid. *Quarterly of Applied Mathematics*, 8 (1) (1950) pp.81 – 94
- [33] P. Greenfield, Cydic – pressure freeze drying. *Chem. Engng. Sci.*, 29, (1974) pp 2115 - 2123
- [34] J. Duda, M. Melone, R. Notter, J. Vrentas, Analysis of two – Dimensional Diffusion – Controlled Moving Boundary Problems. *Int. J. Heat Mass Transfer*, 18, (1975) pp 901 – 910
- [35] R. Furzeland, A Comparative Study of Numerical Methods for Moving Boundary Problems. *J. Inst. Maths. Applics*, 26, (1980) pp 411-429.

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