Generalized Buzano Inequality

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ABSTRACT. If P is an orthogonal projection defined on an inner product space \mathcal{H} , then the inequality

$$|\langle Px, y \rangle| \le \frac{1}{2} [||x|| ||y|| + |\langle x, y \rangle|]$$

fulfills for any $x,y\in\mathcal{H}$ (see [10]). In particular, when P is the identity operator, then it recovers the famous Buzano inequality. We obtain generalizations of such classical inequality, which hold for certain families of bounded linear operators defined on \mathcal{H} . In addition, several new inequalities involving the norm and numerical radius of an operator are established.

1. Introduction

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex numbers field \mathbb{K} . The following inequality is well known in literature as the Cauchy-Schwarz inequality

$$|\langle x, y \rangle| \le ||x|| ||y||, \tag{1.1}$$

for any $x, y \in \mathcal{H}$. The equality in (1.1) holds if and only if there exists a constant $\alpha \in \mathbb{K}$ such that $x = \alpha y$.

In [6], Maria Luisa Buzano gave the following extension of the celebrated Cauchy–Schwarz inequality in \mathcal{H}

$$|\langle x, z \rangle \langle z, y \rangle| \le \frac{1}{2} (|\langle x, y \rangle| + ||x|| ||y||) ||z||^2, \tag{1.2}$$

for any $x, y, z \in \mathcal{H}$. Last inequality is called Buzano inequality.

The original proof of Buzano has it difficulty since it requires some facts about orthogonal decomposition of a complete inner product space.

In [8], Dragomir established a refinement of (1.1) which implies the Buzano inequality. Moreover, Fuji and Kubo [13] gave a simpler proof of (1.2) by using an orthogonal projection on a subspace of \mathcal{H} and (1.1). Furthermore, they characterized when the equality holds.

This paper aims to present new generalizations of Buzano inequality and it is organized as follows. Section 2 contains some definitions and usual results about bounded linear operators defined on a Hilbert space. In Section 3, we present and prove the $\frac{1}{\alpha}$ -Buzano inequality (if $\alpha = 2$ gives the classical Buzano inequality) and it is devoted to

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describing different families of operators which fulfill such inequality for different values of the parameter α . Finally, in Section 4 relates the distinct inequalities previously obtained with the numerical radius, improving new bounds for the last one.

2. Preliminaries

As any pre-Hilbert space can be completed to a Hilbert space, from now on, we suppose that \mathcal{H} is a Hilbert space. Let $\mathcal{B}(\mathcal{H})$ denote the C^* -algebra of all bounded linear operators acting on a separable non trivial complex Hilbert space \mathcal{H} with an inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. The symbol I stands for the identity operator and $\mathcal{GL}(\mathcal{H})$ denotes the group of invertible operators on \mathcal{H} .

The range of every operator is denoted by $\mathcal{R}(T)$, its null space by $\mathcal{N}(T)$. If $T \in \mathcal{B}(\mathcal{H})$, we say that T is a positive operator, $T \geq 0$, whenever $\langle Tx, x \rangle \geq 0$ for all $x \in \mathcal{H}$ and we denote by $\mathcal{B}(\mathcal{H})^+$, the subset of all positive bounded linear operators definded on \mathcal{H} . The definition of positivity induces the order $T \geq S$ for self-adjoint operators if and only if $T - S \geq 0$. For any $T \in \mathcal{B}(\mathcal{H})^+$, there exists a unique positive $T^{1/2} \in \mathcal{B}(\mathcal{H})$ such that $T = (T^{1/2})^2$. Let T^* be the adjoint of T and $|T| = (T^*T)^{1/2}$.

The polar decomposition theorem asserts that for every operator $T \in \mathcal{B}(\mathcal{H})$ there is a partial isometry $V \in \mathcal{B}(\mathcal{H})$ such that can be written as the product T = V|T|. In particular, V satisfying $\mathcal{N}(V) = \mathcal{N}(T)$ exists and is uniquely determined.

For any $T \in \mathcal{B}(\mathcal{H})$, we denote by $\sigma(T)$ its spectrum and by $\sigma_{app}(T)$ its approximate point spectrum, that is

$$\sigma_{app}(T) = \{ \lambda \in \mathbb{C} : \exists \{x_n\}_{n \in \mathbb{N}}, ||x_n|| = 1 \text{ and } \lim_{n \to \infty} ||Tx_n - \lambda x_n|| = 0 \}.$$

For any $T \in \mathcal{B}(\mathcal{H})$, we define $m(T) = \inf\{\|Tx\| : x \in \mathcal{H}, \|x\| = 1\}$. Clearly, $m(T) \geq 0$ and m(T) > 0 if and only if $0 \notin \sigma_{app}(T)$ ([16]).

For a linear operator T on a Hilbert space \mathcal{H} , the numerical range W(T) is the image of the unit sphere of \mathcal{H} under the quadratic form $x \to \langle Tx, x \rangle$. More precisely,

$$W(T) = \{ \langle Tx, x \rangle : x \in \mathcal{H}, ||x|| = 1 \}.$$

The numerical range of an operator is a convex subset of the complex plane ([14]). Then, for any T in $\mathcal{B}(\mathcal{H})$ we define the numerical radius of T,

$$\omega(T) = \sup\{|\lambda| : \lambda \in W(T)\}.$$

It is well-known that $\omega(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$, and we have for all $T \in \mathcal{B}(\mathcal{H})$,

$$\frac{1}{2}||T|| \le \omega(T) \le ||T||. \tag{2.1}$$

Thus, the usual operator norm and the numerical radius are equivalent. Inequalities in (2.1) are sharp if $T^2 = 0$, then the first inequality becomes equality, while the second inequality becomes an equality if T is normal.

For any compact operator $T \in \mathcal{B}(\mathcal{H})$ and $j \in \mathbb{N}$, let $s_j(T) = \lambda_j(|T|)$, be the j-th singular value of T, i.e. the j-th eigenvalue of |T| in decreasing order and repeated according to multiplicity. Let $\operatorname{tr}(\cdot)$ be the trace functional,

$$\operatorname{tr}(T) = \sum_{j=1}^{\infty} \langle Te_j, e_j \rangle,$$

where $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis of \mathcal{H} . Note that this coincides with the usual definition of the trace if \mathcal{H} is finite-dimensional.

Let $T = x \otimes y$ be a rank one operator defined by $T(z) = \langle z, y \rangle x$ with $x, y, z \in \mathcal{H}$. Then, by Lemma 2.1 in [7] and using the well-known fact that $\operatorname{tr}(x \otimes y) = \langle x, y \rangle$, we obtain

$$\omega(x \otimes y) = \frac{1}{2} (|\text{tr}(x \otimes y)| + ||x \otimes y||) = \frac{1}{2} (|\langle x, y \rangle| + ||x|| ||y||).$$

We remark that the numerical radius of the rank one operator $T = x \otimes y$ coincides with the upper bound of Buzano inequality. From this fact, we are able to give a new proof of inequality (1.2) using this fact. If ||z|| = 1, then $\langle Tz, z \rangle = \langle z, y \rangle \langle x, z \rangle \in W(T)$ and

$$|\langle x, z \rangle \langle z, y \rangle| = |\langle Tz, z \rangle| \le \omega(T) = \frac{1}{2} (|\langle x, y \rangle| + ||x|| ||y||).$$

For $T \in \mathcal{B}(\mathcal{H})$, we have, by definition,

$$dist(I, \mathbb{C}T) := \inf_{\gamma \in \mathbb{C}} \|\gamma T - I\| \text{ and } dist(T, \mathbb{C}I) := \inf_{\beta \in \mathbb{C}} \|T - \beta I\|.$$

Evidently there is at least one complex number $\gamma_0 \in \mathbb{C}$ such that $dist(I, \mathbb{C}T) = \|\gamma_0 T - I\|$ and in addition, if m(T) > 0 then the value γ_0 is unique. Following Stampfli [19], we call such scalar as the center of mass of T and we denote by c(T). For $A, T \in \mathcal{B}(\mathcal{H})$ such that m(T) > 0 we consider

$$M_T(A) = \sup_{\|x\|=1} \left[\|Ax\|^2 - \frac{|\langle Ax, Tx \rangle|^2}{\|Tx\|^2} \right]^{1/2}.$$
 (2.2)

In [17], Paul proved that $M_T(A) = dist(A, \mathbb{C}T)$.

Given $T, S \in \mathcal{B}(\mathcal{H})$ we said that T is Birkhoff-James orthogonal to S if and only if $||T|| \leq ||T - \lambda S||$ for every $\lambda \in \mathbb{C}$.

3.
$$\frac{1}{\alpha}$$
-Buzano inequality

In the last decades, several mathematicians presented different proofs of Buzano inequality. We start by presenting a new and simple proof of such inequality using a rank one operator.

Given $z \in \mathcal{H}$ with ||z|| = 1 and $\alpha \in \mathbb{C}$, we consider the rank one operator $T = z \otimes z$. Then, for any $u \in \mathcal{H}$, it holds

$$\|(\alpha T - I)u\|^2 = \|\alpha Tu - u\|^2 = (|\alpha - 1|^2 - 1)|\langle z, u \rangle|^2 + \|u\|^2 \le \max\{1, |\alpha - 1|^2\} \|u\|^2.$$

Hence $\|\alpha T - I\| \leq \max\{1, |\alpha - 1|\}$ and for any $x, y \in \mathcal{H}$ we get

$$|\langle (\alpha T - I) \, x, y \rangle| \leq \|T - I\| \|x\| \|y\| \leq \max\{1, |\alpha - 1|\} \|x\| \|y\|.$$

In conclusion, we have

$$|\alpha\langle x, z\rangle\langle z, y\rangle - \langle x, y\rangle| \le \max\{1, |\alpha - 1|\} ||x|| ||y||, \tag{3.1}$$

for any $x, y, z \in \mathcal{H}$ with ||z|| = 1 and $\alpha \in \mathbb{C}$. If $\alpha \in \mathbb{C} - \{0\}$, then (3.1) is equivalent to

$$\left| \langle x, z \rangle \langle z, y \rangle - \frac{1}{\alpha} \langle x, y \rangle \right| \le \frac{1}{|\alpha|} \max\{1, |\alpha - 1|\} ||x|| ||y||.$$

From the continuity property of modulus for complex numbers, we obtain

$$|\langle x, z \rangle \langle z, y \rangle| \le \frac{1}{|\alpha|} (|\langle x, y \rangle| + \max\{1, |\alpha - 1|\} ||x|| ||y||),$$

for any $x, y, z \in \mathcal{H}$ with ||z|| = 1. The value $\alpha = 2$ gives Buzano inequality.

We note that the inequality (3.1) was previously obtained by Moslehian et al. ([15], Corollary 2.5) using properties of singular values.

The main idea in the previous proof was to obtain a bound for the distance between a rank one operator and the identity operator. On the other hand, Fujii and Kubo in [13] based their proof of Buzano inequality on the fact that $||2P - I|| \le 1$ where P is an orthogonal projection. Because of the above, we are in a position to prove our first result in this paper, which generalizes these previous ideas.

Proposition 3.1. Let $T \in \mathcal{B}(\mathcal{H})$ and $\alpha \in \mathbb{C} - \{0\}$, with $\|\alpha T - I\| \leq 1$. Then, for any $x, y \in \mathcal{H}$

$$\left| \langle Tx, y \rangle - \frac{1}{\alpha} \langle x, y \rangle \right| \le \frac{1}{|\alpha|} ||x|| ||y||,$$

and

$$|\langle Tx, y \rangle| \le \left| \langle Tx, y \rangle - \frac{1}{\alpha} \langle x, y \rangle \right| + \frac{1}{|\alpha|} |\langle x, y \rangle| \le \frac{1}{|\alpha|} (|\langle x, y \rangle| + ||x|| ||y||). \tag{3.2}$$

On the other, if T fulfills

$$|\langle Tx, y \rangle| \le \frac{1}{|\alpha|} (|\langle x, y \rangle| + ||x|| ||y||), \tag{3.3}$$

for any $x, y \in \mathcal{H}$ and for some $\alpha \in \mathbb{C} - \{0\}$. Then, $dist(\alpha T, \mathbb{C}I) \leq 1$.

Proof. Let $x, y \in \mathcal{H}$ and $\alpha \in \mathbb{C} - \{0\}$, with $\|\alpha T - I\| \leq 1$. By (1.1), we have

$$\left| \langle Tx, y \rangle - \frac{1}{\alpha} \langle x, y \rangle \right| = \left| \left\langle \left(T - \frac{1}{\alpha} I \right) x, y \right\rangle \right| \le \frac{1}{|\alpha|} \|\alpha T - I\| \|x\| \|y\|$$
$$\le \frac{1}{|\alpha|} \|x\| \|y\|.$$

Therefore, we obtain

$$|\langle Tx,y\rangle| \leq \left|\langle Tx,y\rangle - \frac{1}{\alpha}\langle x,y\rangle\right| + \frac{1}{|\alpha|}|\langle x,y\rangle| \leq \frac{1}{|\alpha|}(|\langle x,y\rangle| + ||x||||y||).$$

If T satisfies (3.3) and we recall the formula which express the distance from T to the one-dimensional subspace $\mathbb{C}I$ (see [2]),

$$dist(T,\mathbb{C}I) = \sup\{|\langle Tx,y\rangle|: \|x\| = \|y\| = 1, \langle x,y\rangle = 0\},\$$

then there exists $\beta \in \mathbb{C} - \{0\}$ such that $\|\alpha T - \beta I\| \leq 1$.

The next example shows that not all bounded linear operator satisfies the hypothesis of Proposition 3.1.

Example 3.1. Consider the unilateral shift operator, $T: l^2(\mathbb{N}) \to l^2(\mathbb{N})$, defined by $T(x_1, x_2, x_3, \dots,) = (0, x_1, x_2, x_3, \dots)$. Let $e_1 = (1, 0, 0, \dots) \in l^2(\mathbb{N})$, then $||e_1|| = 1$ and $\langle -Te_1, e_1 \rangle = 0$, by Theorem 2.1 in [1], we have that for any $\alpha \in \mathbb{C} - \{0\}$ it holds

$$||I||^2 + |\alpha|^2 m^2 (-T) \le ||I - \alpha T||^2.$$

As -T is a left invertible operator, then m(-T) > 0 and in consequence

$$1 < ||I||^2 + |\alpha|^2 m^2 (-T) \le ||I - \alpha T||^2.$$

For convenience, for any $\alpha \in \mathbb{C} - \{0\}$, we denote by

$$\mathcal{A}_{\alpha} = \{ T \in \mathcal{B}(\mathcal{H}) : \|\alpha T - I\| \le 1 \},$$

the set of bounded operators that fulfills the hypothesis of Proposition 3.1. Since we have proved that each operator belonging to the set \mathcal{A}_{α} satisfies a $\frac{1}{\alpha}$ Buzano-type inequality, we will call to such set the $\frac{1}{\alpha}$ Buzano set.

Next, we collect some properties of \mathcal{A}_{α} .

Proposition 3.2. Let $\alpha \in \mathbb{C} - \{0\}$, then

(1) A_{α} is a non-empty convex and closed set.

For any $T \in \mathcal{A}_{\alpha}$, then

- (2) $||T|| \leq \frac{2}{|\alpha|}$.
- (3) $T^* \in \mathcal{A}_{\overline{\alpha}}$
- (4) If $S \in \mathcal{A}_{\alpha}$, then $T + S \in \mathcal{A}_{\frac{\alpha}{2}}$.
- (5) If $\|\alpha T I\| < 1$, then $T \in \mathcal{GL}(\mathcal{H})$.
- (6) If T is self-adjoint, then $T \ge 0$ or $-T \ge 0$.
- (7) If $T = h \otimes h$, $h \in \mathcal{H}$ and $h \neq 0$, then $\alpha = \frac{te^{i\theta}+1}{\|h\|^2}$ for every $\theta \in [0, 2\pi]$ and $t \in [-1, 1]$.
- (8) $d(T, \mathbb{C}I) \leq \frac{1}{|\alpha|}$ and for every $P \geq 0$ with tr(P) = 1

$$tr(|T|^2P) - |tr(TP)|^2 \le \frac{1}{|\alpha|}.$$

Proof. (1) Let $T, S \in \mathcal{A}_{\alpha}$ and $\lambda \in [0, 1]$, then

$$\begin{aligned} \|\alpha(\lambda T + (1 - \lambda)S) - I\| &\leq \|\alpha\lambda T - \lambda I\| + \|\alpha(1 - \lambda)S - (1 - \lambda)I\| \\ &= \lambda \|\alpha T - I\| + (1 - \lambda)\|\alpha S - I\| \\ &\leq \lambda + 1 - \lambda = 1. \end{aligned}$$

This shows that $\lambda T + (1 - \lambda)S \in \mathcal{A}_{\alpha}$ and therefore, \mathcal{A}_{α} is convex.

Now, let $\{T_n\}$ be a sequence in \mathcal{A}_{α} such that converges to $T \in \mathcal{B}(\mathcal{H})$. We must show that $T \in \mathcal{A}_{\alpha}$. Then, we have for any $n \in \mathbb{N}$ that it hold

$$\|\alpha T - I\| = \|\alpha T - \alpha T_n + \alpha T_n - I\| \le |\alpha| \|T_n - T\| + 1.$$

Taking limit when n tends to infinity we obtain $\|\alpha T - I\| \le 1$, i.e. $T \in \mathcal{A}_{\alpha}$.

The proof of items (2), (3), (4), and (5) are trivial.

(6) If T is normal, then $(\alpha T - I)^*(\alpha T - I) = (\alpha T - I)(\alpha T - I)^*$, for every $\alpha \in \mathbb{C}$. Therefore, $\alpha T - I$ is normal and

$$r(\alpha T - I) = \omega(\alpha T - I) = ||\alpha T - I||,$$

where $r(\alpha T - I) = \sup\{|\beta| : \beta \in \sigma(\alpha T - I)\}$ is the spectral radius. By the spectral theorem

$$\|\alpha T - I\| = r(\alpha T - I) = \sup\{|\alpha \lambda - 1| : \lambda \in \sigma(T)\}.$$

Thus, if T is selfadjoint and $T \in \mathcal{A}_{\alpha}$

$$|\alpha\lambda - 1| \le 1$$
,

for all $\lambda \in \sigma(T)$. Therefore, each $\alpha\lambda$ must lie in the unit disk in the complex plane centered in z = 1. Also,

$$\lambda \in \mathbb{R} \text{ for every } \lambda \in \sigma(T) \Rightarrow \left\{ \begin{array}{ll} Re(\lambda \alpha) = \lambda Re(\alpha) \in & [0,2] \\ Im(\lambda \alpha) = \lambda Im(\alpha) \in & [-1,1] \end{array} \right.$$

Suppose there exist $\lambda_j, \lambda_k \in \sigma(T)$ such that $\lambda_j < 0$ and $\lambda_k > 0$, then

$$Re(\lambda_j \alpha) \in [0, 2] \Rightarrow 0 \ge Re(\alpha) \ge \frac{2}{\lambda_j}$$

and

$$Re(\lambda_k \alpha) \in [0, 2] \Rightarrow 0 \le Re(\alpha) \le \frac{2}{\lambda_k}$$
.

Thus, $Re(\alpha) = 0$, which means that $\lambda_i \alpha$ is pure imaginary and $\alpha = 0$ (because the unit disk in the complex plane centered in z = 1 intersects the imaginary axis only in z=0). This is a contradiction, so λ_i and λ_k have the same sign.

(7) For any non-zero $h \in \mathcal{H}$, $T = h \otimes h$ is a compact, positive operator and its spectrum has only two eigenvalues, $||h||^2$ and 0. Then, by the proof of item (6)

$$\left|\alpha\|h\|^2 - 1\right| \le 1$$

if and only if $\alpha \|h\|^2 = te^{i\theta} + 1$, with $|t| \le 1$ and $\theta \in [0, 2\pi]$. (8) $1 \ge \|\alpha T - I\| = |\alpha| \|T - \frac{1}{\alpha}I\| \ge |\alpha| d(T, \mathbb{C}I) \Rightarrow \frac{1}{|\alpha|} \ge d(T, \mathbb{C}I)$ and using Proposition 3.1 in [4] we obtain that

$$tr(|T|^2P) - |tr(TP)|^2 \le \frac{1}{|\alpha|}$$

for every $P \ge 0$ with tr(P) = 1.

Next, we enumerate other properties of \mathcal{A}_{α} related to $dist(I,\mathbb{C}T)$ and the center of mass of T.

Lemma 3.1. Let $T \in \mathcal{B}(\mathcal{H})$.

- (1) If m(T) > 0 and $c(T) \neq 0$, then $T \in \mathcal{A}_{c(T)}$.
- (2) If m(T) > 0 and c(T) = 0, then I is Birkhoff-James orthogonal to T. We deduce that $1 < \|\alpha T - I\|$ and $T \notin \mathcal{A}_{\alpha}$, for every $\alpha \in \mathbb{C} - \{0\}$.
- (3) If $dist(I, \mathbb{C}T) = 1$ and there exists $\alpha_0 \in \mathbb{C} \{0\}$ such that

$$1 = \|\alpha_0 T - I\| < \|\alpha T - I\|, \text{ for all } \alpha \in \mathbb{C} - \{\alpha_0, 0\},$$

then $0 \in \sigma_{app}(T)$ and $T \in \mathcal{A}_{\alpha_0}$.

Proposition 3.3. Let $T \in \mathcal{B}(\mathcal{H})$ such that m(T) > 0, $c(T) \neq 0$ and $dist(T, \mathbb{C}I) = ||T||$, then ||I - c(T)T|| = 1 and, in particular, $T \in \mathcal{A}_{c(T)}$.

Proof. Let $x \in \mathcal{H}$ such that ||x|| = 1, then $|\langle Tx, x \rangle| \ge ||Tx|| \inf_{||y|| = 1} \frac{|\langle Ty, y \rangle|}{||Ty||}$ and

$$\left[||Tx||^2 - |\langle Tx, x \rangle|^2 \right]^{1/2} \le \left(1 - \inf_{\|y\| = 1} \frac{|\langle Ty, y \rangle|^2}{\|Ty\|^2} \right)^{1/2} \|Tx\|.$$

Calculating the supremum of both sides, and using the equality (2.2), we get

$$||T|| = dist(T, \mathbb{C}I) \le dist(I, \mathbb{C}T)||T||.$$

Then $1 \leq dist(I, \mathbb{C}T) \leq ||I|| = 1$, i.e. $dist(I, \mathbb{C}T) = ||I - c(T)T|| = 1$. This completes the proof.

As we have shown in Proposition 3.2, if $T \in \mathcal{A}_{\alpha}$ with $\|\alpha T - I\| < 1$, then $T \in \mathcal{GL}(\mathcal{H})$ and T verifies Proposition 3.1. Now, we obtain a generalization of such statement for any invertible operator. In order to prove it, we need the following result.

Lemma 3.2 (Corollary 3.7, [5]). If $T \in \mathcal{GL}(\mathcal{H})$, then there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ and a non-zero complex number β such that $\|\beta T^{-1} - U^*\| < 1$.

Observe that in the value β in Lemma 3.2 satisfies that $|\beta| < \frac{2}{\|T^{-1}\|}$.

Theorem 3.1. Let $T \in \mathcal{GL}(\mathcal{H})$, then there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ and a non-zero complex number β as in Lemma 3.2 such that for any $x, y \in \mathcal{H}$

$$\left| \langle T^{-1}x, y \rangle - \frac{1}{\beta} \langle U^*x, y \rangle \right| \le \frac{1}{|\beta|} ||x|| ||y||,$$

and

$$|\langle T^{-1}x,y\rangle| \le \left|\langle T^{-1}x,y\rangle - \frac{1}{\beta}\langle U^*x,y\rangle\right| + \frac{1}{|\beta|}|\langle U^*x,y\rangle| \le \frac{1}{|\beta|}(|\langle U^*x,y\rangle| + ||x|| ||y||).$$

Proof. It is analogous to the proof of Proposition 3.1, so we omit it.

3.1. Bounded Linear operators which belong to \mathcal{A}_{α} . We begin this subsection showing when a normal operator belongs to \mathcal{A}_{α} .

Theorem 3.2. Let T be a normal operator in $\mathcal{B}(\mathcal{H}) - \{0\}$, such that $\sigma(T)$ is fully included into an arc of the disk of radius ||T|| and centered in the origin, with central angle less than π .

Then, $T \in \mathcal{A}_{\alpha}$ for every $\alpha \in \mathbb{C}$ such that

$$\arg(\alpha) + \arg(\lambda) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \text{ for all } \lambda \in \sigma(T)$$
 (3.4)

and

$$|\alpha| \le \frac{2}{\|T\|} \min_{\lambda \in \sigma(T)} \cos(\arg(\alpha) + \arg(\lambda)).$$
 (3.5)

Proof. Since T is normal, $\alpha T - I$ is a normal operator, then

$$\|\alpha T - I\| = r(\alpha T - I) = \sup\{|\alpha \lambda - 1| : \lambda \in \sigma(T)\} \le 1$$

if and only if $|\alpha \lambda - 1| \le 1$ for every $\lambda \in \sigma(T)$. This is equivalent to find if there exists $\alpha \in \mathbb{C}$ such that

$$|\alpha\lambda - 1|^2 \le 1$$
 for every $\lambda \in \sigma(T)$. (3.6)

Taking $\alpha = |\alpha|e^{i\theta}$ and $\lambda = |\lambda|e^{i\varphi_{\lambda}}$, with $|\lambda| \leq ||T||$,

$$\begin{aligned} \left| |\alpha| |\lambda| e^{i(\theta + \varphi_{\lambda})} - 1 \right|^2 &= \left(|\alpha| |\lambda| \cos(\theta + \varphi_{\lambda}) - 1 \right)^2 + \left(|\alpha| |\lambda| \sin(\theta + \varphi_{\lambda}) \right)^2 \\ &= |\alpha|^2 |\lambda|^2 - 2|\alpha| |\lambda| \cos(\theta + \varphi_{\lambda}) + 1 \end{aligned}$$

and we can rewrite (3.6) as follows

$$|\alpha||\lambda|(|\alpha||\lambda| - 2\cos(\theta + \varphi_{\lambda})) \leq 0.$$

Then, we arrive to the following condition

$$|\alpha||\lambda| - 2\cos(\theta + \varphi_{\lambda}) \le 0. \tag{3.7}$$

Take an $\alpha \in \mathbb{C}$ that satisfies (3.4) and (3.5). For $\lambda = 0$ it is immediate that α satisfies condition (3.7). Consider $\lambda \neq 0$, then $\cos(\arg(\alpha) + \varphi_{\lambda}) > 0$ for every $\lambda \in \sigma(T)$ and

$$|\alpha| \le \frac{2}{\|T\|} \min_{\lambda \in \sigma(T)} \cos(\arg(\alpha) + \arg(\lambda)) \le \frac{2}{|\lambda|} \cos(\arg(\alpha) + \arg(\lambda)), \lambda \ne 0.$$

Therefore, α fulfills the condition (3.7) for every $\lambda \in \sigma(T)$ and we conclude that $\|\alpha T - I\| \le 1$ $(T \in \mathcal{A}_{\alpha})$.

In order to fulfill (3.7) and $\cos(\arg(\alpha) + \varphi_{\lambda}) \geq 0$, it is a necessary condition that the spectrum of T lies in into an arc of the disk of radius ||T|| and centered in the origin, with central angle less than π . Otherwise, it is not possible to fix any α such that the property holds. Additionally, we exclude $\arg(\alpha) + \varphi_{\lambda} = \pm \frac{\pi}{2}$, since $|\alpha| |\lambda| = 0$ if and only if $\lambda = 0$ or $\alpha = 0$.

For example, if T is Hermitian $\varphi_{\lambda} \in \{0, \pi\}$ it can be seen that there is no α that (3.7) holds, unless $\lambda \geq 0$ for every $\lambda \in \sigma(T)$ ($\varphi_{\lambda} = 0$), or $\lambda \leq 0$ for every $\lambda \in \sigma(T)$ ($\varphi_{\lambda} = \pi$). Thus, the unique Hermitian operators T that can reach (3.7) are semidefinite positive or semidefinite negative, as we show in item 6 of Proposition 3.2.

In particular, for positive operators, we arrive to the following result.

Corollary 3.1. If $T \in \mathcal{B}(\mathcal{H})^+$, then $T \in \mathcal{A}_{\alpha}$ for every $\alpha \in \mathbb{C}$ such that $|\alpha| \leq \frac{a}{\|T\|} \leq \frac{2}{\|T\|}$ and $\cos(\arg(\alpha)) \geq \frac{a}{2}$.

Proof. As we mention before, in this case $\varphi_{\lambda} = 0$ for every $\lambda \in \sigma(T)$. Then,

$$|\alpha||\lambda| - 2\cos(\arg(\alpha)) \le \frac{a}{\|T\|}|\lambda| - 2\cos(\arg(\alpha)) \le a - 2\cos(\arg(\alpha)) \le 0.$$

The next result is a generalization of Buzano inequality for any bounded linear operator.

Theorem 3.3. Let $T \in \mathcal{B}(\mathcal{H}) - \{0\}$. Then, for any $x, y \in \mathcal{H}$

$$\left| \langle Tx, Ty \rangle - \frac{\|T\|^2}{2} \langle x, y \rangle \right| \le \frac{\|T\|^2}{2} \|x\| \|y\|,$$

and

$$|\langle Tx, Ty \rangle| \le \left| \langle Tx, Ty \rangle - \frac{\|T\|^2}{2} \langle x, y \rangle \right| + \frac{\|T\|^2}{2} |\langle x, y \rangle| \le \frac{\|T\|^2}{2} (|\langle x, y \rangle| + \|x\| \|y\|). \quad (3.8)$$

Proof. By Corollary 3.1, if $S \in \mathcal{B}(\mathcal{H})^+ - \{0\}$, then $S \in \mathcal{A}_{\frac{2}{\|S\|}}$. In particular, if we consider the positive operator $S = T^*T$, then we conclude that $T^*T \in \mathcal{A}_{\frac{2}{\|T\|^2}}$ and the proof is complete as a consequence of Proposition 3.1

The constant $\frac{||T||^2}{2}$ is best possible in (3.8). Now, if we assume that (3.8) holds with a constant C > 0, i.e.

$$|\langle Tx, Ty \rangle| \le C(|\langle x, y \rangle| + ||x|| ||y||),$$

for any $T \in \mathcal{B}(\mathcal{H})$. So, if we choose x = y, then $||Tx||^2 \le 2C||x||^2$ and we deduce that $2C \ge ||T||^2$. Thus, (3.8) is an improvement and refinement of

$$|\langle Tx, Ty \rangle| \le \frac{||T||^2}{2} (|\langle x, y \rangle| + ||x|| ||y||),$$

which was obtained in a different way earlier by Dragomir in [11] using a non-negative Hermitian form on a Hilbert space.

From the polar decomposition of any bounded linear operator and the main idea used in the proof of Theorem 3.3, we have the following statement.

Corollary 3.2. Let $T \in \mathcal{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$. Then,

$$\begin{aligned} |\langle Tx, y \rangle| &= |\langle |T|x, V^*y \rangle| \le \left| \langle Tx, y \rangle - \frac{\|T\|}{2} \langle x, V^*y \rangle \right| + \frac{\|T\|}{2} |\langle x, V^*y \rangle| \\ &\le \frac{\|T\|}{2} (|\langle x, V^*y \rangle| + \|x\| \|V^*y\|) \\ &\le \frac{\|T\|}{2} (|\langle x, V^*y \rangle| + \|x\| \|y\|). \end{aligned} \tag{3.9}$$

where T = V|T| is the polar decomposition of T.

Remark 3.1. Inequality (3.9) is an improvement and refinement of a result recently obtained by Sababheh et al. (see [18, Remark 3.1]).

Recall that T is called a positive contraction if $0 \le T \le I$. As a consequence of Corollary 3.1, we conclude that $T \in \mathcal{A}_{\alpha}$ for every $\alpha \in [0, 2]$.

Now, we obtain a refinement of the classical Cauchy-Schwarz inequality, using positive contractions. The idea of the proof is based in [11, Theorem 2.1]. Recently, in [18] the same result was obtained with a different proof.

Theorem 3.4. Let $T \in \mathcal{B}(\mathcal{H})$ be a positive contraction and $x, y \in \mathcal{H}$, then

$$|\langle x,y\rangle| + \langle Tx,x\rangle^{1/2} \langle Ty,y\rangle^{1/2} - |\langle Tx,y\rangle| \le ||x|| ||y||.$$

Proof. For any $x, y \in \mathcal{H}$ and the elementary inequality $(ac - bd)^2 \ge (a^2 - b^2)(c^2 - d^2)$, which holds for any real numbers a, b, c, d, we have

$$(||x||||y|| - \langle Tx, x \rangle^{1/2} \langle Ty, y \rangle^{1/2})^{2} \geq (||x||^{2} - \langle Tx, x \rangle)(||y||^{2} - \langle Ty, y \rangle)$$

$$= \langle (I - T)x, x \rangle \langle (I - T)y, y \rangle. \tag{3.10}$$

As T is a positive contraction, then $I - T \in \mathcal{B}(\mathcal{H})^+$. By Cauchy-Schwarz inequality for positive operators

$$\langle (I-T)x, x \rangle \langle (I-T)y, y \rangle \ge |\langle (I-T)x, y \rangle|^2 = |\langle x, y \rangle - \langle Tx, y \rangle|^2. \tag{3.11}$$

Now, by (3.10) and (3.11),

$$\left(\|x\|\|y\| - \langle Tx, x \rangle^{1/2} \langle Ty, y \rangle^{1/2}\right)^2 \ge |\langle x, y \rangle - \langle Tx, y \rangle|^2, \tag{3.12}$$

for any $x, y \in \mathcal{H}$. Since $||x|| \ge \langle Tx, x \rangle^{1/2}$ and $||y|| \ge \langle Ty, y \rangle^{1/2}$, by taking the square root, (3.12) is equivalent to

$$||x|||y|| - \langle Tx, x \rangle^{1/2} \langle Ty, y \rangle^{1/2} \ge |\langle x, y \rangle - \langle Tx, y \rangle|. \tag{3.13}$$

On making use of the triangle inequality for the modulus, we have

$$||x|| ||y|| - \langle Tx, x \rangle^{1/2} \langle Ty, y \rangle^{1/2} \ge |\langle x, y \rangle - \langle Tx, y \rangle| \ge |\langle x, y \rangle| - |\langle Tx, y \rangle|,$$
 (3.14)

and this completes the proof.

Remark 3.2. Recall that any orthogonal projection $P = P^2 = P^*$ is a positive contraction with ||P|| = 1 and $P \in \mathcal{A}_2$. Then, for any $x, y \in \mathcal{H}$

$$|\langle Px, y \rangle| \le \left| \langle Px, y \rangle - \frac{1}{2} \langle x, y \rangle \right| + \frac{1}{2} |\langle x, y \rangle| \le \frac{1}{2} (|\langle x, y \rangle| + ||x|| ||y||), \tag{3.15}$$

and

$$|\langle Px, y \rangle - \langle x, y \rangle| \le \frac{1}{2} (|\langle x, y \rangle| + ||x|| ||y||). \tag{3.16}$$

In (3.15) and (3.16) we reach, with a new proof, an improvement and refinement of different statements previously obtained by Dragomir in [10].

Motivated by the previous inequalities valid for orthogonal projections, we establish some vector inequalities for particular projections. Let $T=z\otimes z$, with $z\in \mathcal{H}$ and ||z||=1, as T is an orthogonal projection then by using inequality (3.2) we get

$$|\langle x,z\rangle\langle z,y\rangle| \leq \left|\langle x,z\rangle\langle z,y\rangle - \frac{1}{2}\langle x,y\rangle\right| + \frac{1}{2}|\langle x,y\rangle| \leq \frac{1}{2}(|\langle x,y\rangle| + ||x|| ||y||),$$

for any $x, y \in \mathcal{H}$. This inequality refines the classical Buzano inequality.

On the other hand, using inequality (3.13)

$$|\langle x, y \rangle| \le |\langle x, y \rangle - \langle x, z \rangle \langle z, y \rangle| + |\langle x, z \rangle \langle z, y \rangle| \le ||x|| ||y||,$$

for any $x, y \in \mathcal{H}$. This refinement of (1.1) was also obtained in [8].

Now we are in position to obtain a Buzano type inequality for the sum of two orthogonal projections. It is well-known that given two orthogonal projections on \mathcal{H} , P and Q, then

$$||P + Q|| = 1 + ||PQ||. (3.17)$$

This is usually called Duncan-Taylor equality, and its proof can be found in [12].

Proposition 3.4. Let P,Q be orthogonal projections on \mathcal{H} . Then, $P+Q \in \mathcal{A}_{\alpha}$ for $|\alpha| \leq \frac{2}{1+\|PQ\|}$.

Proof. Note that $P + Q \in \mathcal{B}(\mathcal{H})^+$ and by (3.17), ||P + Q|| = 1 + ||PQ||. Thus, using Corollary 3.1, the proof is complete.

Throughout, S and T denote two closed subspaces of H. The minimal angle or angle of Dixmier between S and T is the angle $\theta_0(S,T) \in [0,\frac{\pi}{2}]$ whose cosine is defined by

$$c_0(\mathcal{S}, \mathcal{T}) = \sup\{|\langle x, y \rangle| : x \in \mathcal{S}, y \in \mathcal{T}; ||x||, ||y|| \le 1\}.$$

A linear operator defined on \mathcal{H} , such that $Q^2 = Q$ is called a projection. Such operators are not necessarily bounded, since on every infinite-dimensional Hilbert space there exist unbounded examples of projections (see [3]). The operator $Q_{\mathcal{M}//\mathcal{N}}$ is an oblique projection along (or parallel to) its null space $\mathcal{N} = \mathcal{N}(Q)$ onto its range $\mathcal{M} = \mathcal{R}(Q)$.

Theorem 3.5. Let H be a Hilbert space such that is the direct sum of closed subspaces \mathcal{M} and \mathcal{N} . Let $Q_{\mathcal{M}//\mathcal{N}}$, be the bounded projection with range \mathcal{M} and null space \mathcal{N} , and $\theta_0(\mathcal{M}, \mathcal{N})$, be the minimal angle between \mathcal{M} and \mathcal{N} . Then, for any $x, y \in \mathcal{H}$

$$\left| \langle Qx, y \rangle - \frac{1}{2} \langle x, y \rangle \right| \le \frac{\cot(\alpha_0)}{2} ||x|| ||y||,$$

and

$$|\langle Qx, y \rangle| \le \frac{\cot(\alpha_0)}{2} (|\langle x, y \rangle| + ||x|| ||y||),$$

where $\alpha_0 = \frac{\theta_0(\mathcal{M}, \mathcal{N})}{2}$.

Proof. By Theorem 2 in [3] we have that $||Q|| = \csc(\theta_0(\mathcal{M}, \mathcal{N}))$ and $||2Q - I|| = \cot(\alpha_0)$. From the boundness of Q we can assert that $0 < \theta_0(\mathcal{M}, \mathcal{N}) \le \frac{\pi}{2}$ and $\cot(\alpha_0) \ge 1$. From these facts and mimicking the proof of Proposition 3.1, we have that for any $x, y \in \mathcal{H}$

$$\left| \langle Qx, y \rangle - \frac{1}{2} \langle x, y \rangle \right| \le \frac{\cot(\alpha_0)}{2} ||x|| ||y||,$$

and

$$\begin{aligned} |\langle Qx, y \rangle| &\leq \left| \langle Qx, y \rangle - \frac{1}{2} \langle x, y \rangle \right| + \frac{1}{2} |\langle x, y \rangle| \\ &\leq \frac{\cot(\alpha_0)}{2} ||x|| ||y|| + \frac{1}{2} |\langle x, y \rangle| \\ &\leq \frac{\cot(\alpha_0)}{2} (|\langle x, y \rangle| + ||x|| ||y||). \end{aligned}$$

This completes the proof.

We finish this section by showing that any operator whose real part is greater than sI for some s > 0, is invertible and its inverse belongs to \mathcal{A}_{2s} .

Theorem 3.6. Let $T \in \mathcal{B}(\mathcal{H}) - \{0\}$ with $Re(T) = \frac{T+T^*}{2} \ge sI$ for some s > 0. Then, $T^{-1} \in \mathcal{A}_{2s}$.

Proof. First, we show that T is invertible. The hypothesis $Re(T) \geq sI$ implies that

$$W(T)\subseteq\{z\in\mathbb{C}:Re(z)\geq s\},$$

since if $z \in W(T)$ then

$$Re(z) = \frac{z + \overline{z}}{2} = \frac{\langle Tx, x \rangle + \overline{\langle Tx, x \rangle}}{2} = \frac{\langle Tx, x \rangle + \langle T^*x, x \rangle}{2}$$
$$= \langle Re(T)x, x \rangle \ge s.$$

Thus $\sigma(T) \subseteq \overline{W(T)} \subseteq \{z \in \mathbb{C} : Re(z) \geq s\}$ and, in particular, we have that $0 \notin \sigma(T)$, which means $T \in \mathcal{GL}(\mathcal{H})$. If $T + T^* \geq 2sI$ and $T \in \mathcal{GL}(\mathcal{H})$, then $2sT^{-1}(T + T^* - 2sI)(T^*)^{-1} \geq 0$ and

$$I \ge I - 2sT^{-1}(T + T^* - 2sI)(T^*)^{-1} = (I - 2sT^{-1})(I - 2sT^{-1})^*$$

which is equivalent to $||2sT^{-1} - I|| \leq 1$. Hence, $T^{-1} \in \mathcal{A}_{2s}$ and the result follows. \square

4. Bounds for the numerical radius using Buzano inequality

In this section, we use (3.9) to obtain a refinement of the classical inequality $\omega(T) \leq ||T||$ and an upper bound for $\omega(T) - \frac{1}{2}||T||$.

Proposition 4.1. Let $T \in \mathcal{B}(\mathcal{H})$ with polar decomposition T = V|T|. Then,

$$\omega(T) \le \frac{\|T\|}{2} (1 + \omega(V)) \le \|T\|$$
 (4.1)

and

$$\omega(T) - \frac{\|T\|}{2} \le \frac{\|T\|}{2} \omega(V).$$
 (4.2)

Proof. Taking x = y in (3.9), and the supremum over all $x \in \mathcal{H}$ with ||x|| = 1, we obtain $\omega(T) \leq \frac{||T||}{2} (1 + \omega(V))$ and this completes the proof.

It is important to note that inequalities (4.1) and (4.2) are not trivial, since $\omega(V)$ may be less than one, depending on the partial isometry V. For instance, let

$$T = \begin{bmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} = V|T|,$$

where V is a partial isometry in \mathbb{C}^2 with $\ker(T) = \ker(V) = \operatorname{span}\{(0,1)\}$ and $\ker(V)^{\perp} = \operatorname{span}\{(1,0)\}$. Then, for any $x = (x_1, x_2) \in \mathbb{C}$ with $||x|| = \sqrt{|x_1|^2 + |x_2|^2} = 1$, we have as a consequence of the arithmetic-geometric mean inequality

$$|\langle Vx, x \rangle| = |x_1 \overline{x_2}| = |x_1||x_2| \le \frac{|x_1|^2 + |x_2|^2}{2} = \frac{1}{2}.$$

Therefore, $W(V) = \{z \in \mathbb{C} : |z| \leq \frac{1}{2}\}$ and $\omega(V) = \frac{1}{2} < 1$.

Proposition 4.2. Let $T \in \mathcal{B}(\mathcal{H}) - \{0\}$ with polar decomposition T = V|T| such that $\omega(T) = ||T||$, then $\omega(V) = ||V|| = 1$.

Proof. From inequality (4.1), we get

$$\omega(T) \le \frac{\|T\|}{2} (1 + \omega(V)) \le \|T\|.$$

Thus, if $\omega(T) = ||T||$, then $1 + \omega(V) = 2$, and hence $\omega(V) = 1$. As V is a nonzero partial isometry, therefore ||V|| = 1 and thus $\omega(V) = ||V|| = 1$, as required.

It should also be mentioned here that the converse of Proposition 4.2 is not true. To see this, consider

$$T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = V|T|.$$

where V|T| is a polar decomposition of T. Then, $\omega(V) = ||V|| = 1$, but $\omega(T) = \frac{1}{\sqrt{2}} < 1 = ||T||$.

In order to estimate how close the numerical radius is from the operator norm, the following reverse inequalities have been obtained under appropriate conditions for the involved operator $T \in \mathcal{B}(\mathcal{H})$. If $T \in \mathcal{A}_{\alpha}$, then by (2.1) and Proposition 3.2 we have that

$$0 \le ||T|| - \omega(T) \le ||T|| - \frac{||T||}{2} \le \frac{1}{|\alpha|}.$$

Motivated by the above inequality, we establish a new upper bound for the non-negative quantity $||T|| - \omega(T)$.

Theorem 4.1. Let $T \in \mathcal{A}_{\alpha}$. Then,

$$||T|| - \omega(T) \le \frac{1}{2|\alpha|}.$$

Proof. For $x \in \mathcal{H}$ with ||x|| = 1, we have

$$\|\alpha Tx - x\|^2 = |\alpha|^2 \|Tx\|^2 - 2Re(\alpha \langle Tx, x \rangle) + 1 \le 1$$

giving

$$|\alpha|^2 ||Tx||^2 + 1 \le 1 + 2Re(\alpha \langle Tx, x \rangle) \le 2|\alpha| |\langle Tx, x \rangle| + 1$$

By arithmetic-geometric mean inequality, we deduce

$$2|\alpha|||Tx|| \le |\alpha|^2||Tx||^2 + 1 \le 1 + 2Re(\alpha\langle Tx, x\rangle) \le 2|\alpha||\langle Tx, x\rangle| + 1. \tag{4.3}$$

Now, taking the supremum over $x \in \mathcal{H}$, ||x|| = 1 in (4.3), we obtain

$$2|\alpha||T|| - 2|\alpha|\omega(T) \le 1.$$

Now, we derive upper bounds for the numerical radius of products of bounded linear operators.

Theorem 4.2. Let $R, S, T \in \mathcal{B}(\mathcal{H})$ such that $T \in \mathcal{A}_{\alpha}$. Then,

$$\omega(STR) \le \frac{1}{|\alpha|} (\|R\| \|S\| + \omega(SR)).$$
 (4.4)

Proof. From inequality (3.2), we have

$$|\langle STRx, y \rangle| = |\langle TRx, S^*y \rangle| \le \frac{1}{|\alpha|} (|\langle Rx, S^*y \rangle| + ||Rx|| ||S^*y||).$$

Taking y = x and the supremum over $x \in \mathcal{H}$ with ||x|| = 1, yields the desired inequality.

We note that the previous result is a generalization of Theorem 3.6 in [11]. In particular, for the sum of two orthogonal projections, we obtain the following result.

Corollary 4.1. Let $P, Q, R, S \in \mathcal{B}(\mathcal{H})$ with P, Q be orthogonal projections. Then,

$$\omega(R(P+Q)S) \le \frac{1+\|PQ\|}{2} (\|S\|\|R\|+\omega(RS)).$$

Proof. As we have already mentioned, $P + Q \in \mathcal{A}_{\frac{2}{1+||PQ||}}$. Then, the statement is a consequence of Theorem 4.2.

On the other hand, in [9], Dragomir obtained, utilizing Buzano's inequality, the following inequality for the numerical radius

$$\omega(S)^2 \le \frac{1}{2} (\|S\|^2 + \omega(S^2)),$$

combining with the following power inequality for the numerical radius, $w(S^n) \leq w(S)^n$ for any natural number n, we have

$$w(S^2) \le \frac{1}{2} (\|S\|^2 + \omega(S^2)).$$
 (4.5)

The following corollary, which is an immediate consequence of Theorem 4.2 considering R = S, gives a generalization of (4.5).

Corollary 4.2. Let $S, T \in \mathcal{B}(\mathcal{H})$ with $T \in \mathcal{A}_{\alpha}$. Then,

$$\omega(STS) \le \frac{1}{|\alpha|} \left(||S||^2 + \omega(S^2) \right).$$

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