PSEUDO-DUAL PAIRS AND BRANCHING OF DISCRETE SERIES

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ABSTRACT. For a semisimple Lie group G, we study Discrete Series representations with admissible branching to a symmetric subgroup H. This is done using a canonical associated symmetric subgroup H_0, forming a pseudo-dual pair with H, and a corresponding branching law for this group with respect to its maximal compact subgroup. This is in analogy with either Blattner's or Kostant-Heckmann multiplicity formulas, and has some resemblance to Frobenius reciprocity. We give several explicit examples and links to Kobayashi-Pevzner theory of symmetry breaking and holographic operators. Our method is well adapted to computer algorithms, such as for example the Atlas program.

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1. Introduction

For a semisimple Lie group G, an irreducible representation (π, V) of G and closed reductive subgroup $H \subset G$ the problem of decomposing the restriction of π to H has received attention ever since number theory or physics and other branches of mathematics required a solution. In this paper, we are concerned with the important particular case of branching representations of the Discrete Series, i.e. those π arising as closed irreducible subspaces of the left regular representation in $L^2(G)$, and breaking the symmetry by a reductive subgroup H. Here much work has been done. Notable is the paper of Gross-Wallach, [9], and the work of Toshiyuki Kobayashi and his school. For further references on the subject, we refer to the overview work of Toshiyuki Kobayashi and references therein. To compute the decomposition of the restriction of π to a symmetric subgroup (see 3.4.2), in [9] it is shown a duality Theorem for Discrete Series representation. Their duality is based on the dual subgroup G^d (this is the dual subgroup which enters the duality introduced by Flensted-Jensen in his study of discrete series for affine symmetric spaces [7]) and, roughly speaking, their formula looks like

$$dimHom_H(\sigma, \pi_{|_H}) = dimHom_{\widetilde{K}}(F_\sigma, \tilde{\pi}).$$

Here, π is a irreducible square integrable representation of G, σ is a irreducible representation of H, F_σ is a irreducible representation of a maximal compact subgroup $\widetilde{K}$ of G^d, and $\tilde{\pi}$ is a finite sum of fundamental representations of G^d attached to π. In [23], B. Speh and the first author noticed a different duality Theorem for restriction to a symmetric subgroup, let H_0 the associated subgroup to H and $L := H_0 \cap H$ a maximal compact subgroup of H. Then,

$$(\ddagger) \qquad dimHom_H(\sigma, \pi_{|_H}) = dimHom_L(\sigma_0, \widetilde{\Pi}).$$

Here, π is certain irreducible representation of G, σ is a irreducible representation of H, σ_0 is the lowest L-type of σ and $\widetilde{\Pi}$ is a finite sum of irreducible representation of H_0 attached to π. The purpose of this paper is, for a H-admissible Discrete Series π for G, to show a formula as the above and to provide an explicit isomorphism between the two vector spaces involved in the equality. This is embodied in Theorem 3.1.

Theorem 3.1 reduces the branching law in two steps (1) For the maximal compact subgroup K of G and the lowest K-type of π, branching this under L (maximal compact in H and also in H_0) (2) branching a Discrete Series of H_0 with respect to L, i.e. finding its L-types with multiplicity. Both of these steps can be implemented in algorithms, as they are available for example in the computer program Atlas, http://atlas.math.umd.edu.

We would like to point out that T. Kobayashi, T. Kobayashi-Pevzner and Nakahama have shown a duality formula as (‡) for holomorphic Discrete Series representation π. In order to achieve their result, they have shown a explicit isomorphism between the two vector spaces in the formula. Further, with respect to analyze $res_H(\pi)$, Kobayashi-Oshima have shown a way to compute the irreducible components of $res_H(\pi)$ in the language of Zuckerman modules $A_{\mathfrak{q}}(\lambda)$ [18][19].

As a consequence of the involved material, we obtain a necessary and sufficient condition for a symmetry breaking operators to be represented via normal derivatives. This is presented in Proposition 6.1.

Another consequence is Proposition 4.9. That is, for the closure of the linear span of the totality of H_0-translates (resp. H-translates) of the isotypic component associated to the lowest K-type of π, we exhibit its explicit decomposition as a finite sum of Discrete Series representations of H_0 (resp. H).

Our proof is heavily based in that Discrete Series representations are realized in reproducing kernel Hilbert spaces. As a consequence, in Lemma 3.6, we obtain a general result on the structure of the kernel of a certain restriction map. The proof also relies on the work of Hecht-Schmid [11], and a result of Schmid in [27].

It follows from the work of Kobayashi-Oshima, else, from Tables 1,2,3, that whenever a Discrete Series for G has admissible restriction to a symmetric subgroup, then, the infinitesimal character of the representation is dominant with respect to either a Borel de Siebenthal system of positive roots or to a system of positive roots so that it has two noncompact simple roots, each of one, has multiplicity one in the highest root. Under the H-admissible hypothesis, the infinitesimal character of each of the irreducible components of $\widetilde{\Pi}$ in formula (‡), has the same property as the infinitesimal character of π. Thus, for most H-admissible Discrete series, to compute the right hand side of (‡), we may appeal to the work of the first author and Wolf [26]. Their results let us compute the highest weight of each irreducible factor in the restriction of π to $K_1(\Psi)$, next, we apply [5, Theorem 5] for the general case.

We may speculate that a formula like (‡) might be true for π whose underlying Harish-Chandra module is equivalent to a unitarizable Zuckerman module. In this case, the definition of σ_0 would be the subspace spanned by the lowest L-type of σ and $\widetilde{\Pi}$ would be a Zuckerman module attached to the lowest K-type of π.

The paper is organized as follows. In Section 2, we introduce facts about Discrete Series representation and notation. In Section 3, we state the main Theorem and begin its proof. As a tool, we obtain information on the kernel of the restriction map.

In Section 4, we complete the proof of the main Theorem. As a subproduct, we obtain information on the kernel of the restriction map, under admissibility hypothesis. We present examples and applications of the Main Theorem in section 5. This includes lists of multiplicity free restriction of representations, many of the multiplicity free representations are non holomorphic Discrete Series representations. We also dealt with quaternionic and generalized quaternionic representations.

In Section 6, we analyze when symmetry breaking operators are represented by means of normal derivatives. Section 7 presents the list of H-admissible Discrete Series and related information.

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2. Preliminaries and some notation

Let G be an arbitrary, matrix, connected semisimple Lie group. Henceforth, we fix a maximal compact subgroup K for G and a maximal torus T for K. Harish-Chandra showed that G admits square integrable irreducible representations if and only if T is a Cartan subgroup of G. For this paper, we always assume T is a Cartan subgroup of G. Under these hypothesis, Harish-Chandra showed that the set of equivalence classes of irreducible square integrable representations is parameterized by a lattice in $i\mathfrak{t}^\star$. In order to state our results we need to make explicit this parametrization and set up some notation. As usual, the Lie algebra of a Lie group is denoted by the corresponding lower case German letter. To avoid notation, the complexification of the Lie algebra of a Lie group is also denoted by the corresponding German letter without any subscript. $V^\star$ denotes the dual space to a vector space V. Let θ be the Cartan involution which corresponds to the subgroup K, the associated Cartan decomposition is denoted by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Let $\Phi(\mathfrak{g}, \mathfrak{t})$ denote the root system attached to the Cartan subalgebra $\mathfrak{t}$. Hence, $\Phi(\mathfrak{g}, \mathfrak{t}) = \Phi_c \cup \Phi_n = \Phi_c(\mathfrak{g}, \mathfrak{t}) \cup \Phi_n(\mathfrak{g}, \mathfrak{t})$ splits up as the union the set of compact roots and the set of noncompact roots. From now on, we fix a system of positive roots Δ for Φ_c. For this paper, either the highest weight or the infinitesimal character of an irreducible representation of K is dominant with respect to Δ. The Killing form gives rise to an inner product $(..., ...)$ in $i\mathfrak{t}^\star$. As usual, let $\rho = \rho_G$ denote half of the sum of the roots for some system of positive roots for $\Phi(\mathfrak{g}, \mathfrak{t})$. *A Harish-Chandra parameter* for G is $\lambda \in i\mathfrak{t}^\star$ such that $(\lambda, \alpha) \neq 0$, for every $\alpha \in \Phi(\mathfrak{g}, \mathfrak{t})$, and so that $\lambda + \rho$ lifts to a character of T. To each Harish-Chandra parameter λ, Harish-Chandra, associates a unique irreducible square integrable representation $(\pi_\lambda^G, V_\lambda^G)$ of G of infinitesimal character λ. Moreover, he showed the map $\lambda \to (\pi_\lambda^G, V_\lambda^G)$ is a bijection from the set of Harish-Chandra parameters dominant with respect to Δ onto the set of equivalence classes of irreducible square integrable representations for G (see [32, Chap 6]). For short, we will refer to an irreducible square integrable representation as a Discrete Series representation.

Each Harish-Chandra parameter λ gives rise to a system of positive roots

$$\Psi_\lambda = \Psi_{G,\lambda} = \{\alpha \in \Phi(\mathfrak{g}, \mathfrak{t}) : (\lambda, \alpha) > 0\}.$$

From now on, we assume that Harish-Chandra parameter for G are dominant with respect to Δ. Whence, $\Delta \subset \Psi_\lambda$. We write $\rho_n^\lambda = \rho_n = \frac{1}{2}\sum_{\beta \in \Psi_\lambda \cap \Phi_n} \beta$, $(\Psi_\lambda)_n := \Psi_\lambda \cap \Phi_n$.

We denote by $(\tau, W) := (\pi_{\lambda+\rho_n}^K, V_{\lambda+\rho_n}^K)$ the lowest $K-$type of $\pi_\lambda := \pi_\lambda^G$. The highest weight of $(\pi_{\lambda+\rho_n}^K, V_{\lambda+\rho_n}^K)$ is $\lambda + \rho_n - \rho_c$. We recall a Theorem of Vogan's thesis [31][6] which states that (τ, W) determines $(\pi_\lambda, V_\lambda^G)$ up to unitary equivalence. We recall the set of square integrable sections of the vector bundle determined by the principal bundle $K \to G \to G/K$ and the representation (τ, W) of K is isomorphic to the space

$$L^2(G\times_\tau W) \\ := \{f\in L^2(G)\otimes W : f(gk)=\tau(k)^{-1}f(g), g\in G, k\in K\}.$$

Here, the action of G is by left translation $L_x, x\in G$. The inner product on $L^2(G)\otimes W$ is given by

$$(f,g)_{V_\lambda}=\int_G (f(x),g(x))_W dx,$$

where $(...,...)_W$ is a $K-$invariant inner product on W. Subsequently, L_D (resp. R_D) denotes the left infinitesimal (resp. right infinitesimal) action on functions from G of an element D in universal enveloping algebra $U(\mathfrak{g})$ for the Lie algebra $\mathfrak{g}$. As usual, Ω_G denotes the Casimir operator for $\mathfrak{g}$. Following Hotta-Parthasarathy [13], Enright-Wallach [6], Atiyah-Schmid [1], we realize $V_\lambda := V_\lambda^G$ as the space

$$H^2(G,\tau)=\{f\in L^2(G)\otimes W : f(gk)=\tau(k)^{-1}f(g) \\ g\in G, k\in K, R_{\Omega_G}f=[(\lambda,\lambda)-(\rho,\rho)]f\}.$$

We also recall, $R_{\Omega_G}=L_{\Omega_G}$ is an elliptic $G-$invariant operator on the vector bundle $W\to G\times_\tau W\to G/K$ and hence, $H^2(G,\tau)$ consists of smooth sections, moreover point evaluation e_x defined by $H^2(G,\tau)\ni f\mapsto f(x)\in W$ is continuous for each $x\in G$ (cf. [25, Appendix A4]). Therefore, the orthogonal projector P_λ onto $H^2(G,\tau)$ is an integral map (integral operator) represented by the smooth *matrix kernel* or *reproducing kernel* [25, Appendix A1, Appendix A4, Appendix A6].

$$K_\lambda : G\times G\to End_{\mathbb{C}}(W) \tag{2.1}$$

which satisfies $K_\lambda(\cdot,x)^\star w$ belongs to $H^2(G,\tau)$ for each $x\in G, w\in W$ and

$$(P_\lambda(f)(x),w)_W=\int_G (f(y),K_\lambda(y,x)^\star w)_W dy,\ f\in L_2(G\times_\tau W).$$

For a closed reductive subgroup H, after conjugation by an inner automorphism of G we may and will assume $L:=K\cap H$ is a maximal compact subgroup for H. That is, H is $\theta-$stable. In this paper for irreducible square integrable representations (π_λ,V_λ) for G we would like to analyze its restriction to H. In particular, we study the irreducible $H-$subrepresentations for π_λ. A known fact is that any irreducible $H-$subrepresentation of V_λ is a square integrable representation for H, for a proof (cf. [9]). Thus, owing to the result of Harish-Chandra on the existence of square integrable representations, from now on, we may and will assume H *admits a compact Cartan subgroup*. After conjugation, we may assume $U:=H\cap T$ is a maximal torus in $L = H \cap K$. From now on, we set a square integrable representation $V_\mu^H \equiv H^2(H,\sigma) \subset L^2(H \times_\sigma Z)$ of lowest $L-$type $(\pi^L_{\mu+\rho_n^\mu}, V^L_{\mu+\rho_n^\mu}) \equiv: (\sigma, Z)$.

For a representation M and irreducible representation N, $M[N]$ denotes the isotypic component of N, that is, $M[N]$ is the linear span of the irreducible subrepresentations of M equivalent to N. If topology is involved $M[N]$ is the closure of the linear span.

For a H-admissible representation π, $Spec_H(\pi)$, denotes the set of Harish-Chandra parameters of the irreducible H-subrepresentations of π.

3. Duality Theorem, explicit isomorphism

3.1. **Statement and proof of the duality result.** The unexplained notation is as in section 2, our hypotheses are $(G, H = (G^\sigma)_0)$ is a symmetric pair and $(\pi_\lambda, V_\lambda^G)$ is a H-admissible, square integrable irreducible representation for G. $K = G^\theta$ is a maximal compact subgroup of G, $H_0 := (G^{\sigma\theta})_0$ and K is so that $L = H \cap K = H_0 \cap K$ is a maximal compact subgroup of both H and H_0. By definition, H_0 is the *associated* subgroup to H.

In this section, under our hypothesis, for V_μ^H a irreducible factor for $res_H(\pi_\lambda)$, we show an explicit isomorphism from the space

$Hom_H(V_\mu^H, V_\lambda^G)$ onto $Hom_L(V^L_{\mu+\rho_n^\mu}, \pi_\lambda(\mathcal{U}(\mathfrak{h}_0))V_\lambda^G[V^K_{\lambda+\rho_n}])$.

We also analyze the restriction map $r_0 : H^2(G,\tau) \to L^2(H_0 \times_\tau W)$.

To follow, we present the necessary definitions and facts involved in the main statement.

3.1.1. We consider the linear subspace $\mathcal{L}_\lambda$ spanned by the lowest L-type subspace of each irreducible H-factor of $res_H((L, H^2(G,\tau)))$. That is,

$$\mathcal{L}_\lambda \text{ is the linear span of } \cup_{\mu \in Spec_H(\pi_\lambda)} H^2(G,\tau)[V_\mu^H][V^L_{\mu+\rho_n^\mu}].$$

We recall that our hypothesis yields that the subspace of L-finite vectors in V_λ^G is equal to the subspace of K-finite vectors [16, Prop. 1.6]. Whence, we have $\mathcal{L}_\lambda$ is a subspace of the space of K-finite vectors in $H^2(G,\tau)$.

3.1.2. We also need the subspace

$$\mathcal{U}(\mathfrak{h}_0)W := L_{\mathcal{U}(\mathfrak{h}_0)} H^2(G,\tau)[V^K_{\lambda+\rho_n^\lambda}] \equiv \pi_\lambda(\mathcal{U}(\mathfrak{h}_0))(V_\lambda^G[V^K_{\lambda+\rho_n^\lambda}]).$$

We write $\mathrm{Cl}(U(\mathfrak{h}_0)W)$ for the closure of $U(\mathfrak{h}_0)W$. Hence, $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)$ is the closure of the left translates by the algebra $\mathcal{U}(\mathfrak{h}_0)$ of the subspace of K-finite vectors

$$H^2(G,\tau)[V^K_{\lambda+\rho_n^\lambda}] = \{K_\lambda(\cdot,e)^\star w : w \in W\} \equiv W.$$

Thus, $\mathcal{U}(\mathfrak{h}_0)W$ consists of analytic vectors for π_λ. Therefore, $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)$ is invariant under left translations by H_0. In Proposition 4.9 we present the decomposition of $\mathcal{U}(\mathfrak{h}_0)W$ as a sum of irreducible representations for H_0.

We point out

The L-module $\mathcal{L}_\lambda$ is equivalent to the underlying L -module in $\mathcal{U}(\mathfrak{h}_0)W$.

This has been proven in [30, (4.5)]. For completeness we present a proof in Proposition 4.9.

Under the extra assumption $res_L(\tau)$ is irreducible, we have $\mathcal{U}(\mathfrak{h}_0)W$ is a irreducible $(\mathfrak{h}_0, L)$-module, and, in this case, the lowest L-type of $\mathcal{U}(\mathfrak{h}_0)W$ is $(res_L(\tau), W)$. That is, $\mathcal{U}(\mathfrak{h}_0)W$ is equivalent to the underlying Harish-Chandra module for $H^2(H_0, res_L(\tau))$. The Harish-Chandra parameter $\eta_0 \in i\mathfrak{u}^\star$ for $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)$ is computed in 3.4.1.

For scalar holomorphic Discrete Series, the classification of the symmetric pairs (G,H) such that the equality $\mathcal{U}(\mathfrak{h}_0)W = \mathcal{L}_\lambda$ holds, is:

$(\mathfrak{su}(m,n), \mathfrak{su}(m,l)+\mathfrak{su}(n-l)+\mathfrak{u}(1))$, $(\mathfrak{so}(2m,2), \mathfrak{u}(m,1))$, $(\mathfrak{so}^\star(2n), \mathfrak{u}(1,n-1))$, $(\mathfrak{so}^\star(2n), \mathfrak{so}(2)+\mathfrak{so}^\star(2n-2))$, $(\mathfrak{e}_{6(-14)}, \mathfrak{so}(2,8)+\mathfrak{so}(2))$. [30, (4.6)]. Thus, there exists scalar holomorphic Discrete Series with $\mathcal{U}(\mathfrak{h}_0)W \neq \mathcal{L}_\lambda$.

3.1.3. To follow, we set some more notation. We fix a representative for (τ, W). We write

$(res_L(\tau), W) = \sum_{1\leq j\leq r} q_j(\sigma_j, Z_j)$, $q_j = \dim Hom_L(Z_j, res_L(W))$ and the decomposition in isotypic components

$$W = \oplus_{1\leq j\leq r} W[(\sigma_j, Z_j)] = \oplus_{1\leq j\leq r} W[\sigma_j].$$

From now on, we fix respective representatives for (σ_j, Z_j) with $Z_j \subset W[(\sigma_j, Z_j)]$.

Henceforth, we denote by

$$\mathbf{H}^2(H_0,\tau) := \sum_j dimHom_L(\tau,\sigma_j)\, H^2(H_0,\sigma_j).$$

We think the later module as a linear subspace of

$$\sum_j L^2(H_0 \times_{\sigma_j} W[\sigma_j])_{H_0-disc} \equiv L^2(H_0 \times_\tau W)_{H_0-disc}.$$

Hence, $\mathbf{H}^2(H_0,\tau) \subset L^2(H_0 \times_\tau W)_{H_0-disc}$. We note that when $res_L(\tau)$ is irreducible, then $\mathbf{H}^2(H_0,\tau) = H^2(H_0, res_L(\tau))$.

3.1.4. Owing to both spaces $H^2(H,\sigma), H^2(G,\tau)$ are reproducing kernel spaces, we represent each $T \in Hom_H(H^2(H,\sigma), H^2(G,\tau))$ by a kernel $K_T : H \times G \to Hom_{\mathbb{C}}(Z,W)$ so that $K_T(\cdot,x)^\star w \in H^2(H,\sigma)$ and $(T(g)(x),w)_W = \int_H (g(h), K_T(h,x)^\star w)_Z dh$. Here, $x \in G, w \in W, g \in H^2(H,\sigma)$. In [25], it is shown: K_T is a smooth function, $K_T(h,\cdot)z = K_{T^\star}(\cdot,h)^\star z \in H^2(G,\tau)$ and

$$K_T(e,\cdot)z \in H^2(G,\tau)[V_\mu^H][V^L_{\mu+\rho_n^H}] \tag{3.1}$$

is a L-finite vector in $H^2(G,\tau)$.

3.1.5. Finally, we recall the restriction map

$$r_0 : H^2(G,\tau) \to L^2(H_0 \times_\tau W),\ r_0(f)(h_0) = f(h_0), h_0 \in H_0,$$

is (L^2, L^2)-continuous [24].

The main result of this section is,

Theorem 3.1. *We assume (G,H) is a symmetric pair and $res_H(\pi_\lambda)$ is admissible. We fix a irreducible factor V_μ^H for $res_H(\pi_\lambda)$. Then, the following statements hold.*
i) The map $r_0 : H^2(G,\tau) \to L^2(H_0 \times_\tau W)$ restricted to $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)$ yields a isomorphism between $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)$ onto $\mathbf{H}^2(H_0,\tau)$.
ii) For each fixed intertwining L-equivalence

$$D : \mathcal{L}_\lambda[V^L_{\mu+\rho_n^\mu}] = H^2(G,\tau)[V_\mu^H][V^L_{\mu+\rho_n^\mu}] \to (\mathcal{U}(\mathfrak{h}_0)W)[V^L_{\mu+\rho_n^\mu}],$$

the map

$$r_0^D : Hom_H(H^2(H,\sigma), H^2(G,\tau)) \to Hom_L(V^L_{\mu+\rho_n^\mu}, \mathbf{H}^2(H_0,\tau))$$

defined by

$$T \overset{r_0^D}{\longmapsto} (V^L_{\mu+\rho_n^\mu} \ni z \mapsto r_0(D(K_T(e,\cdot)z)) \in \mathbf{H}^2(H_0,\tau))$$

is a linear isomorphism.

Remark 3.2. When the natural inclusion $H/L \to G/K$ is a holomorphic map, T. Kobayashi, M. Pevzner and Y. Oshima in [17],[20] has shown a similar dual multiplicity result after replacing the underlying Harish-Chandra module in $\mathbf{H}^2(H_0,\tau)$ by its representation as a Verma module. Also, in the holomorphic setting, Jakobsen-Vergne in [14] has shown the isomorphism $H^2(G,\tau) \equiv \sum_{r\geq 0} H^2(H, \tau_{|_L} \otimes S^{(r)}((\mathfrak{h}_0 \cap \mathfrak{p}^+))^\star)$. On the papers, [22] [21], we find applications of the result of Kobayashi for their work on decomposing holomorphic Discrete Series. H. Sekiguchi [28] has obtained a similar result of branching laws for singular holomorphic representations.

Remark 3.3. The proof of Theorem 3.1 requires to show the map r_0^D is well defined as well as several structure Lemma's. Once we verify the map is well defined, we will show injectivity, Corollary 3.9, Proposition 4.2 and linear algebra will give the surjectivity. In Proposition 4.9, we show i), in the same Proposition we give a proof of the existence of the map D as well as its bijectivity, actually this result has been shown in [30]. However, we sketch a proof in this note. The surjectivity also depends heavily on a result in [30], for completeness we give a proof. We may say that our proof of Theorem 1 is rather long and intricate, involving both linear algebra for finding the multiplicities, and analysis of the kernels of the intertwining operators in question to set up the equivalence of the H-morphisms and the L-morphisms. The structure of the branching and corresponding symmetry breaking is however very convenient to apply in concrete situations, and we give several illustrations.

We explicit the inverse map to the bijection r_0^D in subsection 4.1.

Remark 3.4. When $\mathcal{L}_\lambda = \mathcal{U}(\mathfrak{h}_0)W$ we may take D equal to the identity map. We believe that one choice of D is the orthogonal projector onto $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)$ restricted to $\mathcal{L}_\lambda[V^L_{\mu+\rho_n^\mu}]$.

Remark 3.5. A mirror statement to Theorem 3.1 for symmetry breaking operators is as follows: $Hom_H(H^2(G,\tau), H^2(H,\sigma))$ is isomorphic to $Hom_L(Z, \mathbf{H}^2(H_0,\tau))$ via the map $S \mapsto (z \mapsto (H_0 \ni x \mapsto r_0^D(S^\star)(z)(x) = r_0(D(K_S(\cdot,x)^\star)(z)) \in W)$.

3.1.6. We verify $r_0(D(K_T(e,\cdot)z))(\cdot)$ belongs to $L^2(H_0 \times_\tau W)_{H_0-disc}$.

Indeed, owing to our hypothesis, a result [5] (see [4, Proposition 2.4]) implies π_λ is L-admissible. Hence, [15, Theorem 1.2] implies π_λ is H_0-admissible. Also, [16, Proposition 1.6] shows the subspace of L-finite vectors in $H^2(G,\tau)$ is equal to the subspace of K-finite vectors and $res_{\mathcal{U}(\mathfrak{h}_0)}(H^2(G,\tau)_{K-fin})$ is a admissible, completely algebraically decomposable representation. Thus, the subspace $H^2(G,\tau)[W] \equiv W$ is contained in a finite sum of irreducible $\mathcal{U}(\mathfrak{h}_0)$-factors. Hence, $\mathcal{U}(\mathfrak{h}_0)W$ is a finite sum of irreducible $\mathcal{U}(\mathfrak{h}_0)$ factors. In [9], we find a proof that the irreducible factors for $res_{H_0}(\pi_\lambda)$ are square integrable representations for H_0, whence, the equivariance and continuity of r_0 yields $r_0(\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)$ is contained in $L^2(H_0 \times_\tau W)_{H_0-disc}$. 3.1.4 shows $K_T(e,\cdot) \in V_\lambda[V^H_\mu][V^L_{\mu+\rho_n^H}]$, hence, $D(K_T(e,\cdot)z))(\cdot) \in \mathcal{U}(\mathfrak{h}_0)W$, and the claim follows.

3.1.7. The map $Z \ni z \mapsto r_0(D(K_T(e,\cdot)z))(\cdot) \in L^2(H_0 \times_\tau W)$ is a L-map.

For this, we recall the equalities

$$K_T(hl, gk) = \tau(k^{-1})K_T(h, g)\sigma(l), k \in K, l \in L, g \in G, h \in H.$$

$$K_T(hh_1, hx) = K_T(h_1, x), h, h_1 \in H, x \in G.$$

Therefore, $K_T(e, hl_1)\sigma(l_2)z = \tau(l_1^{-1})K_T(l_2, h)z = \tau(l_1^{-1})K_T(e, l_2^{-1}h)z$ for $l_1, l_2 \in L, h \in H_0$ and we have shown the claim.

We have enough information to verify the injectivity we have claimed in $i)$ as well as the injectivity of the map r_0^D. For these, we show a fact valid for a arbitrary reductive pair (G, H) and arbitrary Discrete Series representation.

3.2. **Kernel of the restriction map.** In this paragraph we show a fact valid for any reductive pair (G, H) and arbitrary representation π_λ. The objects involved in the fact are the restriction map r from $H^2(G, \tau)$ into $L^2(H \times_\tau W)$ and the subspace

$$\mathcal{U}(\mathfrak{h})W := L_{\mathcal{U}(\mathfrak{h})}H^2(G, \tau)[V^K_{\lambda+\rho_n^\lambda}] \equiv \pi_\lambda(\mathcal{U}(\mathfrak{h}))(V^G_\lambda[V^K_{\lambda+\rho_n^\lambda}]), \tag{3.2}$$

We write $\mathrm{Cl}(U(\mathfrak{h})W)$ for the closure of $U(\mathfrak{h})W$. The subspace $Cl(\mathcal{U}(\mathfrak{h})W)$ is the closure of the left translates by the algebra $\mathcal{U}(\mathfrak{h})$ of the subspace of K-finite vectors

$$\{K_\lambda(\cdot, e)^\star w : w \in W\} = H^2(G, \tau)[W].$$

Thus, $\mathcal{U}(\mathfrak{h})W$ consists of analytic vectors for π_λ. Hence, $\mathrm{Cl}(\mathcal{U}(\mathfrak{h})W)$ is invariant by left translations by H. Therefore the subspace

$$L_H(H^2(G, \tau)[W]) = \{K_\lambda(\cdot, h)^\star w = L_h(K_\lambda(\cdot, e)^\star w) : w \in W, h \in H\}$$

is contained in $\mathrm{Cl}(\mathcal{U}(\mathfrak{h})W))$. Actually,

$$\mathrm{Cl}(L_H(H^2(G, \tau)[W])) = \mathrm{Cl}(\mathcal{U}(\mathfrak{h})W).$$

The other inclusion follows from that $\mathrm{Cl}(L_H(H^2(G, \tau)[W]))$ is invariant by left translation by H and $\{K_\lambda(\cdot, e)^\star w : w \in W\}$ is contained in the subspace of smooth vectors in $\mathrm{Cl}(L_H(H^2(G, \tau)[W]))$.

The result pointed out in the title of the this paragraph is:

Lemma 3.6. *Let (G, H) be a arbitrary reductive pair and a arbitrary representation $(\pi_\lambda, H^2(G, \tau))$. Then, $Ker(r)$ is equal to the orthogonal subspace to* $\mathrm{Cl}(\mathcal{U}(\mathfrak{h})W)$.

Proof. Since, [24], $r : H^2(G, \tau) \to L^2(H \times_\tau W)$ is a continuous map, we have $Ker(r)$ is a closed subspace of $H^2(G, \tau)$. Next, for $f \in H^2(G, \tau)$, it holds the identity

$$(f(x), w)_W = \int_G (f(y), K_\lambda(y, x)^\star w)_W dy, \forall x \in G, \forall w \in W.$$

Thus, $r(f) = 0$ if and only if f is orthogonal to the subspace spanned by $\{K_\lambda(\cdot, h)^\star w : w \in W, h \in H\}$.

Hence, $\mathrm{Cl}(Ker(r)) = (\mathrm{Cl}(L_H(H^2(G,\tau)[W]))^{\perp}$. Applying the considerations after the definition of $\mathrm{Cl}(\mathcal{U}(\mathfrak{h})W)$ we obtain $Ker(r)^{\perp} = \mathrm{Cl}(\mathcal{U}(\mathfrak{h})W)$. Thus $Ker(r) = (Ker(r)^{\perp})^{\perp} = \mathrm{Cl}(\mathcal{U}(\mathfrak{h})W)$. □

Corollary 3.7. *Any irreducible H-discrete factor M for* $\mathrm{Cl}(\mathcal{U}(\mathfrak{h})W)$ *contains a L-type in $res_L(\tau)$. That is, $M[res_L(\tau)] \neq \{0\}$.*

The corollary follows from that r restricted to $\mathrm{Cl}(\mathcal{U}(\mathfrak{h})W)$ is injective and that Frobenius reciprocity for $L^2(H \times_\tau W)$ holds.

3.3. **The map r_0^D is injective.** As a consequence of the general fact shown in the previous subsection, we obtain the injectivity in $i)$ and the map r_0^D is injective.

Corollary 3.8. *Let (G,H) be a symmetric pair and $H_0 = G^{\sigma\theta}$. Then, the restriction map $r_0 : H^2(G,\tau) \to L^2(H_0 \times_\tau W)$ restricted to the subspace* $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)$ *is injective.*

Corollary 3.9. *Let (G,H) be a symmetric pair, $H_0 = G^{\sigma\theta}$ and we assume $res_H(\pi_\lambda)$ is H-admissible. Then, the map r_0^D is injective.*

In fact, for $T \in Hom_H(H^2(H,\sigma), H^2(G,\tau))$, if $r_0(D(K_T(e,\cdot)z)) = 0\, \forall z \in Z$, then, since $D(K_T(e,x)z) \in \mathcal{U}(\mathfrak{h}_0)W$, the previous corollary implies $D(K_T(e,x)z) = 0\, \forall z, x \in G$. Since, $K_T(e,\cdot)z \in V_\lambda^G[V_\mu^H][V_{\mu+\rho_n^H}^L]$, and D is injective we obtain $K_T(e,x)z = 0\, \forall z, \forall x$. Lastly, we recall equality $K_T(h,x) = K_T(e,h^{-1}x)$. Whence we have verified the corollary.

Before we show the surjectivity for the map r_0^D we would like to comment other works on the topic object of this note.

3.4. **Previous work on duality formula and Harish-Chandra parameters.** The setting for this subsection is: (G,H) is a symmetric pair and $(\pi_\lambda, V_\lambda^G)$ is a irreducible square integrable representation of G and H-admissible. As before, we fix $K, L = H \cap K, T, U = H \cap T$. The following Theorem has been shown by [9], a different proof is in [18].

Theorem 3.10 (Gross-Wallach, T. Kobayashi-Y. Oshima)**.** *We assume (G,H) is symmetric pair, π_λ^G-is H-admissible, then*

a) $res_H(\pi_\lambda^G)$ is the Hilbert sum of **inequivalent** *Square integrable representations for H, $\pi_{\mu_j}^H, j = 1, 2, \ldots,$ with respective finite multiplicity $0 < m_j < \infty$.*

b) The Harish-Chandra parameters of the totality of discrete factors for $res_H(\pi_\lambda^G)$ belong to a "unique" Weyl Chamber in $i\mathfrak{u}^\star$.

That is, $V_\lambda^G = \oplus_{1 \leq j < \infty} V_\lambda^G[V_{\mu_j}^H] \equiv \oplus_j \, Hom_H(V_{\mu_j}^H, V_\lambda^G) \otimes V_{\mu_j}^H,$

$$dimHom_H(V^H_{\mu_j},V^G_\lambda)=m_j, \qquad \pi^H_{\mu_j}\neq\pi^H_{\mu_i} \text{ iff } i\neq j,$$
and there exists a system of positive roots $\Psi_{H,\lambda}\subset\Phi(\mathfrak{h},\mathfrak{u})$, *such that for all* j, $(\alpha,\mu_j)>0$ *for all* $\alpha\in\Psi_{H,\lambda}$.

In [30][18] (see Tables 1,2,3) we find the list of pairs $(\mathfrak{g},\mathfrak{h})$, as well as systems of positive roots $\Psi_G\subset\Phi(\mathfrak{g},\mathfrak{t})$, $\Psi_{H,\lambda}\subset\Phi(\mathfrak{h},\mathfrak{u})$ such that,

- λ dominant with respect to Ψ_G implies $res_H(\pi^G_\lambda)$ is admissible.
- For all μ_j in $a)$ we have $(\mu_j,\Psi_{H,\lambda})>0$.
- When $U=T$, we have $\Psi_{H,\lambda}=\Psi_\lambda\cap\Phi(\mathfrak{h},\mathfrak{t})$.

Since (G,H_0) is a symmetric pair, Theorem 3.10 as well as its comments apply to (G,H_0) and π_λ. Here, when $U=T$, $\Psi_{H_0,\lambda}=\Psi_\lambda\cap\Phi(\mathfrak{h}_0,\mathfrak{t})$.

From the tables in [30] it follows that any of the system Ψ_λ, $\Psi_{H,\lambda}$, $\Psi_{H_0,\lambda}$ has, at most, two noncompact simple roots, and the sum of the respective multiplicity of each noncompact simple root in the highest root is less or equal than two.

3.4.1. *Computing Harish-Chandra parameters from Theorem 3.1.* As usual, $\rho_n=\frac12\sum_{\beta\in\Psi_\lambda\cap\Phi_n}\beta$, $\rho^H_n=\frac12\sum_{\beta\in\Psi_{H,\lambda}\cap\Phi_n}\beta$, $\rho_K=\frac12\sum_{\alpha\in\Psi_\lambda\cap\Phi_c}\alpha$, $\rho_L=\frac12\sum_{\alpha\in\Psi_{H,\lambda}\cap\Phi_c}\alpha$. We write $res_L(\tau)=res_L(V^K_{\lambda+\rho_n})=\oplus_{1\leq j\leq r}\,q_j\,\pi^L_{\nu_j}=\sum_j q_j\sigma_j$, with ν_j dominant with respect to $\Psi_{H,\lambda}\cap\Phi_c$. we recall ν_j is the infinitesimal character (Harish-Chandra parameter) of $\pi^L_{\nu_j}$. Then, the Harish-Chandra parameter for $H^2(H_0,\pi^L_{\nu_j})$ is $\eta_j=\nu_j-\rho^{H_0}_n$.

According to [27, Lemma 2.22](see Remark 4.8), the infinitesimal character of a L-type of $H^2(H_0,\pi^L_{\nu_j})$ is equal to $\nu_j+B=\eta_j+\rho^{H_0}_n+B$ where B is a sum of roots in $\Psi_{H_0,\lambda}\cap\Phi_n$.

The isomorphism r^D_0 in Theore 3.1, let us conclude:

For each subrepresentation $V^H_{\mu_s}$ of $res_H(\pi_\lambda)$, we have $\mu_s+\rho^H_n$ is a L-type of
$$\mathbf{H}^2(H_0,\tau)\equiv\oplus_j\,q_j\,H^2(H_0,\pi^L_{\nu_j})\equiv\oplus_j\,\underbrace{V^{H_0}_{\eta_j}\oplus\cdots\oplus V^{H_0}_{\eta_j}}_{q_j},$$
and the multiplicity of $V^H_{\mu_s}$ is equal to the multiplicity of $V^L_{\mu_s+\rho^H_n}$ in $\mathbf{H}^2(H_0,\tau)$.

3.4.2. *Gross-Wallach multiplicity formula.* To follow we describe the duality Theorem due to [9]. (G,H) is a symmetric pair. For this paragraph, in order to avoid subindexes we write $\mathfrak{g}=Lie(G),\mathfrak{h}=Lie(H)$ etc. We recall $\mathfrak{h}_0=\mathfrak{g}^{\sigma\theta}$. We have the decompositions $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}=\mathfrak{h}+\mathfrak{q}=\mathfrak{h}_0+\mathfrak{p}\cap\mathfrak{h}+\mathfrak{q}\cap\mathfrak{k}$. The *dual* real Lie algebra to $\mathfrak{g}$ is $\mathfrak{g}^d=\mathfrak{h}_0+i(\mathfrak{p}\cap\mathfrak{h}+\mathfrak{q}\cap\mathfrak{k})$, the algebra $\mathfrak{g}^d$ is a real form for $\mathfrak{g}_{\mathbb{C}}$. A maximal compactly embedded subalgebra for $\mathfrak{g}^d$ is $\widetilde{\mathfrak{k}}=\mathfrak{h}\cap\mathfrak{k}+i(\mathfrak{h}\cap\mathfrak{p})$. Let π_λ

be a H-admissible Discrete Series for G. One of the main results of [9] attach to π_λ a finite sum of underlying Harish-Chandra modules of fundamental representation for G^d, $(\Gamma^{\widetilde{K}}_{H\cap L})^{p_0+q_0}(N(\Lambda))$, so that for each subrepresentation V^H_μ of V_λ we compute the multiplicity $m^{G,H}(\lambda,\mu)$ of V^H_μ by means of Blattner's formula [11] applied to $(\Gamma^{\widetilde{K}}_{H\cap L_1})^{p_0+q_0}(N(\Lambda)$. In more detail, since $Lie(H)_\mathbb{C} = Lie(\widetilde{K})_\mathbb{C}$, and the center of H is equal to the center of $\widetilde{K}$, for the infinitesimal character μ and the central character χ of V^H_μ, we may associate a finite dimensional irreducible representation $F_{\mu,\chi}$ for $\widetilde{K}$. Then, they show

$$dimHom_{\mathfrak{h},H\cap K}(V^H_\mu, V^G_\lambda) = dimHom_{\widetilde{K}}(F_{\mu,\chi}, (\Gamma^{\widetilde{K}}_{H\cap L_1})^{p_0+q_0}(N(\Lambda)).$$

$$m^{G,H}(\lambda,\mu) = (-1)^{\frac{1}{2}\dim(H/H\cap L_1)} \sum_{i=1}^{d} \sum_{s\in W_{\widetilde{K}}} \epsilon(s) p(\Lambda_i + \rho_{\widetilde{K}} + s s_{H\cap K}\mu).$$

where $\tau = F^\Lambda = \sum_i M^{\Lambda_i}$ as a sum of irreducible $H\cap L_1$-module and p is the partition function associated to $\Phi(\mathfrak{u}_1/\mathfrak{u}_1\cap\mathfrak{h}_\mathbb{C}, \mathfrak{u})$, here, $\mathfrak{u}_1$ is the nilpotent radical of certain parabolic subalgebra $\mathfrak{q} = \mathfrak{l}_1 + \mathfrak{u}_1$ used to define the $A_\mathfrak{q}(\lambda)$-presentation for π_λ. *Explicit example* V presents the result of [9] for the pair $(SO(2m,2n), SO(2m,2n-1))$.

3.4.3. *Duflo-Vargas multiplicity formula,* [5]. We keep notation and hypothesis as in the previous paragraph. Then,

$$m^{G,H}(\lambda,\mu) = \pm \sum_{w\in W_K} \epsilon(w) p_{S^H_w}(\mu - q_\mathfrak{u}(w\lambda)).$$

Here, $q_\mathfrak{u} : \mathfrak{t}^\star \to \mathfrak{u}^\star$ is the restriction map. $p_{S^H_w}$ is the partition function associated to the multiset

$$S^H_w := S^L_w \backslash \Phi(\mathfrak{h}/\mathfrak{l},\mathfrak{u}), \text{ where, } S^L_w := q_\mathfrak{u}(w(\Psi_\lambda)_n) \cup \Delta(\mathfrak{k}/\mathfrak{l},\mathfrak{u}).$$

We recall for a strict multiset of elements in vector space V the partition function attached to S, roughly speaking, is the function that counts the number of ways of expressing each vector as a nonnegative integral linear combinations of elements of S. For a precise definition see [5] or the proof of Lemma 4.4.

3.4.4. *Harris-He-Olafsson multiplicity formula,* [10]. Notation and hypothesis as in the previous paragraphs. Let

$$r_m : H^2(G,\tau) \to L^2(H \times_{S^m(Ad)\boxtimes\tau} (S^m(\mathfrak{p}\cap\mathfrak{q})^\star \otimes W)).$$

the normal derivative map defined in [24]. Let $\Theta_{\pi^H_\mu}$ denote the Harish-Chandra character of π^H_μ. For f a tempered function in $H^2(G,\tau)$, they define $\phi_{\pi_\lambda,\pi_\mu^H,m}(f) = \Theta_{\pi_\mu^H} \star r_m(f)$. They show:

$$m^{G,H}(\lambda,\mu) = \lim_{m\to\infty} \dim \phi_{\pi_\lambda,\pi_\mu^H,m}((H^2(G,\tau)\cap \mathcal{C}(G,\tau))[V_{\mu+\rho_n^H}]).$$

3.5. **Completion of the Proof of Theorem 3.1, the map r_0^D is surjective.** Item i) in Theorem 3.1 is shown in Proposition 4.9 c). The existence of the map D is shown in Proposition 4.9 e).

To show the surjectivity of r_0^D we appeal to Theorem 4.2, [30, Theorem 1], where we show the initial space and the target space are equidimensional, linear algebra concludes de proof of Theorem 3.1. Thus, we conclude the proof of Theorem 3.1 as soon as we complete the proof of Theorem 4.2 and Proposition 4.9.

4. Duality Theorem, proof of dimension equality

The purpose of this subsection is to sketch a proof of the equality of dimensions in the duality formula presented in Theorem 3.1 as well as some consequences. Part of the notation has already been introduced in the previous section. Sometimes notation will be explained after it has been used. Unexplained notation is as in [5], [25], [30].

The setting is as follows, (G,H) is a symmetric pair,$(\pi_\lambda, V_\lambda^G) = (L, H^2(G,\tau))$ a H-admissible irreducible square integrable representation. Then, the Harish-Chandra parameter λ gives rise to systems of positive roots Ψ_λ in $\Phi(\mathfrak{g},\mathfrak{t})$ and by mean of Ψ_λ, in [5] is defined a nontrivial normal connected subgroup $K_1(\Psi_\lambda) = K_1$ of K, it is shown that the H-admissibility yields $K_1 \subset H$[1]. Thus, $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$, $\mathfrak{l} = \mathfrak{k}_1 \oplus \mathfrak{l}\cap\mathfrak{k}_2$ (as ideals), and $\mathfrak{t} = \mathfrak{t}\cap\mathfrak{k}_1 + \mathfrak{t}\cap\mathfrak{k}_2$, $\mathfrak{u} := \mathfrak{t}\cap\mathfrak{l} = \mathfrak{u}\cap\mathfrak{k}_1 + \mathfrak{u}\cap\mathfrak{k}_2$ is a Cartan subalgebra of $\mathfrak{l}$. Let $q_\mathfrak{u}$ denote restriction map from $\mathfrak{t}^\star$ onto $\mathfrak{u}^\star$. Let K_2 denote the analytic subgroup corresponding to $\mathfrak{k}_2$. We recall $H_0 := (G^{\sigma\theta})_0$, $L = K\cap H = K\cap H_0$. We have $K = K_1K_2$, $L = K_1(K_2\cap L)$. We set $\Delta := \Psi_\lambda \cap \Phi(\mathfrak{k},\mathfrak{t})$. Applying Theorem 3.10 to both H and H_0 we obtain respective systems of positive roots $\Psi_{H,\lambda}$ in $\Phi(\mathfrak{h},\mathfrak{u})$, $\Psi_{H_0,\lambda}$ in $\Phi(\mathfrak{h}_0,\mathfrak{u})$. For a list of six-tuples $(G, H, \Psi_\lambda, \Psi_{H,\lambda}, \Psi_{H_0,\lambda}, K_1)$ we refer to [30, Table 1, Table 2, Table 3]. Always, $\Psi_{H,\lambda} \cap \Phi_c(\mathfrak{l},\mathfrak{u}) = \Psi_{H_0,\lambda}) \cap \Phi_c(\mathfrak{l},\mathfrak{u})$. As usual, either $\Phi_n(\mathfrak{g},\mathfrak{t})$ or Φ_n denotes the subset of noncompact roots in $\Phi(\mathfrak{g},\mathfrak{t})$, ρ_n^λ (resp. $\rho_n^H, \rho_n^{H_0}$) denotes one half of the sum of the elements in $\Psi_\lambda\cap\Phi_n(\mathfrak{g},\mathfrak{t})$ (resp. $\Phi_n\cap\Psi_{H,\lambda}, \Phi_n\cap\Psi_{H_0,\lambda}$). When $\mathfrak{u} = \mathfrak{t}$, $\rho_n^\lambda = \rho_n^H + \rho_n^{H_0}$. From now on, the infinitesimal character of an irreducible representation of K (resp L) is dominant with respect to Δ (resp. $\Psi_{H,\lambda}\cap\Phi(\mathfrak{l},\mathfrak{u})$). The lowest K-type (τ, W) of π_λ decomposes $\pi^K_{\lambda+\rho_n^\lambda} = \pi^{K_1}_{\Lambda_1} \boxtimes \pi^{K_2}_{\Lambda_2}$, with $\pi^{K_s}_{\Lambda_s}$ an irreducible representation for $K_s, s = 1, 2$. We express $\gamma =$

[1] This also follows from the tables in [18]

$(\gamma_1, \gamma_2) \in \mathfrak{t}^\star = \mathfrak{t}_1^\star + \mathfrak{t}_2^\star$. Hence, [11][3], $\Lambda_1 = \lambda_1 + (\rho_n^\lambda)_1, \Lambda_2 = \lambda_2 + (\rho_n^\lambda)_2$. Sometimes $(\rho_n^\lambda)_2 \neq 0$. This happens only for $\mathfrak{su}(m,n)$ and some particular systems Ψ_λ (see proof of Lemma 4.7). Harish-Chandra parameters for the irreducible factors of either $res_H(\pi_\lambda)$ (resp. $res_{H_0}(\pi_\lambda))$) will always be dominant with respect to $\Psi_{H,\lambda} \cap \Phi(\mathfrak{l}, \mathfrak{u})$ (resp. $\Psi_{H_0,\lambda} \cap \Phi(\mathfrak{l}, \mathfrak{u})$).

For short, we write $\pi_{\Lambda_2} := \pi_{\Lambda_2}^{K_2}$. We write

$$res_{L \cap K_2}(\pi_{\Lambda_2}) = res_{L \cap K_2}(\pi_{\Lambda_2}^{K_2}) = \sum_{\nu_2 \in (\mathfrak{u} \cap \mathfrak{k}_2)^\star} m^{K_2, L \cap K_2}(\Lambda_2, \nu_2)\, \pi_{\nu_2}^{L \cap K_2},$$

as a sum of irreducible representations of $L \cap K_2$.
The set of ν_2 so that $m^{K_2, L \cap K_2}(\Lambda_2, \nu_2) \neq 0$ is denoted by $Spec_{L \cap K_2}(\pi_{\Lambda_2}^{K_2})$. Thus,

$$res_L(\pi_{\Lambda_1}^{K_1} \boxtimes \pi_{\Lambda_2}^{K_2}) = \sum_{\nu_2 \in Spec_{L \cap K_2}(\pi_{\Lambda_2}^{K_2})} m^{K_2, L \cap K_2}(\Lambda_2, \nu_2)\, \pi_{\Lambda_1}^{K_1} \boxtimes \pi_{\nu_2}^{L \cap K_2},$$

as a sum of irreducible representations of L. Besides, for a Harish-Chandra parameter $\eta = (\eta_1, \eta_2)$ for H_0, we write

$$res_L(\pi_{(\eta_1,\eta_2)}^{H_0}) = \sum_{(\theta_1,\theta_2) \in Spec_L(\pi_{(\eta_1,\eta_2)}^{H_0})} m^{H_0, L}((\eta_1, \eta_2), (\theta_1, \theta_2))\, \pi_{(\theta_1,\theta_2)}^L.$$

The restriction of π_λ to H is expressed by (see 3.10)

$$res_H(\pi_\lambda) = res_H(\pi_\lambda^G) = \sum_{\mu \in Spec_H(\pi_\lambda)} m^{G,H}(\lambda, \mu)\, \pi_\mu^H.$$

In the above formulaes, $m^{\cdot,\cdot}(\cdot, \cdot)$ are non negative integers and represent multiplicities; for $\nu_2 \in Spec_{L \cap K_2}(\pi_{\Lambda_2}^{K_2})$, ν_2 is dominant with respect to $\Psi_{H,\lambda} \cap \Phi(\mathfrak{k}_2, \mathfrak{u} \cap \mathfrak{k}_2)$, and (Λ_1, ν_2) is $\Psi_{H_0,\lambda}$-dominant (see [30]); in the third formulae, (η_1, η_2) is dominant with respect to $\Psi_{H_0,\lambda}$ and (θ_1, θ_2) is dominant with respect to $\Psi_{H_0,\lambda} \cap \Phi_c(\mathfrak{h}_0, \mathfrak{u})$; in the fourth formula, μ is dominant with respect to $\Psi_{H,\lambda}$. Sometimes, for $\mu \in Spec_H(\pi_\lambda^G)$, we replace ρ_n^μ by ρ_n^H.

We make a change of notation:

$$\sigma_j = \pi_{\Lambda_1}^{K_1} \boxtimes \pi_{\nu_2}^{L \cap K_2} \text{ and } q_j = m^{K_2, L \cap K_2}(\Lambda_2, \nu_2).$$

Then, in order to show either the existence of the map D or the surjectivity of the map r_0^D, we need to show:

Theorem 4.1.

$$\begin{aligned} m^{G,H}(\lambda,\mu) &= \dim Hom_H(H^2(H, V^L_{\mu+\rho_n^H}), H^2(G, V^K_{\lambda+\rho_n^G})) \\ &= \sum_{\nu_2 \in Spec_{L\cap K_2}(\pi^{K_2}_{\Lambda_2})} m^{K_2, L\cap K_2}(\Lambda_2,\nu_2) \\ &\qquad \times \dim Hom_L(V^L_{\mu+\rho_n^H}, H^2(H_0, \pi^{K_1}_{\Lambda_1} \boxtimes \pi^{L\cap K_2}_{\nu_2})). \end{aligned}$$

A complete proof of the result is in [30]. However, for sake of completeness and clarity we would like to sketch a proof. We also present some consequences of the Theorem.

Next, we compute the infinitesimal character (Harish-Chandra parameter) for $H^2(H_0, \pi^{K_1}_{\Lambda_1} \boxtimes \pi^{L\cap K_2}_{\nu_2})$ and restate the previous Theorem.

ic$(H^2(H_0, \pi^{K_1}_{\Lambda_1} \boxtimes \pi^{L\cap K_2}_{\nu_2})) = (\Lambda_1, \nu_2) - \rho_n^{H_0} = (\lambda_1 + \rho_n^G - \rho_n^{H_0}, \nu_2) = (\lambda_1,\nu_2) + \rho_n^H$. This equality is obviously true when $(\rho_n^\lambda)_2 = 0$.

To follow, we state Theorem 4.1 regardless of the realization of the involved Discrete Series.

Theorem 4.2. *Duality, dimension formula. The hypothesis is (G,H) is a symmetric pair and π_λ is a H-admissible representation. Then,*

$$\begin{aligned} m^{G,H}(\lambda,\mu) &= \dim Hom_H(V^H_\mu, V^G_\lambda) \\ &= \sum_{\nu_2 \in Spec_{L\cap K_2}(\pi^{K_2}_{\Lambda_2})} m^{K_2, L\cap K_2}(\Lambda_2,\nu_2) \\ &\qquad \times \dim Hom_L(V^L_{\mu+\rho_n^H}, V^{H_0}_{(\lambda_1,\nu_2)+\rho_n^H}). \end{aligned}$$

After Lemma 4.5 the formula simplifies to

$$m^{G,H}(\lambda,\mu) = \sum_{\nu_2 \in Spec_{L\cap K_2}(\pi^{K_2}_{\lambda_2})} m^{K_2, L\cap K_2}(\lambda_2,\nu_2) \dim Hom_L(V^L_\mu, V^{H_0}_{(\lambda_1,\nu_2)}).$$

The following diagram helps to understand the equalities in the Theorem and in the next three Lemmas.

$$\begin{tikzcd}
Spec_H(V^G_{(\lambda_1,\lambda_2)}) \arrow[r, "\mu\mapsto\mu"] \arrow[dr, "\nu\mapsto\nu+\rho_n^H"'] & \cup_{\nu_2 \in Spec_{L\cap K_2}(\pi_{\Lambda_2})} Spec_L(V^{H_0}_{(\lambda_1,\nu_2)}) \arrow[d, "\nu\mapsto\nu+\rho_n^H"] \\
& Spec_L(\mathbf{H}^2(H_0,\tau)) = \cup_{\nu_2 \in Spec_{L\cap K_2}(\pi_{\Lambda_2})} Spec_L(V^{H_0}_{(\lambda_1,\nu_2)+\rho_n^H})
\end{tikzcd}$$

A consequence of Theorem 4.2, Lemma 4.5 and Lemma 4.4 is:

Corollary 4.3.

$$Spec_H(\pi_\lambda) + \rho_n^H$$
$$= Spec_L(\mathbf{H}^2(H_0,\tau)) = \cup_{\nu_2 \in Spec_{L\cap K_2}(\pi_{\Lambda_2})} Spec_L(V^{H_0}_{(\lambda_1,\nu_2)+\rho_n^H}).$$
$$Spec_H(\pi_\lambda) = \cup_{\nu_2 \in Spec_{L\cap K_2}(\pi_{\Lambda_2})} Spec_L(V^{H_0}_{(\lambda_1,\nu_2)}).$$

Theorem 4.2 follows after we verify the next two Lemmas.

Lemma 4.4. *The hypothesis is (G,H) is a symmetric space and π_λ is H-admissible. Then*

$$\dim Hom_H(V_\mu^H, V_\lambda^G)$$
$$= \sum_{\nu_2 \in Spec_{L\cap K_2}(\pi^{K_2}_{\lambda_2})} m^{K_2, L\cap K_2}(\lambda_2,\nu_2) \dim Hom_L(V_\mu^L, V^{H_0}_{(\lambda_1,\nu_2)}).$$

Proof of Lemma 4.4. The hypothesis (G,H) is a symmetric pair and π_λ is H-admissible, let us to apply notation and facts in [5], [30] as well as in [12] [3] [9] [18]. The proof is based on an idea in [3] of piling up multiplicities by means of Dirac delta distributions. That is, let δ_ν denote the Dirac delta distribution at $\nu \in i\mathfrak{u}^\star$. Under our hypothesis, the function $m^{G,H}(\lambda,\mu)$ has polynomial growth in μ, whence, the series $\sum_\mu m^{G,H}(\lambda,\mu)\ \delta_\mu$ converges in the space of distributions in $i\mathfrak{u}^\star$. Since Harish-Chandra parameter is regular, we may and will extend the function $m^{G,H}(\lambda,\cdot)$ to a W_L-skew symmetric function by the rule $m^{G,H}(\lambda, w\mu) = \epsilon(w)m^{G,H}(\lambda,\mu), w \in W_L$. Thus, the series $\sum_{\mu \in HC-param(H)} m^{G,H}(\lambda,\mu)\delta_\mu$ converges in the space of distributions in $i\mathfrak{u}^\star$. Next, for $0 \neq \gamma \in i\mathfrak{u}^\star$ we consider the discrete Heaviside distribution $y_\gamma := \sum_{n\geq 0} \delta_{\frac{\gamma}{2}+n\gamma}$, and for a strict, finite, multiset $S = \{\gamma_1,\dots,\gamma_r\}$ of elements in $i\mathfrak{u}^\star$, we set

$$y_S := y_{\gamma_1} \star \cdots \star y_{\gamma_r} = \sum_{\mu \in i\mathfrak{u}^\star} p_S(\mu)\delta_\mu.$$

Here, $\star$ is the convolution product in the space of distributions on $i\mathfrak{u}^\star$. p_S is called the *partition* function attached to the set S. Then, in [5] there is presented the equality

$$\sum_{\mu \in HC-param(H)} m^{G,H}(\lambda,\mu)\ \delta_\mu = \sum_{w \in W_K} \epsilon(w)\ \delta_{q_\mathfrak{u}(w\lambda)} \bigstar y_{S_w^H}.$$

Here, W_S is the Weyl group of the compact connected Lie group S; for a $ad(\mathfrak{u})$-invariant linear subspace R of $\mathfrak{g}_\mathbb{C}$, $\Phi(R,\mathfrak{u})$ denotes the multiset of elements in $\Phi(\mathfrak{g},\mathfrak{u})$ such that its root space is contained in R, and $S_w^H = [q_\mathfrak{u}(w(\Psi_\lambda)_n) \cup \Delta(\mathfrak{k}/\mathfrak{l},\mathfrak{u})]\backslash\Phi(\mathfrak{h}/\mathfrak{l},\mathfrak{u})$.

Since, $K = K_1K_2$, $W_K = W_{K_1} \times W_{K_2}$, we write $W_K \ni w = st, s \in W_{K_1}, t \in W_{K_2}$. We recall the hypothesis yields $K_1 \subset L$. It readily follows: $s\Phi(\mathfrak{h}/\mathfrak{l},\mathfrak{u}) = \Phi(\mathfrak{h}/\mathfrak{l},\mathfrak{u})$, $s\Delta(\mathfrak{k}/\mathfrak{l},\mathfrak{u}) = \Delta(\mathfrak{k}/\mathfrak{l},\mathfrak{u}), t(\Psi_\lambda)_n = (\Psi_\lambda)_n, t\eta_1 = \eta_1, s\eta_2 = \eta_2 \, for \, \eta_j \in \mathfrak{k}_j \cap \mathfrak{u}$, $sq_\mathfrak{u}(\cdot) = q_\mathfrak{u}(s\cdot)$. Hence,

$$S_w^H = s([q_\mathfrak{u}((\Psi_\lambda)_n) \cup \Delta(\mathfrak{k}/\mathfrak{l},\mathfrak{u})]\backslash\Phi(\mathfrak{h}/\mathfrak{l},\mathfrak{u})) = s(\Psi_n^{H_0}) \cup \Delta(\mathfrak{k}/\mathfrak{l},\mathfrak{u}).$$

Thus,

$$\begin{aligned}\sum_{w\in W_K} \epsilon(w)\ \delta_{q_\mathfrak{u}(w\lambda)} \bigstar y_{S_w^H} &= \sum_{s,t} \epsilon(st)\delta_{q_\mathfrak{u}(st\lambda)} \bigstar y_{s(\Psi_n^{H_0})\cup\Delta(\mathfrak{k}/\mathfrak{l},\mathfrak{u})} \\ &= \sum_{s,t} \epsilon(st)\delta_{q_\mathfrak{u}(s\cancel{t}\lambda_1+\cancel{s}t\lambda_2)} \bigstar y_{s(\Psi_n^{H_0})} \bigstar y_{\Delta(\mathfrak{k}/\mathfrak{l},\mathfrak{u})} \\ &= \sum_s \epsilon(s)\delta_{(s\lambda_1,0)} \bigstar y_{s(\Psi_n^{H_0})} \bigstar \sum_t \epsilon(t)\delta_{q_\mathfrak{u}(t\lambda_2)} \bigstar y_{\Delta(\mathfrak{k}/\mathfrak{l},\mathfrak{u})}.\end{aligned}$$

Following [12], we write the restriction of $\pi_{\lambda_2}^{K_2}$ to $L \cap K_2$ in the language of Dirac, Heaviside distributions in $i\mathfrak{u}^\star$, whence

$$\begin{aligned}&\sum_{t\in W_{K_2}} \epsilon(t)\delta_{q_\mathfrak{u}(t\lambda_2)} \bigstar y_{\Delta(\mathfrak{k}_2/(\mathfrak{k}_2\cap\mathfrak{l}),\mathfrak{u})} \\ &\qquad = \sum_{\nu_2\in Spec_{L\cap K_2}(\pi_{\lambda_2}^{K_2})} m^{K_2,L\cap K_2}(\lambda_2,\nu_2) \sum_{w_2\in W_{K_2\cap L}} \epsilon(w_2)\delta_{(0,w_2\nu_2)}.\end{aligned}$$

In the previous formula, we will apply $\Delta(\mathfrak{k}_2/(\mathfrak{k}_2\cap\mathfrak{l}),\mathfrak{u}) = \Delta(\mathfrak{k}/\mathfrak{l},\mathfrak{u})$. We also write in the same language the restriction to L of a Discrete Series $\pi_{(\lambda_1,\nu_2)}^{H_0}$ for H_0. This is.

$$\sum_{\nu\in i\mathfrak{u}^\star} m^{H_0,L}((\lambda_1,\nu_2),\nu)\,\delta_\nu = \sum_{s\in W_{K_1}, t\in W_{K_2\cap L}} \epsilon(st)\delta_{st(\lambda_1,\nu_2)} \bigstar y_{st(\Psi_n^{H_0})}.$$

Putting together the previous equalities, we obtain

$$\begin{aligned}&\sum_\mu m^{G,H}(\lambda,\mu)\ \delta_\mu \\ &\qquad = \textstyle\sum_{\nu_2\in Spec_{L\cap K_2}(\pi_{\lambda_2}^{K_2})} m^{K_2,L\cap K_2}(\lambda_2,\nu_2) \\ &\qquad\qquad \times \textstyle\sum_{s\in W_{K_1}, t\in W_{K_2\cap L}} \epsilon(st)\delta_{(st\lambda_1,st\nu_2)} \bigstar y_{st(\Psi_n^{H_0})} \\ &\qquad = \textstyle\sum_\nu (\sum_{\nu_2\in Spec_{L\cap K_2}(\pi_{\lambda_2}^{K_2})} m^{K_2,L\cap K_2}(\lambda_2,\nu_2) m^{H_0,L}((\lambda_1,\nu_2),\nu))\ \delta_\nu.\end{aligned}$$

Since, the family $\{\delta_\nu\}_{\nu\in i\mathfrak{u}^\star}$ is linearly independent, we have shown Lemma 4.4. □

In order to conclude the proof of the dimension equality we state and prove a translation invariant property of multiplicity.

Lemma 4.5. *For a dominant integral $\mu \in i\mathfrak{u}^\star$, it holds:*

$$m^{H_0,L}((\lambda_1,\nu_2)+\rho_n^H,\mu+\rho_n^H)=m^{H_0,L}((\lambda_1,\nu_2),\mu).$$

Proof. We recall that the hypothesis of the Lemma 4.5 is: (G,H) is a symmetric pair and π_λ is H-admissible. The proof of Lemma 4.5 is an application of Blattner's multiplicity formula, facts from [11] and observations from [30, Table 1,2,3]. In the next paragraphs we only consider systems Ψ_λ so that $res_H(\pi_\lambda)$ is admissible. We check the following statements by means of case by case analysis and the tables in [9] and [30].

OBS0. Every quaternionic system of positive roots that we are dealing with, satisfies the Borel de Siebenthal property, except for the algebra $\mathfrak{su}(2,2n)$ and the systems Ψ_1 (see 4.6). Its Dynkin diagram is $\bullet$ —— $\circ$ —— $\circ$ —— $\bullet$. Bullet represents non compact roots, circle compact.

OBS1. Always the systems $\Psi_{H,\lambda},\Psi_{H_0,\lambda}$ have the same compact simple roots.

OBS2. When Ψ_λ satisfies the Borel de Siebenthal property, it follows that both systems $\Psi_{H,\lambda},\Psi_{H_0,\lambda}$ satisfy the Borel de Siebenthal property.

OBS3. Ψ_λ satisfies the Borel de Siebenthal property except for two families of algebras: a) the algebra $\mathfrak{su}(m,n)$ and the systems $\Psi_a, a=1,\cdots,m-1$, $\tilde{\Psi}_b, b=1,\cdots,n-1$, the corresponding systems $\Psi_{H_0,\lambda},\Psi_{H,\lambda}$ do not satisfy the Borel de Siebenthal property. They have two noncompact simple roots; b) For the algebra $\mathfrak{so}(2m,2)$ each system $\Psi_\pm$ does not satisfy the Borel de Siebenthal property, however, each associated system $\Psi_{SO(2m,1),\lambda},\Psi_{H_0,\lambda}$ satisfies the Borel de Siebenthal property.

OBS4. For the pair $(\mathfrak{su}(2,2n),\mathfrak{sp}(1,n))$. Ψ_1 does not satisfy the Borel de Siebenthal property. Here, $\Psi_{H,\lambda}=\Psi_{H_0,\lambda}$ and they have Borel de Siebenthal property.

OBS5. Summing up. Both systems $\Psi_{H,\lambda},\Psi_{H_0,\lambda}$ satisfy the Borel de Siebenthal property except for $(\mathfrak{su}(m,n),\mathfrak{su}(m,k)+\mathfrak{su}(n-k)+\mathfrak{u}(1))$, $(\mathfrak{su}(m,n),\mathfrak{su}(k,n)+\mathfrak{su}(m-k)+\mathfrak{u}(1))$ and the systems $\Psi_a, a=1,\cdots,m-1$, $\tilde{\Psi}_b, b=1,\cdots,n-1$.

To continue, we explicit Blattner's formula according to our setting, we recall fact's from [11] and finish the proof of Lemma 4.5 under the assumption $\Psi_{H_0,\lambda}$ satisfies the property of Borel de Siebenthal. Later on, we consider other systems.

Blattner's multiplicity formula applied to the L-type $V^L_{\mu+\rho_n^H}$ of $V^{H_0}_{(\lambda_1,\nu_2)+\rho_n^H}$ yields

$$dimHom_L(V^L_{\mu+\rho_n^H}, V^{H_0}_{(\lambda_1,\nu_2)+\rho_n^H}) = \sum_{s\in W_L} \epsilon(s) Q_0(s(\mu+\rho_n^H) - ((\lambda_1,\nu_2)+\rho_n^H+\rho_n^{H_0})). \tag{4.1}$$

Here, Q_0 is the partition function associated to the set $\Phi_n(\mathfrak{h}_0)\cap\Psi_{H_0,\lambda}$.

We recall a fact that allows to simplify the formula of above under our setting.

Fact 1: [11, Statement 4.31]. For a system $\Psi_{H_0,\lambda}$ having the Borel de Siebenthal property, it is shown that in the above sum, if the summand attached to $s\in W_L$ contributes nontrivially, then s belongs to the subgroup $W_U(\Psi_{H_0,\lambda})$ spanned by the reflections about the compact simple roots in $\Psi_{H_0,\lambda}$.

From OBS1 we have $W_U(\Psi_{H_0,\lambda}) = W_U(\Psi_{H,\lambda})$. Owing that either $\Psi_{H_0,\lambda}$ or $\Psi_{H,\lambda}$ has the Borel de Siebenthal property we apply [11, Lemma 3.3], whence $W_U(\Psi_{H,\lambda}) = \{s\in W_L : s(\Psi_{H,\lambda}\cap\Phi_n(\mathfrak{h},\mathfrak{u})) = \Psi_{H,\lambda}\cap\Phi_n(\mathfrak{h},\mathfrak{u})\}$. Thus, for $s\in W_U(\Psi_{H_0,\lambda})$ we have $s\rho_n^H=\rho_n^H$. We apply the equality $s\rho_n^H=\rho_n^H$ in 4.1 and we obtain

$$\dim Hom_L(V^L_{\mu+\rho_n^H}, V^{H_0}_{(\lambda_1,\nu_2)+\rho_n^H}) = \sum_{s\in W_U(\Psi_{H_0,\lambda})} \epsilon(s) Q_0(s\mu-((\lambda_1,\nu_2)+\rho_n^{H_0})).$$

Blattner's formula and the previous observations gives us that the right hand side of the above equality is

$$\dim Hom_L(V^L_\mu, V^{H_0}_{(\lambda_1,\nu_2)}) = m^{H_0,L}((\lambda_1,\nu_2),\mu),$$

whence, we have shown Lemma 4.5 when $\Psi_{H_0,\lambda}$ has the Borel de Siebenthal property.

In order to complete the proof of Lemma 4.5, owing to OBS5, we are left to consider the pair $(\mathfrak{su}(m,n), \mathfrak{su}(m,k)+\mathfrak{su}(n-k)+\mathfrak{u}(1))$ as well as $(\mathfrak{su}(m,n), \mathfrak{su}(k,n)+\mathfrak{su}(m-k)+\mathfrak{u}(1))$ and the systems $\Psi_a, a=1,\cdots,m-1$, $\tilde{\Psi}_b, b=1,\cdots,n-1$. The previous reasoning says we are left to extend Fact 1, [11, Statement (4.31)], for the pair $(\mathfrak{su}(m,n), \mathfrak{su}(m,k)+\mathfrak{su}(n-k)+\mathfrak{u}(1))$ (resp. $(\mathfrak{su}(m,n), \mathfrak{su}(k,n)+\mathfrak{su}(m-k)+\mathfrak{u}(1))$) and the systems $(\Psi_a)_{a=1,\cdots,m-1}$ (resp. $(\tilde{\Psi}_b)_{b=1,\cdots,n-1}$). Under this setting we first verify:

Remark 4.6. If $w\in W_L$ and $Q_0(w\mu-(\lambda+\rho_n))\neq 0$, then $w\in W_U(\Psi_{H_0,\lambda})$.

To show *Remark* 4.6 we follow [11]. We fix as Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{su}(m,n)$ the set of diagonal matrices in $\mathfrak{su}(m,n)$. For certain orthogonal basis $\epsilon_1,\ldots,\epsilon_m,\delta_1,\ldots,\delta_n$ of the dual vector space to the subspace of diagonal matrices in $\mathfrak{gl}(m+n,\mathbb{C})$, we may, and will choose $\Delta = \{\epsilon_r - \epsilon_s, \delta_p - \delta_q, 1 \leq r < s \leq m, 1 \leq p < q \leq n\}$, the set of noncompact roots is $\Phi_n = \{\pm(\epsilon_r - \delta_q)\}$. We recall the positive roots systems for $\Phi(\mathfrak{g},\mathfrak{t})$ containing Δ are in a bijective correspondence with the totality of lexicographic orders for the basis $\epsilon_1,\dots,\epsilon_m,\delta_1,\dots,\delta_n$ which contains the "suborder" $\epsilon_1 > \cdots > \epsilon_m,\ \delta_1 > \cdots > \delta_n$. The two holomorphic systems correspond to the orders $\epsilon_1 > \cdots > \epsilon_m > \delta_1 > \cdots > \delta_n;\ \delta_1 > \cdots > \delta_n > \epsilon_1 > \cdots > \epsilon_m$. We fix $1 \leq a \leq m-1$, and let Ψ_a denote the set of positive roots associated to the order $\epsilon_1 > \cdots > \epsilon_a > \delta_1 > \cdots > \delta_m > \epsilon_{a+1} > \cdots > \epsilon_m$. We fix $1 \leq b \leq n-1$ and let $\tilde{\Psi}_b$ denote the set of positive roots associated to the order $\delta_1 > \cdots > \delta_b > \epsilon_1 > \cdots > \epsilon_m > \delta_{b+1} > \cdots > \delta_n$. Since, $\mathfrak{h} = \mathfrak{su}(n,k) + \mathfrak{u}(m-k)$, $\mathfrak{h}_0 = \mathfrak{su}(n,n-k) + \mathfrak{u}(k)$. The root systems for $(\mathfrak{h},\mathfrak{t})$ and $(\mathfrak{h}_0,\mathfrak{t})$ respectively are:

$$\begin{aligned}\Phi(\mathfrak{h},\mathfrak{t}) = \{&\pm(\epsilon_r - \epsilon_s), \pm(\delta_p - \delta_q), \pm(\epsilon_i - \delta_j), 1 \leq r < s \leq m,\\ &1 \leq p < q \leq k,\ or,\ k+1 \leq p < q \leq n, 1 \leq i \leq m, 1 \leq j \leq k\}.\end{aligned}$$

$$\begin{aligned}\Phi(\mathfrak{h}_0,\mathfrak{t}) = \{&\pm(\epsilon_r - \epsilon_s), \pm(\delta_p - \delta_q), \pm(\epsilon_i - \delta_j), 1 \leq r < s \leq m,\\ &1 \leq p < q \leq k\ or\ k+1 \leq p < q \leq n, 1 \leq i \leq m, k+1 \leq j \leq n\}.\end{aligned}$$

The system $\Psi_{H,\lambda} = \Psi_\lambda \cap \Phi(\mathfrak{h},\mathfrak{t})$, $\Psi_{H_0,\lambda} = \Psi_\lambda \cap \Phi(\mathfrak{h}_0,\mathfrak{t})$ which correspond to Ψ_a are the system associated to the respective lexicographic orders

$$\epsilon_1 > \cdots > \epsilon_a > \delta_1 > \cdots > \delta_k > \epsilon_{a+1} > \cdots > \epsilon_m,\ \delta_1 > \cdots > \delta_n.$$

$$\epsilon_1 > \cdots > \epsilon_a > \delta_{k+1} > \cdots > \delta_n > \epsilon_{a+1} > \cdots > \epsilon_m,\ \delta_1 > \cdots > \delta_n.$$

For the time being we set $k=n$ and we show *Remark* 4.6 for $\mathfrak{su}(m,n)$ and Ψ_a. Q denotes the partition function for $\Psi_a \cap \Phi_n$.

The subroot system spanned by the compact simple roots in Ψ_a is

$\Phi_U = \{\epsilon_i - \epsilon_j, 1 \leq i \leq a, 1 \leq j \leq a \,\mathrm{or}\, a+1 \leq i \leq m, a+1 \leq j \leq m\} \cup \{\delta_i - \delta_j, 1 \leq i \neq j \leq n\}$.

$\Psi_a \cap \Phi_c \backslash \Phi_U = \{\epsilon_i - \epsilon_j, 1 \leq i \leq a, a+1 \leq j \leq m\}$.

$\Psi_a \cap \Phi_n = \{\epsilon_i - \delta_j, \delta_j - \epsilon_r, 1 \leq i \leq a, a+1 \leq r \leq m, 1 \leq j \leq n\}$.

$2\rho_n^H = n(\epsilon_1 + \cdots + \epsilon_a) - n(\epsilon_{a+1} + \cdots + \epsilon_m) + (a-(m-a))(\delta_1 + \cdots + \delta_n)$.

A finite sum of non compact roots in Ψ_a is equal a to

$B = \sum_{1 \leq j \leq a} A_j \epsilon_j - \sum_{a+1 \leq i \leq m} B_i \epsilon_i + \sum_r C_r \delta_r$ with A_j, B_i non negative numbers.

Let $w \in W$ so that $Q(w\mu - (\lambda + \rho_n)) \neq 0$. Hence, $\mu = w^{-1}(\lambda + \rho_n + B)$, with B a sum of roots in $\Psi_a \cap \Phi_n$. Thus, w^{-1} is the unique element in W_L that takes $\lambda + \rho_n + B$ to the Weyl chamber determined by $\Psi_a \cap \Phi_c$.

Let $w_1 \in W_U(\Psi_a)$ so that $w_1(\lambda + \rho_n + B)$ is $\Psi_a \cap \Phi_U$-dominant. Next we verify $w_1(\lambda + \rho_n + B)$ is $\Psi_a \cap \Phi_c$-dominant. For this, we fix

$\alpha \in \Psi_a \cap \Phi_c \backslash \Phi_U$ and check $(w_1(\lambda+\rho_n+B),\alpha) > 0$. $\alpha = \epsilon_i - \epsilon_j, i \leq a < j$, and $w_1 \in W_U(\Psi_a)$, hence, $w_1^{-1}(\alpha) = e_r - e_s, r \leq a < s$ belongs to Ψ_a. Thus, $(w_1(\lambda+\rho_n+B),\alpha) = (\lambda+\rho_n+B, w_1^{-1}\alpha) = (\lambda, w_1^{-1}\alpha) + (\rho_n, w_1^{-1}\alpha) + (B, w_1^{-1}\alpha) = (\lambda, w_1^{-1}\alpha) + n - (-n) + A_r + B_s$, the first summand is positive because λ is Ψ_a-dominant, the third and fourth are nonnegative. Therefore, $w^{-1} = w_1$ and we have shown *Rematk* 4.6, whence, we have concluded the proof of Lemma 4.5. $\square$

Lemma 4.7. *We recall $\rho_n^G = \rho_n^\lambda$ and $\Lambda_2 = \lambda_2 + (\rho_n^G)_2$. We claim:*
$$m^{K_2, L\cap K_2}(\Lambda_2, \nu_2) = m^{K_2, L\cap K_2}(\lambda_2, \nu_2).$$

In fact, when Ψ_λ is holomorphic, ρ_n^G is in $\mathfrak{z}_\mathfrak{k} = \mathfrak{k}_1$ hence $(\rho_n^G)_2 = 0$. In [30] it is shown that when K is semisimple $(\rho_n^G)_2 = 0$. Actually, this is so, owing that the simple roots for $\Psi_\lambda \cap \Phi(\mathfrak{k}_2, \mathfrak{t}_2)$ are simple roots for Ψ_λ and that ρ_n^G is orthogonal to every compact simple root for Ψ_λ. For general $\mathfrak{g}$, the previous considerations together with that $(\rho_n^G)_2$ is orthogonal to $\mathfrak{k}_1$ yields that $(\rho_n^G)_2$ belongs to the dual of the center of $\mathfrak{l} \cap \mathfrak{k}_2$. From Tables 1,2,3 we deduce we are left to analyze $(\rho_n^G)_2$ for $\mathfrak{su}(m,n), \mathfrak{so}(m,2)$. For $\mathfrak{so}(m,2)$ we follow the notation in 4.0.3, then $\mathfrak{t}_1 = span(e_1, \ldots, e_m), \mathfrak{t}_2 = span(\delta_1)$ and $\rho_n^{\Psi_{\pm m}} = c(e_1 + \cdots + e_m) \in \mathfrak{t}_1$. For $\mathfrak{su}(m,n)$ we follow the notation in Lemma 4.5. It readily follows that for $1 \leq a < m$, $\rho_n^{\Psi_a} = \frac{n}{m}((m-a)(e_1+\cdots+e_a) - a(e_{a+1}+\cdots+e_m)) + \frac{2a-m}{2m}((n(e_1+\cdots+e_m) - m(d_1+\cdots+d_n))$. The first summand is in $\mathfrak{t} \cap \mathfrak{su}(m)$, the second summand belongs to $\mathfrak{z}_\mathfrak{k}$, thus, $(\rho_n^{\Psi_a})_2 = 0$ if and only if $2a = m$. Whence, for $(\mathfrak{su}(2,m), \mathfrak{sp}(1,m))$, we have $(\rho_n^{\Psi_1})_2 = 0$. For $(\mathfrak{su}(m,n), \mathfrak{su}(m,k) + \mathfrak{su}(n-k) + \mathfrak{z}_\mathfrak{l})$, always, $(\rho_n^{\Psi_a})_2$ determines a character of the center of $\mathfrak{k}$. In this case, $\lambda_2 = \Lambda_2$ except for $(\mathfrak{su}(m,n), \mathfrak{su}(m,k) + \mathfrak{su}(n-k) + \mathfrak{z}_\mathfrak{l}), \Psi_a$ and $a \neq 2m$, hence, $\pi_{\Lambda_2}^{K_2}$ is equal to $\pi_{\lambda_2}^{K_2}$ times a central character of K. Thus, the equality holds.

4.0.1. *Conclusion proof of Theorem 4.2.* We just put together Lemma 4.5, Lemma 4.4 and Lemma 4.7, hence, we obtain the equalities we were searching for. This concludes the proof of Theorem 4.2.

4.0.2. *Existence of D.* To follow we show the existence of the isomorphism D in Theorem 4.2 *i)* and derive the decomposition into irreducible factors of the semisimple $\mathfrak{h}_0$-module $\mathcal{U}(\mathfrak{h}_0)W$. On the mean time, we also consider some particular cases of Theorem 4.2. Before, we proceed we comment on the structure of the representation τ.

4.0.3. *Representations π_λ so that $res_L(\tau)$ is irreducible.* Under our H-admissibility hypothesis of π_λ we analyze the cases so that the representation $res_L(\tau)$ is irreducible. The next structure statements are verified in [30]. To begin with, we recall the decomposition $K = K_1 Z_K K_2$, (this is not a direct product!, Z_K connected center of K) and the direct product $K = K_1 K_2$, we also recall that actually, either K_1 or K_2 depend on Ψ_λ. When π_λ is a holomorphic representation $K_1 = Z_K$ and $\mathfrak{k}_2 = [\mathfrak{k}, \mathfrak{k}]$; when, Z_K is nontrivial and π_λ is not a holomorphic representation we have $Z_K \subset K_2$; for $\mathfrak{g} = \mathfrak{su}(m,n), \mathfrak{h} = \mathfrak{su}(m,k) + \mathfrak{su}(n-k) + \mathfrak{z}_L$, we have $\mathbf{T} \equiv Z_K \subset Z_L \equiv \mathbf{T}^2$. Here, $Z_K \subset L$, and, $\tau_{|_L}$ irreducible, forces $\tau = \pi_{\Lambda_1}^{SU(m)} \boxtimes \pi_\chi^{Z_K} \boxtimes \pi_{\rho_{SU(n)}}^{SU(n)}$; for $\mathfrak{g} \ncong \mathfrak{su}(m,n)$ and G/K a Hermitian symmetric space, we have to consider the next two examples.

For both cases we have $K_2 = Z_K (K_2)_{ss}$ and $Z_K \nsubseteq L$.

1) When $\mathfrak{g} = \mathfrak{so}(m,2), \mathfrak{h} = \mathfrak{so}(m,1)$ and $\Psi_\lambda = \Psi_{\pm m}$, then $\mathfrak{k}_1 = \mathfrak{so}(m)$, $\mathfrak{k}_2 = \mathfrak{z}_K$ and obviously $res_L(\tau)$ is always an irreducible representation. Here, $\pi_{\Lambda_2}^{K_2}$ is one dimensional representation.

2) When $\mathfrak{g} = \mathfrak{su}(2,n), \mathfrak{h} = \mathfrak{sp}(1,n)$, $\Psi_\lambda = \Psi_1$, then $\mathfrak{k}_1 = \mathfrak{su}_2(\alpha_{max})$, $\mathfrak{k}_2 = \mathfrak{sp}(n) + \mathfrak{z}_\mathfrak{k}$, $L = K_1(L \cap (K_2)_{ss})$. Here, $\tau_{|_L}$ irreducible forces, $\tau = \pi_{\Lambda_1}^{K_1} \boxtimes \pi_\chi^{Z_K} \boxtimes \pi_{\rho_{(K_2)_{ss}}}^{(K_2)_{ss}}$.

We would like to point out, for $\mathfrak{g} = \mathfrak{so}(2m,2n), \mathfrak{h} = \mathfrak{so}(2m,2n-1), n > 1$, $\Psi_\lambda \cap \Phi_n = \{\epsilon_i \pm \delta_j\}$, $\mathfrak{k}_1 = \mathfrak{so}(2m)$, and if λ is so that $\lambda + \rho_n^\lambda = ic(\tau) = (\sum c_i \epsilon_i, k(\delta_1 + \cdots + \delta_{n-1} \pm \delta_n)) + \rho_K$, then $res_L(\tau)$ is irreducible and $\pi_{\Lambda_2}^{K_2} = \pi_{k(\delta_1+\cdots+\delta_{n-1}\pm\delta_n)+\rho_{K_2}}^{K_2}$ is not a one dimensional representation for $k > 0$. It follows from the classical branching laws that these are the unique $\tau' s$ such that $res_L(\tau)$ is irreducible.

We believe, if $res_L(\tau)$ is irreducible and $\mathfrak{g} \ncong \mathfrak{so}(2m,2n)$ we may conclude that τ is the tensor product of a irreducible representation of K_1 times a one dimensional representation of K_2. That is, $\tau \equiv \pi_{\Lambda_1}^{K_1} \boxtimes \pi_{\rho_{K_2}}^{K_2} \otimes \pi_\chi^{Z_{K_2}}$.

In 5.3.1 we show that whenever a symmetric pair (G,H) is so that some Discrete Series is H-admissible, then there exists H-admissible Discrete Series so that its lowest K-type restricted to L is irreducible.

4.0.4. *Analysis of* $\mathcal{U}(\mathfrak{h}_0)W$, $\mathcal{L}_\lambda$, *existence of* D, *case* $\tau_{|_L}$ *is irreducible.* As before, our hypothesis is (G,H) is a symmetric space and π_λ^G is H-admissible. For this paragraph we add the hypothesis $\tau_{|_L} = res_L(\tau)$ is irreducible. We recall that $\mathcal{U}(\mathfrak{h}_0)W = L_{\mathcal{U}(\mathfrak{h}_0)}(H^2(G,\tau)[W])$, $\mathcal{L}_\lambda = \oplus_{\mu \in Spec_H(\pi_\lambda)} H^2(G,\tau)[V_\mu^H][V_{\mu+\rho_n^\mu}^L]$. We claim:

a) if a H-irreducible discrete factor of V_λ contains a copy of $\tau_{|L}$, then $\tau_{|L}$ is the lowest L-type of such factor.

b) the multiplicity of $res_L(\tau)$ in $H^2(G,\tau)$ is one.

c) $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)$ is equivalent to $H^2(H_0,\tau)$.

d) $\mathcal{L}_\lambda$ is equivalent to $H^2(H_0,\tau)_{L-fin}$.

e) $\mathcal{L}_\lambda$ is equivalent to $\mathcal{U}(\mathfrak{h}_0)W$. Thus, D exists.

We rely on:

Remark 4.8. 1) Two Discrete Series are equivalent if and only if their respective lowest L-types are equivalent. [31].

2) For any Discrete Series π_λ, the highest weight (resp. infinitesimal character) of any K-type is equal to the highest weight of the lowest K-type (resp. the infinitesimal character of the lowest K-type) plus a sum of noncompact roots in Ψ_λ [27, Lemma 2.22].

From now on $ic(\phi)$ denotes the infinitesimal character (Harish-Chandra parameter) of the representation ϕ

Let V_μ^H a discrete factor for $res_H(\pi_\lambda)$ so that $\tau_{|_L}$ is a L-type. Then, Theorem 4.2 implies $V_{\mu+\rho_n^H}$ is a L-type for $H^2(H_0,\tau)$. Hence, after we apply *Remark* 4.8, we obtain

$\mu+\rho_n^H+B_1=ic(\tau_{|_L})$ with B_1 a sum of roots in $\Psi_{H,\lambda}\cap\Phi_n$.

$\mu+\rho_n^H=ic(\tau_{|_L})+B_0$ with B_0 a sum of roots in $\Psi_{H_0,\lambda}\cap\Phi_n$.

Thus, $B_0+B_1=0$, whence $B_0=B_1=0$ and $\mu+\rho_n^H=ic(\tau_{|_L})$, we have verified a).

Due to H-admissibility hypothesis, we have $\mathcal{U}(\mathfrak{h})W$ is a finite sum of irreducible underlying modules of Discrete Series for H. Now, Corollary 1 to Lemma 3.6, yields that a copy of a V_μ^H contained in $\mathcal{U}(\mathfrak{h})W$ contains a copy of $V_\lambda[W]$. Thus, a) implies $\tau_{|_L}$ is the lowest L-type of such V_μ^H. Hence, $H^2(H,\tau)$ is nonzero. Now, Theorem 4.2 together with that the lowest L-type of a Discrete Series has multiplicity one yields that $\dim Hom_H(H^2(H,\tau),V_\lambda)=1$. Also, we obtain $\dim Hom_{H_0}(H^2(H_0,\tau),V_\lambda)=1$. Thus, whenever $\tau_{|L}$ occurs in $res_L(V_\lambda)$, we have $\tau_{|_L}$ is realized in $V_\lambda[W]$. In other words, the isotypic compoent $V_\lambda[\tau_{|_L}]\subset V_\lambda[W]$. Hence, b) holds.

Owing our hypothesis, we may write $U(\mathfrak{h}_0)W=N_1+...+N_k$, with each N_j being the underlying Harish-Chandra module of a irreducible square integrable representation for H_0. Since Lemma 3.6 shows r_0 is injective in $\mathcal{U}(\mathfrak{h}_0)W$, we have $r_0(\mathrm{Cl}(N_j))$ is a Discrete Series in $L^2(H_0\times_{res_L(\tau)}W)$, whence Frobenius reciprocity implies $\tau_{|_L}$ is a L-type for N_j. Hence, b) and a) forces $\mathcal{U}(\mathfrak{h}_0)W$ is $\mathfrak{h}_0$-irreducible and c) follows.

By definition, the subspace $\mathcal{L}_\lambda$ is the linear span of $V_\lambda[V_\mu^H][V^L_{\mu+\rho_n^H}]$ with $\mu\in Spec_H(\pi_\lambda)$. Since, $\dim V_\lambda[V_\mu^H][V^L_{\mu+\rho_n^H}]=\dim Hom_H(V_\mu^H,V_\lambda)$ $=\dim Hom_L(V^L_{\mu+\rho_n^H},H^2(H_0,\tau))=\dim\, H^2(H_0,\tau)[V^L_{\mu+\rho_n^H}]$, and, both L-modules are isotypical, and it follows d). Finally, e) follows from c) and d).

Under the assumption $\pi^{K_2}_{\Lambda_2}$ is the trivial representation, the formulae in Theorem 4.2 becomes:

$$\begin{aligned}\dim Hom_H(V^H_\mu, V^G_\lambda) &= \dim Hom_L(V^L_{\mu+\rho^H_n}, V^{H_0}_{(\lambda_1,\rho_{K_2\cap L})+\rho^H_n})\\ &= \dim Hom_L(V^L_{\mu+\rho^H_n}, H^2(H_0,\tau)) = \dim Hom_L(V^L_\mu, V^{H_0}_{(\lambda_1,\rho_{K_2\cap L})}),\end{aligned}$$

the infinitesimal character of $H^2(H_0,\tau)$ is $(\lambda_1+\rho^\lambda_n, \rho_{K_2\cap L}) - \rho^{H_0}_n = (\lambda_1,\rho_{K_2\cap L}) + \rho^H_n$. Thus, $H^2(H_0,\tau) \equiv V^{H_0}_{(\lambda_1,\rho_{K_2\cap L})+\rho^H_n}$.

4.0.5. *Analysis of $\mathcal{U}(\mathfrak{h}_0)W$, $\mathcal{L}_\lambda$, existence of D, for general (τ, W).* We recall that by definition, $\mathcal{L}_\lambda = \oplus_{\mu\in Spec_H(\pi_\lambda)} H^2(G,\tau)[V^H_\mu][V^L_{\mu+\rho^\mu_n}]$, $\mathcal{U}(\mathfrak{h}_0)W = L_{\mathcal{U}(\mathfrak{h}_0)}(H^2(G,\tau)[W])$.

Proposition 4.9. *The hypothesis is: (G,H) is a symmetric pair and π_λ a H-admissible square integrable representation of lowest K-type (τ, W). We write*

$res_L(\tau) = q_1\sigma_1 + \cdots + q_r\sigma_r$, with $(\sigma_j, Z_j) \in \widehat{L}, q_j > 0$. Then,

a) if a H-irreducible discrete factor for $res_H(\pi_\lambda)$ contains a copy of σ_j, then σ_j is the lowest L-type of such factor.

b) the multiplicity of σ_j in $res_L(H^2(G,\tau))$ is equal to q_j.

c) $r_0 : \mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W) \to \mathbf{H}^2(H_0,\tau)$ is a equivalence.

d) $\mathcal{L}_\lambda$ is L-equivalent to $\mathbf{H}^2(H_0,\tau)_{L-fin}$.

e) $\mathcal{L}_\lambda$ is L-equivalent to $\mathcal{U}(\mathfrak{h}_0)W$. Whence, D exists.

Proof. Let V^H_μ a discrete factor for $res_H(\pi_\lambda)$ so that some irreducible factor of $\tau_{|_L}$ is a L-type. Then, Theorem 4.2 implies $V^L_{\mu+\rho^H_n}$ is a L-type for $\mathbf{H}^2(H_0,\tau) = \oplus_j q_j H^2(H_0,\sigma_j)$. Let's say $V^L_{\mu+\rho^H_n}$ is a subrepresentation of $H^2(H_0,\sigma_i)$. We recall $ic(\phi)$ denotes the infinitesimal character (Harish-Chandra parameter) of the representation ϕ. Hence, after we apply *Remark* 4.8 we obtain

$\mu + \rho^H_n + B_1 = ic(\sigma_j)$ with B_1 a sum of roots in $\Psi_{H,\lambda}\cap\Phi_n$.

$\mu + \rho^H_n = ic(\sigma_i) + B_0$ with B_0 a sum of roots in $\Psi_{H_0,\lambda}\cap\Phi_n$.

Thus, $B_0 + B_1 = ic(\sigma_j) - ic(\sigma_i)$. Now, since $\mathfrak{k} = \mathfrak{k}_1 + \mathfrak{k}_2$, $\mathfrak{k}_1 \subset \mathfrak{l}$, $\tau = \pi^{K_1}_{\Lambda_1}\boxtimes\pi^{K_2}_{\Lambda_2}$, we may write $\sigma_s = \pi^{K_1}_{\Lambda_1}\boxtimes\phi_s$, with $\phi_s \in \widehat{L\cap K_2}$, whence, $ic(\sigma_j) - ic(\sigma_i) = ic(\phi_j) - ic(\phi_i)$. Since, each ϕ_t is a irreducible factor of $res_{L\cap K_2}(\pi^{K_2}_{\Lambda_2})$, we have $ic(\phi_j) - ic(\phi_i)$ is equal to the difference of two sum of roots in $\Phi(\mathfrak{k}_2, \mathfrak{t}\cap\mathfrak{k}_2)$. The hypothesis forces that the simple roots for $\Psi_\lambda\cap\Phi(\mathfrak{k}_2,\mathfrak{t}\cap\mathfrak{k}_2)$ are compact simple roots for Ψ_λ (see [5]) whence $ic(\sigma_j) - ic(\sigma_i)$ is a linear combination of compact simple roots for Ψ_λ. On the other hand, $B_0 + B_1$ is a sum of noncompact roots in Ψ_λ. Now $B_0 + B_1$ can not be a linear combination of compact simple roots, unless $B_0 = B_1 = 0$. Whence, $ic(\sigma_i) = ic(\sigma_j)$ and $Z_j \equiv V^L_{\sigma_j}$ is the lowest L-type of V^H_μ, we have verified a).

Due to H-admissibility hypothesis, we have $\mathcal{U}(\mathfrak{h})W$ is a finite sum of irreducible underlying Harish-Chandra modules of Discrete Series for H. Thus, a copy of certain V^H_μ contained in $\mathcal{U}(\mathfrak{h})W$ contains $W[\sigma_j]$. Whence, σ_j is the lowest L-type of such V^H_μ. Whence, $H^2(H, \sigma_j)$ is nonzero and it is equivalent to a subrepresentation of $\mathrm{Cl}(\mathcal{U}(\mathfrak{h})W)$.

We claim, for $i \neq j$, no σ_j is a L-type of $\mathrm{Cl}(\mathcal{U}(\mathfrak{h})W)[H^2(H, \sigma_i)]$.

Indeed, if σ_j were a L-type in $\mathrm{Cl}(\mathcal{U}(\mathfrak{h})W)[H^2(H, \sigma_i)]$, then, σ_j would be a L-type of a Discrete Series of lowest L-type equal to σ_i, according to a) this forces $i = j$, a contradiction. Now, we compute the multiplicity of $H^2(H, \sigma_j)$ in $H^2(G, \tau)$. For this, we apply Theorem 4.2. Thus, $\dim Hom_H(V_\lambda, H^2(H, \sigma_j)) = \sum_i q_i \dim Hom_L(\sigma_j, H^2(H, \sigma_i)) = q_j$

In order to realize the isotypic component corresponding to $H^2(H, \sigma_j)$ we write $V_\lambda[W][\sigma_j] = R_1 + \cdots + R_{q_j}$ a explicit sum of L-irreducibles modules. Then, owing to a), $L_{\mathcal{U}(\mathfrak{h})}(R_r)$ contains a copy N_r of $H^2(H, \sigma_j)$ and R_r is the lowest L-type of N_r. Therefore, the multiplicity computation yields $H^2(G, \tau)[H^2(H, \sigma_j)] = N_1 + \cdots + N_{q_j}$. Hence, b) holds. A corollary of this computation is:

$$Hom_H(H^2(H, \sigma_j), (\mathrm{Cl}(\mathcal{U}(\mathfrak{h})W))^\perp) = \{0\}.$$

Verification of c). After we recall Lemma 3.6, we have $r_0 : \mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0) \to L^2(H_0 \times_\tau W)$ is injective and we apply to the algebra $\mathfrak{h} := \mathfrak{h}_0$, the statement b) together with the computation to show b), we make the choice of the $q_j' s$ subspaces Z_j as a lowest L-type subspace of $W[Z_j]$. Thus, the image via r_0 of $\mathcal{U}(\mathfrak{h}_0)Z_j$ is a subspace of $L^2(H_0 \times \sigma_j)$. Since, either Atiyah-Schmid or Enright-Wallach [6] have shown $H^2(H_0, \sigma_j)$ has multiplicity one in $L^2(H_0 \times \sigma_j)$ we obtain the image of r_0 is equal to $\mathbf{H}^2(H_0, \tau)$.

The proof of d) and e) are word by word as the one for 4.0.4. $\square$

Corollary 4.10. *The multiplicity of $H^2(H, \sigma_j)$ in $res_H(H^2(G, \tau))$ is equal to*

$$q_j = dimHom_L(\sigma_j, H^2(G, \tau)).$$

Corollary 4.11. *For each σ_j, $\mathcal{L}_\lambda[Z_j] = \mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)[Z_j] = H^2(G, \tau)[W][Z_j] =$ "W"$[Z_j]$. Thus, we may fix $D = I_{"W"[Z_j]} : \mathcal{L}_\lambda[Z_j] \to \mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)[Z_j]$.*

4.1. **Explicit inverse map to r_0^D.** We consider three cases: $res_L(\tau)$ is irreducible, $res_L(\tau)$ is multiplicity free, and general case. Formally, they are quite alike, however, for us it has been illuminating to consider the three cases. As a byproduct, we obtain information on the compositions $r^\star r, r_0^\star r_0$; a functional equation that must satisfy the kernel of a holographic operator; for some particular discrete factor $H^2(H, \sigma)$ of

$res_H(\pi_\lambda)$ the reproducing kernel for $H^2(G,\tau)$ is a extension of the reproducing kernel for $H^2(H,\sigma)$ as well as that the holographic operator from $H^2(H,\sigma)$ into $H^2(G,\tau)$ is just plain extension of functions.

4.1.1. *Case (τ,W) restricted to L is irreducible.* In Tables 1,2,3, we show the list of the triples (G,H,π_λ) such that (G,H) is a symmetric pair, and π_λ is H-admissible. In 5.3.1 we show that if there exists (G,H,π_λ) so that π_λ is H-admissible, then there exists a H-admissible $\pi_{\lambda'}$ so that its lowest K-type restricted to L is irreducible and λ' is dominant with respect to π_λ. We denote by η_0 the Harish-Chandra parameter for $H^2(H_0,\tau)\equiv \mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)$.

We set $c=d(\pi_\lambda)dimW/d(\pi_{\eta_0}^{H_0})$. Next, we show

Proposition 4.12. *We assume the setting as well as the hypothesis in Theorem 3.1, and further (τ,W) restricted to L is irreducible.*
Let $T_0\in Hom_L(Z,H^2(H_0,\tau))$, then the kernel K_T corresponding to $T:=(r_0^D)^{-1}(T_0)\in Hom_H(H^2(H,\sigma),H^2(G,\tau))$ is

$$K_T(h,x)z=(D^{-1}[\int_{H_0}\frac{1}{c}K_\lambda(h_0,\cdot)(T_0(z)(h_0))dh_0])(h^{-1}x).$$

Proof. We systematically apply Theorem 4.9. Under our assumptions, we have: $\mathbf{H}^2(H_0,\tau)$ is a irreducible representation and $\mathbf{H}^2(H_0,\tau)=H^2(H_0,\tau)$;
$\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)(H^2(G,\tau)[W]))$ is H_0-irreducible; We define
$\tilde r_0:=rest(r_0):\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)H^2(G,\tau)[W])\to H^2(H_0,\tau)$ is a isomorphism. To follow, we notice the inverse of $\tilde r_0$, is up to a constant, equal to $r_0^\star$ restricted to $H^2(H_0,\tau)$. This is so, because functional analysis yields the equalities $\mathrm{Cl}(Im(r_0^\star))=ker(r_0)^\perp=\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)$, $Ker(r_0^\star)=Im(r_0)^\perp=H^2(H_0,\tau)^\perp$. Thus, Schur's lemma applied to the irreducible modules $H^2(H_0,\tau),\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)$ implies there exists non zero constants b,d so that $(\tilde r_0r_0^\star)_{|_{H^2(H_0,\tau)}}=bI_{H^2(H_0,\tau)}$, $r_0^\star\tilde r_0=dI_{\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)}$. Whence, the inverse to $\tilde r_0$ follows. In 4.1.2, we show $b=d=d(\pi_\lambda)dimW/d(\pi_{\eta_0}^{H_0})=c$.

For $x\in G, f\in H^2(G,\tau)$, the identity $f(x)=\int_G K_\lambda(y,x)f(y)dy$ holds. Thus, $r_0(f)(p)=f(p)=\int_G K_\lambda(y,p)f(y)dy$, for $p\in H_0, f\in H^2(G,\tau)$, and, we obtain

$$K_{r_0}(x,h_0)=K_\lambda(x,h_0),\ \ K_{r_0^*}(h_0,x)=K_{r_0}(x,h_0)^\star=K_\lambda(h_0,x).$$

Hence, for $g\in H^2(H_0,\tau)$ we have,

$$\tilde r_0^{-1}(g)(x)=\tfrac{1}{c}\int_{H_0}K_{r_0^\star}(h_0,x)g(h_0)dh_0=\tfrac{1}{c}\int_{H_0}K_\lambda(h_0,x)g(h_0)dh_0.$$

Therefore, for $T_0\in Hom_L(Z,H^2(H_0,\tau))$, the kernel K_T of the element T in $Hom_H(H^2(H,\sigma),H^2(G,\tau))$ such that $r_0^D(T)=T_0$, satisfies for $z\in Z$

$$D^{-1}([r_0^{-1}(T_0(z)(\cdot))])(\cdot)=K_T(e,\cdot)z\in V_\lambda^G[H^2(H,\sigma)][Z]\subset H^2(G,\tau).$$

More explicitly, after we recall $K_T(e,h^{-1}x)=K_T(h,x)$,

$$K_T(h,x)z=(D^{-1}[\int_{H_0}\frac{1}{c}K_\lambda(h_0,\cdot)(T_0(z)(h_0))dh_0])(h^{-1}x).$$

$\square$

Corollary 4.13. *For any T in $Hom_H(H^2(H,\sigma),H^2(G,\tau))$ we have*

$$K_T(h,x)z=(D^{-1}[\int_{H_0}\frac{1}{c}K_\lambda(h_0,\cdot)(r_0(D(K_T(e,\cdot)z))(h_0))dh_0])(h^{-1}x).$$

Corollary 4.14. *When D is the identity map, we obtain*

$$\begin{aligned}K_T(h,x)z&=\int_{H_0}\frac{1}{c}K_\lambda(h_0,h^{-1}x)(T_0(z)(h_0))dh_0\\&=\int_{H_0}\frac{1}{c}K_\lambda(hh_0,x)K_T(e,h_0)zdh_0.\end{aligned}$$

The equality in the conclusion of Proposition 4.12 is equivalent to
$D(K_T(e,\cdot))(y)=\int_{H_0}\frac{1}{c}K_\lambda(h_0,y)D(K_T(e,\cdot))(h_0)dh_0, y\in G.$
Whence, we have derived a formula that let us to recover the kernel K_T (resp. $D(K_T(e,\cdot))(\cdot)$) from $K_T(e,\cdot)$ (resp. $D(K_T(e,\cdot))(\cdot)$) restricted to H_0!

Remark 4.15. We notice,

$$r_0^\star r_0(f)(y)=\int_{H_0}K_\lambda(h_0,y)f(h_0)dh_0,\ f\in H^2(G,\tau),\ y\in G. \tag{4.2}$$

Since we are assuming $\tau_{|_L}$ is irreducible, we have $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)$ is irreducible, hence, Lemma 3.6 let us to obtain that a scalar multiple of $r_0^\star r_0$ is the orthogonal projector onto the irreducible factor $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W$. Whence, the orthogonal projector onto $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)$ is given by $\frac{d(\pi_{\eta_0}^{H_0})}{d(\pi_\lambda)dimW}\ r_0^\star r_0$.

Thus, the kernel K_{λ,η_0} of the orthogonal projector onto $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)$ is

$$K_{\lambda,\eta_0}(x,y):=\frac{d(\pi_{\eta_0}^{H_0})}{d(\pi_\lambda)dimW}\int_{H_0}K_\lambda(p,y)K_\lambda(x,p)dp.$$

Doing $H:=H_0$ we obtain a similar result for the kernel of the orthogonal projector onto $\mathrm{Cl}(\mathcal{U}(\mathfrak{h})W)$.

The equality $(r_0r_0^\star)_{|_{H^2(H_0,\tau)}}=cI_{H^2(H_0,\tau)}$ yields the first claim in:

Proposition 4.16. *Assume $res_L(\tau)$ is irreducible. Then,*
a) for every $g\in H^2(H_0,\tau_{|_L})$ (resp. $g\in H^2(H,\tau_{|_L})$), the function $r_0^\star(g)$ (resp. $r^\star(g)$), is an extension of a scalar multiple of g.
b) The kernel K_λ^G is a extension of a scalar multiple of $K_{\tau_{|_L}}^H$.

When we restrict holomorphic Discrete Series, this fact naturally happens, see [22], [25, Example 10.1] and references therein.

Proof. Let $r : H^2(G,\tau) \to L^2(H \times_\tau W)$ the restriction map. The duality H, H_0, and Theorem 3.1 applied to $H := H_0$ implies $H^2(H,\tau) = r(\mathrm{Cl}(\mathcal{U}(\mathfrak{h})W))$, as well as that there exists, up to a constant, a unique $T \in Hom_H(H^2(H,\tau), H^2(G,\tau)) \equiv Hom_L(W, H^2(H,\tau)) \equiv \mathbb{C}$. It follows from the proof of Proposition 4.12, that, up to a constant, $T = r^*$ restricted to $H^2(H,\tau)$. After we apply the equality $T(K_\mu^H(\cdot,e)^\star z)(x) = K_T(e,x)z$, (see [24]), we obtain,

$$r^\star(K_\mu^H(\cdot,e)^* z)(y) = K_\lambda(y,e)^\star z.$$

Also, Schur's lemma implies rr^* restricted to $H^2(H,\tau)$ is a constant times the identity map. Thus, for $h \in H$, we have $rr^*(K_\mu^H(\cdot,e)^\star w)(h) = qK_\mu^H(h,e)^\star w$. For the value of q see 4.1.2. Putting together, we obtain,

$$K_\lambda(h,e)^\star z = r(K_\lambda(\cdot,e)^\star z)(h) = qK_\mu^H(h,e)^\star z.$$

Whence, for $h, h_1 \in H$ we have

$K_\lambda(h_1,h)^\star z = K_\lambda(h^{-1}h_1,e)^\star z = qK_\mu^H(h^{-1}h_1,e)^\star z = qK_\mu^H(h_1,h)^\star z$ as we have claimed. □

By the same token, after we set $H := H_0$ we obtain:

For $res_L(\tau)$ irreducible, $(\sigma, Z) = (res_L(\tau), W)$, and $V_{\eta_0}^{H_0} = H^2(H_0,\sigma)$, the kernel K_λ extends a scalar multiple of $K_{\eta_0}^{H_0}$. Actually, $r_0(K_\lambda(\cdot,e)^\star w) = cK_{\eta_0}^{H_0}(\cdot,e)^\star w$.

Remark 4.17. We would like to point out that the equality

$$r^\star(K_\mu^H(\cdot,e)^*(z))(y) = qK_\lambda(y,e)^\star z$$

implies $res_H(\pi_\lambda)$ is H-algebraically discretely decomposable. Indeed, we apply a Theorem shown by Kobayashi [16, Lemma 1.5], the Theorem says that when $(V_\lambda^G)_{K-fin}$ contains an irreducible $(\mathfrak{h}, L)$ irreducible submodule, then V_λ is discretely decomposable. We know $K_\lambda(y,e)^\star z$ is a K-finite vector, the equality implies $K_\lambda(y,e)^\star z$ is $\mathfrak{z}(\mathcal{U}(\mathfrak{h}))$-finite. Hence, owing to Harish-Chandra [32, Corollary 3.4.7 and Theorem 4.2.1], $H^2(G,\tau)_{K-fin}$ contains a nontrivial irreducible $(\mathfrak{h}, L)$-module and the fact shown by Kobayashi applies.

4.1.2. *Value of $b = d = c$ when $res_L(\tau)$ is irreducible.* We show $b = d = d(\pi_\lambda)dimW/d(\pi_{\eta_0}^{H_0}) = c$. In fact, the constant b, d satisfies $(r_0^\star r_0)_{\mathcal{U}(\mathfrak{h}_0)W} = dI_{\mathcal{U}(\mathfrak{h}_0)W}$, $(r_0 r_0^\star)_{|_{H^2(H_0,\tau)}} = bI_{H^2(H_0,\tau)}$. Now, it readily follows $b = d$. To evaluate $r_0^\star r_0$ at $K_\lambda(\cdot,e)^\star w$, for $h_1 \in H_0$ we compute, for $h_1 \in H_0$,

$$\begin{aligned} bK_\lambda(h_1,e)^\star w = r_0^* r_0(K_\lambda(\cdot,e)^\star w)(h_1) &= \int_{H_0} K_\lambda(h_0,h_1)K_\lambda(h_0,e)^\star dh_0 w \\ &= d(\pi_\lambda)^2 \int_{H_0} \Phi(h_1^{-1}h_0)\Phi(h_0)^\star dh_0 w. \end{aligned}$$

Here, Φ is the spherical function attached to the lowest K-type of

π_λ. Since, we are assuming $res_L(\tau)$ is a irreducible representation, we have $\mathcal{U}(\mathfrak{h}_0)W$ is a irreducible $(\mathfrak{h}_0, L)$-module and it is equivalent to the underlying Harish-Chandra module for $H^2(H_0, res_L(\tau))$. Thus, the restriction of Φ to H_0 is the spherical function attached to the lowest L-type of the irreducible square integrable representation $Cl(\mathcal{U}(\mathfrak{h}_0)W) \equiv H^2(H_0, res_L(\tau))$. We fix a orthonormal basis $\{w_i\}$ for $\mathcal{U}(\mathfrak{h}_0)W[W]$. We recall,

$\Phi(x)w = P_W\pi(x)P_W w = \sum_{1\leq i\leq \dim W}(\pi(x)w, w_i)_{L^2} w_i$,

$\Phi(x^{-1}) = \Phi(x)^\star$.

For $h_1 \in H_0$, we compute, to justify steps we appel to the invariance of Haar measure and to the orthogonality relations for matrix coefficients of irreducible square integrable representations and we recall $d(\pi_{\eta_0}^{H_0})$ denotes the formal degree for $H^2(H_0, res_L(\tau))$.

$$\begin{aligned}
\int_{H_0} \Phi(h_1^{-1}h)\Phi(h)^\star w dh &= \sum_{i,j}\int_{H_0} (\pi(h_1^{-1}h)w_j, w_i)_{L^2}(\pi(h^{-1})w, w_j)_{L^2} w_i \\
&= \sum_{i,j}\int_{H_0} (\pi(h)w_j, h_1 w_i)_{L^2}\overline{(\pi(h)w_j, w)_{L^2}} w_i \\
&= 1/d(\pi_{\eta_0}^{H_0})\sum_{i,j}(w_j, w_j)_{L^2}\overline{(h_1 w_i, w)_{L^2}} \\
&= dimW/d(\pi_{\eta_0}^{H_0})\sum_{i}(h_1^{-1}w, w_i)_{L^2} w_i \\
&= dimW/d(\pi_{\eta_0}^{H_0})\Phi(h_1)^\star w.
\end{aligned}$$

Thus,

$$\begin{aligned}
r_0^\star r_0(K_\lambda(\cdot, e)^\star w)(h_1) &= d(\pi_\lambda)^2 \int_{H_0} \Phi(h_1^{-1}h)\Phi(h)^\star w dh \\
&= d(\pi_\lambda)^2 dimW/d(\pi_{\eta_0}^{H_0})/d(\pi_\lambda)K_\lambda(h_1, e)^\star w.
\end{aligned}$$

The functions $K_\lambda(\cdot, e)^\star w, r_0^\star r_0(K_\lambda(\cdot, e)^\star w)(\cdot)$ belong to $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)$, the injectivity of r_0 on $Cl(\mathcal{U}(\mathfrak{h}_0)W)$, forces, for every $x \in G$

$$r_0^\star r_0(K_\lambda(\cdot, e)^\star w)(x) = d(\pi_\lambda)dimW/d(\pi_{\eta_0}^{H_0})K_\lambda(x, e)^\star w.$$

Hence, we have computed $b = d = c$.

4.1.3. *Analysis of r_0^D for arbitrary $(\tau, W), (\sigma, Z)$.* We recall the decomposition $W = \sum_{\nu_2 \in Spec_{L\cap K_2}(\pi_{\Lambda_2}^{K_2})} W[\pi_{\Lambda_1}^{K_1} \boxtimes \pi_{\nu_2}^{L\cap K_2}]$.

A consequence of Proposition 4.9 is that $r_0^\star$ maps $\mathbf{H}^2(H_0, W[\pi_{\Lambda_1}^{K_1} \boxtimes \pi_{\nu_2}^{L\cap K_2}])$ into $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W[\pi_{\Lambda_1}^{K_1} \boxtimes \pi_{\nu_2}^{L\cap K_2}])$. In consequence, $r_0 r_0^\star$ restricted to $\mathbf{H}^2(H_0, W[\pi_{\Lambda_1}^{K_1} \boxtimes \pi_{\nu_2}^{L\cap K_2}])$ is a bijective H_0-endomorphism C_j. Hence, the inverse map of r_0 restricted to $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W[\pi_{\Lambda_1}^{K_1} \boxtimes \pi_{\nu_2}^{L\cap K_2}])$ is $r_0^\star C_j^{-1}$.

Since, $H^2(H_0, \pi_{\Lambda_1}^{K_1} \boxtimes \pi_{\nu_2}^{L\cap K_2})$ has a unique lowest L-type, we conclude C_j is determined by an element of $Hom_L(\pi_{\Lambda_1}^{K_1} \boxtimes \pi_{\nu_2}^{L\cap K_2}, H^2(H, \pi_{\Lambda_1}^{K_1} \boxtimes \pi_{\nu_2}^{L\cap K_2})[\pi_{\Lambda_1}^{K_1} \boxtimes \pi_{\nu_2}^{L\cap K_2}])$. Since for $D \in \mathcal{U}(\mathfrak{h}_0), w \in W$ we have $C_j(L_D w) = L_D C_j(w)$, we obtain C_j is a zero order differential operator on the underlying Harish-Chandra module of $H^2(H_0, \pi_{\Lambda_1}^{K_1} \boxtimes \pi_{\nu_2}^{L\cap K_2})$. Summing up, we have that the inverse to $r_0 : \mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W) \to \mathbf{H}^2(H_0, \tau)$ is the function $r_0^\star(\oplus_j C_j^{-1})$.
For $T \in Hom_H(H^2(H, \sigma), H^2(G, \tau))$ and $T_0 \in Hom_L(Z, \mathbf{H}^2(H, \tau))$ so that $r_0^D(T) = T_0$ we obtain the equalities

$$K_T(e, x)z = (D^{-1}[\int_{H_0} K_\lambda(h_0, \cdot)((\oplus_j C_j^{-1})T_0(z))(h_0)dh_0])(x).$$

$$\begin{aligned} &K_T(h, x)z \\ &\qquad = (D^{-1}[\int_{H_0} K_\lambda(h_0, \cdot) \\ &\qquad\qquad \times ((\oplus_j C_j^{-1})(r_0(D(K_T(e, \cdot)z))(\cdot))(h_0)dh_0])(h^{-1}x). \end{aligned}$$

When D is the identity the formula simplifies as the one in the second Corollary to Proposition 4.12.

4.1.4. *Eigenvalues of $r_0^\star r_0$.* For general case, we recall $r_0^\star r_0$ intertwines the action of H_0. Moreover, Proposition 4.9 and its Corollary gives that for each L-isotypic component $Z_1 \subseteq W$ of $res_L(\tau)$, we have $\mathcal{U}(\mathfrak{h}_0)W[Z_1] = Z_1$. Thus, each isotypic component of $res_L((\mathcal{U}(\mathfrak{h}_0)W)[W])$ is invariant by $r_0^\star r_0$, in consequence, $r_0^\star r_0$ leaves invariant the subspace "W" $= H^2(G, \tau)[W] = \{K_\lambda(\cdot, e)^\star w, w \in W\}$. Since, $Ker(r_0) = (\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W))^\perp$, we have $r_0^\star r_0$ is determined by the values it takes on "W". Now, we assume $res_L(\tau)$ is a multiplicity free representation, we write $Z_1^\perp = Z_2 \oplus \cdots \oplus Z_q$, where Z_j are L-invariant and L-irreducible. Thus, Proposition 4.9 implies $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W) = \mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)Z_1) \oplus \cdots \oplus \mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)Z_q)$. This a orthogonal decomposition, each summand is irreducible and no irreducible factor is equivalent to other. For $1 \leq i \leq q$, let η_i denote the Harish-Chandra parameter for $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)Z_i)$.

Proposition 4.18. *When $res_L(\tau)$ is a multiplicity free representation, the linear operator $r_0^\star r_0$ on $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)Z_i)$ is equal to $\frac{d(\pi_\lambda) dim Z_i}{d(\pi_{\eta_i}^{H_0})}$ times the identity map.*

Proof. For the subspace $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)W)[W]$, we choose a $L^2(G)$-orthonormal basis $\{w_j\}_{1 \leq j \leq dim W}$ equal to the union of respective $L^2(G)$-orthonormal basis for $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)Z_i)[Z_i]$. Next, we compute and freely make use of notation in 4.1.2. Owing to our multiplicity free hypothesis, we have that $r_0^\star r_0$ restricted to $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)Z_i)$ is equal to a constant d_i

times the identity map. Hence, on $w \in \mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)Z_i)[Z_i]$ we have $d_i w = d(\pi_\lambda)^2 \int_{H_0} \Phi(h_0)\Phi(h_0)^\star w dh_0$.

Now, $\Phi(h_0) = (a_{ij}) = ((\pi_\lambda(h_0)w_j, w_i)_{L^2(G)})$, Whence, the pq-coefficient of the product $\Phi(h_0)\Phi(h_0)^\star$ is equal to

$\sum_{1\leq j\leq dimW} (\pi_\lambda(h_0)w_j, w_p)_{L^2(G)} \overline{(\pi_\lambda(h_0)w_q, w_j)}_{L^2(G)}$

Let I_i denote the set of indexes j so that $w_j \in Z_i$. Thus, $\{1, \dots, dimW\}$ is equal to the disjoint union $\cup_{1\leq i\leq q} I_i$. A consequence of Proposition 4.9 is the $L^2(G)$-orthogonality of the subspaces $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)Z_j)$, hence, for $t \in I_a, q \in I_d$ and $a \neq d$ we have $(\pi_\lambda(h_0)w_q, w_t)_{L^2(G)} = 0$. Therefore, the previous observation and the disjointness of the sets I_r, let us obtain that for $i \neq d, p \in I_i, q \in I_d$ each summand in

$\sum_{1\leq j\leq dimW} \int_{H_0} (\pi_\lambda(h_0)w_j, w_p)_{L^2(G)} \overline{(\pi_\lambda(h_0)w_q, w_j)}_{L^2(G)} dh_0$

is equal to zero.

For $p, q \in I_i$, we apply the previous computation and the orthogonality relations to the irreducible representation $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)Z_i$. We obtain

$\sum_{1\leq j\leq dimW} \int_{H_0} (\pi_\lambda(h_0)w_j, w_p)_{L^2(G)} \overline{(\pi_\lambda(h_0)w_q, w_j)}_{L^2(G)} dh_0$

$= \sum_{j\in I_i} \int_{H_0} (\pi_\lambda(h_0)w_j, w_p)_{L^2(G)} \overline{(\pi_\lambda(h_0)w_q, w_j)}_{L^2(G)} dh_0$

$= \sum_{j\in I_i} \frac{1}{d(\pi^{H_0}_{\eta_i})} (w_j, w_q)_{L^2(G)} (w_j, w_p)_{L^2(G)} = \frac{dimZ_i}{d(\pi^{H_0}_{\eta_i})}$.

Thus, we have shown Proposition 4.18. □

Remark 4.19. Even, when $res_L(\tau)$ is not multiplicity free, the conclusion in Proposition 4.18 holds. In fact, let us denote the L-isotypic component of $res_L(\tau)$ again by Z_i. Now, the proof goes as the one for Proposition 4.18 till we need to compute

$= \sum_{j\in I_i} \int_{H_0} (\pi_\lambda(h_0)w_j, w_p)_{L^2(G)} \overline{(\pi_\lambda(h_0)w_q, w_j)}_{L^2(G)} dh_0$

For this, we decompose "Z_i" $= \sum_s Z_{i,s}$ as a $L^2(G)$-orthogonal sum of irreducible L-modules and we choose the orthonormal basis for "Z_i" as a union of orthonormal basis for each $Z_{i,s}$. Then, we have the $L^2(G)$-orthogonal decomposition $\mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)Z_i) = \sum_s \mathrm{Cl}(\mathcal{U}(\mathfrak{h}_0)Z_{i,s})$. Then, the proof follows as in the case $res_L(\tau)$ is multiplicity free.

5. Examples

We present three type of examples. The first is: *Multiplicity free representations.* A simple consequence of the duality theorem is that it readily follows examples of symmetric pair (G, H) and square integrable representation π^G_λ so that $res_H(\pi_\lambda)$ is H-admissible and the multiplicity of each irreducible factor is one. This is equivalent to determinate when the representation $res_H(\mathbf{H}^2(H_0, \tau))$ is multiplicity free. The second is: *Explicit examples.* Here, we compute the Harish-Chandra parameters of the irreducible factors for some $res_H(H^2(G,\tau))$. The third is: *Existence of representations so that its lowest K-types restricted to L is a irreducible representation.*

In order to present the examples we need information on certain families of representations.

5.1. **Multiplicity free representations.** In this paragraph we generalize work of T. Kobayashi and his coworkers in the setting of Hermitian symmetric spaces and holomorphic Discrete Series.

Before we present the examples, we would like to comment.

a) Assume a Discrete Series π_λ has admissible restriction to a subgroup H. Then, any Discrete Series $\pi_{\lambda'}$ for λ' dominant with respect to Ψ_λ is H-admissible [15].

b) If $res_H(\pi_\lambda)$ is H-admissible and a multiplicity free representation. Then the restriction to L of the lowest K-type for π_λ is multiplicity free. This follows from the duality theorem.

c) In the next paragraphs we will list families $\mathcal{F}$ of Harish-Chandra parameters of Discrete Series for G so that each representation in the family has multiplicity free restriction to H. We find that it may happen that $\mathcal{F}$ is the whole set of Harish-Chandra parameters on a Weyl chamber or $\mathcal{F}$ is a proper subset of a Weyl Chamber. Information on $\mathcal{F}$ for holomorphic reprentations is in [18], [19].

d) Every irreducible $(\mathfrak{g}, K)$-module for either $\mathfrak{g} \equiv \mathfrak{su}(n,1)$ or $\mathfrak{g} \equiv \mathfrak{so}(n,1)$, restricted to K, is a multiplicity free representation.

5.1.1. *Holomorphic representations.* For G so that G/K is a Hermitian symmetric space, it has been shown by Harish-Chandra that G admits Discrete Series representations with one dimensional lowest K-type. For this paragraph we further assume that the smooth imbedding $H/L \to G/K$ is holomorphic, equivalently the center of K is contained in L, and π_λ is a holomorphic representation. Under this hypothesis, it was shown by Kobayashi [17] that a holomorphic Discrete Series for G has a multiplicity free restriction to the subgroup H whenever the it is a scalar holomorphic Discrete Series. Moreover, in [17, Theorem 8.8] computes the Harish-Chandra parameter of each irreducible factor. Also, from the work of Kobayashi and Nakahama we find a description of the restriction to H of a arbitrary holomorphic Discrete Series representations. As a consequence, we find restrictions which are not multiplicity free.

In [19] we find a complete list of the pairs $(\mathfrak{g}, \mathfrak{h})$ so that $H/L \to G/K$ is a holomorphic embedding. From the list in [17], it can be constructed the list bellow.

Also, Theorem 3.1 let us verify that the following pairs $(\mathfrak{g},\mathfrak{h})$ are so that $res_H(\pi_\lambda)$ is multiplicity free for any holomorphic π_λ. For this, we list the corresponding $\mathfrak{h}_0$.

$(\mathfrak{su}(m,n), \mathfrak{u}(\mathfrak{u}(m-1,n)+\mathfrak{u}(1)))$, $\mathfrak{h}_0 = \mathfrak{su}(1,n)+\mathfrak{su}(m-1)+\mathfrak{u}(1)$.
$(\mathfrak{su}(m,n), \mathfrak{s}(\mathfrak{u}(m,n-1)+\mathfrak{u}(1)))$, $\mathfrak{h}_0 = \mathfrak{su}(n-1)+\mathfrak{su}(m,1)+\mathfrak{u}(1)$.
$(\mathfrak{so}(2m,2), \mathfrak{u}(m,1))$, $\mathfrak{h}_0 = \mathfrak{u}(m,1)$.
$(\mathfrak{so}^\star(2n), \mathfrak{so}^\star(2)+\mathfrak{so}^\star(2n-2))$, $\mathfrak{h}_0 = \mathfrak{u}(1,n-1)$.
$(\mathfrak{sp}(n,\mathbb{R}), \mathfrak{sp}(n-1,\mathbb{R})+\mathfrak{sp}(1,\mathbb{R})), \mathfrak{h}_0 = \mathfrak{u}(1,n-1)$.
$(\mathfrak{e}_{6(-14)}, \mathfrak{so}^\star(10)+\mathfrak{so}(2))$, $\mathfrak{h}_0 = \mathfrak{su}(5,1)+\mathfrak{sl}_2(\mathbb{R})$ (Prasad).

The list is correct, owing to any Discrete Series for $SU(n,1)$ restricted to K is a multiplicity free representation.

5.1.2. *Quaternionic real forms, quaternionic representations.* In [9], the authors considered and classified quaternionic real forms as well as they made a careful study of quaternionic representations. To follow we bring out the essential facts for us. From [9] we read that the list of Lie algebra of quaternionic groups is: $\mathfrak{su}(2,n)$, $\mathfrak{so}(4,n)$, $\mathfrak{sp}(1,n)$, $\mathfrak{e}_{6(2)}$, $\mathfrak{e}_{7(-5)}$, $\mathfrak{e}_{8(-24)}$, $\mathfrak{f}_{4(4)}$, $\mathfrak{g}_{2(2)}$. For each quaternionic real form G, there exists a system of positive roots $\Psi \subset \Phi(\mathfrak{g},\mathfrak{t})$ so that the maximal root α_{max} in Ψ is compact, α_{max} is orthogonal to all compact simple roots and α_{max} is not orthogonal to each noncompact simple roots. Hence, $\mathfrak{k}_1(\Psi) \equiv \mathfrak{su}_2(\alpha_{max})$. The system Ψ is not unique. We appel such a system of positive roots *a quaternionic system.*

Let us recall that a *quaternionic representation* is a Discrete Series for a quaternionic real form G so that its Harish-Chandra parameter is dominant with respect to a quaternionic system of positive roots, and so that its lowest K-type is equivalent to a irreducible representation for $K_1(\Psi)$ times the trivial representation for K_2. A fact shown in [9] is: Given a quaternionic system of positive roots, for all but finitely many representations (τ, W) equivalent to the tensor of a nontrivial representation for $K_1(\Psi)$ times the trivial representation of K_2, it holds: τ is the lowest K-type of a quaternionic (unique) irreducible square integrable representation $H^2(G,\tau)$. We define a *generalized quaternionic representation* to be a Discrete Series representation π_λ so that its Harish-Chandra parameter is dominant with respect to a quaternionic system of positive roots.

From Table 1,2 we readily read the pairs $(\mathfrak{g},\mathfrak{h})$ so that $\mathfrak{g}$ is a quaternionic Lie algebra and hence, we have a list of generalized quaternionic representations of G with admissible restriction to H.

Let (G,H) denote a symmetric pair so that a quaternionic representation $(\pi_\lambda, H^2(G,\tau))$ is H-admissible. Then, from [30] [5] [4] we have:

$\mathfrak{k}_1(\Psi_\lambda) \equiv \mathfrak{su}_2(\alpha_{max}) \subset \mathfrak{l}$ and π_λ is L-admissible. In consequence, [16], π_λ is H_0-admissible. By definition, for a quaternionic representation π_λ, we have $\tau_{|_L}$ is irreducible, hence, $\mathbf{H}^2(H_0,\tau)$ is irreducible. Moreover, after checking on [30] or Tables 1,2, the list of systems $\Psi_{H_0,\lambda}$, it follows that $H^2(H_0,\tau)$ is again a quaternionic representation. Finally, in order to present a list of quaternionic representations with multiplicity free restriction to H we recall that it follows from the duality Theorem that $res_H(H^2(G,\tau))$ is multiplicity free if and only if $res_L(H^2(H_0,\tau))$ is a multiplicity free representation, and that on [9, Page 88] it is shown that a quaternionic representation for H_0 is L-multiplicity free if and only if $\mathfrak{h}_0 = \mathfrak{sp}(n,1), n \geq 1$.

To follow, we list pairs $(\mathfrak{g},\mathfrak{h})$ where multiplicity free restriction holds for all quaternionic representations.

$(\mathfrak{su}(2,2n),\mathfrak{sp}(1,n))$, $\mathfrak{h}_0 = \mathfrak{sp}(1,n)$, $n \geq 1$.
$(\mathfrak{so}(4,n),\mathfrak{so}(4,n-1))$, $\mathfrak{h}_0 = \mathfrak{so}(4,1)+\mathfrak{so}(n-1)$ (n even or odd).
$(\mathfrak{sp}(1,n),\mathfrak{sp}(1,k)+\mathfrak{sp}(n-k))$, $\mathfrak{h}_0 = \mathfrak{sp}(1,n-k)+\mathfrak{sp}(k)$.
$(\mathfrak{f}_{4(4)},\mathfrak{so}(5,4))$, $\mathfrak{h}_0 = \mathfrak{sp}(1,2)\oplus\mathfrak{su}(2)$.
$(\mathfrak{e}_{6(2)},\mathfrak{f}_{4(4)})$, $\mathfrak{h}_0 = \mathfrak{sp}(3,1)$.

A special pair is:
$(\mathfrak{su}(2,2),\mathfrak{sp}(1,1))$, $\mathfrak{h}_0 = \mathfrak{sp}(1,1)$.

Here, multiplicity free holds for any π_λ so that λ is dominant with respect to a system of positive roots that defines a quaternionic structure on G/K. For details see [30, Table 2] or Explicit example II.

5.1.3. *More examples of multiplicity free restriction.* Next, we list pairs $(\mathfrak{g},\mathfrak{h})$ and systems of positive roots $\Psi \subset \Phi(\mathfrak{g},\mathfrak{t})$ so that $\pi_{\lambda'}$ is H- admissible and multiplicity free for every element λ' dominant with respect to Ψ. We follow either Table 1,2,3 or [19]. For each $(\mathfrak{g},\mathfrak{h})$ we list the corresponding $\mathfrak{h}_0$.

$(\mathfrak{su}(m,n),\mathfrak{su}(m,n-1)+\mathfrak{u}(1))$, $\Psi_a, \tilde{\Psi}_b$,
$\mathfrak{h}_0 = \mathfrak{su}(m,1)+\mathfrak{su}(n-1)+\mathfrak{u}(1)$.
$(\mathfrak{so}(2m,2n+1),\mathfrak{so}(2m,2n))$, $\Psi_{\pm m}$, $\mathfrak{h}_0 = \mathfrak{so}(2m,1)+\mathfrak{so}(2n)$.
$(\mathfrak{so}(2m,2),\mathfrak{so}(2m,1))$, $\Psi_{\pm m}$, $\mathfrak{h}_0 = \mathfrak{so}(2m,1)$.
$(\mathfrak{so}(2m,2n),\mathfrak{so}(2m,2n-1)), n>1$, $\Psi_\pm$, $\mathfrak{h}_0 = \mathfrak{so}(2m,1)+\mathfrak{so}(2n-1)$.

5.2. **Explicit examples.**

5.2.1. *Quaternionic representations for $Sp(1,b)$.* For further use we present a intrinsic description for the $Sp(1)\times Sp(b)$-types of a quaternionic representation for $Sp(1,b)$, a proof of the statements is in [8]. The quaternionic representations for $Sp(1,b)$ are the representations of lowest $Sp(1)\times Sp(b)$-type $S^n(\mathbb{C}^2)\boxtimes\mathbb{C}, n\geq 1$. We label the simple roots for the quaternionic system of positive roots Ψ as in [9], $\beta_1,\dots,\beta_{b+1}$, the long root is β_{b+1}, β_1 is adjacent to just one simple root and the maximal root β_{max} is adjacent to $-\beta_1$. Let $\Lambda_1,\dots,\Lambda_{d+1}$ the associated fundamental weights. Thus, $\Lambda_1=\frac{\beta_{max}}{2}$. Let $\tilde\Lambda_1,\dots,\tilde\Lambda_b$ denote the fundamental weights for "$\Psi\cap\Phi(\mathfrak{sp}(b))$". The irreducible $L=Sp(1)\times Sp(b)$-factors of

$$H^2(Sp(1,b),\pi^{Sp(1)}_{n\frac{\beta_{max}}{2}}\boxtimes\pi^{Sp(b)}_{\rho_{Sp(b)}})=H^2(Sp(1,b),S^{n-1}(\mathbb{C}^2)\boxtimes\mathbb{C})$$

are

$$\{S^{n-1+m}(\mathbb{C}^2)\boxtimes S^m(\mathbb{C}^{2b})$$
$$\equiv\pi^{Sp(1)}_{(n+m)\frac{\beta_{max}}{2}}\boxtimes\pi^{Sp(b)}_{m\tilde\Lambda_1+\rho_{Sp(b)}},\ m\geq 0\}.$$

The multiplicity of each L-type in $H^2(Sp(1,b),S^{n-1}(\mathbb{C}^2)\boxtimes\mathbb{C})$ is one.

5.2.2. *Explicit example I.* We develop this example in detail. We restrict quaternionic representations for $Sp(1,d)$ to $Sp(1,k)\times Sp(d-k)$. For this, we need to review definitions and facts in [8][19] [30]. The group $G:=Sp(1,d)$ is a subgroup of $GL(\mathbb{C}^{2+2d})$. A maximal compact subgroup of $Sp(1,d)$ is the usual immersion of $Sp(1)\times Sp(d)$. Actually, $Sp(1,d)$ is a quaternionic real form for $Sp(\mathbb{C}^{1+d})$. $Sp(1,d)$ has a compact Cartan subgroup T and there exists a orthogonal basis $\delta,\epsilon_1,\dots,\epsilon_d$ for $i\mathfrak{t}^\star$ so that

$\Phi(\mathfrak{sp}(d+1,\mathbb{C}),\mathfrak{t})=\{\pm2\delta,\pm2\epsilon_1,\dots,\pm2\epsilon_d,\pm(\epsilon_i\pm\epsilon_j),1\leq i\neq j\leq d,\pm(\delta\pm\epsilon_s),1\leq s\leq d\}$.

We fix $1\leq k<d$. We consider the usual immersion of $H:=Sp(1,k)\times Sp(d-k)$ into $Sp(1,d)$.

Thus, $\Phi(\mathfrak{h},\mathfrak{t}):=\{\pm2\delta,\pm2\epsilon_1,\dots,\pm2\epsilon_d,\pm(\epsilon_i\pm\epsilon_j),1\leq i\neq j\leq k\,\text{or}\,k+1\leq i\neq j\leq d,\pm(\delta\pm\epsilon_s),1\leq s\leq k\}$.

Then, H_0 is isomorphic to $Sp(1,d-k)\times Sp(k)$. We have

$\Phi(\mathfrak{h}_0,\mathfrak{t}):=\{\pm2\delta,\pm2\epsilon_1,\dots,\pm2\epsilon_d,\pm(\epsilon_i\pm\epsilon_j),k+1\leq i\neq j\leq d\,\text{or}\,1\leq i\neq j\leq k,\pm(\delta\pm\epsilon_s),k+1\leq s\leq d\}$.

From now on, we fix the quaternionic system of positive roots $\Psi:=\{2\delta,2\epsilon_1,\dots,2\epsilon_d,(\epsilon_i\pm\epsilon_j),1\leq i<j\leq d,(\delta\pm\epsilon_s),1\leq s\leq d\}$. Here, $\alpha_{max}=2\delta$, $\rho_n^\Psi=d\delta$. The Harish-Chandra parameter λ of a quaternionic representation π_λ is dominant with respect to Ψ. Whence, $\Psi_\lambda=\Psi$. The systems in Theorem 3.10 are $\Psi_{H,\lambda}=\Phi(\mathfrak{h},\mathfrak{t})\cap\Psi$, $\Psi_{H_0,\lambda}=\Phi(\mathfrak{h}_0,\mathfrak{t})\cap\Psi$. Also, [5], $\Phi(\mathfrak{k}_1:=\mathfrak{k}_1(\Psi),\mathfrak{t}_1:=\mathfrak{t}\cap\mathfrak{k}_1)=\{\pm2\delta\}$, $\Phi(\mathfrak{k}_2:=\mathfrak{k}_2(\Psi),\mathfrak{t}_2:=\mathfrak{t}\cap\mathfrak{k}_2)=\{\pm2\epsilon_1,\dots,\pm2\epsilon_d,\pm(\epsilon_i\pm\epsilon_j),1\leq i\neq j\leq d\}$. Thus, $K_1(\Psi)\equiv SU_2(2\delta)\equiv Sp(1)\subset H$, $K_2\equiv Sp(d)$. Hence, for a Harish-Chandra parameter $\lambda=(\lambda_1,\lambda_2),\lambda_j\in i\mathfrak{t}_j^\star$ dominant with respect to Ψ, the representation π_λ is H-admissible.

The lowest K-type of a generalized quaternionic representation π_λ is the representation $\tau = \pi^K_{\lambda+\rho_n^\lambda} = \pi^{K_1}_{\lambda_1+d\delta} \boxtimes \pi^{K_2}_{\lambda_2}$. Since, $\rho_{K_2} = d\epsilon_1 + (d-1)\epsilon_2 + \cdots + \epsilon_d$, for $n \geq 2d+1$, the functional $\mathfrak{t}^\star \ni \lambda_n := n\delta + \rho_{K_2}$ is a Harish-Chandra parameter dominant with respect to Ψ and the lowest K-type τ_n of π_{λ_n} is $\pi^{K_1}_{(n+d)\delta} \boxtimes \pi^{K_2}_{\rho_{K_2}}$. That is, $\pi^K_{\lambda_n+\rho_n^\lambda}$ is equal to a irreducible representation of $K_1 \equiv Sp(1) = SU(2\delta)$ times the trivial representation of $K_2 \equiv Sp(d)$. The family $(\pi_{\lambda_n})_n$ exhausts, up to equivalence, the set of quaternionic representations for $Sp(1,d)$. Now, $\mathbf{H}^2(H_0, \tau_n)$ is the irreducible representation of lowest L-type equal to the irreducible representation $\pi^{K_1}_{(n+d)\delta}$ of K_1 times the trivial representation of $K_2 \cap L$. Actually, $\mathbf{H}^2(H_0, \pi^{K_1}_{(n+d)\delta} \boxtimes \pi^{K_2}_{\rho_{K_2}})$ is a realization of the quaternionic representation $H^2(Sp(1,n-k), \pi^{Sp(1)}_{(n+d)\delta} \boxtimes \pi^{Sp(n-k)}_{\rho_{Sp(n-k)}})$ for $Sp(1,d-k)$ times the trivial representation of $Sp(k)$. In [8, Proposition 6.3] it is shown that the representation $H^2(Sp(1,n-k), \pi^{Sp(1)}_{(n+d)\delta} \boxtimes \pi^{Sp(n-k)}_{\rho_{Sp(n-k)}})$ restricted to L is a multiplicity free representation as well as it is computed the highest weight of the totality of L-irreducible factors. To follow we explicit such a computation. For this we recall 5.2.1 and notice $b = d-k$; $\Lambda_1 = \delta$, $\beta_{max} = 2\delta$, $\tilde{\Lambda}_1 = \epsilon_1$; as $Sp(1)$-module, $S^p(\mathbb{C}^2) \equiv \pi^{SU(2\delta)}_{(p+1)\delta}$; for $p \geq 1$, as $Sp(p)$-module $S^m(\mathbb{C}^{2p}) \equiv \pi^{Sp(p)}_{m\epsilon_1+\rho_{Sp(p)}}$. Then,
the irreducible $L = Sp(1) \times Sp(d-k) \times Sp(k)$-factors of

$$\mathbf{H}^2(H_0, \pi^{K_1}_{(n+d)\delta} \boxtimes \pi^{K_2}_{\rho_{K_2}}) \equiv H^2(Sp(1,d-k), \pi^{Sp(1)}_{(n+d)\delta} \boxtimes \pi^{Sp(d-k)}_{\rho_{Sp(d-k)}}) \boxtimes \mathbb{C}.$$

are multiplicity free and it is the set of inequivalent representations

$$\begin{aligned}\{S^{n+d-1+m}(\mathbb{C}^2) \boxtimes S^m(\mathbb{C}^{2(d-k)}) \boxtimes \mathbb{C} \\ \equiv \pi^{Sp(1)}_{(n+d+m)\delta} \boxtimes \pi^{Sp(d-k)}_{m\epsilon_1+\rho_{Sp(d-k)}} \boxtimes \pi^{Sp(k)}_{\rho_{Sp(k)}}, \; m \geq 0\}.\end{aligned}$$

Here, $\rho_{Sp(d-k)} = (d-k)\epsilon_{k+1} + (d-k-1)\epsilon_{k+2} + \cdots + \epsilon_d$ and $\rho_{Sp(k)} = k\epsilon_1 + (k-1)\epsilon_2 + \cdots + \epsilon_k$.

We compute $\Psi_{H,\lambda} = \{2\delta, 2\epsilon_1, \ldots, 2\epsilon_d, (\epsilon_i \pm \epsilon_j), 1 \leq i \neq j \leq k \,\text{or}\, k+1 \leq i \neq j \leq d, (\delta \pm \epsilon_s), 1 \leq s \leq k\}$. $\rho_n^\mu = \rho_n^H = k\delta$. Now, from Theorem 3.1 we have $Spec_H(\pi_\lambda) + \rho_n^H = Spec_L(\mathbf{H}^2(H_0, \tau))$, whence, we conclude:

The representation $res_{Sp(1,k)\times Sp(d-k)}(\pi^{Sp(1,d)}_{\lambda_n})$ is a multiplicity free representation and the totality of Harish-Chandra parameters of the $Sp(1,k) \times Sp(d-k)$-irreducible factors is the set

$$\begin{aligned}\{(n+d+m)\delta + m\epsilon_1 + \rho_{Sp(k)+\rho_{Sp(d-k)}} - \rho_n^H = \\ (n+d+m-k)\delta + m\epsilon_1 + (d-k)\epsilon_{k+1} + \cdots + \epsilon_d + k\epsilon_1 + \cdots + \epsilon_k, m \geq 0\}.\end{aligned}$$

Whence, $res_{Sp(1,k)\times Sp(d-k)}(\pi^{Sp(1,d)}_{\lambda_n})$ is equivalent to the Hilbert sum

$$\oplus_{m\geq 0} V^{Sp(1,k)\times Sp(d-k)}_{(n+d+m-k)\delta+m\epsilon_1+\rho_{Sp(k)}+\rho_{Sp(d-k)}}$$
$$\equiv \oplus_{m\geq 0} H^2(Sp(1,k)\times Sp(d-k), \pi^{Sp(1)\times Sp(k)\times Sp(d-k)}_{(n+d+m)\delta+m\epsilon_1+\rho_{Sp(k)}+\rho_{Sp(d-k)}}).$$

A awkward point of our decomposition is that not provide a explicit description of the H-isotypic components for $res_H(V^G_\lambda)$.

5.2.3. *Explicit example II.* We restrict from $Spin(2m,2), m\geq 2$, to $Spin(2m,1)$. We notice the isomorphism between $(Spin(4,2), Spin(1,1))$ and the pair $(SU(2,2), Sp(1,1))$. In this setting $K = Spin(2m)\times Z_K$, $L = Spin(2m)$, $Z_K\equiv \mathbb{T}$. Obviously, we may conclude that any irreducible representation of K is irreducible when restricted to L. In this case $H_0\equiv Spin(2m,1)$, and (for $m=2$, $H_0\equiv Sp(1,1)$) and $\mathbf{H}^2(H_0,\tau)$ is irreducible. Therefore, the duality theorem together with that any irreducible representation for $Spin(2m,1)$ is L-multiplicity free, we obtain:

Any $Spin(2m,1)$-admissible representation $(\pi^{Spin(2m,2)}_\lambda, V^{Spin(2m,2)}_\lambda)$ is multiplicity free.

For $(Spin(2m,2), Spin(2m,1))$ in [30, Table 2], [19] it is verified that any π_λ, with λ dominant with respect to one of the systems $\Psi_{\pm m}$ (see proof of 4.7) has admissible restriction to $Spin(2m,1)$ and no other π_λ has admissible restriction to $Spin(2m,1)$.

In [30, Table 2] [15] [16] it is verified that any square integrable representation π_λ with λ dominant with respect to a quaternionic system for $SU(2,2)$, has admissible restriction to $Sp(1,1)$. As in 5.2.2, we may compute the Harish-Chandra parameters for the irreducible components of $res_{Sp(1,1)}(\pi^{SU(2,2)}_\lambda)$.

5.2.4. *Explicit example III.* To follow, G is so that its Lie algebra is $\mathfrak{sp}(m,n)$, $n\geq 2, m>1$. The aim of this example is twofold. One is to produce Discrete Series representations so that the lowest K-type restricted to $K_1(\Psi)$ is still irreducible and secondly to produce another multiplicity free examples. Here, $\mathfrak{k}=\mathfrak{sp}(m)+\mathfrak{sp}(n)$. We fix maximal torus $T\subset K$ and describe the root system as in [30]. For the system of positive roots $\Psi := \{\epsilon_i\pm\epsilon_j, i<j, \delta_r\pm\delta_s, r<s, \epsilon_a\pm\delta_b, 1\leq a,i,j\leq m, 1\leq b,r,s\leq n\}$, we have $K_1(\Psi)=K_1\equiv Sp(m), K_2(\Psi)=K_2\equiv Sp(n)$. Obviously, there exists a system of positive roots $\tilde{\Psi}$ so that $K_1(\tilde{\Psi})\equiv Sp(n), K_2(\tilde{\Psi})\equiv Sp(m)$. For any other system of positive roots in $\Phi(\mathfrak{g},\mathfrak{t})$ we have that the associated subgroup K_1 is equal to K.

It readily follows that $\lambda := \sum_{1\leq j\leq m} a_j\epsilon_j + \rho_{K_2}$ is a Ψ-dominant Harish-Chandra parameter when the coefficients a_j are all integers so that $a_1 > \cdots > a_m >> 0$. Since ρ_n^λ belongs to $span_{\mathbb{C}}\{e_1, \ldots, e_m\}$, it follows that the lowest K type of π_λ is equivalent to a irreducible representation for $Sp(m)$ times the trivial representation for $Sp(n)$. Next, we consider $\mathfrak{h} = \mathfrak{sp}(m, n-1) + \mathfrak{sp}(1)$ in the usual embedding. Here, $\mathfrak{h}_0 \equiv \mathfrak{sp}(m, 1) + \mathfrak{sp}(n-1)$. Whence, after we proceed as in *Explicit example I* we may conclude $res_{Sp(m,n-1)\times Sp(1)}(\pi_\lambda^{Sp(m,n)})$ is a multiplicity free representation and we may compute the Harish-Chandra parameters of each $Sp(m, n-1) \times Sp(1)$-irreducible factor for π_λ.

5.2.5. *Explicit example IV.* $(\mathfrak{e}_{6(2)}, \mathfrak{f}_{4(4)})$. We fix a compact Cartan subgroup $T \subset K$ so that $U := T \cap H$ is a compact Cartan subgroup of $L = K \cap H$. Then, there exist a quaternionic and Borel de Siebenthal positive root system Ψ_{BS} for $\Phi(\mathfrak{e}_6, \mathfrak{t})$ so that, after we write the simple roots as in Bourbaki (see [8][30]), the compact simple roots are $\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ (They determinate the A_5-Dynkin sub-diagram) and α_2 is noncompact. α_2 is adjacent to α_{max} and to α_4. In [30], it is verified Ψ_{BS} is the unique system of positive roots such that $\mathfrak{k}_1(\Psi_{BS}) = \mathfrak{su}_2(\alpha_{max})$.

The automorphism σ of $\mathfrak{g}$ acts on the simple roots as follows

$$\sigma(\alpha_2) = \alpha_2,\ \sigma(\alpha_1) = \alpha_6,\ \sigma(\alpha_3) = \alpha_5,\ \sigma(\alpha_4) = \alpha_4.$$

Hence, $\sigma(\Psi_{BS}) = \Psi_{BS}$. Let $h_2 \in i\mathfrak{t}^\star$ be so that $\alpha_j(h_2) = \delta_{j2}$ for $j = 1, \ldots, 6$. Then, $h_2 = \frac{2H_{\alpha_m}}{(\alpha_m,\alpha_m)}$ and $\theta = Ad(exp(\pi i h_2))$. A straightforward computation yields: $\mathfrak{k} \equiv \mathfrak{su}_2(\alpha_{max}) + \mathfrak{sp}(3)$, $\mathfrak{l} \equiv \mathfrak{su}_2(\alpha_{max}) + \mathfrak{sp}(1) + \mathfrak{sp}(2)$; the fix point subalgebra for $\theta\sigma$ is isomorphic to $\mathfrak{sp}(1, 3)$. Thus, the pair $(\mathfrak{e}_{6(2)}, \mathfrak{sp}(1, 3))$ is the associated pair to $(\mathfrak{e}_{6(2)}, \mathfrak{f}_{4(4)})$. Let $q_{\mathfrak{u}}$ denote the restriction map from $\mathfrak{t}^\star$ into $\mathfrak{u}^\star$. Then, then, for λ dominant with respect to Ψ_{BS}, the simple roots for $\Psi_{H,\lambda} = \Psi_{\mathfrak{f}_{4(4)},\lambda}$, respectively $\Psi_{\mathfrak{sp}(1,3),\lambda}$, are:

$$\alpha_2,\ \alpha_4,\ q_{\mathfrak{u}}(\alpha_3) = q_{\mathfrak{u}}(\alpha_5),\ q_{\mathfrak{u}}(\alpha_1) = q_{\mathfrak{u}}(\alpha_6).$$
$$\beta_1 = q_{\mathfrak{u}}(\alpha_2 + \alpha_4 + \alpha_5) = q_{\mathfrak{u}}(\alpha_2 + \alpha_4 + \alpha_3),\ \beta_2 = q_{\mathfrak{u}}(\alpha_1) = q_{\mathfrak{u}}(\alpha_6),$$
$$\beta_3 = q_{\mathfrak{u}}(\alpha_3) = q_{\mathfrak{u}}(\alpha_5),\ \beta_4 = \alpha_4.$$

The fundamental weight $\tilde{\Lambda}_1$ associated to β_1 is equal to $\frac{1}{2}\beta_{max}$. Hence, $\tilde{\Lambda}_1 = \beta_1 + \beta_2 + \beta_3 + \frac{1}{2}\beta_4 = \alpha_2 + \frac{3}{2}\alpha_4 + \alpha_3 + \alpha_5 + \frac{1}{2}(\alpha_1 + \alpha_6)$.

Thus, from the Duality Theorem, for the quaternionic representation

$$H^2(E_{6(2)}, \pi^{SU_2(\alpha_{max})\times SU(6)}_{n\frac{\alpha_{max}}{2}+\rho_{SU(6)}})$$

the set of Harish-Chandra parameters of the irreducible $F_{4(4)}$-factors is equal to:

$-\rho_n^H$ plus the set of infinitesimal characters of the $L \equiv SU(\alpha_{max}) \times Sp(3)$-irreducible factors for

$$res_{SU_2(\alpha_{max})\times Sp(3)}(H^2(Sp(1,3), \pi^{SU_2(\alpha_{max})\times Sp(3)}_{n\frac{\alpha_{max}}{2}+\rho_{Sp(3)}})).$$

Here, $-\rho_n^H = -d_H\frac{\alpha_{max}}{2}$, $d_H = d_{\mathfrak{f}_{4(4)}} = 7$ (see [9]).

Therefore, from the computation in 5.2.1, we obtain:

$$res_{F_{4(4)}}(\pi^{E_{6(2)}}_{n\frac{\alpha_{max}}{2}+\rho_{SU(6)}}) = \oplus_{m\geq 0}\ V^{F_{4(4)}}_{(n-7+m)\frac{\alpha_{max}}{2}+m\tilde{\Lambda}_1+\rho_{Sp(3)}}.$$

Here, $\rho_{Sp(3)} = 3\beta_2 + 5\beta_3 + 3\beta_4 = \frac{3}{2}(\alpha_5+\alpha_3) + \frac{5}{2}(\alpha_1+\alpha_6) + 3\alpha_4$.

5.2.6. *Comments on admissible restriction of quaternionic representations.* As usual (G,H) is a symmetric pair and $(\pi_\lambda, H^2(G,\tau))$ a H-admissible, non-holomorphic, square integrable representation. We further assume G/K holds a quaternionic structure. Then, from Tables 1,2,3 it follows:

a) λ is dominant with respect to a quaternionic system of positive roots. That is, π_λ is a generalized quaternionic representation.

b) H/L has a quaternionic structure.

c) Each system $\Psi_{H,\lambda}$, $\Psi_{H_0,\lambda}$ is a quaternionic system.

d) The representation $\mathbf{H}^2(H_0,\tau)$ is a sum of generalized quaternionic rep's.

e) When π_λ is quaternionic, then the representation $\mathbf{H}^2(H_0,\tau)$ is equal to $H^2(H_0, res_L(\tau))$, hence, it is quaternionic. Moreover, in [8], it is computed the highest weight and the respective multiplicity of each of its L-irreducible factors.

f) Thus, the duality Theorem 3.1 together with a)—e) let us compute the Harish-Chandra parameters of the irreducible H-factors for a quaternionic representation π_λ. Actually, the computation of the Harish-Chandra parameters is quite similar to the computation in *Explicit example I, Explicit example IV.*

To follows we consider particular quaternionic symmetric pairs. One pair is $(\mathfrak{f}_{4(4)}, \mathfrak{so}(5,4))$. Here, $\mathfrak{h}_0 \equiv \mathfrak{sp}(1,2)+\mathfrak{su}(2)$. Thus, for any Harish-Chandra parameter λ dominant with respect to the quaternionic system of positive roots, we have π_λ restricted to $SO(5,4)$ is a admissible representation and the Duality theorem let us compute either multiplicities or Harish-Chandra parameters of the restriction. Moreover, since quaternionic Discrete Series for $Sp(1,2)\times SU(2)$ are multiplicity free, [8], we have that quaternionic Discrete Series for $\mathfrak{f}_{4(4)}$, restricted to $SO(5,4)$ are multiplicity free. It seems that it can be deduced from the branching rules for the pair $(Sp(3), Sp(1) \times Sp(2))$ that a generalized quaternionic representation, $res_{SO(5,4)}(\pi_\lambda)$ is multiplicity free if and only π_λ is quaternionic.

For the pair $(\mathfrak{f}_{4(4)}, \mathfrak{so}(5,4))$, if we attempt to deduce our decomposition result from the work of [9], we have to consider the group of Lie algebra $\mathfrak{g}^d \equiv \mathfrak{f}_{4(-20)}$, its maximal compactly embedded subalgebra is isomorphic to $\mathfrak{so}(9)$, a simple Lie algebra, hence no Discrete Series for G^d has admissible restriction to H_0 (see [18] [5]). Thus, it is not clear to us how to deduce our Duality result from the Duality Theorem in [8].

For the pairs $(\mathfrak{e}_{6(-14)}, \mathfrak{su}(2,4)+\mathfrak{su}(2))$, $(\mathfrak{e}_{6(2)}, \mathfrak{so}(6,4)+\mathfrak{so}(2))$, $(\mathfrak{e}_{7(-5)}, \mathfrak{e}_{6(2)}+\mathfrak{so}(2))$, for each G, generalized quaternionic representations do exist and they are H-admissible. For these pairs, the respective $\mathfrak{h}_0$ are: $\mathfrak{su}(2,4) + \mathfrak{su}(2), \mathfrak{su}(2,4) + \mathfrak{su}(2), \mathfrak{su}(6,2)$. In these three cases, the Maple soft developed by Silva-Vergne[2], allows to compute the L-Harish-Chandra parameters and respective multiplicity for each Discrete Series for $H_0 \equiv SU(p,q) \times SU(r)$, hence, the duality formula yields the Harish-Parameters for $res_H(\pi_\lambda)$ and their multiplicity.

5.2.7. *Explicit example V.* The pair $(SO(2m,n), SO(2m,n-1))$. This pair is considered in [9]. We recall their result and we sketch how to derive the result from our duality Theorem. We only consider the case $\mathfrak{g} = \mathfrak{so}(2m, 2n+1)$. Here, $\mathfrak{k} = \mathfrak{so}(2m)+\mathfrak{so}(2n+1)$, $\mathfrak{h} = \mathfrak{so}(2m,2n)$, $\mathfrak{h}_0 = \mathfrak{so}(2m,1) + \mathfrak{so}(2n)$, $\mathfrak{l} = \mathfrak{so}(2m) + \mathfrak{so}(2n)$. We fix a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{l} \subset \mathfrak{k}$. Then, there exists a orthogonal basis $\epsilon_1, \dots, \epsilon_m, \delta_1, \dots, \delta_n$ for $i\mathfrak{t}^\star$ so that

$$\Delta = \{(\epsilon_i \pm \epsilon_j), 1 \le i < j \le m, (\delta_r \pm \delta_s), 1 \le r < s \le n\} \cup \{\delta_j\}_{1 \le j \le m}.$$

$$\Phi_n = \{\pm(\epsilon_r \pm \delta_s), r = 1, \dots, m, s = 1, \dots, n\} \cup \{\pm\epsilon_j, j = 1, \dots, m\}.$$

The systems of positive roots Ψ_λ so that π_λ^G is an admissible representation of H are the systems $\Psi_\pm$ associated to the lexicographic orders $\epsilon_1 > \cdots > \epsilon_m > \delta_1 > \cdots > \delta_n$, $\epsilon_1 > \cdots > \epsilon_{m-1} > -\epsilon_m > \delta_1 > \cdots > \delta_{n-1} > -\delta_n$. Here, for $m \ge 3$, $\mathfrak{k}_1(\Psi_\pm) = \mathfrak{so}(2m)$. For $m = 2, \mathfrak{k}_1(\Psi_\pm) = \mathfrak{su}_2(\epsilon_1 \pm \epsilon_2)$. Then,

$\Psi_{H,+} = \{(\epsilon_i \pm \epsilon_j), 1 \le i < j \le m, (\delta_r \pm \delta_s), 1 \le r < s \le n\} \cup \{(\epsilon_r \pm \delta_s), r = 1, \dots, m, s = 1, \dots, n\}$,

$\Psi_{H_0,+} = \{(\epsilon_i \pm \epsilon_j), 1 \le i < j \le m, (\delta_r \pm \delta_s), 1 \le r < s \le n\} \cup \{\epsilon_j, j = 1, \dots, m\}$.

$\mathfrak{g}^d = \mathfrak{so}(2m+2n, 1)$. Thus, from either our duality Theorem or from [9], we infer that whenever $res_H(\pi_\lambda)$ is H-admissible, then, $res_H(\pi_\lambda)$

is a multiplicity free representation. Whence, we are left to compute the Harish-Chandra parameters for $res_{SO(2m,2n)}(H^2(SO(2m,2n+1),\pi_{\Lambda_1}^{SO(2m)}\boxtimes\pi_{\Lambda_2}^{SO(2m+1)}))$. For this, according to the duality Theorem, we have to compute the infinitesimal characters of each irreducible factor of the underlying L-module in
$\mathbf{H}^2(H_0,\tau)=\sum_{\nu\in Spec_{SO(2m)}(\pi_{\Lambda_2}^{SO(2m+1)})} H^2(SO(2m,1),\pi_{\Lambda_1}^{SO(2m)})\boxtimes V_\nu^{SO(2m)}$.
The branching rules for $res_{SO(2m)}(H^2(SO(2m,1),\pi_{\Lambda_1}^{SO(2m)})$ are found in [29] and other references, the branching rule for $res_{SO(2m)}(\pi_{\Lambda_2}^{SO(2m+1)})$ can be found in [29]. From both computations, we deduce: [9, Proposition 3], for $\lambda=\sum_{1\le i\le m}\lambda_i\epsilon_i+\sum_{1\le j\le n}\lambda_{m+j}\delta_j$, then V_μ^H is a H-subrepresentation of $H^2(G,\tau)\equiv V_\lambda^{SO(2m,2n+1)}$ ($\mu=\sum_{1\le i\le m}\mu_i\epsilon_i+\sum_{1\le j\le n}\mu_{j+m}\delta_j$) if and only if

$$\mu_1>\lambda_1>\cdots>\mu_m>\lambda_m,\lambda_{m+1}>\mu_{m+1}>\ldots\lambda_{m+n}>|\mu_{m+n}|.$$

5.3. **Existence of Discrete Series whose lowest K-type restricted to $K_1(\Psi)$ is irreducible.** Let G a semisimple Lie group that admits square integrable representations. This hypothesis allows to fix a compact Cartan subgroup $T\subset K$ of G. In [5] it is defined for each system of positive roots $\Psi\subset\Phi(\mathfrak{g},\mathfrak{t})$ a normal subgroup $K_1(\Psi)\subset K$ so that for a symmetric pair (G,H), with H a θ-invariant subgroup, it holds: for any Harish-Chandra parameter dominant with respect to Ψ, the representation $res_H(\pi_\lambda)$ is H-admissible if and only if $K_1(\Psi)$ is a subgroup of H. For a holomorphic system Ψ, $K_1(\Psi)$ is equal to the center of K; for a quaternionic system of positive roots $K_1(\Psi)\equiv SU_2(\alpha_{max})$. Either for the holomorphic family or for a quaternionic real forms we find that among the H-admissible Discrete Series for G, there are many examples of the following nature: the lowest K-type of π_λ is equal to a irreducible representation of $K_1(\Psi)$ tensor with the trivial representation for K_2, [9]. To follow, under the general setting at the beginning of this paragraph, we verify.

5.3.1. *For each system of positive roots $\Psi\subset\Phi(\mathfrak{g},\mathfrak{t})$, there exists Discrete Series with Harish-Chandra parameter dominant with respect to Ψ and so that its lowest K-type is equal to a irreducible representation of $K_1(\Psi)$ tensor with the trivial representation for $K_2(\Psi)$.*

We may assume $K_1(\Psi)$ is a proper subgroup of K. Then, when $K_1(\Psi)=Z_K$, Harish-Chandra showed there exists such a representation. For G a quaternionic real form, Ψ a quaternionic system of positive roots, $K_1(\Psi)=SU_2(\alpha_{max})$, then, in [9] we find a proof of the statement. From the tables in [5][30], we are left to consider the triples $(G,K,K_1(\Psi))$ so that their respective Lie algebras is the triple

$(\mathfrak{su}(m,n), \mathfrak{su}(m)+\mathfrak{su}(n)+\mathfrak{u}(1), \mathfrak{su}(m)), m > 2,$
$(\mathfrak{sp}(m,n), \mathfrak{sp}(m)+\mathfrak{sp}(n), \mathfrak{sp}(m)).$
$(\mathfrak{so}(2m,n), \mathfrak{so}(2m)+\mathfrak{so}(n), \mathfrak{so}(2m)).$

In *Explicit example III* we already analyzed the second triple of the list. With the same proof it is verified that the statement holds for the third triple. For the first triple, we further assume $G = SU(p,q)$. Thus, K is the product of two simply connected subgroups times a one dimensional torus Z_K, we notice $\rho_n^{\Psi_a} = \rho_{\mathfrak{g}}^{\lambda} - \rho_K$, hence, $\rho_n^{\Psi_a}$ lifts to a character of K. Thus, as in Explicit example III, we obtain π_λ with λ dominant with respect to Ψ_a so that its lowest $K = SU(p)SU(q)Z_K$-type is the tensor product of a irreducible representation for $SU(p)Z_K$ times the trivial representation for $SU(q)$. Since $\rho_n^{\Psi_a}$ lifts to a character of K, after some computation the claim follows.

6. Symmetric breaking operators and normal derivatives

For this subsection (G, H) is a symmetric pair and π_λ is a square integrable representation. Our aim is to generalize a result in [22, Theorem 5.1]. In [20] it is considered symmetry breaking operators expressed by means of normal derivatives, they obtain results for holomorphic embedding of a rank one symmetric pairs. As before, $H_0 = G^{\sigma\theta}$ is the dual subgroup. We recall $\mathfrak{h} \cap \mathfrak{p}$ is orthogonal to $\mathfrak{h}_0 \cap \mathfrak{p}$ and that $\mathfrak{h} \cap \mathfrak{p} \equiv T_{eL}(H/L)$, $\mathfrak{h}_0 \cap \mathfrak{p} \equiv T_{eL}(H_0/L)$. Hence, for $X \in \mathfrak{h}_0 \cap \mathfrak{p}$, more generally for $X \in \mathcal{U}(\mathfrak{h}_0)$, we say L_X is a normal derivative to H/K differential operator. For short, *normal derivative.* Other ingredient necessary for the next Proposition are the subspaces $\mathcal{L}_\lambda$ and $\mathcal{U}(\mathfrak{h}_0)W$. The later subspace is contained in the subspace of K-finite vectors, whereas, the former subspace, it is believed, that when $res_H(\pi_\lambda)$ is not discretely decomposable it is disjoint to the subspace of G-smooth vectors. When, $res_H(\pi_\lambda)$ is H-admissible $\mathcal{L}_\lambda$ is contained in the subspace of K-finite vectors. However, it might not be equal to $\mathcal{U}(\mathfrak{h}_0)W$ as we have pointed out. The next Proposition and its converse, dealt with consequences of the equality $\mathcal{L}_\lambda = \mathcal{U}(\mathfrak{h}_0)W$.

Proposition 6.1. *We assume (G, H) is a symmetric pair. We also assume there exists a irreducible representation (σ, Z) of L so that $H^2(H,\sigma)$ is a irreducible factor of $H^2(G,\tau)$ and $H^2(G,\tau)[H^2(H,\sigma)][Z] = \mathcal{L}_\lambda[Z] = \mathcal{U}(\mathfrak{h}_0)W[Z] = L_{\mathcal{U}(\mathfrak{h}_0)}(H^2(G,\tau)[W])[Z]$. Then, $res_H(\pi_\lambda)$ is H-admissible. Moreover, any symmetry breaking operator from $H^2(G,\tau)$ into $H^2(H,\sigma)$ is represented by a normal derivative differential operator.*

We show a converse to Proposition 6.1 in 6.0.1.

Proof. To begin with we recall $H^2(G,\tau)[W] = \{K_\lambda(\cdot,e)^\star w, w \in W\}$ is a subspace of $H^2(G,\tau)_{K-fin}$, whence $L_{\mathcal{U}(\mathfrak{h}_0)}(H^2(G,\tau)[W])[Z]$ is a subspace of $H^2(G,\tau)_{K-fin}$. Owing to our hypothesis we then have $\mathcal{L}_\lambda[Z]$ is a subspace of $H^2(G,\tau)_{K-fin}$. Next, we quote a result of Harish-Chandra: a $U(\mathfrak{h})-$finitely generated, $\mathfrak{z}(U(\mathfrak{h}))-$finite, module has a finite composition series. Thus, $H^2(G,\tau)_{K-fin}$ contains an irreducible $(\mathfrak{h}, L)$-submodule. For a proof (cf. [32, Corollary 3.4.7 and Theorem 4.2.1]). Now, in [15, Lemma 1.5] we find a proof of: if a $(\mathfrak{g}, K)-$module contains an irreducible $(\mathfrak{h}, L)-$submodule, then the $(\mathfrak{g}, K)-$module is $\mathfrak{h}-$algebraically decomposable. Thus, $res_H(\pi_\lambda)$ is algebraically discretely decomposable. In [16, Theorem 4.2], it is shown that under the hypothesis (G,H) is a symmetric pair, for Discrete Series, $\mathfrak{h}$-algebraically discrete decomposable is equivalent to H-admissibility, whence $res_H(\pi_\lambda)$ is H-admissible. Let $S : H^2(G,\tau) \to H^2(H,\sigma) = V_\mu^H$ a continuous intertwining linear map. Then, we have shown in 3.1, for $z \in Z$, $K_S(\cdot,e)^\star z \in H^2(G,\tau)[V_\mu^H][Z]$. We fix a orthonormal basis $\{z_p\}$, $p = 1,\dots,dimZ$ for Z. The hypothesis

$$H^2(G,\tau)[V_\mu^H][Z] = L_{\mathcal{U}(\mathfrak{h}_0)}(H^2(G,\tau)[W])[Z]$$

implies for each p, there exists $D_p \in \mathcal{U}(\mathfrak{h}_0)$ and $w_p \in W$ so that $K_S(\cdot,e)^\star z_p = L_{D_p}K_\lambda(\cdot,e)^\star w_p$. Next, we fix $f_1 \in H^2(G,\tau)^\infty, h \in H$ and set $f := L_{h^{-1}}(f_1)$, then $f(e) = f_1(h)$. We have,

$$\begin{aligned}
(S(f)(e), z_p)_Z &= \int_G (f(y), K_S(y,e)^\star z_p)_W dy \\
&= \int_G (L_{D_p^\star} f(y), K_\lambda(y,e)^\star w_p)_W dy \\
&= (L_{D_p^\star} f(e), w_p)_W \\
&= (R_{\check{D}_p^\star} f(e), w_p)_W
\end{aligned} \tag{6.1}$$

Thus, for each $z \in Z$ and f_1 smooth vector we obtain $(S(f_1)(h), z)_Z = \sum_p (S(f_1)(h), (z,z_p)_Z z_p)_Z = (\sum_p (R_{\check{D}_p^\star} f_1(h), w_p)_W (z_p, z)_Z$. As in [25, Proof of Lemma 2] we conclude for any $f \in H^2(G,\tau)$ that

$$S(f)(h) = \sum_{1 \le p \le dimZ} (R_{\check{D}_p^\star} f(h), w_p)_W \ z_p \tag{6.2}$$

Since $D_p \in \mathcal{U}(\mathfrak{h}_0)$ such a expression of $S(f)$ is a representation in terms of normal derivatives. □

6.0.1. *Converse to Proposition 6.1.* We want to show: If every element in $Hom_H(H^2(G,\tau), H^2(H,\sigma))$ has a expression as differential operator by means of "normal derivatives", then, the equality $\mathcal{L}_\lambda[Z] = H^2(G,\tau)[H^2(H,\sigma)][Z] = \mathcal{U}(\mathfrak{h}_0)W[Z]$ holds.

In fact, the hypothesis $S(f)(h) = \sum_{1\leq p\leq dimZ}(R_{\check{D}_p^\star}f(h), w_p)_W\ z_p$, $D_p \in \mathcal{U}(\mathfrak{h}_0)$, yields $K_S(\cdot,e)^\star z = L_{D_z}K_\lambda(\cdot,e)^\star w_z$, $D_z \in \mathcal{U}(\mathfrak{h}_0)$, $w_z \in W$. The fact that (σ, Z) has multiplicity one in $H^2(H,\sigma)$ gives

$$dimHom_H(H^2(G,\tau), H^2(H,\sigma)) = dimH^2(G,\tau)[H^2(H,\sigma)][Z].$$

Hence, the functions

$$\{K_S(\cdot,e)^\star z, z\in Z, S\in Hom_H(H^2(G,\tau), H^2(H,\sigma))\}$$

span $H^2(G,\tau)[H^2(H,\sigma)][Z]$. Therefore, $H^2(G,\tau)[H^2(H,\sigma)][Z]$ is contained in $\mathcal{U}(\mathfrak{h}_0)W[Z] = L_{\mathcal{U}(\mathfrak{h}_0)}H^2(G,\tau)[W][Z]$. Owing to Theorem 3.1, both spaces have the same dimension, whence, the equality holds.

The pairs so that Proposition 6.1 holds for scalar holomorphic Discrete Series are $(\mathfrak{su}(m,n), \mathfrak{su}(m,l)+\mathfrak{su}(n-l)+\mathfrak{u}(1))$, $(\mathfrak{so}(2m,2), \mathfrak{u}(m,1))$, $(\mathfrak{so}^\star(2n), \mathfrak{u}(1,n-1))$, $(\mathfrak{so}^\star(2n), \mathfrak{so}(2)+\mathfrak{so}^\star(2n-2))$, $(\mathfrak{e}_{6(-14)}, \mathfrak{so}(2,8)+\mathfrak{so}(2))$. See [30, (4.6)].

6.0.2. *Comments on the interplay among the subspaces, $\mathcal{L}_\lambda$, $\mathcal{U}(\mathfrak{h}_0)W$, $H^2(G,\tau)_{K-fin}$ and symmetry breaking operators.* It readily follows that the subspace $\mathcal{L}_\lambda[Z] = V_\lambda^G[H^2(H,\sigma)][Z]$ is equal to the closure of the linear span of

$$\mathcal{K}_{Sy}(G,H) := \{K_{S^*}(e,\cdot)z = K_S(\cdot,e)^\star z, z\in Z, S\in Hom_H(V_\lambda^G, V_\mu^H)\}.$$

(1) $H^2(G,\tau)_{K-fin}\cap\mathcal{K}_{Sy}(G,H)$ is equal to the linear span of elements in $\mathcal{K}_{Sy}(G,H)$ so that the corresponding symmetry breaking operator is represented by a differential operator. See [25, Lemma 4.2].

(2) $\mathcal{U}(\mathfrak{h}_0)W\cap\mathcal{K}_{Sy}(G,H)$ is equal to the linear span corresponding to element K_S in $\mathcal{K}_{Sy}(G,H)$ so that S is represented by normal derivative differential, operator. This is shown in Proposition 6.1 and its converse.

(3) The set of symmetry breaking operators represented by a differential operator is not the null space if and only if $res_H(\pi_\lambda)$ is H-admissible. See [25, Theorem 4.3] and the proof of Proposition 6.1.

(4) We believe that from Nakahama's thesis, it is possible to construct examples of $V_\lambda^G[H^2(H,\sigma)][Z]\cap\mathcal{U}(\mathfrak{h}_0)W[Z] \neq \{0\}$, so that the equality $V_\lambda^G[H^2(H,\sigma)][Z] = \mathcal{U}(\mathfrak{h}_0)W[Z]$ does not hold! That is, there are symmetry breaking represented by plain differential operators and some of them are not represented by normal derivative operators.

6.0.3. *A functional equation for symmetry breaking operators.* Notation is as in Theorem 3.1. We assume (G,H) is a symmetric pair and $res_H(\pi_\lambda)$ is admissible. The objects involved in the equation are: $H_0 = G^{\sigma\theta}$, $Z = V^H_{\mu+\rho_n^H}$ the lowest L-type for V_μ^H, $\mathcal{L}_\lambda = \sum_\mu H^2(G,\tau)[V_\mu^H][V^L_{\mu+\rho_n^H}]$, $\mathcal{U}(\mathfrak{h}_0)W = L_{\mathcal{U}(\mathfrak{h}_0)}H^2(G,\tau)[W]$, L-isomorphism $D : \mathcal{L}_\lambda[Z] \to \mathcal{U}(\mathfrak{h}_0)W[Z]$, a H-equivariant continuous linear map $S : H^2(G,\tau) \to H^2(H,\sigma)$, the kernel $K_S : G\times H \to Hom_{\mathbb{C}}(W,Z)$ corresponding to S, 3.1 implies

$K_S(\cdot,e)^\star z \in \mathcal{L}_\lambda[Z]$, finally, we recall $K_\lambda : G \times G \to Hom_{\mathbb{C}}(W,W)$ the kernel associated to the orthogonal projector onto $H^2(G,\tau)$. Then,

Proposition 6.2. *For $z \in Z$, $y \in G$ we have*

$$D(K_S(e,\cdot)^\star(z))(y) = \int_{H_0} K_\lambda(h_0,y) D(K_S(e,\cdot)^\star(z))(h_0)dh_0$$

When, D is the identity map, the functional equation turns into

$$K_S(x,h) = \int_{H_0} K_S(h_0,e)K_\lambda(x,hh_0)dh_0$$

The functional equation follows from Proposition 4.12 applied to $T := S^\star$. The second equation follows after we compute the adjoint of the first equation.

We note, that as in the case of holographic operators, a symmetry breaking operator can be recovered from its restriction to H_0.

We also note that [22] has shown a different functional equation for K_S for scalar holomorphic Discrete Series and holomorphic embedding $H/L \to G/K$.

7. Tables

For an arbitrary symmetric pair (G,H), whenever π_λ^G is an admissible representation of H, we define,

$$K_1 = \begin{cases} Z_K & \text{if } \Psi_\lambda \text{ holomorphic} \\ K_1(\Psi_\lambda) & \text{otherwise} \end{cases}$$

In the next tables we present the 5-tuple so that: (G,H) is a symmetric pair, H_0 is the associated group to H, Ψ_λ is a system of positive such that π_λ^G is an admissible representation of H, and $K_1 = Z_1(\Psi_\lambda)K_1(\Psi_\lambda)$. Actually, instead of writing Lie groups we write their respective Lie algebras. Each table is in part a reproduction of tables in [18] [30]. The tables can also be computed by means of the techniques presented in [5]. Note that each table is "symmetric" when we replace H by H_0. As usual, α_m denotes the highest root in Ψ_λ. Unexplained notation is as in [30].

Table 1. Case $U = T$, Ψ_λ nonholomorphic

G	H	H_0	Ψ_λ	K_1
$\mathfrak{su}(m,n)$	$\mathfrak{su}(m,k) \oplus \mathfrak{su}(n-k) \oplus \mathfrak{u}(1)$	$\mathfrak{su}(m,n-k) \oplus \mathfrak{su}(k) \oplus \mathfrak{u}(1)$	Ψ_a	$\mathfrak{su}(m)$
$\mathfrak{su}(m,n)$	$\mathfrak{su}(k,n) \oplus \mathfrak{su}(m-k) \oplus \mathfrak{u}(1)$	$\mathfrak{su}(m-k,n) \oplus \mathfrak{su}(k) \oplus \mathfrak{u}(1)$	$\tilde{\Psi}_b$	$\mathfrak{su}(n)$
$\mathfrak{so}(2m,2n), m > 2$	$\mathfrak{so}(2m,2k) \oplus \mathfrak{so}(2n-2k)$	$\mathfrak{so}(2m,2n-2k) \oplus \mathfrak{so}(2k)$	$\Psi_\pm$	$\mathfrak{so}(2m)$
$\mathfrak{so}(4,2n)$	$\mathfrak{so}(4,2k) \oplus \mathfrak{so}(2n-2k)$	$\mathfrak{so}(4,2n-2k) \oplus \mathfrak{so}(2k)$	$\Psi_\pm$	$\mathfrak{su}_2(\alpha_m)$
$\mathfrak{so}(2m,2n+1), m > 2$	$\mathfrak{so}(2m,k) \oplus \mathfrak{so}(2n+1-k)$	$\mathfrak{so}(2m,2n+1-k) \oplus \mathfrak{so}(k)$	$\Psi_\pm$	$\mathfrak{so}(2m)$
$\mathfrak{so}(4,2n+1)$	$\mathfrak{so}(4,k) \oplus \mathfrak{so}(2n+1-k)$	$\mathfrak{so}(4,2n+1-k) \oplus \mathfrak{so}(k)$	$\Psi_\pm$	$\mathfrak{su}_2(\alpha_m)$
$\mathfrak{so}(4,2n), n > 2$	$\mathfrak{u}(2,n)_1$	$w\mathfrak{u}(2,n)_1$	$\Psi_{1\,-1}$	$\mathfrak{su}_2(\alpha_m)$
$\mathfrak{so}(4,2n), n > 2$	$\mathfrak{u}(2,n)_2$	$w\mathfrak{u}(2,n)_2$	$\Psi_{1\,1}$	$\mathfrak{su}_2(\alpha_m)$
$\mathfrak{so}(4,4)$	$\mathfrak{u}(2,2)_{1\,1}$	$w\mathfrak{u}(2,2)_{11}$	$\Psi_{1\,-1},\ w_{\epsilon,\delta}\Psi_{1\,-1}$	$\mathfrak{su}_2(\alpha_m)$
$\mathfrak{so}(4,4)$	$\mathfrak{u}(2,2)_{12}$	$w\mathfrak{u}(2,2)_{12}$	$\Psi_{1\,-1},\ w_{\epsilon,\delta}\Psi_{1\,1}$	$\mathfrak{su}_2(\alpha_m)$
$\mathfrak{so}(4,4)$	$\mathfrak{u}(2,2)_{21}$	$w\mathfrak{u}(2,2)_{21}$	$\Psi_{1\,1},\ w_{\epsilon,\delta}\Psi_{1\,-1}$	$\mathfrak{su}_2(\alpha_m)$
$\mathfrak{so}(4,4)$	$\mathfrak{u}(2,2)_{22}$	$w\mathfrak{u}(2,2)_{22}$	$\Psi_{1\,1},\ w_{\epsilon,\delta}\Psi_{1\,1}$	$\mathfrak{su}_2(\alpha_m)$
$\mathfrak{sp}(m,n)$	$\mathfrak{sp}(m,k) \oplus \mathfrak{sp}(n-k)$	$\mathfrak{sp}(m,n-k) \oplus \mathfrak{sp}(k)$	Ψ_+	$\mathfrak{sp}(m)$
$\mathfrak{f}_{4(4)}$	$\mathfrak{sp}(1,2) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(5,4)$	Ψ_{BS}	$\mathfrak{su}_2(\alpha_m)$
$\mathfrak{e}_{6(2)}$	$\mathfrak{so}(6,4) \oplus \mathfrak{so}(2)$	$\mathfrak{su}(4,2) \oplus \mathfrak{su}(2)$	Ψ_{BS}	$\mathfrak{su}_2(\alpha_m)$
$\mathfrak{e}_{7(-5)}$	$\mathfrak{so}(8,4) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(8,4) \oplus \mathfrak{su}(2)$	Ψ_{BS}	$\mathfrak{su}_2(\alpha_m)$
$\mathfrak{e}_{7(-5)}$	$\mathfrak{su}(6,2)$	$\mathfrak{e}_{6(2)} \oplus \mathfrak{so}(2)$	Ψ_{BS}	$\mathfrak{su}_2(\alpha_m)$
$\mathfrak{e}_{8(-24)}$	$\mathfrak{so}(12,4)$	$\mathfrak{e}_{7(-5)} \oplus \mathfrak{su}(2)$	Ψ_{BS}	$\mathfrak{su}_2(\alpha_m)$

Table 2, Case $U \neq T$, Ψ_λ non holomorphic

G	H	H_0	Ψ_λ	K_1
$\mathfrak{su}(2,2n),\ n>2$	$\mathfrak{sp}(1,n)$	$\mathfrak{sp}(1,n)$	Ψ_1	$\mathfrak{su}_2(\alpha_m)$
$\mathfrak{su}(2,2)$	$\mathfrak{sp}(1,1)$	$\mathfrak{sp}(1,1)$	Ψ_1	$\mathfrak{su}_2(\alpha_m)$
$\mathfrak{su}(2,2)$	$\mathfrak{sp}(1,1)$	$\mathfrak{sp}(1,1)$	$\tilde{\Psi}_1$	$\mathfrak{su}_2(\alpha_m)$
$\mathfrak{so}(2m,2n), m>2$	$\mathfrak{so}(2m,2k+1)+\mathfrak{so}(2n-2k-1)$	$\mathfrak{so}(2m,2n-2k-1)+\mathfrak{so}(2k+1)$	$\Psi_\pm$	$\mathfrak{so}(2m)$
$\mathfrak{so}(4,2n),$	$\mathfrak{so}(4,2k+1)+\mathfrak{so}(2n-2k-1)$	$\mathfrak{so}(4,2n-2k-1)+\mathfrak{so}(2k+1)$	$\Psi_\pm$	$\mathfrak{su}_2(\alpha_m)$
$\mathfrak{so}(2m,2), m>2$	$\mathfrak{so}(2m,1)$	$\mathfrak{so}(2m,1)$	$\Psi_\pm$	$\mathfrak{so}(2m)$
$\mathfrak{so}(4,2),$	$\mathfrak{so}(4,1)$	$\mathfrak{so}(4,1)$	$\Psi_\pm$	$\mathfrak{su}_2(\alpha_m)$
$\mathfrak{e}_{6(2)}$	$\mathfrak{f}_{4(4)}$	$\mathfrak{sp}(3,1)$	Ψ_{BS}	$\mathfrak{su}_2(\alpha_m)$

G	H (a)	H_0 (b)
$\mathfrak{su}(m,n), m \neq n$	$\mathfrak{su}(k,l)+\mathfrak{su}(m-k,n-l)+\mathfrak{u}(1)$	$\mathfrak{su}(k,n-l)+\mathfrak{su}(m-k,l)+\mathfrak{u}(1)$
$\mathfrak{su}(n,n)$	$\mathfrak{su}(k,l)+\mathfrak{su}(n-k,n-l)+\mathfrak{u}(1)$	$\mathfrak{su}(k,n-l)+\mathfrak{su}(n-k,l)+\mathfrak{u}(1)$
$\mathfrak{so}(2,2n)$	$\mathfrak{so}(2,2k)+\mathfrak{so}(2n-2k)$	$\mathfrak{so}(2,2n-2k)+\mathfrak{so}(2k)$
$\mathfrak{so}(2,2n)$	$\mathfrak{u}(1,n)$	$\mathfrak{u}(1,n)$
$\mathfrak{so}(2,2n+1)$	$\mathfrak{so}(2,k)+\mathfrak{so}(2n+1-k)$	$\mathfrak{so}(2,2n+1-k)+\mathfrak{so}(k)$
$\mathfrak{so}^\star(2n)$	$\mathfrak{u}(m,n-m)$	$\mathfrak{so}^\star(2m)+\mathfrak{so}^\star(2n-2m)$
$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{u}(m,n-m)$	$\mathfrak{sp}(m,\mathbb{R})+\mathfrak{sp}(n-m,\mathbb{R})$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(2,8)+\mathfrak{so}(2)$	$\mathfrak{so}(2,8)+\mathfrak{so}(2)$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{su}(2,4)+\mathfrak{su}(2)$	$\mathfrak{su}(2,4)+\mathfrak{su}(2)$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}^\star(10)+\mathfrak{so}(2)$	$\mathfrak{su}(5,1)+\mathfrak{sl}(2,\mathbb{R})$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{so}^\star(12)+\mathfrak{su}(2)$	$\mathfrak{su}(6,2)$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{so}(2,10)+\mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{e}_{6(-14)}+\mathfrak{so}(2)$
$\mathfrak{su}(n,n)$	$\mathfrak{so}^\star(2n)$	$\mathfrak{sp}(n,\mathbb{R})$
$\mathfrak{so}(2,2n)$	$\mathfrak{so}(2,2k+1)+\mathfrak{so}(2n-2k-1)$	$\mathfrak{so}(2,2n-2k-1)+\mathfrak{so}(2k+1)$

Table 3, π_λ^G holomorphic Discrete Series.
The last two lines show the unique holomorphic pairs so that $U \neq T$.

8. Partial list of symbols and definitions

- (τ, W), (σ, Z), $L^2(G\times_\tau W), L^2(H\times_\sigma Z)$ (cf. Section 2).
- $H^2(G,\tau)=V_\lambda=V_\lambda^G$, $H^2(H,\sigma)=V_\mu^H, \pi_\mu^H, \pi_\nu^K$. (cf. Section 2).
-$\pi_\lambda=\pi_\lambda^G$, $d_\lambda=d(\pi_\lambda)$ dimension of π_λ, $P_\lambda, P_\mu, K_\lambda, K_\mu$, (cf. Section 2).
-P_X orthogonal projector onto subspace X.
-$\Phi(x)=P_W\pi(x)P_W$ spherical function attached to the lowest K-type W of π_λ.
-$K_\lambda(y,x)=d(\pi_\lambda)\Phi(x^{-1}y)$.
-$M_{K-fin}(resp. M^\infty)$ $K-$finite vectors in M (resp. smooth vectors in M).
-dg, dh Haar measures on G, H.
-A unitary representation is *square integrable*, equivalently a *Discrete Series* representation, (resp. *integrable*) if some nonzero matrix coefficient is square integrable (resp. integrable) with respect to Haar measure.
-$\Theta_{\pi_\mu^H}(...)$ Harish-Chandra character of the representation π_μ^H.
-For a module M and a simple submodule N, $M[N]$ denotes the *isotypic component* of N in M. That is, $M[N]$ is the sum of all irreducible submodules isomorphic to N. If topology is involved, we define $M[N]$ to be the closure of $M[N]$.
-M_{H-disc} is the closure of the linear subspace spanned by the totality of $H-$irreducible submodules. $M_{disc} := M_{G-disc}$

-A representation M is $H-$*discretely decomposable* if $M_{H-disc}=M$.
-A representation is $H-$*admissible* if it is $H-$discretely decomposable and each isotypic component is equal to a finite sum of $H-$irreducible representations.
-$U(\mathfrak{g})$ (resp. $\mathfrak{z}(U(\mathfrak{g})=\mathfrak{z}_{\mathfrak{g}})$ universal enveloping algebra of the Lie algebra $\mathfrak{g}$(resp. center of universal enveloping algebra).
-Cl(X) =closure of the set X.
-I_X identity function on set X.
-$\mathbb{T}$ one dimensional torus.
-Z_S identity connected component of the center of the group S.
$S^{(r)}(V)$ the r^{th}-symmetric power of the vector space V.

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