# Generalized Buzano Inequality 

Tamara Bottazzi ${ }^{1}{ }^{1 a, b}$ and Cristian Conde ${ }^{1},{ }^{2}$


#### Abstract

If $P$ is an orthogonal projection defined on an inner product space $\mathcal{H}$, then the inequality $$
|\langle P x, y\rangle| \leq \frac{1}{2}[\|x\|\|y\|+|\langle x, y\rangle|]
$$ fulfills for any $x, y \in \mathcal{H}$ (see [10]). In particular, when $P$ is the identity operator, then it recovers the famous Buzano inequality. We obtain generalizations of such classical inequality, which hold for certain families of bounded linear operators defined on $\mathcal{H}$. In addition, several new inequalities involving the norm and numerical radius of an operator are established.


## 1. Introduction

Let $(\mathcal{H},\langle\cdot, \cdot\rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. The following inequality is well known in literature as the Cauchy-Schwarz inequality

$$
\begin{equation*}
|\langle x, y\rangle| \leq\|x\|\|y\|, \tag{1.1}
\end{equation*}
$$

for any $x, y \in \mathcal{H}$. The equality in (1.1) holds if and only if there exists a constant $\alpha \in \mathbb{K}$ such that $x=\alpha y$.

In [6], Maria Luisa Buzano gave the following extension of the celebrated CauchySchwarz inequality in $\mathcal{H}$

$$
\begin{equation*}
|\langle x, z\rangle\langle z, y\rangle| \leq \frac{1}{2}(|\langle x, y\rangle|+\|x\|\|y\|)\|z\|^{2}, \tag{1.2}
\end{equation*}
$$

for any $x, y, z \in \mathcal{H}$. Last inequality is called Buzano inequality.
The original proof of Buzano has it difficulty since it requires some facts about orthogonal decomposition of a complete inner product space.

In [8], Dragomir established a refinement of (1.1) which implies the Buzano inequality. Moreover, Fuji and Kubo [13] gave a simpler proof of (1.2) by using an orthogonal projection on a subspace of $\mathcal{H}$ and (1.1). Furthermore, they characterized when the equality holds.

This paper aims to present new generalizations of Buzano inequality and it is organized as follows. Section 2 contains some definitions and usual results about bounded linear operators defined on a Hilbert space. In Section 3, we present and prove the $\frac{1}{\alpha}$-Buzano inequality (if $\alpha=2$ gives the classical Buzano inequality) and it is devoted to

[^0]describing different families of operators which fulfill such inequality for different values of the parameter $\alpha$. Finally, in Section 4 relates the distinct inequalities previously obtained with the numerical radius, improving new bounds for the last one.

## 2. Preliminaries

As any pre-Hilbert space can be completed to a Hilbert space, from now on, we suppose that $\mathcal{H}$ is a Hilbert space. Let $\mathcal{B}(\mathcal{H})$ denote the $C^{*}$-algebra of all bounded linear operators acting on a separable non trivial complex Hilbert space $\mathcal{H}$ with an inner product $\langle\cdot, \cdot\rangle$ and the corresponding norm $\|\cdot\|$. The symbol $I$ stands for the identity operator and $\mathcal{G} \mathcal{L}(\mathcal{H})$ denotes the group of invertible operators on $\mathcal{H}$.

The range of every operator is denoted by $\mathcal{R}(T)$, its null space by $\mathcal{N}(T)$. If $T \in \mathcal{B}(\mathcal{H})$, we say that $T$ is a positive operator, $T \geq 0$, whenever $\langle T x, x\rangle \geq 0$ for all $x \in \mathcal{H}$ and we denote by $\mathcal{B}(\mathcal{H})^{+}$, the subset of all positive bounded linear operators definded on $\mathcal{H}$. The definition of positivity induces the order $T \geq S$ for self-adjoint operators if and only if $T-S \geq 0$. For any $T \in \mathcal{B}(\mathcal{H})^{+}$, there exists a unique positive $T^{1 / 2} \in \mathcal{B}(\mathcal{H})$ such that $T=\left(T^{1 / 2}\right)^{2}$. Let $T^{*}$ be the adjoint of $T$ and $|T|=\left(T^{*} T\right)^{1 / 2}$.

The polar decomposition theorem asserts that for every operator $T \in \mathcal{B}(\mathcal{H})$ there is a partial isometry $V \in \mathcal{B}(\mathcal{H})$ such that can be written as the product $T=V|T|$. In particular, $V$ satisfying $\mathcal{N}(V)=\mathcal{N}(T)$ exists and is uniquely determined.

For any $T \in \mathcal{B}(\mathcal{H})$, we denote by $\sigma(T)$ its spectrum and by $\sigma_{\text {app }}(T)$ its approximate point spectrum, that is

$$
\sigma_{a p p}(T)=\left\{\lambda \in \mathbb{C}: \exists\left\{x_{n}\right\}_{n \in \mathbb{N}},\left\|x_{n}\right\|=1 \text { and } \lim _{n \rightarrow \infty}\left\|T x_{n}-\lambda x_{n}\right\|=0\right\}
$$

For any $T \in \mathcal{B}(\mathcal{H})$, we define $m(T)=\inf \{\|T x\|: \quad x \in \mathcal{H},\|x\|=1\}$. Clearly, $m(T) \geq 0$ and $m(T)>0$ if and only if $0 \notin \sigma_{\text {app }}(T)$ ([16]).

For a linear operator $T$ on a Hilbert space $\mathcal{H}$, the numerical range $W(T)$ is the image of the unit sphere of $\mathcal{H}$ under the quadratic form $x \rightarrow\langle T x, x\rangle$. More precisely,

$$
W(T)=\{\langle T x, x\rangle: x \in \mathcal{H},\|x\|=1\} .
$$

The numerical range of an operator is a convex subset of the complex plane ([14]). Then, for any $T$ in $\mathcal{B}(\mathcal{H})$ we define the numerical radius of $T$,

$$
\omega(T)=\sup \{|\lambda|: \lambda \in W(T)\}
$$

It is well-known that $\omega(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$, and we have for all $T \in \mathcal{B}(\mathcal{H})$,

$$
\begin{equation*}
\frac{1}{2}\|T\| \leq \omega(T) \leq\|T\| \tag{2.1}
\end{equation*}
$$

Thus, the usual operator norm and the numerical radius are equivalent. Inequalities in (2.1) are sharp if $T^{2}=0$, then the first inequality becomes equality, while the second inequality becomes an equality if $T$ is normal.

For any compact operator $T \in \mathcal{B}(\mathcal{H})$ and $j \in \mathbb{N}$, let $s_{j}(T)=\lambda_{j}(|T|)$, be the $j$-th singular value of $T$, i.e. the $j$-th eigenvalue of $|T|$ in decreasing order and repeated according to multiplicity. Let $\operatorname{tr}(\cdot)$ be the trace functional,

$$
\operatorname{tr}(T)=\sum_{j=1}^{\infty}\left\langle T e_{j}, e_{j}\right\rangle
$$

where $\left\{e_{j}\right\}_{j=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$. Note that this coincides with the usual definition of the trace if $\mathcal{H}$ is finite-dimensional.

Let $T=x \otimes y$ be a rank one operator defined by $T(z)=\langle z, y\rangle x$ with $x, y, z \in \mathcal{H}$. Then, by Lemma 2.1 in [7] and using the well-known fact that $\operatorname{tr}(x \otimes y)=\langle x, y\rangle$, we obtain

$$
\omega(x \otimes y)=\frac{1}{2}(|\operatorname{tr}(x \otimes y)|+\|x \otimes y\|)=\frac{1}{2}(|\langle x, y\rangle|+\|x\|\|y\|) .
$$

We remark that the numerical radius of the rank one operator $T=x \otimes y$ coincides with the upper bound of Buzano inequality. From this fact, we are able to give a new proof of inequality (1.2) using this fact. If $\|z\|=1$, then $\langle T z, z\rangle=\langle z, y\rangle\langle x, z\rangle \in W(T)$ and

$$
|\langle x, z\rangle\langle z, y\rangle|=|\langle T z, z\rangle| \leq \omega(T)=\frac{1}{2}(|\langle x, y\rangle|+\|x\|\|y\|) .
$$

For $T \in \mathcal{B}(\mathcal{H})$, we have, by definition,

$$
\operatorname{dist}(I, \mathbb{C} T):=\inf _{\gamma \in \mathbb{C}}\|\gamma T-I\| \text { and } \operatorname{dist}(T, \mathbb{C} I):=\inf _{\beta \in \mathbb{C}}\|T-\beta I\|
$$

Evidently there is at least one complex number $\gamma_{0} \in \mathbb{C}$ such that $\operatorname{dist}(I, \mathbb{C} T)=$ $\left\|\gamma_{0} T-I\right\|$ and in addition, if $m(T)>0$ then the value $\gamma_{0}$ is unique. Following Stampfli [19], we call such scalar as the center of mass of $T$ and we denote by $c(T)$. For $A, T \in$ $\mathcal{B}(\mathcal{H})$ such that $m(T)>0$ we consider

$$
\begin{equation*}
M_{T}(A)=\sup _{\|x\|=1}\left[\|A x\|^{2}-\frac{|\langle A x, T x\rangle|^{2}}{\|T x\|^{2}}\right]^{1 / 2} \tag{2.2}
\end{equation*}
$$

In [17], Paul proved that $M_{T}(A)=\operatorname{dist}(A, \mathbb{C} T)$.
Given $T, S \in \mathcal{B}(\mathcal{H})$ we said that $T$ is Birkhoff-James orthogonal to $S$ if and only if $\|T\| \leq\|T-\lambda S\|$ for every $\lambda \in \mathbb{C}$.

## 3. $\frac{1}{\alpha}$-BUZANO INEQUALITY

In the last decades, several mathematicians presented different proofs of Buzano inequality. We start by presenting a new and simple proof of such inequality using a rank one operator.

Given $z \in \mathcal{H}$ with $\|z\|=1$ and $\alpha \in \mathbb{C}$, we consider the rank one operator $T=z \otimes z$. Then, for any $u \in \mathcal{H}$, it holds

$$
\|(\alpha T-I) u\|^{2}=\|\alpha T u-u\|^{2}=\left(|\alpha-1|^{2}-1\right)|\langle z, u\rangle|^{2}+\|u\|^{2} \leq \max \left\{1,|\alpha-1|^{2}\right\}\|u\|^{2} .
$$

Hence $\|\alpha T-I\| \leq \max \{1,|\alpha-1|\}$ and for any $x, y \in \mathcal{H}$ we get

$$
|\langle(\alpha T-I) x, y\rangle| \leq\|T-I\|\|x\|\|y\| \leq \max \{1,|\alpha-1|\}\|x\|\|y\| .
$$

In conclusion, we have

$$
\begin{equation*}
|\alpha\langle x, z\rangle\langle z, y\rangle-\langle x, y\rangle| \leq \max \{1,|\alpha-1|\}\|x\|\|y\|, \tag{3.1}
\end{equation*}
$$

for any $x, y, z \in \mathcal{H}$ with $\|z\|=1$ and $\alpha \in \mathbb{C}$. If $\alpha \in \mathbb{C}-\{0\}$, then (3.1) is equivalent to

$$
\left|\langle x, z\rangle\langle z, y\rangle-\frac{1}{\alpha}\langle x, y\rangle\right| \leq \frac{1}{|\alpha|} \max \{1,|\alpha-1|\}\|x\|\|y\| .
$$

From the continuity property of modulus for complex numbers, we obtain

$$
|\langle x, z\rangle\langle z, y\rangle| \leq \frac{1}{|\alpha|}(|\langle x, y\rangle|+\max \{1,|\alpha-1|\}\|x\|\|y\|)
$$

for any $x, y, z \in \mathcal{H}$ with $\|z\|=1$. The value $\alpha=2$ gives Buzano inequality.
We note that the inequality (3.1) was previously obtained by Moslehian et al. ([15], Corollary 2.5) using properties of singular values.

The main idea in the previous proof was to obtain a bound for the distance between a rank one operator and the identity operator. On the other hand, Fujii and Kubo in [13] based their proof of Buzano inequality on the fact that $\|2 P-I\| \leq 1$ where $P$ is an orthogonal projection. Because of the above, we are in a position to prove our first result in this paper, which generalizes these previous ideas.

Proposition 3.1. Let $T \in \mathcal{B}(\mathcal{H})$ and $\alpha \in \mathbb{C}-\{0\}$, with $\|\alpha T-I\| \leq 1$. Then, for any $x, y \in \mathcal{H}$

$$
\left|\langle T x, y\rangle-\frac{1}{\alpha}\langle x, y\rangle\right| \leq \frac{1}{|\alpha|}\|x\|\|y\|,
$$

and

$$
\begin{equation*}
|\langle T x, y\rangle| \leq\left|\langle T x, y\rangle-\frac{1}{\alpha}\langle x, y\rangle\right|+\frac{1}{|\alpha|}|\langle x, y\rangle| \leq \frac{1}{|\alpha|}(|\langle x, y\rangle|+\|x\|\|y\|) . \tag{3.2}
\end{equation*}
$$

On the other, if $T$ fulfills

$$
\begin{equation*}
|\langle T x, y\rangle| \leq \frac{1}{|\alpha|}(|\langle x, y\rangle|+\|x\|\|y\|) \tag{3.3}
\end{equation*}
$$

for any $x, y \in \mathcal{H}$ and for some $\alpha \in \mathbb{C}-\{0\}$. Then, $\operatorname{dist}(\alpha T, \mathbb{C} I) \leq 1$.
Proof. Let $x, y \in \mathcal{H}$ and $\alpha \in \mathbb{C}-\{0\}$, with $\|\alpha T-I\| \leq 1$. By (1.1), we have

$$
\begin{aligned}
\left|\langle T x, y\rangle-\frac{1}{\alpha}\langle x, y\rangle\right| & =\left|\left\langle\left(T-\frac{1}{\alpha} I\right) x, y\right\rangle\right| \leq \frac{1}{|\alpha|}\|\alpha T-I\|\|x\|\|y\| \\
& \leq \frac{1}{|\alpha|}\|x\|\|y\| .
\end{aligned}
$$

Therefore, we obtain

$$
|\langle T x, y\rangle| \leq\left|\langle T x, y\rangle-\frac{1}{\alpha}\langle x, y\rangle\right|+\frac{1}{|\alpha|}|\langle x, y\rangle| \leq \frac{1}{|\alpha|}(|\langle x, y\rangle|+\|x\|\|y\|) .
$$

If $T$ satisfies (3.3) and we recall the formula which express the distance from $T$ to the one-dimensional subspace $\mathbb{C} I$ (see [2]),

$$
\operatorname{dist}(T, \mathbb{C} I)=\sup \{|\langle T x, y\rangle|:\|x\|=\|y\|=1,\langle x, y\rangle=0\}
$$

then there exists $\beta \in \mathbb{C}-\{0\}$ such that $\|\alpha T-\beta I\| \leq 1$.
The next example shows that not all bounded linear operator satisfies the hypothesis of Proposition 3.1.

Example 3.1. Consider the unilateral shift operator, $T: l^{2}(\mathbb{N}) \rightarrow l^{2}(\mathbb{N})$, defined by $T\left(x_{1}, x_{2}, x_{3}, \cdots,\right)=\left(0, x_{1}, x_{2}, x_{3}, \cdots\right)$. Let $e_{1}=(1,0,0, \cdots) \in l^{2}(\mathbb{N})$, then $\left\|e_{1}\right\|=1$ and $\left\langle-T e_{1}, e_{1}\right\rangle=0$, by Theorem 2.1 in [1], we have that for any $\alpha \in \mathbb{C}-\{0\}$ it holds

$$
\|I\|^{2}+|\alpha|^{2} m^{2}(-T) \leq\|I-\alpha T\|^{2}
$$

As $-T$ is a left invertible operator, then $m(-T)>0$ and in consequence

$$
1<\|I\|^{2}+|\alpha|^{2} m^{2}(-T) \leq\|I-\alpha T\|^{2} .
$$

For convenience, for any $\alpha \in \mathbb{C}-\{0\}$, we denote by

$$
\mathcal{A}_{\alpha}=\{T \in \mathcal{B}(\mathcal{H}):\|\alpha T-I\| \leq 1\}
$$

the set of bounded operators that fulfills the hypothesis of Proposition 3.1. Since we have proved that each operator belonging to the set $\mathcal{A}_{\alpha}$ satisfies a $\frac{1}{\alpha}$ Buzano-type inequality, we will call to such set the $\frac{1}{\alpha}$ Buzano set.

Next, we collect some properties of $\mathcal{A}_{\alpha}$.
Proposition 3.2. Let $\alpha \in \mathbb{C}-\{0\}$, then
(1) $\mathcal{A}_{\alpha}$ is a non-empty convex and closed set.

For any $T \in \mathcal{A}_{\alpha}$, then
(2) $\|T\| \leq \frac{2}{|\alpha|}$.
(3) $T^{*} \in \mathcal{A}_{\bar{\alpha}}$.
(4) If $S \in \mathcal{A}_{\alpha}$, then $T+S \in \mathcal{A}_{\frac{\alpha}{2}}$.
(5) If $\|\alpha T-I\|<1$, then $T \in \mathcal{G} \mathcal{L}(\mathcal{H})$.
(6) If $T$ is self-adjoint, then $T \geq 0$ or $-T \geq 0$.
(7) If $T=h \otimes h, h \in \mathcal{H}$ and $h \neq 0$, then $\alpha=\frac{t e^{i \theta}+1}{\|h\|^{2}}$ for every $\theta \in[0,2 \pi]$ and $t \in[-1,1]$.
(8) $d(T, \mathbb{C} I) \leq \frac{1}{|\alpha|}$ and for every $P \geq 0$ with $\operatorname{tr}(P)=1$

$$
\operatorname{tr}\left(|T|^{2} P\right)-|\operatorname{tr}(T P)|^{2} \leq \frac{1}{|\alpha|}
$$

Proof. (1) Let $T, S \in \mathcal{A}_{\alpha}$ and $\lambda \in[0,1]$, then

$$
\begin{aligned}
\|\alpha(\lambda T+(1-\lambda) S)-I\| & \leq\|\alpha \lambda T-\lambda I\|+\|\alpha(1-\lambda) S-(1-\lambda) I\| \\
& =\lambda\|\alpha T-I\|+(1-\lambda)\|\alpha S-I\| \\
& \leq \lambda+1-\lambda=1
\end{aligned}
$$

This shows that $\lambda T+(1-\lambda) S \in \mathcal{A}_{\alpha}$ and therefore, $\mathcal{A}_{\alpha}$ is convex.
Now, let $\left\{T_{n}\right\}$ be a sequence in $\mathcal{A}_{\alpha}$ such that converges to $T \in \mathcal{B}(\mathcal{H})$. We must show that $T \in \mathcal{A}_{\alpha}$. Then, we have for any $n \in \mathbb{N}$ that it hold

$$
\|\alpha T-I\|=\left\|\alpha T-\alpha T_{n}+\alpha T_{n}-I\right\| \leq|\alpha|\left\|T_{n}-T\right\|+1
$$

Taking limit when $n$ tends to infinity we obtain $\|\alpha T-I\| \leq 1$, i.e. $T \in \mathcal{A}_{\alpha}$.
The proof of items (2), (3), (4), and (5) are trivial.
(6) If $T$ is normal, then $(\alpha T-I)^{*}(\alpha T-I)=(\alpha T-I)(\alpha T-I)^{*}$, for every $\alpha \in \mathbb{C}$. Therefore, $\alpha T-I$ is normal and

$$
r(\alpha T-I)=\omega(\alpha T-I)=\|\alpha T-I\|
$$

where $r(\alpha T-I)=\sup \{|\beta|: \beta \in \sigma(\alpha T-I)\}$ is the spectral radius. By the spectral theorem

$$
\|\alpha T-I\|=r(\alpha T-I)=\sup \{|\alpha \lambda-1|: \lambda \in \sigma(T)\} .
$$

Thus, if $T$ is selfadjoint and $T \in \mathcal{A}_{\alpha}$

$$
|\alpha \lambda-1| \leq 1
$$

for all $\lambda \in \sigma(T)$. Therefore, each $\alpha \lambda$ must lie in the unit disk in the complex plane centered in $z=1$. Also,

$$
\lambda \in \mathbb{R} \text { for every } \lambda \in \sigma(T) \Rightarrow\left\{\begin{array}{l}
\operatorname{Re}(\lambda \alpha)=\lambda \operatorname{Re}(\alpha) \in \quad[0,2] \\
\operatorname{Im}(\lambda \alpha)=\lambda \operatorname{Im}(\alpha) \in[-1,1]
\end{array}\right.
$$

Suppose there exist $\lambda_{j}, \lambda_{k} \in \sigma(T)$ such that $\lambda_{j}<0$ and $\lambda_{k}>0$, then

$$
\operatorname{Re}\left(\lambda_{j} \alpha\right) \in[0,2] \Rightarrow 0 \geq \operatorname{Re}(\alpha) \geq \frac{2}{\lambda_{j}}
$$

and

$$
\operatorname{Re}\left(\lambda_{k} \alpha\right) \in[0,2] \Rightarrow 0 \leq \operatorname{Re}(\alpha) \leq \frac{2}{\lambda_{k}}
$$

Thus, $\operatorname{Re}(\alpha)=0$, which means that $\lambda_{j} \alpha$ is pure imaginary and $\alpha=0$ (because the unit disk in the complex plane centered in $z=1$ intersects the imaginary axis only in $z=0)$. This is a contradiction, so $\lambda_{j}$ and $\lambda_{k}$ have the same sign.
(7) For any non-zero $h \in \mathcal{H}, T=h \otimes h$ is a compact, positive operator and its spectrum has only two eigenvalues, $\|h\|^{2}$ and 0 . Then, by the proof of item (6)

$$
\left|\alpha\|h\|^{2}-1\right| \leq 1
$$

if and only if $\alpha\|h\|^{2}=t e^{i \theta}+1$, with $|t| \leq 1$ and $\theta \in[0,2 \pi]$.
(8) $1 \geq\|\alpha T-I\|=|\alpha|\left\|T-\frac{1}{\alpha} I\right\| \geq|\alpha| d(T, \mathbb{C} I) \Rightarrow \frac{1}{|\alpha|} \geq d(T, \mathbb{C} I)$ and using Proposition 3.1 in [4] we obtain that

$$
\operatorname{tr}\left(|T|^{2} P\right)-|\operatorname{tr}(T P)|^{2} \leq \frac{1}{|\alpha|}
$$

for every $P \geq 0$ with $\operatorname{tr}(P)=1$.

Next, we enumerate other properties of $\mathcal{A}_{\alpha}$ related to $\operatorname{dist}(I, \mathbb{C} T)$ and the center of mass of $T$.

Lemma 3.1. Let $T \in \mathcal{B}(\mathcal{H})$.
(1) If $m(T)>0$ and $c(T) \neq 0$, then $T \in \mathcal{A}_{c(T)}$.
(2) If $m(T)>0$ and $c(T)=0$, then $I$ is Birkhoff-James orthogonal to $T$. We deduce that $1<\|\alpha T-I\|$ and $T \notin \mathcal{A}_{\alpha}$, for every $\alpha \in \mathbb{C}-\{0\}$.
(3) If $\operatorname{dist}(I, \mathbb{C} T)=1$ and there exists $\alpha_{0} \in \mathbb{C}-\{0\}$ such that

$$
1=\left\|\alpha_{0} T-I\right\|<\|\alpha T-I\|, \text { for all } \alpha \in \mathbb{C}-\left\{\alpha_{0}, 0\right\}
$$

then $0 \in \sigma_{\text {app }}(T)$ and $T \in \mathcal{A}_{\alpha_{0}}$.
Proposition 3.3. Let $T \in \mathcal{B}(\mathcal{H})$ such that $m(T)>0, c(T) \neq 0$ and dist $(T, \mathbb{C} I)=\|T\|$, then $\|I-c(T) T\|=1$ and, in particular, $T \in \mathcal{A}_{c(T)}$.

Proof. Let $x \in \mathcal{H}$ such that $\|x\|=1$, then $|\langle T x, x\rangle| \geq\|T x\| \inf _{\|y\|=1} \frac{|\langle T y, y\rangle|}{\|T y\|}$ and

$$
\left[\|T x\|^{2}-|\langle T x, x\rangle|^{2}\right]^{1 / 2} \leq\left(1-\inf _{\|y\|=1} \frac{|\langle T y, y\rangle|^{2}}{\|T y\|^{2}}\right)^{1 / 2}\|T x\|
$$

Calculating the supremum of both sides, and using the equality (2.2), we get

$$
\|T\|=\operatorname{dist}(T, \mathbb{C} I) \leq \operatorname{dist}(I, \mathbb{C} T)\|T\|
$$

Then $1 \leq \operatorname{dist}(I, \mathbb{C} T) \leq\|I\|=1$, i.e. $\operatorname{dist}(I, \mathbb{C} T)=\|I-c(T) T\|=1$. This completes the proof.

As we have shown in Proposition 3.2, if $T \in \mathcal{A}_{\alpha}$ with $\|\alpha T-I\|<1$, then $T \in \mathcal{G} \mathcal{L}(\mathcal{H})$ and $T$ verifies Proposition 3.1. Now, we obtain a generalization of such statement for any invertible operator. In order to prove it, we need the following result.

Lemma 3.2 (Corollary 3.7, [5]). If $T \in \mathcal{G} \mathcal{L}(\mathcal{H})$, then there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ and a non-zero complex number $\beta$ such that $\left\|\beta T^{-1}-U^{*}\right\|<1$.

Observe that in the value $\beta$ in Lemma 3.2 satisfies that $|\beta|<\frac{2}{\left\|T^{-1}\right\|}$.
Theorem 3.1. Let $T \in \mathcal{G} \mathcal{L}(\mathcal{H})$, then there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ and a non- zero complex number $\beta$ as in Lemma 3.2 such that for any $x, y \in \mathcal{H}$

$$
\left|\left\langle T^{-1} x, y\right\rangle-\frac{1}{\beta}\left\langle U^{*} x, y\right\rangle\right| \leq \frac{1}{|\beta|}\|x\|\|y\|,
$$

and

$$
\left|\left\langle T^{-1} x, y\right\rangle\right| \leq\left|\left\langle T^{-1} x, y\right\rangle-\frac{1}{\beta}\left\langle U^{*} x, y\right\rangle\right|+\frac{1}{|\beta|}\left|\left\langle U^{*} x, y\right\rangle\right| \leq \frac{1}{|\beta|}\left(\left|\left\langle U^{*} x, y\right\rangle\right|+\|x\|\|y\|\right) .
$$

Proof. It is analogous to the proof of Proposition 3.1, so we omit it.
3.1. Bounded Linear operators which belong to $\mathcal{A}_{\alpha}$. We begin this subsection showing when a normal operator belongs to $\mathcal{A}_{\alpha}$.
Theorem 3.2. Let $T$ be a normal operator in $\mathcal{B}(\mathcal{H})-\{0\}$, such that $\sigma(T)$ is fully included into an arc of the disk of radius $\|T\|$ and centered in the origin, with central angle less than $\pi$.
Then, $T \in \mathcal{A}_{\alpha}$ for every $\alpha \in \mathbb{C}$ such that

$$
\begin{equation*}
\arg (\alpha)+\arg (\lambda) \in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \text { for all } \lambda \in \sigma(T) \tag{3.4}
\end{equation*}
$$

and

$$
\begin{equation*}
|\alpha| \leq \frac{2}{\|T\|} \min _{\lambda \in \sigma(T)} \cos (\arg (\alpha)+\arg (\lambda)) \tag{3.5}
\end{equation*}
$$

Proof. Since $T$ is normal, $\alpha T-I$ is a normal operator, then

$$
\|\alpha T-I\|=r(\alpha T-I)=\sup \{|\alpha \lambda-1|: \lambda \in \sigma(T)\} \leq 1
$$

if and only if $|\alpha \lambda-1| \leq 1$ for every $\lambda \in \sigma(T)$. This is equivalent to find if there exists $\alpha \in \mathbb{C}$ such that

$$
\begin{equation*}
|\alpha \lambda-1|^{2} \leq 1 \text { for every } \lambda \in \sigma(T) \tag{3.6}
\end{equation*}
$$

Taking $\alpha=|\alpha| e^{i \theta}$ and $\lambda=|\lambda| e^{i \varphi_{\lambda}}$, with $|\lambda| \leq\|T\|$,

$$
\begin{aligned}
\| \alpha| | \lambda\left|e^{i\left(\theta+\varphi_{\lambda}\right)}-1\right|^{2} & =\left(|\alpha||\lambda| \cos \left(\theta+\varphi_{\lambda}\right)-1\right)^{2}+\left(|\alpha||\lambda| \sin \left(\theta+\varphi_{\lambda}\right)\right)^{2} \\
& =|\alpha|^{2}|\lambda|^{2}-2|\alpha||\lambda| \cos \left(\theta+\varphi_{\lambda}\right)+1
\end{aligned}
$$

and we can rewrite (3.6) as follows

$$
|\alpha||\lambda|\left(|\alpha||\lambda|-2 \cos \left(\theta+\varphi_{\lambda}\right)\right) \leq 0
$$

Then, we arrive to the following condition

$$
\begin{equation*}
|\alpha||\lambda|-2 \cos \left(\theta+\varphi_{\lambda}\right) \leq 0 . \tag{3.7}
\end{equation*}
$$

Take an $\alpha \in \mathbb{C}$ that satisfies (3.4) and (3.5). For $\lambda=0$ it is immediate that $\alpha$ satisfies condition (3.7). Consider $\lambda \neq 0$, then $\cos \left(\arg (\alpha)+\varphi_{\lambda}\right)>0$ for every $\lambda \in \sigma(T)$ and

$$
|\alpha| \leq \frac{2}{\|T\|} \min _{\lambda \in \sigma(T)} \cos (\arg (\alpha)+\arg (\lambda)) \leq \frac{2}{|\lambda|} \cos (\arg (\alpha)+\arg (\lambda)), \lambda \neq 0
$$

Therefore, $\alpha$ fulfills the condition (3.7) for every $\lambda \in \sigma(T)$ and we conclude that $\|\alpha T-I\| \leq 1\left(T \in \mathcal{A}_{\alpha}\right)$.

In order to fulfill (3.7) and $\cos \left(\arg (\alpha)+\varphi_{\lambda}\right) \geq 0$, it is a necessary condition that the spectrum of $T$ lies in into an arc of the disk of radius $\|T\|$ and centered in the origin, with central angle less than $\pi$. Otherwise, it is not possible to fix any $\alpha$ such that the property holds. Additionally, we exclude $\arg (\alpha)+\varphi_{\lambda}= \pm \frac{\pi}{2}$, since $|\alpha||\lambda|=0$ if and only if $\lambda=0$ or $\alpha=0$.

For example, if $T$ is Hermitian $\varphi_{\lambda} \in\{0, \pi\}$ it can be seen that there is no $\alpha$ that (3.7) holds, unless $\lambda \geq 0$ for every $\lambda \in \sigma(T)\left(\varphi_{\lambda}=0\right)$, or $\lambda \leq 0$ for every $\lambda \in \sigma(T)\left(\varphi_{\lambda}=\pi\right)$. Thus, the unique Hermitian operators $T$ that can reach (3.7) are semidefinite positive or semidefinite negative, as we show in item 6 of Proposition 3.2.

In particular, for positive operators, we arrive to the following result.
Corollary 3.1. If $T \in \mathcal{B}(\mathcal{H})^{+}$, then $T \in \mathcal{A}_{\alpha}$ for every $\alpha \in \mathbb{C}$ such that $|\alpha| \leq \frac{a}{\|T\|} \leq \frac{2}{\|T\|}$ and $\cos (\arg (\alpha)) \geq \frac{a}{2}$.
Proof. As we mention before, in this case $\varphi_{\lambda}=0$ for every $\lambda \in \sigma(T)$. Then,

$$
|\alpha \| \lambda|-2 \cos (\arg (\alpha)) \leq \frac{a}{\|T\|}|\lambda|-2 \cos (\arg (\alpha)) \leq a-2 \cos (\arg (\alpha)) \leq 0 .
$$

The next result is a generalization of Buzano inequality for any bounded linear operator.

Theorem 3.3. Let $T \in \mathcal{B}(\mathcal{H})-\{0\}$. Then, for any $x, y \in \mathcal{H}$

$$
\left|\langle T x, T y\rangle-\frac{\|T\|^{2}}{2}\langle x, y\rangle\right| \leq \frac{\|T\|^{2}}{2}\|x\|\|y\|,
$$

and

$$
\begin{equation*}
|\langle T x, T y\rangle| \leq\left|\langle T x, T y\rangle-\frac{\|T\|^{2}}{2}\langle x, y\rangle\right|+\frac{\|T\|^{2}}{2}|\langle x, y\rangle| \leq \frac{\|T\|^{2}}{2}(|\langle x, y\rangle|+\|x\|\|y\|) . \tag{3.8}
\end{equation*}
$$

Proof. By Corollary 3.1, if $S \in \mathcal{B}(\mathcal{H})^{+}-\{0\}$, then $S \in \mathcal{A}_{\frac{2}{S S \|}}$. In particular, if we consider the positive operator $S=T^{*} T$, then we conclude that $T^{*} T \in \mathcal{A}_{\frac{2}{} \frac{2}{\|T\|^{2}} \text { and the }}$ proof is complete as a consequence of Proposition 3.1

The constant $\frac{\|T\|^{2}}{2}$ is best possible in (3.8). Now, if we assume that (3.8) holds with a constant $C>0$, i.e.

$$
|\langle T x, T y\rangle| \leq C(|\langle x, y\rangle|+\|x\|\|y\|),
$$

for any $T \in \mathcal{B}(\mathcal{H})$. So, if we choose $x=y$, then $\|T x\|^{2} \leq 2 C\|x\|^{2}$ and we deduce that $2 C \geq\|T\|^{2}$. Thus, (3.8) is an improvement and refinement of

$$
|\langle T x, T y\rangle| \leq \frac{\|T\|^{2}}{2}(|\langle x, y\rangle|+\|x\|\|y\|)
$$

which was obtained in a different way earlier by Dragomir in [11] using a non-negative Hermitian form on a Hilbert space.

From the polar decomposition of any bounded linear operator and the main idea used in the proof of Theorem 3.3, we have the following statement.

Corollary 3.2. Let $T \in \mathcal{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$. Then,

$$
\begin{align*}
|\langle T x, y\rangle| & \left.=|\langle | T| x, V^{*} y\right\rangle \left.\left|\leq\left|\langle T x, y\rangle-\frac{\|T\|}{2}\left\langle x, V^{*} y\right\rangle\right|+\frac{\|T\|}{2}\right|\left\langle x, V^{*} y\right\rangle \right\rvert\, \\
& \leq \frac{\|T\|}{2}\left(\left|\left\langle x, V^{*} y\right\rangle\right|+\|x\|\left\|V^{*} y\right\|\right) \\
& \leq \frac{\|T\|}{2}\left(\left|\left\langle x, V^{*} y\right\rangle\right|+\|x\|\|y\|\right) . \tag{3.9}
\end{align*}
$$

where $T=V|T|$ is the polar decomposition of $T$.
Remark 3.1. Inequality (3.9) is an improvement and refinement of a result recently obtained by Sababheh et al. (see [18, Remark 3.1]).

Recall that $T$ is called a positive contraction if $0 \leq T \leq I$. As a consequence of Corollary 3.1, we conclude that $T \in \mathcal{A}_{\alpha}$ for every $\alpha \in[0,2]$.

Now, we obtain a refinement of the classical Cauchy-Schwarz inequality, using positive contractions. The idea of the proof is based in [11, Theorem 2.1]. Recently, in [18] the same result was obtained with a different proof.

Theorem 3.4. Let $T \in \mathcal{B}(\mathcal{H})$ be a positive contraction and $x, y \in \mathcal{H}$, then

$$
|\langle x, y\rangle|+\langle T x, x\rangle^{1 / 2}\langle T y, y\rangle^{1 / 2}-|\langle T x, y\rangle| \leq\|x\|\|y\| .
$$

Proof. For any $x, y \in \mathcal{H}$ and the elementary inequality $(a c-b d)^{2} \geq\left(a^{2}-b^{2}\right)\left(c^{2}-d^{2}\right)$, which holds for any real numbers $a, b, c, d$, we have

$$
\begin{align*}
\left(\|x\|\|y\|-\langle T x, x\rangle^{1 / 2}\langle T y, y\rangle^{1 / 2}\right)^{2} & \geq\left(\|x\|^{2}-\langle T x, x\rangle\right)\left(\|y\|^{2}-\langle T y, y\rangle\right) \\
& =\langle(I-T) x, x\rangle\langle(I-T) y, y\rangle . \tag{3.10}
\end{align*}
$$

As $T$ is a positive contraction, then $I-T \in \mathcal{B}(\mathcal{H})^{+}$. By Cauchy-Schwarz inequality for positive operators

$$
\begin{equation*}
\langle(I-T) x, x\rangle\langle(I-T) y, y\rangle \geq|\langle(I-T) x, y\rangle|^{2}=|\langle x, y\rangle-\langle T x, y\rangle|^{2} . \tag{3.11}
\end{equation*}
$$

Now, by (3.10) and (3.11),

$$
\begin{equation*}
\left(\|x\|\|y\|-\langle T x, x\rangle^{1 / 2}\langle T y, y\rangle^{1 / 2}\right)^{2} \geq|\langle x, y\rangle-\langle T x, y\rangle|^{2} \tag{3.12}
\end{equation*}
$$

for any $x, y \in \mathcal{H}$. Since $\|x\| \geq\langle T x, x\rangle^{1 / 2}$ and $\|y\| \geq\langle T y, y\rangle^{1 / 2}$, by taking the square root, (3.12) is equivalent to

$$
\begin{equation*}
\|x\|\|y\|-\langle T x, x\rangle^{1 / 2}\langle T y, y\rangle^{1 / 2} \geq|\langle x, y\rangle-\langle T x, y\rangle| . \tag{3.13}
\end{equation*}
$$

On making use of the triangle inequality for the modulus, we have

$$
\begin{align*}
\|x\|\|y\|-\langle T x, x\rangle^{1 / 2}\langle T y, y\rangle^{1 / 2} & \geq|\langle x, y\rangle-\langle T x, y\rangle| \\
& \geq|\langle x, y\rangle|-|\langle T x, y\rangle| \tag{3.14}
\end{align*}
$$

and this completes the proof.
Remark 3.2. Recall that any orthogonal projection $P=P^{2}=P^{*}$ is a positive contraction with $\|P\|=1$ and $P \in \mathcal{A}_{2}$. Then, for any $x, y \in \mathcal{H}$

$$
\begin{equation*}
|\langle P x, y\rangle| \leq\left|\langle P x, y\rangle-\frac{1}{2}\langle x, y\rangle\right|+\frac{1}{2}|\langle x, y\rangle| \leq \frac{1}{2}(|\langle x, y\rangle|+\|x\|\|y\|), \tag{3.15}
\end{equation*}
$$

and

$$
\begin{equation*}
|\langle P x, y\rangle-\langle x, y\rangle| \leq \frac{1}{2}(|\langle x, y\rangle|+\|x\|\|y\|) . \tag{3.16}
\end{equation*}
$$

In (3.15) and (3.16) we reach, with a new proof, an improvement and refinement of different statements previously obtained by Dragomir in [10].

Motivated by the previous inequalities valid for orthogonal projections, we establish some vector inequalities for particular projections. Let $T=z \otimes z$, with $z \in \mathcal{H}$ and $\|z\|=1$, as $T$ is an orthogonal projection then by using inequality (3.2) we get

$$
|\langle x, z\rangle\langle z, y\rangle| \leq\left|\langle x, z\rangle\langle z, y\rangle-\frac{1}{2}\langle x, y\rangle\right|+\frac{1}{2}|\langle x, y\rangle| \leq \frac{1}{2}(|\langle x, y\rangle|+\|x\|\|y\|)
$$

for any $x, y \in \mathcal{H}$. This inequality refines the classical Buzano inequality.
On the other hand, using inequality (3.13)

$$
|\langle x, y\rangle| \leq|\langle x, y\rangle-\langle x, z\rangle\langle z, y\rangle|+|\langle x, z\rangle\langle z, y\rangle| \leq\|x\|\|y\|,
$$

for any $x, y \in \mathcal{H}$. This refinement of (1.1) was also obtained in [8].
Now we are in position to obtain a Buzano type inequality for the sum of two orthogonal projections. It is well-known that given two orthogonal projections on $\mathcal{H}$, $P$ and $Q$, then

$$
\begin{equation*}
\|P+Q\|=1+\|P Q\| \tag{3.17}
\end{equation*}
$$

This is usually called Duncan-Taylor equality, and its proof can be found in [12].
Proposition 3.4. Let $P, Q$ be orthogonal projections on $\mathcal{H}$. Then, $P+Q \in \mathcal{A}_{\alpha}$ for $|\alpha| \leq \frac{2}{1+\|P Q\|}$.

Proof. Note that $P+Q \in \mathcal{B}(\mathcal{H})^{+}$and by (3.17), $\|P+Q\|=1+\|P Q\|$. Thus, using Corollary 3.1, the proof is complete.

Throughout, $\mathcal{S}$ and $\mathcal{T}$ denote two closed subspaces of $\mathcal{H}$. The mininal angle or angle of Dixmier between $\mathcal{S}$ and $\mathcal{T}$ is the angle $\theta_{0}(\mathcal{S}, \mathcal{T}) \in\left[0, \frac{\pi}{2}\right]$ whose cosine is defined by

$$
c_{0}(\mathcal{S}, \mathcal{T})=\sup \{|\langle x, y\rangle|: x \in \mathcal{S}, y \in \mathcal{T} ;\|x\|,\|y\| \leq 1\}
$$

A linear operator defined on $\mathcal{H}$, such that $Q^{2}=Q$ is called a projection. Such operators are not necessarily bounded, since on every infinite-dimensional Hilbert space there exist unbounded examples of projections (see [3]). The operator $Q_{\mathcal{M} / / \mathcal{N}}$ is an oblique projection along (or parallel to) its null space $\mathcal{N}=\mathcal{N}(Q)$ onto its range $\mathcal{M}=\mathcal{R}(Q)$.

Theorem 3.5. Let $H$ be a Hilbert space such that is the direct sum of closed subspaces $\mathcal{M}$ and $\mathcal{N}$. Let $Q_{\mathcal{M} / / \mathcal{N}}$, be the bounded projection with range $\mathcal{M}$ and null space $\mathcal{N}$, and $\theta_{0}(\mathcal{M}, \mathcal{N})$, be the minimal angle between $\mathcal{M}$ and $\mathcal{N}$. Then, for any $x, y \in \mathcal{H}$

$$
\left|\langle Q x, y\rangle-\frac{1}{2}\langle x, y\rangle\right| \leq \frac{\cot \left(\alpha_{0}\right)}{2}\|x\|\|y\|,
$$

and

$$
|\langle Q x, y\rangle| \leq \frac{\cot \left(\alpha_{0}\right)}{2}(|\langle x, y\rangle|+\|x\|\|y\|)
$$

where $\alpha_{0}=\frac{\theta_{0}(\mathcal{M}, \mathcal{N})}{2}$.
Proof. By Theorem 2 in [3] we have that $\|Q\|=\csc \left(\theta_{0}(\mathcal{M}, \mathcal{N})\right)$ and $\|2 Q-I\|=\cot \left(\alpha_{0}\right)$. From the boundness of $Q$ we can assert that $0<\theta_{0}(\mathcal{M}, \mathcal{N}) \leq \frac{\pi}{2}$ and $\cot \left(\alpha_{0}\right) \geq 1$. From these facts and mimicking the proof of Proposition 3.1, we have that for any $x, y \in \mathcal{H}$

$$
\left|\langle Q x, y\rangle-\frac{1}{2}\langle x, y\rangle\right| \leq \frac{\cot \left(\alpha_{0}\right)}{2}\|x\|\|y\|,
$$

and

$$
\begin{aligned}
|\langle Q x, y\rangle| & \leq\left|\langle Q x, y\rangle-\frac{1}{2}\langle x, y\rangle\right|+\frac{1}{2}|\langle x, y\rangle| \\
& \leq \frac{\cot \left(\alpha_{0}\right)}{2}\|x\|\|y\|+\frac{1}{2}|\langle x, y\rangle| \\
& \leq \frac{\cot \left(\alpha_{0}\right)}{2}(|\langle x, y\rangle|+\|x\|\|y\|) .
\end{aligned}
$$

This completes the proof.
We finish this section by showing that any operator whose real part is greater than $s I$ for some $s>0$, is invertible and its inverse belongs to $\mathcal{A}_{2 s}$.
Theorem 3.6. Let $T \in \mathcal{B}(\mathcal{H})-\{0\}$ with $\operatorname{Re}(T)=\frac{T+T^{*}}{2} \geq$ sI for some $s>0$. Then, $T^{-1} \in \mathcal{A}_{2 s}$.

Proof. First, we show that $T$ is invertible. The hypothesis $\operatorname{Re}(T) \geq s I$ implies that

$$
W(T) \subseteq\{z \in \mathbb{C}: \operatorname{Re}(z) \geq s\}
$$

since if $z \in W(T)$ then

$$
\begin{aligned}
\operatorname{Re}(z) & =\frac{z+\bar{z}}{2}=\frac{\langle T x, x\rangle+\overline{\langle T x, x\rangle}}{2}=\frac{\langle T x, x\rangle+\left\langle T^{*} x, x\right\rangle}{2} \\
& =\langle\operatorname{Re}(T) x, x\rangle \geq s .
\end{aligned}
$$

Thus $\sigma(T) \subseteq \overline{W(T)} \subseteq\{z \in \mathbb{C}: \operatorname{Re}(z) \geq s\}$ and, in particular, we have that $0 \notin \sigma(T)$, which means $T \in \mathcal{G} \mathcal{L}(\mathcal{H})$. If $T+T^{*} \geq 2 s I$ and $T \in \mathcal{G \mathcal { L }}(\mathcal{H})$, then $2 s T^{-1}\left(T+T^{*}-2 s I\right)\left(T^{*}\right)^{-1} \geq 0$ and

$$
I \geq I-2 s T^{-1}\left(T+T^{*}-2 s I\right)\left(T^{*}\right)^{-1}=\left(I-2 s T^{-1}\right)\left(I-2 s T^{-1}\right)^{*}
$$

which is equivalent to $\left\|2 s T^{-1}-I\right\| \leq 1$. Hence, $T^{-1} \in \mathcal{A}_{2 s}$ and the result follows.

## 4. Bounds for the numerical radius using Buzano inequality

In this section, we use (3.9) to obtain a refinement of the classical inequality $\omega(T) \leq$ $\|T\|$ and an upper bound for $\omega(T)-\frac{1}{2}\|T\|$.
Proposition 4.1. Let $T \in \mathcal{B}(\mathcal{H})$ with polar decomposition $T=V|T|$. Then,

$$
\begin{equation*}
\omega(T) \leq \frac{\|T\|}{2}(1+\omega(V)) \leq\|T\| \tag{4.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\omega(T)-\frac{\|T\|}{2} \leq \frac{\|T\|}{2} \omega(V) . \tag{4.2}
\end{equation*}
$$

Proof. Taking $x=y$ in (3.9), and the supremum over all $x \in \mathcal{H}$ with $\|x\|=1$, we obtain $\omega(T) \leq \frac{\|T\|}{2}(1+\omega(V))$ and this completes the proof.

It is important to note that inequalities (4.1) and (4.2) are not trivial, since $\omega(V)$ may be less than one, depending on the partial isometry $V$. For instance, let

$$
T=\left[\begin{array}{cc}
0 & 0 \\
\sqrt{2} & 0
\end{array}\right]=\left[\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right]\left[\begin{array}{cc}
\sqrt{2} & 0 \\
0 & 0
\end{array}\right]=V|T|
$$

where $V$ is a partial isometry in $\mathbb{C}^{2}$ with $\operatorname{ker}(T)=\operatorname{ker}(V)=\operatorname{span}\{(0,1)\}$ and $\operatorname{ker}(V)^{\perp}=$ $\operatorname{span}\{(1,0)\}$. Then, for any $x=\left(x_{1}, x_{2}\right) \in \mathbb{C}$ with $\|x\|=\sqrt{\left|x_{1}\right|^{2}+\left|x_{2}\right|^{2}}=1$, we have as a consequence of the arithmetic-geometric mean inequality

$$
|\langle V x, x\rangle|=\left|x_{1} \overline{x_{2}}\right|=\left|x_{1}\right|\left|x_{2}\right| \leq \frac{\left|x_{1}\right|^{2}+\left|x_{2}\right|^{2}}{2}=\frac{1}{2} .
$$

Therefore, $W(V)=\left\{z \in \mathbb{C}:|z| \leq \frac{1}{2}\right\}$ and $\omega(V)=\frac{1}{2}<1$.
Proposition 4.2. Let $T \in \mathcal{B}(\mathcal{H})-\{0\}$ with polar decomposition $T=V|T|$ such that $\omega(T)=\|T\|$, then $\omega(V)=\|V\|=1$.

Proof. From inequality (4.1), we get

$$
\omega(T) \leq \frac{\|T\|}{2}(1+\omega(V)) \leq\|T\|
$$

Thus, if $\omega(T)=\|T\|$, then $1+\omega(V)=2$, and hence $\omega(V)=1$. As $V$ is a nonzero partial isometry, therefore $\|V\|=1$ and thus $\omega(V)=\|V\|=1$, as required.

It should also be mentioned here that the converse of Proposition 4.2 is not true. To see this, consider

$$
T=\left[\begin{array}{lll}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{array}\right]=\left[\begin{array}{lll}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{array}\right]\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{array}\right]=V|T| .
$$

where $V|T|$ is a polar decomposition of $T$. Then, $\omega(V)=\|V\|=1$, but $\omega(T)=\frac{1}{\sqrt{2}}<$ $1=\|T\|$.

In order to estimate how close the numerical radius is from the operator norm, the following reverse inequalities have been obtained under appropriate conditions for the involved operator $T \in \mathcal{B}(\mathcal{H})$. If $T \in \mathcal{A}_{\alpha}$, then by (2.1) and Proposition 3.2 we have that

$$
0 \leq\|T\|-\omega(T) \leq\|T\|-\frac{\|T\|}{2} \leq \frac{1}{|\alpha|}
$$

Motivated by the above inequality, we establish a new upper bound for the non-negative quantity $\|T\|-\omega(T)$.

Theorem 4.1. Let $T \in \mathcal{A}_{\alpha}$. Then,

$$
\|T\|-\omega(T) \leq \frac{1}{2|\alpha|}
$$

Proof. For $x \in \mathcal{H}$ with $\|x\|=1$, we have

$$
\|\alpha T x-x\|^{2}=|\alpha|^{2}\|T x\|^{2}-2 \operatorname{Re}(\alpha\langle T x, x\rangle)+1 \leq 1
$$

giving

$$
|\alpha|^{2}\|T x\|^{2}+1 \leq 1+2 \operatorname{Re}(\alpha\langle T x, x\rangle) \leq 2|\alpha \||\langle T x, x\rangle|+1 .
$$

By arithmetic-geometric mean inequality, we deduce

$$
\begin{equation*}
2|\alpha|\|T x\| \leq|\alpha|^{2}\|T x\|^{2}+1 \leq 1+2 \operatorname{Re}(\alpha\langle T x, x\rangle) \leq 2|\alpha \||\langle T x, x\rangle|+1 \tag{4.3}
\end{equation*}
$$

Now, taking the supremum over $x \in \mathcal{H},\|x\|=1$ in (4.3), we obtain

$$
2|\alpha|\|T\|-2|\alpha| \omega(T) \leq 1
$$

Now, we derive upper bounds for the numerical radius of products of bounded linear operators.
Theorem 4.2. Let $R, S, T \in \mathcal{B}(\mathcal{H})$ such that $T \in \mathcal{A}_{\alpha}$. Then,

$$
\begin{equation*}
\omega(S T R) \leq \frac{1}{|\alpha|}(\|R\|\|S\|+\omega(S R)) \tag{4.4}
\end{equation*}
$$

Proof. From inequality (3.2), we have

$$
|\langle S T R x, y\rangle|=\left|\left\langle T R x, S^{*} y\right\rangle\right| \leq \frac{1}{|\alpha|}\left(\left|\left\langle R x, S^{*} y\right\rangle\right|+\|R x\|\left\|S^{*} y\right\|\right)
$$

Taking $y=x$ and the supremum over $x \in \mathcal{H}$ with $\|x\|=1$, yields the desired inequality.

We note that the previous result is a generalization of Theorem 3.6 in [11]. In particular, for the sum of two orthogonal projections, we obtain the following result.

Corollary 4.1. Let $P, Q, R, S \in \mathcal{B}(\mathcal{H})$ with $P, Q$ be orthogonal projections. Then,

$$
\omega(R(P+Q) S) \leq \frac{1+\|P Q\|}{2}(\|S\|\|R\|+\omega(R S))
$$

Proof. As we have already mentioned, $P+Q \in \mathcal{A}_{\frac{2}{1+\|P Q\|}}$. Then, the statement is a consequence of Theorem 4.2.

On the other hand, in [9], Dragomir obtained, utilizing Buzano's inequality, the following inequality for the numerical radius

$$
\omega(S)^{2} \leq \frac{1}{2}\left(\|S\|^{2}+\omega\left(S^{2}\right)\right)
$$

combining with the following power inequality for the numerical radius, $w\left(S^{n}\right) \leq w(S)^{n}$ for any natural number $n$, we have

$$
\begin{equation*}
w\left(S^{2}\right) \leq \frac{1}{2}\left(\|S\|^{2}+\omega\left(S^{2}\right)\right) \tag{4.5}
\end{equation*}
$$

The following corollary, which is an immediate consequence of Theorem 4.2 considering $R=S$, gives a generalization of (4.5).

Corollary 4.2. Let $S, T \in \mathcal{B}(\mathcal{H})$ with $T \in \mathcal{A}_{\alpha}$. Then,

$$
\omega(S T S) \leq \frac{1}{|\alpha|}\left(\|S\|^{2}+\omega\left(S^{2}\right)\right)
$$

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## References

[1] M. Barraa and M. Boumazgour, A note on the orthogonality of bounded linear operators. Funct. Anal. Approx. Comput. 4 (2012), no. 1, 65-70.
[2] R. Bhatia and P. Šemrl, Orthogonality of matrices and some distance problems, Linear Algebra Appl. 287 (1999), no. 1-3, 77-85.
[3] D. Buckholtz, Hilbert space idempotents and involutions, Proc. Amer. Math. Soc. 128 (2000), 1415-1418.
[4] T. Bottazzi and C. Conde, A Grüss type operator inequality. Ann. Funct. Anal. 8 (2017), no. 1, 124-132.
[5] J. Bračič and C. Diogo, Relative numerical ranges, Linear Algebra Appl. 485 (2015), 208-221.
[6] M. L. Buzano, Generalizzazione della diseguaglianza di Cauchy-Schwarz (Italian), Rend. Sem. Mat. Univ. e Politech. Torino 31 (1974), 405-409.
[7] M-T. Chien, H-L. Gau, C-K. Li, M-C. Tsai, and K-Z. Wang, Product of operators and numerical range. Linear Multilinear Algebra 64 (2016), no. 1, 58-67.
[8] S. S. Dragomir, Some refinements of Schwartz inequality, Simpozionul de Matematici şi Aplicaţii, Timişoara, Romania 1-2 (1985), 13-16.
[9] S. S. Dragomir, Inequalities for the norm and the numerical radius of linear operators in Hilbert spaces, Demonstratio Math. 40 (2) (2007), 411-417.
[10] S.S. Dragomir, Buzano inequality holds for any projection Bull. Aust. Math. Soc. 93 (2016), no. 3, 504-510.
[11] S.S. Dragomir, A Buzano type inequality for two Hermitian forms and applications, Linear Multilinear Algebra 65 (2017), no. 3, 514-525.
[12] J. Duncan and P. J. Taylor, Norm inequalities for $C^{*}$-algebras, Proc. Roy. Soc. Edinburgh Sect. A $75(1975 / 76)$, no. 2, 119-129.
[13] M. Fujii and F. Kubo, Buzano inequality and bounds for roots of algebraic equations, Proc. Amer. Math. Soc. 117 (1993), no. 2, 359-361.
[14] P. R. Halmos, A Hilbert space problem book, Van Nostrand, Princeton, 1967.
[15] M. S. Moslehian, M. Khosravi, and R. Drnovs̆ek, A commutator approach to Buzano inequality, Filomat 26 (2012), no. 4, 827-832.
[16] K. Paul, S. Hossein and K. Das, Orthogonality on $B(H, H)$ and minimal-norm operator, J. Anal. Appl. 6 (2008), no. 3, 169-178.
[17] K. Paul, Translatable radii of an operator in the direction of another operator, Sci. Math. 2 (1999), no. 1, 119-122.
[18] M. Sababheh, H. R. Moradi and Z. Heydarbeygi, Buzano, Krein and Cauchy-Schwarz inequalitites, Oper. Matrices, 16, (2022) 1,239-250.
[19] J. G. Stampfli, The norm of a derivation, Pacific J. Math. 33 (1970), 737-747.
${ }^{\left[11_{a}\right]}$ Universidad Nacional de Río Negro. Centro Interdisciplinario de Telecomunicaciones, Electrónica, Computación Y Ciencia Aplicada (CitecCa), Sede Andina (8400) S.C. de Bariloche, Argentina.
${ }^{\left[11_{b}\right]}$ Consejo Nacional de Investigaciones Científicas y Técnicas, (1425) Buenos Aires, Argentina.

Email address: tbottazzi@unrn.edu.ar
${ }^{[2]}$ Instituto de Ciencias, Universidad Nacional de Gral. Sarmiento, J. M. Gutierrez 1150, (B1613GSX) Los Polvorines, Argentina

Email address: cconde@campus.ungs.edu.ar


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