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BEST SIMULTANEOUS MONOTONE APPROXIMANTS IN ORLICZ SPACES

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□ *Let* $\mathbf{f} = (f_1, \ldots, f_m)$, *where* f_j *belongs to the Orlicz space* $\mathcal{L}_\phi[0,1]$, *and let* $\mathbf{w} = (w_1, \ldots, w_m)$ *be an m-tuple of m positive weights. If* $\mathcal{D} \subset \mathcal{L}_\phi[0,1]$ *is the class of nondecreasing functions, we denote by* $\mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathcal{D})$ *the set of best simultaneous monotone approximants to* $\mathbf{f}$, *that is, all the elements* $g \in \mathcal{D}$ *minimizing* $\sum_{j=1}^m \int_0^1 \phi(|f_j - g|) w_j$, *where* ϕ *is a convex function,* $\phi(t) > 0$ *for* $t > 0$, *and* $\phi(0) = 0$. *In this work, we show an explicit formula to calculate the maximum and minimum elements in* $\mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathcal{D})$. *In addition, we study the continuity of the best simultaneous monotone approximants.*

Keywords Monotone approximation; Orlicz spaces; Simultaneous approximation.

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1. INTRODUCTION

Let $\mathcal{M}_0$ be the set of all real extended μ-measurable functions on $[0,1]$, where μ is the Lebesgue measure, and let $\phi : [0,\infty) \to [0,\infty)$ be a differentiable and convex function, $\phi(0) = 0$, $\phi(t) > 0$ for $t > 0$. For $f \in \mathcal{M}_0$, let

$$\Psi_\phi(f) := \int_0^1 \phi(|f(x)|)\, d\mu(x).$$

We will deal with the Orlicz space

$$\mathcal{L}_\phi[0,1] := \{f \in \mathcal{M}_0 : \Psi_\phi(\lambda f) < \infty \text{ for some } \lambda > 0\}.$$

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Under the Luxemburg norm, $\mathscr{L}_\phi[0,1] =: \mathscr{L}_\phi$ is a Banach space. It is easy to see that if $\phi(t) = t^p$, $1 \le p < \infty$, we obtain the Lebesgue space L_p and $\Psi_\phi(f) = \|f\|_p^p$.

We assume that ϕ satisfies the Δ_2-condition, that is, there exists $K > 0$ such that $\phi(2t) \le K\phi(t)$ for all $t \ge 0$. So,

$$\mathscr{L}_\phi = \{f \in \mathscr{M}_0 : \Psi_\phi(\lambda f) < \infty \quad \text{for all } \lambda > 0\}.$$

We refer to [7, 13] for a detailed treatment of this subject.

Throughout this paper, $f_j \in \mathscr{L}_\phi$, $j = 1, 2, \dots, m$, and we write $\mathbf{f} = (f_1, \dots, f_m)$. Given $\mathscr{D} \subset \mathscr{L}_\phi$, we consider the problem of finding $g \in \mathscr{D}$ such that

$$\sum_{j=1}^{m} \Psi_\phi(f_j - g) w_j = \inf_{h \in \mathscr{D}} \sum_{j=1}^{m} \Psi_\phi(f_j - h) w_j =: \mathbf{E}, \tag{1}$$

where w_j are positive real numbers. We denote by $\mathscr{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathscr{D})$ the set of elements $g \in \mathscr{D}$ satisfying (1), where $\mathbf{w} = (w_1, \dots, w_m)$. Each element of $\mathscr{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathscr{D})$ is called a *best (simultaneous) approximant to* $\mathbf{f}$ *from* $\mathscr{D}$.

When $\mathscr{D}$ is the convex cone of nondecreasing functions in $\mathscr{L}_\phi$ and $\mathbf{f}$ is a single function ($m = 1$), it is known that $\mathscr{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathscr{D}) \neq \emptyset$ and there exist $\min \mathscr{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathscr{D})$ and $\max \mathscr{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathscr{D})$ in $\mathscr{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathscr{D})$, that is, there exist elements in $\mathscr{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathscr{D})$ that satisfy

$$\min \mathscr{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathscr{D}) \le g \le \max \mathscr{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathscr{D})$$

almost everywhere on $[0,1]$ for all $g \in \mathscr{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathscr{D})$ ([8], Theorems 4 and 14). These results of existence can be obtained for $m > 1$ with analogous proofs. Thus, when $m \ge 1$, in Section 4 of this paper we give a characterization of best simultaneous approximants to $\mathbf{f}$ from $\mathscr{D}$, as well as an explicit formula to calculate $\min \mathscr{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathscr{D})$ and $\max \mathscr{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathscr{D})$. In Section 5 we discuss the continuity of a g in $\mathscr{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathscr{D})$ when each f_j is approximately continuous, $j = 1, 2, \dots, m$.

Best monotone approximation to a single function has been studied extensively in the literature [2–4, 18, 19, 21]. In [11] and [20], there are explicit formulas to compute the best monotone approximant to a single function defined on an interval when a p-norm is used. The $\mathscr{L}_\phi$-approximation case was considered in [9].

The problem of best simultaneous monotone approximation to two functions with $\phi(t) = t^p$, $1 \le p < \infty$, it was an early subject of study. In [14], the author gives an algorithm to calculate the best approximant in the discrete case and $p > 1$, and they study the continuity of the best approximant. Similar results can be seen in [16], and in [17] results of characterization are proved in L_1-approximation from convex sets.

Simultaneous monotone approximation to two functions when the measure of deviation of f_1 and f_2 to an element h is $\max\{\|f_1 - h\|_p, \|f_2 - h\|_p\}$, $1 \le p \le \infty$, can be seen in [5, 6, 15].

For $f, g, h \in \mathscr{L}_\phi$, we write

$$N(g) = \{x \in [0,1] : g(x) \neq 0\} \quad \text{and} \quad Z(g) = \{x \in [0,1] : g(x) = 0\},$$

and, throughout this article, we will denote the one-sided Gateaux derivative of Ψ_ϕ at f in the direction of h by

$$\begin{aligned} \gamma_\phi^+(f,h) &:= \lim_{s \to 0^+} \frac{\Psi_\phi(f+sh) - \Psi_\phi(f)}{s} \\ &= \int_{N(f)} \phi'(|f|)\operatorname{sgn}(f) h \, d\mu + \phi'(0) \int_{Z(f)} |h| \, d\mu, \end{aligned} \tag{2}$$

where $\phi'(0)$ is the right derivative of ϕ at 0. Observe that, for $h \ge 0$, (2) can be written

$$\gamma_\phi^+(f,h) = \int_0^1 \phi'(|f|)\overline{\operatorname{sgn}}(f) h \, d\mu, \tag{3}$$

where $\overline{\operatorname{sgn}} = \operatorname{sgn} + \chi_{\{0\}}$, χ_A being the characteristic function of the set A.

2. SIMULTANEOUS APPROXIMATION FROM CONVEX SETS

The following theorem is an immediate consequence of (2) and a modified version of Theorem 1.6 in [12] for convex functionals.

Theorem 1. *Let $\mathscr{K}$ be a convex set in $\mathscr{L}_\phi$. Then $g \in \mathscr{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathscr{K})$ if only if*

$$\sum_{j=1}^m \gamma_\phi^+(f_j - g, g - h) w_j \ge 0 \quad \textit{for all } h \in \mathscr{K}. \tag{4}$$

The next corollary generalizes Lemma 3.2 in [17].

Corollary 2. *Let $\mathscr{K}$ be a convex set in $\mathscr{L}_\phi$. Assume $h \in \mathscr{K}$ and $g \in \mathscr{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathscr{K})$. Then $h \in \mathscr{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathscr{K})$ if only if $\sum_{j=1}^m \gamma_\phi^+(f_j - h, h - g) w_j = 0$.*

Proof. Let $r : [0,1] \to [0,\infty)$ be the convex function defined by

$$r(t) = \sum_{j=1}^m \Psi_\phi(f_j - h + t(h-g)) w_j.$$

Then $r'(0) \le r(1) - r(0)$, that is,

$$\sum_{j=1}^{m} \gamma_\phi^+(f_j - h, h - g)w_j \le \sum_{j=1}^{m} \Psi_\phi(f_j - g)w_j - \sum_{j=1}^{m} \Psi_\phi(f_j - h)w_j. \qquad (5)$$

On the other hand, if $g \in \mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathcal{K})$, we get

$$\sum_{j=1}^{m} \Psi_\phi(f_j - g)w_j - \sum_{j=1}^{m} \Psi_\phi(f_j - h)w_j \le 0.$$

So, (5) implies

$$\sum_{j=1}^{m} \gamma_\phi^+(f_j - h, h - g)w_j \le 0. \qquad (6)$$

If $h \in \mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathcal{K})$, by (6) and Theorem 1 we have $\sum_{j=1}^{m} \gamma_\phi^+(f_j - h, h - g)w_j = 0$.

Reciprocally, if $\sum_{j=1}^{m} \gamma_\phi^+(f_j - h, h - g)w_j = 0$, from (5) and the fact that $g \in \mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathcal{K})$, we get

$$\sum_{j=1}^{m} \Psi_\phi(f_j - g)w_j = \sum_{j=1}^{m} \Psi_\phi(f_j - h)w_j.$$

In consequence, $h \in \mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathcal{K})$. □

We now turn our attention to the uniqueness of the simultaneous approximation from a convex set. This means that if g, h are in $\mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathcal{K})$, then $g = h$ a.e. on $[0, 1]$.

Theorem 3. *Let $\mathcal{K}$ be a convex set in $\mathcal{L}_\phi$. If ϕ is a strictly convex function and g, h are in $\mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathcal{K})$, then $g = h$ a.e. on $[0, 1]$.*

Proof. Assume that there exist $g, h \in \mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathcal{K})$ with $\mu\{g \ne h\} > 0$. Since ϕ is a strictly convex function we have

$$\Psi_\phi\left(f_j - \frac{g+h}{2}\right)w_j < \frac{1}{2}\Psi_\phi(f_j - g)w_j + \frac{1}{2}\Psi_\phi(f_j - h)w_j, \quad j = 1, 2, \ldots, m.$$

So,

$$\sum_{j=1}^{m} \Psi_\phi\left(f_j - \frac{g+h}{2}\right)w_j < \frac{1}{2}\sum_{j=1}^{m} \Psi_\phi(f_j - g)w_j + \frac{1}{2}\sum_{j=1}^{m} \Psi_\phi(f_j - h)w_j = \mathbf{E},$$

which yields a contradiction because $\frac{g+h}{2} \in \mathcal{K}$. □

3. SIMULTANEOUS APPROXIMATION BY CONSTANT FUNCTIONS

Let $f_j \in \mathcal{L}_\phi$, $j = 1, 2, \dots, m$. Throughout this section, $A \subset [0, 1]$ stands for any measurable set with $\mu(A) > 0$. Let $\mathcal{C}_A := \{c\chi_A : c \in \mathbb{R}\}$, and we write $\mathbf{f_A} = (f_1\chi_A, \dots, f_m\chi_A)$.

Since Ψ_ϕ is a convex functional, the function $E : \mathbb{R} \to [0, \infty)$ defined by

$$E(c) = \sum_{j=1}^{m} \Psi_\phi((f_j - c)\chi_A)w_j$$

is convex. Moreover, $\lim_{c\to\pm\infty} E(c) = +\infty$. So, $\mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f_A}, \mathcal{C}_A)$, the set of best constant approximants to $\mathbf{f}$ on A, is a nonempty compact interval. We call

$$\underline{m}(\mathbf{f}, A) = \min \mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f_A}, \mathcal{C}_A) \quad \text{and} \quad \overline{m}(\mathbf{f}, A) = \max \mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f_A}, \mathcal{C}_A).$$

Lemma 4. *A constant c is in $\mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f_A}, \mathcal{C}_A)$ if and only if*

$$\sum_{j=1}^{m} \gamma_\phi^+(f_j - c, \chi_A)w_j \geq 0 \quad \textit{and} \quad \sum_{j=1}^{m} \gamma_\phi^+(c - f_j, \chi_A)w_j \geq 0. \tag{7}$$

Proof. Take $\mathcal{K} = \mathcal{C}_A$ in Theorem 1; now, since $-\mathrm{sgn}(f_j - c) = \mathrm{sgn}(c - f_j)$, the lemma follows immediately from that theorem and (2). □

Lemma 5. *Let $g, h \in \mathcal{L}_\phi$.*

(a) *If $g \leq h$ a.e. on A, then $\gamma_\phi^+(g, \chi_A) \leq \gamma_\phi^+(h, \chi_A)$;*
(b) *If $g < h$ a.e. on A, then $\gamma_\phi^+(g, \chi_A) \leq -\gamma_\phi^+(-h, \chi_A)$.*

Proof. (a) We have

$$\begin{aligned}
&\gamma_\phi^+(g, \chi_A) \\
&\quad = \int_0^1 \phi'(|g|)\overline{\mathrm{sgn}}(g)\chi_A\, d\mu \\
&\quad = \int_0^1 \phi'(|g|)\chi_{A\cap\{g\geq 0\}}\, d\mu - \int_0^1 \phi'(|g|)\chi_{A\cap\{g<0\}\cap\{h\geq 0\}}\, d\mu - \int_0^1 \phi'(|g|)\chi_{A\cap\{h<0\}}\, d\mu \\
&\quad \leq \int_0^1 \phi'(|h|)\chi_{A\cap\{g\geq 0\}}\, d\mu + \int_0^1 \phi'(|h|)\chi_{A\cap\{g<0\}\cap\{h\geq 0\}}\, d\mu - \int_0^1 \phi'(|h|)\chi_{A\cap\{h<0\}}\, d\mu \\
&\quad = \int_0^1 \phi'(|h|)\overline{\mathrm{sgn}}(h)\chi_A\, d\mu = \gamma_\phi^+(h, \chi_A).
\end{aligned}$$

(b) There holds

$$\gamma_\phi^+(g,\chi_A) = \int_0^1 \phi'(|g|)\chi_{A\cap\{g\geq 0\}}\,d\mu - \int_0^1 \phi'(|g|)\chi_{A\cap\{g<0\}}\,d\mu$$
$$\leq \int_0^1 \phi'(|h|)\chi_{A\cap\{h>0\}}\,d\mu - \int_0^1 \phi'(|h|)\chi_{A\cap\{h\leq 0\}}\,d\mu = -\gamma_\phi^+(-h,\chi_A).$$

□

The next Corollary follows immediately from Lemma 5.

Corollary 6. *For $g \in \mathcal{L}_\phi$, the application that assigns $\gamma_\phi^+(g-u,\chi_A)$ to $u \in \mathbb{R}$ is nonincreasing.*

As a consequence of Lemma 5, we have the following theorem of characterization.

Theorem 7. *We have the following relations:*

(a) $\underline{m}(\mathbf{f},A) = \min\{c \in \mathbb{R} : \sum_{j=1}^m \gamma_\phi^+(c-f_j,\chi_A)w_j \geq 0\}$; *and*
(b) $\overline{m}(\mathbf{f},A) = \max\{c \in \mathbb{R} : \sum_{j=1}^m \gamma_\phi^+(f_j-c,\chi_A)w_j \geq 0\}$.

In addition, if ϕ is a strictly convex function, then $\underline{m}(\mathbf{f},A) = \overline{m}(\mathbf{f},A)$.

Proof. (a) From Lemma 4, $\sum_{j=1}^m \gamma_\phi^+(\underline{m}(\mathbf{f},A)-f_j,\chi_A)w_j \geq 0$. Suppose that there exists $u \in \mathbb{R}$, $u < \underline{m}(\mathbf{f},A)$, such that

$$\sum_{j=1}^m \gamma_\phi^+(u-f_j,\chi_A)w_j \geq 0. \tag{8}$$

By Lemma 5 (a) and Lemma 4,

$$\sum_{j=1}^m \gamma_\phi^+(f_j-u,\chi_A)w_j \geq \sum_{j=1}^m \gamma_\phi^+(f_j-\underline{m}(\mathbf{f},A),\chi_A)w_j \geq 0. \tag{9}$$

Then, Lemma 4, (8) and (9) imply $u \in \mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f_A},\mathcal{C}_A)$, a contradiction. We can prove (b) in a similar way. Finally, if ϕ is a strictly convex function, the equality $\underline{m}(\mathbf{f},A) = \overline{m}(\mathbf{f},A)$ follows from Theorem 3. □

4. SIMULTANEOUS APPROXIMATION BY NONDECREASING FUNCTIONS

Henceforth, $\mathcal{D}$ is the convex cone of nondecreasing functions in $\mathcal{L}_\phi$. In this section, we give a characterization of best approximants to $\mathbf{f}$ from $\mathcal{D}$. Moreover, we show an explicit formula to calculate the maximum and minimum elements in $\mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f},\mathcal{D})$.

Definition 8. For $x \in (0,1)$, we define

$$\underline{f}(x) = \inf_{b>x} \sup_{a<x} \underline{m}(\mathbf{f},(a,b)) \quad \text{and} \quad \overline{f}(x) = \sup_{a<x} \inf_{b>x} \overline{m}(\mathbf{f},(a,b)).$$

Lemma 9. *The functions $\underline{f}$ and $\overline{f}$ are nondecreasing.*

Proof. Let $x, y \in (0,1)$ such that $x < y$. Then

$$\inf_{b>x} \sup_{a<x} \underline{m}(\mathbf{f},(a,b)) \le \inf_{b>x} \sup_{a<y} \underline{m}(\mathbf{f},(a,b)) \le \inf_{b>y} \sup_{a<y} \underline{m}(\mathbf{f},(a,b)).$$

Therefore, $\underline{f}(x) \le \underline{f}(y)$. The proof that $\overline{f}(x) \le \overline{f}(y)$ is analogous. □

That $\overline{f}$ and $\underline{f}$ are in $\mathcal{L}_\phi$ is a consequence of Theorems 16 and 18, respectively.

4.1. Characterization of Best Simultaneous Monotone Approximants

The following is a characterization theorem. Similar results can be seen in [1, 10].

Theorem 10. *The following statements are equivalent:*

(a) $g \in \mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f},\mathcal{D})$;
(b) *For every $u \in \mathbb{R}$ we have*

(b1) $\sum_{j=1}^{m} \gamma_\phi^+(g - f_j, \chi_{\{g<u\}\cap(a,1)})w_j \ge 0$, *for* $0 \le a < 1$; *and*
(b2) $\sum_{j=1}^{m} \gamma_\phi^+(f_j - g, \chi_{\{g>u\}\cap(0,b)})w_j \ge 0$, *for* $0 < b \le 1$.

Proof. (a) ⇒ (b). Take a $g \in \mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f},\mathcal{D})$, and let $u \in \mathbb{R}$. We prove (b1). The proof of (b2) is similar. Let $0 \le a < 1$. If $\mu(\{g < u\} \cap (a,1)) = 0$, then (b1) is obvious. Suppose $\mu(\{g < u\} \cap (a,1)) > 0$. So, $\chi_{\{g<u\}\cap(a,1)} = \chi_{(a,b_u)}$ a.e. on $[0,1]$, where

$$b_u = \sup\{g < u\}. \tag{10}$$

Assume $b_u = 1$, and let $h \in \mathcal{D}$ be given by $h = g$ on $[0,a]$ and $h = g+1$ on $(a,1]$. From (4) with this function h we get (b1).

Suppose now $b_u < 1$. We consider the following three cases:

- g is continuous at b_u, and $g(x) = g(b_u)$ for some $x > b_u$. Let $\{x_n\}_{n\in\mathbb{N}} \subset (a,b_u)$ be such that $x_n \uparrow b_u$. Since g is continuous at b_u,

$$g(b_u) = u. \tag{11}$$

Therefore, $y_n := g(b_u) - g(x_n) > 0$. Consider the function $h_n \in \mathscr{D}$ given by

$$h_n = g \text{ on } [0,a] \cup (b_u, 1], \quad h_n = g + y_n \text{ on } (a, x_n], \quad \text{and}$$
$$h_n = g(b_u) \text{ on } (x_n, b_u].$$

Applying (4) with $h = h_n$, we deduce that

$$0 \le \sum_{j=1}^{m} \gamma_\phi^+ \left(g - f_j, \chi_{(a,x_n)}\right) w_j + \sum_{j=1}^{m} \int_{x_n}^{b_u} \phi'(|g - f_j|)\overline{\mathrm{sgn}}(g - f_j) \frac{g(b_u) - g}{y_n} w_j \, d\mu.$$

Since $0 \le \frac{g(b_u)-g}{y_n} \le 1$ on (x_n, b_u), by passing to the limit as $n \to \infty$, we get (b1).

- g is continuous at b_u, and $g(x) > g(b_u)$ for all $x > b_u$.
Let $\{x_n\}_{n\in\mathbb{N}} \subset (b_u, 1)$ be such that $x_n \downarrow b_u$. Then $y_n := g(x_n) - g(b_u) > 0$. Consider the function $h_n \in \mathscr{D}$ given by

$$h_n = g \text{ on } [0,a] \cup (x_n, 1], \quad h_n = g + y_n \text{ on } (a, b_u], \quad \text{and}$$
$$h_n = g(x_n) \text{ on } (b_u, x_n].$$

Applying (4) with $h = h_n$, we have

$$0 \le \sum_{j=1}^{m} \gamma_\phi^+ \left(g - f_j, \chi_{(a,b_u)}\right) w_j + \sum_{j=1}^{m} \int_{b_u}^{x_n} \phi'(|g - f_j|)\overline{\mathrm{sgn}}(g - f_j) \frac{g(x_n) - g}{y_n} w_j \, d\mu.$$

Since $0 \le \frac{g(x_n)-g}{y_n} \le 1$ on (b_u, x_n), by passing to the limit as $n \to \infty$, we get (b1).

- $g(b_u^+) - g(b_u^-) = 2\delta$.

Taking in (4) the function $h \in \mathscr{D}$ given by $h = g$ on $[0,a] \cup (b_u, 1]$ and $h = g + \delta$ on $(a, b_u]$, we obtain (b1).

(b) $\Rightarrow$ (a) Let $u \in \mathbb{R}$, $h \in \mathscr{D}$, and $b = \sup\{h < u\}$. If $\mu(\{h < u < g\}) > 0$ then $0 < b \le 1$ and $\chi_{\{h<u<g\}} = \chi_{\{g>u\}\cap(0,b)}$ a.e. on $[0,1]$. Therefore, by (b2),

$$\sum_{j=1}^{m} \gamma_\phi^+ \left(f_j - g, \chi_{\{h<u<g\}}\right) w_j \ge 0. \tag{12}$$

If $\mu(\{h < u < g\}) = 0$, then (12) is obvious. Since u is arbitrary, integrating on u in the inequality (12) we have

$$\sum_{j=1}^{m} \int_{-\infty}^{\infty} \left(\int_0^1 \phi'(|f_j - g|)\overline{\mathrm{sgn}}(f_j - g)\chi_{\{h<u<g\}} w_j \, d\mu \right) du \ge 0.$$

Applying Fubini's theorem, we get

$$\sum_{j=1}^{m}\int_{0}^{1}\left(\phi'(|f_j-g|)\overline{\operatorname{sgn}}(f_j-g)w_j\int_{-\infty}^{\infty}\chi_{\{h<u<g\}}\,du\right)d\mu\geq 0,$$

that is,

$$\sum_{j=1}^{m}\gamma_{\phi}^{+}\left(f_j-g,\chi_{\{g>h\}}(g-h)\right)w_j\geq 0. \tag{13}$$

The inequality

$$\sum_{j=1}^{m}\gamma_{\phi}^{+}\left(f_j-g,\chi_{\{h>g\}}(g-h)\right)w_j\geq 0 \tag{14}$$

follows from (b1) in a similar way. Now, according to (13) and (14), we have

$$\sum_{j=1}^{m}\gamma_{\phi}^{+}\left(f_j-g,g-h\right)w_j\geq 0. \tag{15}$$

Since $h\in\mathcal{D}$ is arbitrary, (a) follows from (15) and Theorem 1. □

Corollary 11. *If $g\in\mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f},\mathcal{D})$, then for every $u\in\mathbb{R}$ we have*

(a) $\sum_{j=1}^{m}\gamma_{\phi}^{+}(g-f_j,\chi_{\{g\leq u\}\cap(a,1)})w_j\geq 0$, *for* $0\leq a<1$; *and*
(b) $\sum_{j=1}^{m}\gamma_{\phi}^{+}(f_j-g,\chi_{\{g\geq u\}\cap(0,b)})w_j\geq 0$, *for* $0<b\leq 1$.

Proof. For every $u\in\mathbb{R}$ and $\epsilon>0$, Theorem 10 implies

$$\sum_{j=1}^{m}\gamma_{\phi}^{+}\left(g-f_j,\chi_{\{g<u+\epsilon\}\cap(a,1)}\right)w_j\geq 0,\quad\text{for } 0\leq a<1,\quad\text{and}$$

$$\sum_{j=1}^{m}\gamma_{\phi}^{+}\left(f_j-g,\chi_{\{g>u-\epsilon\}\cap(0,b)}\right)w_j\geq 0,\quad\text{for } 0<b\leq 1.$$

As $\lim_{\epsilon\to 0^{+}}\chi_{\{g<u+\epsilon\}}=\chi_{\{g\leq u\}}$ and $\lim_{\epsilon\to 0^{+}}\chi_{\{g>u-\epsilon\}}=\chi_{\{g\geq u\}}$, both (a) and (b) hold. □

Remark 12. Under the same hypothesis of Corollary 11, observe that if $\mu(\{g=u\})>0$, then this Corollary and Lemma 4 show that u is a best constant approximant to $\mathbf{f}$ on $\{g=u\}$.

Theorem 13. *If* $g \in \mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathcal{D})$, *then* $\underline{f} \leq g \leq \overline{f}$ *a.e. on* $[0,1]$.

Proof. Let $x \in (0,1)$ be a continuity point of g. Let $\lambda > 0$ and $u = g(x) + \lambda$. For $0 \leq a < b_u$, where b_u is defined in (10), we have $x < b_u$ and $\chi_{(a,b_u)} = \chi_{\{g<u\}\cap(a,1)}$ a.e. on $[0,1]$. By Theorem 10, we get

$$\sum_{j=1}^{m} \gamma_\phi^+ \left(g - f_j, \chi_{(a,b_u)}\right) w_j \geq 0. \tag{16}$$

Since $g - f_j \leq g(x) + \lambda - f_j$ on (a, b_u) for all $j = 1, 2, \ldots, m$, Lemma 5 (a) implies

$$\sum_{j=1}^{m} \gamma_\phi^+ \left(g(x) + \lambda - f_j, \chi_{(a,b_u)}\right) w_j \geq \sum_{j=1}^{m} \gamma_\phi^+ \left(g - f_j, \chi_{(a,b_u)}\right) w_j. \tag{17}$$

From (16), (17), and Theorem 7 (a) we have

$$\underline{m}(\mathbf{f}, (a, b_u)) \leq g(x) + \lambda, \quad 0 \leq a < b_u.$$

So, $\sup_{a<x} \underline{m}(\mathbf{f}, (a, b_u)) \leq g(x) + \lambda$. Consequently, as $b_u > x$,

$$\underline{f}(x) = \inf_{b>x} \sup_{a<x} \underline{m}(\mathbf{f}, (a, b)) \leq g(x) + \lambda.$$

As λ is arbitrary, we obtain $\underline{f}(x) \leq g(x)$. A similar argument shows that $\overline{f}(x) \geq g(x)$. Since g is continuous a.e. on $[0,1]$, the proof is complete. □

Corollary 14. *If* ϕ *is a strictly convex function, then* $\underline{f} = \overline{f}$ *a.e. on* $[0,1]$, *and* $g = \overline{f}$ *a.e. on* $[0,1]$ *for any* g *in* $\mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathcal{D})$.

Proof. Let $x \in (a, b)$ with $0 < a < b < 1$. Since ϕ is a strictly convex function, it follows that

$$\inf_{c>x} \overline{m}(\mathbf{f}, (a, c)) \leq \overline{m}(\mathbf{f}, (a, b)) = \underline{m}(\mathbf{f}, (a, b)),$$

where the equality is due to Theorem 7. Then $\overline{f}(x) = \sup_{a<x} \inf_{c>x} \overline{m}(\mathbf{f}, (a, c)) \leq \sup_{a<x} \underline{m}(\mathbf{f}, (a, b))$ and, consequently,

$$\overline{f}(x) \leq \inf_{b>x} \sup_{a<x} \underline{m}(\mathbf{f}, (a, b)) = \underline{f}(x).$$

So, Theorem 13 completes the proof. □

4.2. Maximum and Minimum of Best Simultaneous Monotone Approximants

We now prove that $\max \mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f},\mathcal{D}) = \overline{f}$ a.e. on $[0,1]$ and $\min \mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f},\mathcal{D}) = \underline{f}$ a.e. on $[0,1]$. For $u \in \mathbb{R}$, observe that the function $x \longrightarrow \sum_{j=1}^{m} \gamma_{\phi}^{+}\left(f_j - u, \chi_{(x,1)}\right) w_j$ is continuous on $[0,1]$. Let

$$Q_u = \max\left\{\sum_{j=1}^{m} \gamma_{\phi}^{+}\left(f_j - u, \chi_{(x,1)}\right) w_j : x \in [0,1]\right\} \quad \text{and}$$

$$y_u = \min\left\{x \in [0,1] : \sum_{j=1}^{m} \gamma_{\phi}^{+}\left(f_j - u, \chi_{(x,1)}\right) w_j = Q_u\right\}.$$

Lemma 15. *Let $u \in \mathbb{R}$. If $0 < x < y_u < y < 1$, then*

(a) $\sum_{j=1}^{m} \gamma_{\phi}^{+}(f_j - u, \chi_{(y_u,y)}) w_j \geq 0$ *and* $\sum_{j=1}^{m} \gamma_{\phi}^{+}(f_j - u, \chi_{(x,y_u)}) w_j < 0$;
(b) $\overline{f}(x) \leq u \leq \overline{f}(y)$.

Proof. (a) Let $0 < x < y_u < y < 1$. By definition of y_u and Q_u,

$$\sum_{j=1}^{m} \gamma_{\phi}^{+}\left(f_j - u, \chi_{(y,1)}\right) w_j \leq Q_u = \sum_{j=1}^{m} \gamma_{\phi}^{+}\left(f_j - u, \chi_{(y_u,1)}\right) w_j \quad \text{and}$$

$$\sum_{j=1}^{m} \gamma_{\phi}^{+}\left(f_j - u, \chi_{(x,1)}\right) w_j < Q_u = \sum_{j=1}^{m} \gamma_{\phi}^{+}\left(f_j - u, \chi_{(y_u,1)}\right) w_j.$$

So, (3) and the additivity of the integral imply (a).

(b) Let $0 < v < y_u < z < 1$. If $b > z$, then Theorem 7 (b) and the first inequality in (a) (with $y = b$) imply $\overline{m}(\mathbf{f},(y_u,b)) \geq u$. Thus, $\inf_{b>z} \overline{m}(\mathbf{f},(y_u,b)) \geq u$. As $y_u < z$, we obtain

$$\overline{f}(z) = \sup_{a<z} \inf_{b>z} \overline{m}(\mathbf{f},(a,b)) \geq u.$$

On the other hand, if $a < v$ then $\inf_{b>v} \overline{m}(\mathbf{f},(a,b)) \leq \overline{m}(\mathbf{f},(a,y_u)) < u$, where the first inequality follows from the hypothesis $v < y_u$, and the second inequality is due to Theorem 7 (b), Corollary 6 and the second inequality in (a) (with $x = a$). Then

$$\overline{f}(v) = \sup_{a<v} \inf_{b>v} \overline{m}(\mathbf{f},(a,b)) \leq u.$$

□

Theorem 16. *We have $\overline{f} = \max \mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f},\mathcal{D})$ a.e. on $[0,1]$.*

Proof. Let $g = \max \mathscr{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathscr{D})$. By Theorem 13, $g \le \overline{f}$ a.e. on $[0,1]$. Suppose that there exists $z_0 \in (0,1)$ such that $g(z_0) < u < \overline{f}(z_0)$, where z_0 is a point of continuity of g and $\overline{f}$. Clearly $z_0 < b_u$, where b_u is defined in (10). In addition, $y_u \le z_0$; otherwise Lemma 15 (b) implies $\overline{f}(z_0) \le u$.

Let $R(g) = g\left([y_u, b_u]\right)$. Since g is a nondecreasing function on $(y_u, b_u]$, for $c \in R(g)$ the set

$$I_g(c) := \{z \in (y_u, b_u] : g(z) = c\}$$

is either a singleton, or an interval with endpoints $\underline{c} < \overline{c}$. We observe that the second case can occur for at most countable many values of c, say $\{c_n\}_{n\in I}$, $I \subseteq \mathbb{N}$. Let

$$C := (y_u, b_u) \setminus \left(\bigcup_{n\in I} \left(\underline{c_n}, \overline{c_n}\right) \right)$$

and let $\beta : (y_u, b_u] \to \mathbb{R}$ be the continuous function defined by

$$\beta(x) := \sum_{j=1}^{m} \gamma_\phi^+ \left(f_j - u, \chi_{(y_u,x)}\right) w_j = \sum_{j=1}^{m} \int_{y_u}^{x} \phi'(|f_j - u|)\overline{\operatorname{sgn}}(f_j - u) w_j \, d\mu. \quad (18)$$

We next prove that

$$\beta(x) = 0 \quad \text{for all } x \in C. \quad (19)$$

Let $z \in C$; we consider two cases.

- $z \ne \underline{c_n}$.
  Clearly, $\{g \le g(z)\} \cap (y_u, 1) = (y_u, z]$, because $g(y) > g(z)$ for $y > z$. Since $g < u$ on (y_u, z), from Lemma 15 (a), Lemma 5 (b) and Corollary 11 (a) we have

$$0 \le \beta(z) = \sum_{j=1}^{m} \gamma_\phi^+ \left(f_j - u, \chi_{(y_u,z)}\right) w_j \le -\sum_{j=1}^{m} \gamma_\phi^+ \left(g - f_j, \chi_{(y_u,z)}\right) w_j$$
$$= -\sum_{j=1}^{m} \gamma_\phi^+ \left(g - f_j, \chi_{\{g \le g(z)\} \cap (y_u,1)}\right) w_j \le 0.$$

- $z = \underline{c_n}$.

As $\chi_{\{g<c_n\}\cap(y_u,1)} = \chi_{(y_u,z)}$ a.e. on $[0,1]$, and $g < u$ on (y_u, z), Lemma 15 (a), Lemma 5 (b) and Theorem 10 imply

$$0 \le \beta(z) = \sum_{j=1}^{m} \gamma_\phi^+ \left(f_j - u, \chi_{(y_u,z)}\right) w_j \le -\sum_{j=1}^{m} \gamma_\phi^+ \left(g - f_j, \chi_{(y_u,z)}\right) w_j$$
$$= -\sum_{j=1}^{m} \gamma_\phi^+ \left(g - f_j, \chi_{\{g<c_n\}\cap(y_u,1)}\right) w_j \le 0.$$

Therefore, (19) holds.

On the other hand, β has a derivative β' at almost every point $x \in (y_u, b_u)$. Indeed, from (18),

$$\beta' = \sum_{j=1}^{m} \phi'(|f_j - u|)\overline{\operatorname{sgn}}(f_j - u) w_j \quad \text{a.e. on } (y_u, b_u).$$

Let D be the set of points $x \in C$ such that x is a density point of C and it satisfies the above equation. Since $\mu(D) = \mu(C)$, by (19) we get $\beta' = 0$ a.e. on C. Further, $g < u$ on C; thus

$$\sum_{j=1}^{m} \gamma_\phi^+ \left(f_j - u, (u-g)\chi_C\right) w_j = \int_C (u-g)\beta' \, d\mu = 0. \tag{20}$$

If $(y_u, b_u) \setminus C \neq \emptyset$, then

$$\sum_{j=1}^{m} \gamma_\phi^+ \left(f_j - u, (u-g)\chi_{(y_u,b_u)\setminus C}\right) w_j = \sum_{n\in I} \sum_{j=1}^{m} \gamma_\phi^+ \left(f_j - u, (u-g)\chi_{(\underline{c_n},\overline{c_n})}\right) w_j$$
$$= \sum_{n\in I} \sum_{j=1}^{m} (u - c_n) \gamma_\phi^+ \left(f_j - u, \chi_{(\underline{c_n},\overline{c_n})}\right) w_j$$
$$= \sum_{n\in I} (u - c_n) \left(\beta(\overline{c_n}) - \beta(\underline{c_n})\right) = 0,$$

where the last equality is due to (19). Therefore, by (20) we get

$$\sum_{j=1}^{m} \gamma_\phi^+ (f_j - u, (u-g)\chi_{(y_u,b_u)}) w_j = 0. \tag{21}$$

Now we consider the function $h \in \mathcal{D}$ given by

$$h = g \text{ on } [0, y_u] \cup (b_u, 1] \quad \text{and} \quad h = u \text{ on } (y_u, b_u].$$

It follows from (21) and Corollary 2 that $h \in \mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathcal{D})$, which contradicts the definition of g. So, $\overline{f}(z_0) = g(z_0)$ at every continuity point z_0 of g and $\overline{f}$. Since almost every point in $(0, 1)$ is a continuity point of both g and $\overline{f}$, we conclude that $g = \overline{f}$ a.e. on $[0, 1]$. □

Analogously to the previous case, for $u \in \mathbb{R}$, let

$$M_u = \max\left\{\sum_{j=1}^{m} \gamma_\phi^+\left(u - f_j, \chi_{(0,x)}\right) w_j : x \in [0, 1]\right\} \quad \text{and}$$

$$x_u = \max\left\{x \in [0, 1] : \sum_{j=1}^{m} \gamma_\phi^+\left(u - f_j, \chi_{(0,x)}\right) w_j = M_u\right\}.$$

With similar proofs to those of Lemma 15 and Theorem 16 we obtain the following two results, respectively.

Lemma 17. *Let $u \in \mathbb{R}$. If $0 < x < x_u < y < 1$ then*

(a) $\sum_{j=1}^{m} \gamma_\phi^+(u - f_j, \chi_{(x,x_u)}) w_j \geq 0$ *and* $\sum_{j=1}^{m} \gamma_\phi^+(u - f_j, \chi_{(x_u,y)}) w_j < 0$*;*
(b) $\underline{f}(x) \leq u \leq \underline{f}(y)$.

Theorem 18. *We have* $\underline{f} = \min \mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathcal{D})$ *a.e. on* $[0, 1]$.

5. CONTINUITY OF BEST SIMULTANEOUS MONOTONE APPROXIMANTS

In this section, we study the continuity of best simultaneous monotone approximants to $\mathbf{f}$. Note that ϕ' is a continuous function, since ϕ is convex and differentiable.

A function $f \in \mathcal{M}_0$ is said to be approximately continuous at $x_0 \in (0, 1)$ if, for each $\epsilon > 0$, x_0 is a point of density of $\{|f - f(x_0)| < \epsilon\} =: A_\epsilon(f, x_0)$.

Lemma 19. *Let $g \in \mathcal{D}$, $f \in \mathcal{L}_\phi$, $x_0 \in (0, 1)$, $w > 0$ and*

$$L_\epsilon(\delta, f, w) := \frac{1}{\delta} \int_{x_0-\delta}^{x_0} \chi_{A_\epsilon(f,x_0)} \phi'(|g - f|) \overline{\mathrm{sgn}}(g - f) w \, d\mu, \quad 0 < \delta < x_0.$$

Assume that f is approximately continuous at x_0.

(a) *If* $0 < \epsilon < |g(x_0^-) - f(x_0)|$, *then*

$$\begin{aligned}\overline{L}_\epsilon(f, w) &:= \limsup_{\delta \downarrow 0} L_\epsilon(\delta, f, w) \\ &\leq \phi'(|g(x_0^-) - f(x_0) + \epsilon|) \overline{\mathrm{sgn}}(g(x_0^-) - f(x_0)) w;\end{aligned}$$

(b) *If* $f(x_0) = g(x_0^-)$ *and* $\epsilon > 0$, *then* $\overline{L}_\epsilon(f, w) \le \phi'(\epsilon)w$.

Consequently,

$\overline{L}_\epsilon(f, w) \le \phi'(|g(x_0^-) - f(x_0) + \epsilon|)\overline{\text{sgn}}(g(x_0^-) - f(x_0))w$ *for all* ϵ *small enough.*

Proof. (a) Assume $0 < \epsilon < |g(x_0^-) - f(x_0)|$. Then

$$L_\epsilon(\delta, f, w) \le \frac{\mu\left([x_0 - \delta, x_0] \cap A_\epsilon(f, x_0)\right)}{\delta} \times \phi'\left(|g(x_0^-) - f(x_0) + \epsilon|\right)\overline{\text{sgn}}(g(x_0^-) - f(x_0))w,$$

for all sufficiently small $\delta > 0$. Since $\lim_{\delta \downarrow 0} \frac{\mu([x_0-\delta, x_0] \cap A_\epsilon(f, x_0))}{\delta} = 1$, by passing to the limit as $\delta \downarrow 0$ we get (a).

(b) Suppose now $g(x_0^-) = f(x_0)$ and let $\epsilon > 0$. Since

$$|L_\epsilon(\delta, f, w)| \le \frac{\mu([x_0 - \delta, x_0] \cap A_\epsilon(f, x_0))}{\delta}\phi'(\max\{|\epsilon + f(x_0) - g(x_0 - \delta)|, \epsilon\})w,$$

by passing to the limit as $\delta \downarrow 0$ we have (b). □

With a similar proof to that of Lemma 19, we get the next lemma.

Lemma 20. *Let* $g \in \mathscr{D}$, $f \in \mathscr{L}_\phi$, $x_0 \in (0, 1)$, $w > 0$ *and*

$$N_\epsilon(\delta, f, w) := \frac{1}{\delta}\int_{x_0}^{x_0+\delta} \chi_{A_\epsilon(f, x_0)}\phi'(|f - g|)\overline{\text{sgn}}(f - g)w\, d\mu, \quad \delta > 0.$$

Assume that f *is approximately continuous at* x_0.

(a) *If* $0 < \epsilon < |g(x_0^+) - f(x_0)|$, *then*

$$\begin{aligned}\overline{N}_\epsilon(f, w) &:= \limsup_{\delta \downarrow 0} N_\epsilon(\delta, f, w)\\ &\le \phi'(|f(x_0) - g(x_0^+) + \epsilon|)\overline{\text{sgn}}(f(x_0) - g(x_0^+))w;\end{aligned}$$

(b) *If* $f(x_0) = g(x_0^+)$ *and* $\epsilon > 0$, *then* $\overline{N}_\epsilon(f, w) \le \phi'(\epsilon)w$.

Consequently,

$$\overline{N}_\epsilon(f, w) \le \phi'(|g(x_0^+) - f(x_0) + \epsilon|)\overline{\text{sgn}}(g(x_0^+) - f(x_0))w$$
for all ϵ *small enough.*

Theorem 21. *Let $g \in \mathscr{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathscr{D})$. Assume that ϕ is a strictly convex function. If f_j is approximately continuous at $x_0 \in (0,1)$ for each j, and either ϕ' is bounded, or f_j is essentially bounded on a neighborhood of x_0 for every j, then*

(a) *g is continuous at x_0; and*
(b) *If g is not constant on a neighborhood of x_0, then $g(x_0)$ satisfies*

$$\sum_{j=1}^{m} \phi(|f_j(x_0) - g(x_0)|) w_j = \min_{c \in \mathbb{R}} \sum_{j=1}^{m} \phi(|f_j(x_0) - c|) w_j. \tag{22}$$

Proof. (a) If g is constant on a neighborhood of x_0, then g is continuous at x_0. Otherwise, let $\epsilon > 0$, and for each $j = 1, 2, \ldots, m$ let $A_{j,\epsilon} = A_\epsilon(f_j, x_0)$ and $A_{j,\epsilon}^c = (0,1) \setminus A_{j,\epsilon}$. We consider the case $g(x) > g(x_0)$ for $x > x_0$; the case where $g(x) < g(x_0)$ for $x < x_0$ is proved in a similar way. For each $0 < \delta < x_0$, from Corollary 11 (a) we have

$$\begin{aligned} 0 &\leq \sum_{j=1}^{m} \gamma_\phi^+ \left(g - f_j, \chi_{\{g \leq g(x_0)\} \cap (x_0 - \delta, 1)} \right) w_j \\ &= \sum_{j=1}^{m} \int_{x_0-\delta}^{x_0} \phi'(|g - f_j|) \overline{\operatorname{sgn}}(g - f_j) w_j \, d\mu. \end{aligned} \tag{23}$$

Since g is bounded on $[x_0 - \delta, x_0]$, by hypothesis there exists a constant $M > 0$ such that

$$\sum_{j=1}^{m} \int_{x_0-\delta}^{x_0} \chi_{A_{j,\epsilon}^c} \phi'(|g - f_j|) w_j \, d\mu \leq M \sum_{j=1}^{m} \mu([x_0 - \delta, x_0] \cap A_{j,\epsilon}^c),$$

for all sufficiently small δ. As f_j is approximately continuous at x_0 for each j, we deduce that $\lim_{\delta \downarrow 0} \frac{\mu([x_0 - \delta, x_0] \cap A_{j,\epsilon}^c)}{\delta} = 0$ for $j = 1, 2, \ldots, m$. Thus

$$\limsup_{\delta \downarrow 0} \sum_{j=1}^{m} \frac{1}{\delta} \int_{x_0-\delta}^{x_0} \chi_{A_{j,\epsilon}^c} \phi'(|g - f_j|) \overline{\operatorname{sgn}}(g - f_j) w_j \, d\mu = 0. \tag{24}$$

According to (23) and (24), and applying the additivity of the integral, we get

$$\sum_{j=1}^{m} \overline{L}_\epsilon(f_j, w_j) \geq \limsup_{\delta \downarrow 0} \sum_{j=1}^{m} \frac{1}{\delta} \int_{x_0-\delta}^{x_0} \chi_{A_{j,\epsilon}} \phi'(|g - f_j|) \overline{\operatorname{sgn}}(g - f_j) w_j \, d\mu \geq 0.$$

From Lemma 19,

$$\sum_{j=1}^{m} \phi'(|g(x_0^-) - f_j(x_0) + \epsilon|)\overline{\text{sgn}}(g(x_0^-) - f_j(x_0))w_j \geq 0$$

for all ϵ small enough. Therefore,

$$\sum_{j=1}^{m} \phi'(|g(x_0^-) - f_j(x_0)|)\overline{\text{sgn}}(g(x_0^-) - f_j(x_0))w_j \geq 0. \tag{25}$$

On the other hand, (b2) in Theorem 10 implies

$$0 \leq \sum_{j=1}^{m} \gamma_\phi^+ \left(f_j - g, \chi_{\{g>g(x_0)\}\cap(0,x_0+\delta)}\right) w_j = \sum_{j=1}^{m} \int_{x_0}^{x_0+\delta} \phi'(|f_j - g|)\overline{\text{sgn}}(f_j - g)w_j \, d\mu.$$

In the same manner as before, and using Lemma 20, we can see that

$$\sum_{j=1}^{m} \phi'(|f_j(x_0) - g(x_0^+)|)\overline{\text{sgn}}(f_j(x_0) - g(x_0^+))w_j \geq 0. \tag{26}$$

Suppose now $g(x_0^-) < g(x_0^+)$. Due to (26) the set of indexes $J_1 = \{j : f_j(x_0) \geq g(x_0^+)\}$ cannot be empty. Analogously, by (25) $J_2 = \{j : f_j(x_0) \leq g(x_0^-)\} \neq \emptyset$. Applying again (26) and (25), we deduce that

$$\begin{aligned}
&\sum_{j\in J_1} \phi'(|f_j(x_0) - g(x_0^+)|)\overline{\text{sgn}}(f_j(x_0) - g(x_0^+))w_j \\
&\quad \geq \sum_{j\in J_2} \phi'(|f_j(x_0) - g(x_0^+)|)\overline{\text{sgn}}(g(x_0^+) - f_j(x_0))w_j \\
&\quad > \sum_{j\in J_2} \phi'(|f_j(x_0) - g(x_0^-)|)\overline{\text{sgn}}(g(x_0^-) - f_j(x_0))w_j \\
&\quad \geq \sum_{j\in J_1} \phi'(|f_j(x_0) - g(x_0^-)|)\overline{\text{sgn}}(f_j(x_0) - g(x_0^-))w_j \\
&\quad > \sum_{j\in J_1} \phi'(|f_j(x_0) - g(x_0^+)|)\overline{\text{sgn}}(f_j(x_0) - g(x_0^+))w_j,
\end{aligned}$$

a contradiction. We are using (26) in the first inequality, and (25) in the third inequality. The strict inequalities follow from the fact that ϕ is strictly convex. Hence, $g(x_0^-) = g(x_0^+)$ and g is continuous at x_0. The same reasoning applies to the case $g(x) < g(x_0)$ for $x < x_0$.

(b) According to (a), (25), and (26), we have

$$\sum_{j=1}^{m} \phi'(|g(x_0) - f_j(x_0)|)\overline{\text{sgn}}(g(x_0) - f_j(x_0))w_j \geq 0 \quad \text{and}$$

$$\sum_{j=1}^{m} \phi'(|f_j(x_0) - g(x_0)|)\overline{\text{sgn}}(f_j(x_0) - g(x_0))w_j \geq 0,$$

and these two inequalities are precisely the characterization of the minimum $g(x_0)$ in the discrete problem of (22). □

Remark 22. Under the same hypothesis of Theorem 21, $m = 2$ and $w_1 = w_2 = 1$, we conclude that if g is not constant on a neighborhood of x_0, then $g(x_0) = \frac{f_1(x_0)+f_2(x_0)}{2}$.

The following example shows that if ϕ is not a strictly convex function, then both (a) and (b) in Theorem 21 are not true.

Example 23. Let $\phi(t) = t$ and $w_1 = w_2 = 1$. Take $f_1 \equiv 0$ and $f_2 \equiv 1$ on $[0, 1]$. Then for all $c \in [0, \frac{1}{2}]$, the function

$$g_c(x) = \begin{cases} c & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1 - c & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

is an element of $\mathcal{M}_{\phi,\mathbf{w}}(\mathbf{f}, \mathcal{D})$. Moreover, for $c \in [0, \frac{1}{2})$, g_c is not constant on any neighborhood of $\frac{1}{2}$, and $g_c\left(\frac{1}{2}\right) = c < \frac{1}{2} = \frac{f_1\left(\frac{1}{2}\right)+f_2\left(\frac{1}{2}\right)}{2}$.

In [9], best monotone $\mathcal{L}_\phi$-approximation to a single function f is considered. In Theorem 3 the authors prove, without assuming that ϕ is strictly convex, that if f is approximately continuous at every point in $(0, 1)$, then uniqueness holds. The above example also shows that this result is not true in simultaneous approximation.

6. FINAL REMARK

Let $1 < p < \infty$ and $1 \leq q < \infty$. For a convex set $\mathcal{K} \subset L_p$, and $f_j \in L_p[0, 1] \setminus \mathcal{K}$ for $j = 1, 2, \ldots, m$, consider the problem of finding a g in $\mathcal{K}$ satisfying

$$\sum_{j=1}^{m} \|f_j - g\|_p^q = \inf_{h \in \mathcal{K}} \sum_{j=1}^{m} \|f_j - h\|_p^q.$$

A straightforward computation shows that every solution $g_{p,q}$ of this problem is characterized by

$$\sum_{j=1}^{m} \int_0^1 |f_j - g_{p,q}|^{p-1} \text{sgn}(f_j - g_{p,q})(g_{p,q} - h) w_j \, d\mu \geq 0 \quad \text{for every } h \in \mathcal{K},$$

where $w_j = \|f_j - g_{p,q}\|_p^{q-p}$, $j = 1, 2, \ldots, m$. From Theorems 1 and 3 we deduce that $g_{p,q} \in \mathcal{K}$ is the solution of (1) taking $\phi(t) = t^p$ and the weights w_j given above. Thus, whenever $\mathcal{K}$ is the set of nondecreasing functions in L_p, Corollary 14 shows that $g_{p,q} = \overline{f}$ a.e. on $[0, 1]$, where $\overline{f}$ is given in Definition 8.

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