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The classical limit of non-integrable quantum systems, a route to quantum chaos

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Abstract

The classical limit of non-integrable quantum systems is studied. We define non-integrable quantum systems as those, which have, as their classical limit, a non-integrable classical system. This quantum systems will be the candidates to be the models of quantum chaos. In order to obtain this limit, the *self-induced decoherence* approach and the corresponding classical limit are generalized from integrable to non-integrable systems. In this approach, the lost of information, usually conceived as the result of a coarse-graining or the trace of an environment, is produced by a particular choice of the algebra of observables and the systematic use of mean values, that project the unitary evolution onto an effective non-unitary one. By means of our method, we can obtain the classical limit of the quantum state of a non-integrable system, which turns out to be a set of unstable, potentially chaotic classical trajectories contained in the Wigner transformation of the quantum state.

1. Introduction

The problem of the classical limit of quantum mechanics has a long history. In the beginning, on the basis of the analogy with special relativity where the limit $c \to \infty$ leads to classical behavior, it was thought that the classical limit was just the limit $\hbar \to 0$. But it was soon realized that this was only one element of the problem, namely, *macroscopicity*, and that other elements must be taken into account: e.g., quantum mechanics has a probabilistic non-Boolean structure while classical mechanics has a non-probabilistic and Boolean one. Thus, necessarily two new elements must come into play:

- *Decoherence*, that transforms the non-Boolean structure into a Boolean one, and
- *Localization* (*actualization or the choice of a trajectory*) that, with macroscopicity—which circumvents the uncertainty principle—, turns the probabilistic structure into a non-probabilistic one.

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In general, decoherence in quantum systems is defined as a process that leads to the diagonalization of a density matrix. In a first period, decoherence was explained as the result of the destructive interference of the off-diagonal elements of the density matrix (see [1,2]); however, this line of research was abandoned due to technical difficulties derived from the formalism used to describe the process. As a consequence, decoherence begun to be conceived as produced by the interaction between a system and its environment. This approach gave rise to the einselection program, based on the works of Zeh [3–5] and later developed by Zurek and coworkers [6–12]. Although many relevant results have been obtained by means of einselection, this approach still involves certain unsolved problems, as those related with the explanation of the emergence of classicality in closed quantum systems, the criterion for introducing the 'cut' between the system and its environment, and the definition of the preferred ('pointer') basis where the system behaves classically (see [13]). As the result of these and other difficulties, a number of alternative accounts of decoherence have been proposed (see [14–20]).

On the basis of the formalism introduced in papers [21–25], in a series of papers [13,26–35] we have returned to the initial idea of the destructive interference of the off-diagonal terms of the density matrix, but now by means of a different formalism: the formalism introduced by van Hove [36–39]. We have called this new approach 'self-induced decoherence' [13] because, from this viewpoint, decoherence is not produced by the interaction between a system and its environment, but it results from the own dynamics of the whole quantum system governed by a Hamiltonian with continuous spectrum. In this approach, the difficulties derived from the einselection program are absent: self-induced decoherence can be used in closed systems as the universe [30], the definition of a convenient subalgebra plays the role of the coarse-graining induced by the environment, avoiding the 'cut' problem [13], and the pointer basis is perfectly defined [13,34].

Self-induced decoherence is capable of addressing relevant problems from a general perspective, e.g., the problem of supplying a good definition of the classical limit *in all cases*. We will apply this main idea to the non-integrable case, and present the computations in all detail in Section 3.2, by using our previous results on quantum systems with continuous spectrum contained in papers [22–24,27–29]. With this strategy we have already obtained, in paper [26], the classical limit for *integrable* systems. We have also presented this result in more rigorous mathematical basis in [33] and explained the physical foundations of the method in papers [13,34]. The mathematical basis of the theory is explained in papers [21,32]. Decoherence characteristic times were obtained in [35]. But, of course, the big challenge to prove the consistency and generality of the method is to find its version for *non-integrable* systems, obtaining potentially chaotic classical trajectories as a final result, which could explain models as those of Ref. [16].

In the case of integrable systems, the classical limit was obtained by a combination of the van Hove formalism and the Weyl–Wigner–Moyal isomorphism in a globally defined pointer basis. But in the non-integrable case, such a global basis does not exist. Nevertheless, the just quoted isomorphism is what allows us to relax the global condition and to generalize the formalism: quantum mechanics is formulated in a phase space that is covered with charts where *local pointer bases can be defined*. The set of all these local pointer bases will yield decomposition of Eq. (3.18), which is the essential tool of this paper.

The formalism of the theory is presented in a self-comprehensive way, with a mathematics as simple as possible and in the simplest possible case; this seems enough for the physical purposes of this paper. In Section 2, a brief review of the Weyl–Wigner–Moyal mapping is developed, and in Section 3, the theory of decoherence in non-integrable systems is explained. In Section 4, the classical limit of quantum non-integrable system is obtained. In Section 5, the localization phenomena is briefly discussed. In the conclusion (Section 6), we list the possible future applications of the theory and explain why our formalism could be considered as a *minimal formalism for quantum chaos*. Finally, in Appendix A we give an example of a non-integrable system.

2. Weyl–Wigner–Moyal mapping

Let $\mathscr{M} = \mathscr{M}_{2(N+1)} \equiv \mathbb{R}^{2(N+1)}$ be the phase space of our classical system. The functions over this phase space will be called $f(\phi)$, where ϕ symbolizes the coordinates over $\mathscr{M}$

$$\phi^a = \left(q^1, \ldots, q^{N+1}, p_q^1, \ldots, p_q^{N+1}\right) \quad a = 1, 2, \ldots, 2(N+1). \tag{2.1}$$

As it is known (see [40,41]), we can map $\widehat{\mathscr{A}}$, the algebra of regular operators $\widehat{O}$ of our quantum system, on $\mathscr{A}_q$, the algebra of $\mathbb{L}_1$ functions over $\mathscr{M}$, via the *Wigner symbol*

$$symb : \widehat{\mathscr{A}} \to \mathscr{A}_q, \qquad symb\, \widehat{O} = O(\phi). \tag{2.2}$$

Precisely: let us consider that $\mathscr{M}$ has a symplectic form

$$\omega_{ab} = \begin{pmatrix} 0 & I_{N+1} \\ -I_{N+1} & 0 \end{pmatrix}, \qquad \omega^{ab} = \begin{pmatrix} 0 & -I_{N+1} \\ I_{N+1} & 0 \end{pmatrix}. \tag{2.3}$$

Then,

$$symb\,\hat{f} \stackrel{\circ}{=} f(\phi) = \int \mathrm{d}^{2(N+1)}\psi \exp\left(\frac{\mathrm{i}}{\hbar}\psi^a \omega_{ab}\psi^b\right) \mathrm{Tr}\left(\widehat{T}(\psi)\hat{f}\right), \tag{2.4}$$

where $\hat{f} \in \widehat{\mathscr{A}}$, $f(\phi) \in \mathscr{A}_q$, and

$$\widehat{T}(\psi) = \exp\left(\frac{\mathrm{i}}{\hbar}\psi^a \omega_{ab}\widehat{\phi}^b\right). \tag{2.5}$$

On $\mathscr{A}_q$ we can define the *star product* (i.e., the classical operator related with the multiplication on $\widehat{\mathscr{A}}$ and, therefore, defining the corresponding operation on $\mathscr{A}_q$) as

$$symb(\hat{f}\hat{g}) = symb\,\hat{f} * symb\,\hat{g} = (f * g)(\phi). \tag{2.6}$$

It can be proved [40, Eq. (2.59)] that

$$(f * g)(\phi) = f(\phi)\exp\left(-\frac{\mathrm{i}\hbar}{2}\overleftarrow{\partial}_a\,\omega^{ab}\,\overrightarrow{\partial}_b\right)g(\phi) = g(\phi)\exp\left(\frac{\mathrm{i}\hbar}{2}\overleftarrow{\partial}_a\,\omega^{ab}\,\overrightarrow{\partial}_b\right)f(\phi). \tag{2.7}$$

We also define the *Moyal bracket* as the symbol corresponding to the commutator in $\widehat{\mathscr{A}}$

$$\{f,g\}_{mb} = \frac{1}{\mathrm{i}\hbar}(f * g - g * f) = symb\left(\frac{1}{\mathrm{i}\hbar}[f,g]\right) = \frac{1}{\mathrm{i}\hbar}f(\phi)\sin\left(-\frac{\mathrm{i}\hbar}{2}\overleftarrow{\partial}_a\,\omega^{ab}\,\overrightarrow{\partial}_b\right)g(\phi). \tag{2.8}$$

In the limit $\hbar \to 0$, the star product becomes the ordinary product, and the Moyal bracket becomes the Poisson bracket[1]

$$(f * g)(\phi) = f(\phi)g(\phi) + 0(\hbar), \tag{2.9}$$

$$\{f,g\}_{mb} = \{f,g\}_{pb} + 0(\hbar^2). \tag{2.10}$$

By means of Eq. (2.7) it can be proved that, if $\hat{f}$ commutes with $\hat{g}$, Eq. (2.9) changes to

$$(f * g)(\phi) = f(\phi)g(\phi) + 0(\hbar^2). \tag{2.11}$$

Finally, if we want that the mapping *symb* be one-to-one, we must define a unique inverse of *symb*, namely, the usual quantization rule $q \to \hat{q}$, $p \to \hat{p}$ endowed with a unique ordering prescription, e.g., the symmetrical or *Weyl ordering prescription* that maps

$$symb^{-1}(qp) = \tfrac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q}). \tag{2.12}$$

Then, we have

$$symb^{-1} : \mathscr{A}_q \to \widehat{\mathscr{A}}, \qquad symb : \widehat{\mathscr{A}} \to \mathscr{A}_q. \tag{2.13}$$

The one-to-one mapping so defined is the *Weyl–Wigner–Moyal symbol*. With $symb^{-1}$ we can 'deform' the classical system and obtain a quantum mechanical system. With *symb* we go from usual quantum mechanics to a quantum mechanics 'alla classica', formulated over a phase space $\mathscr{M}$. The relation between the two structures, given by Eq. (2.13) (and Eq. (4.9) below), is an isomorphism that we will call *Weyl–Wigner–Moyal isomorphism*, the only one we will use in this paper.

Since $\widehat{\mathscr{A}}$ is a space of operators on a Hilbert space $\mathscr{H}$, so it is its dual $\widehat{\mathscr{A}}'$; then, as it is known, the symbol for any $\hat{\rho} \in \widehat{\mathscr{A}}'$ is defined as[2]

$$\rho(\phi) = symb\,\hat{\rho} = (2\pi\hbar)^{-(N+1)} symb_{(\text{for operators})}\hat{\rho}, \tag{2.14}$$

[1] From Eq. (2.7) it is clear that the $0(\hbar)$ and $0(\hbar^2)$ of Eqs. (2.9) and (2.10) are continuous functions in the limit $\hbar = 0$. This fact will be important in Section 5.

[2] In the case of states, we must add a new factor $(2\pi\hbar)^{-(N+1)}$ to definition (2.4) in order to preserve the usual normalization of $\rho(\phi)$. However, $\rho(\phi)$ is not non-negatively defined. With decoherence and $\hbar \to 0$ we will obtain a non-negatively defined $\rho(\phi)$, and $\mathscr{A}_q \to \mathscr{A}$, the classical boolean algebra of $\mathbb{L}_1$ operators over $\mathscr{M}$.

where the *symb* for operators is defined by Eqs. (2.4) and (2.5). From this definition, we have (see [40], Eq. (2.13))

$$(\hat{\rho}|\widehat{O}) = (symb\,\hat{\rho}|symb\,\widehat{O}) = \int d\phi^{2(N+1)}\rho(\phi)O(\phi). \tag{2.15}$$

Let us remark that the last equation is the cornerstone of our theory of the classical limit. In fact, as we will see, *it will remain the same when we go from regular to singular objects*. Once this statement is understood, the translation from the quantum language to the classical one will be easy.

3. Decoherence in non-integrable systems

3.1. Local CSCO

(a) We will begin with demonstrating an important theorem: when our quantum system is endowed with a CSCO of $N+1$ observables containing $\widehat{H}$ that defines a basis in terms of which the state of the system can be expressed, the underlying classical system is *integrable*. In fact, let a classical system be defined in a phase space $\mathscr{M} \equiv \mathbb{R}^{2(N+1)}$ that can be deformed 'alla Weyl'. If our quantum system is endowed with a $(N+1)$-CSCO $\{\widehat{H}, \widehat{O}_1, \ldots, \widehat{O}_N\}$, the Moyal brackets of these quantities are

$$\{O_I(\phi), O_J(\phi)\}_{mb} = symb\left(\frac{1}{i\hbar}[\widehat{O}_I, \widehat{O}_J]\right) = 0, \tag{3.1}$$

where $I,J,\ldots = 0,1,\ldots,N$ and $\widehat{H} = \widehat{O}_0$. Then, when $\hbar \to 0$, from Eq. (2.10) we know that

$$\{O_I(\phi), O_J(\phi)\}_{pb} = 0. \tag{3.2}$$

Thus, as $H(\phi) = O_0(\phi)$, the set $\{O_I(\phi)\}$ is a complete set of $N+1$ constants of the motion in involution, globally defined over all $\mathscr{M}$ and, therefore, the system is integrable.

As a consequence, *non-integrable* classical systems, in their quantum version, cannot have a CSCO of $N+1$ observables globally defined containing $\widehat{H}$. But, according to the self-induced approach, the pointer basis is precisely the eigenbasis of a global $(N+1)$-CSCO containing $\widehat{H}$ (in such a way that the vectors of the pointer basis turn out to be stationary states, see [26]). Therefore, pointer bases cannot be globally defined in non-integrable systems. These systems can be adequately quantized, but it is impossible (at least globally) to define a complete stationary eigenbasis of $(N+1)$-CSCO and, a fortiori, a pointer $(N+1)$-CSCO or pointer basis where the system would decohere according to the self-induced approach.[3] This is the main problem with non-integrable quantum systems.

(b) We will now prove that $N+1$ constants of the motion in involution always exist locally.[4] Let us consider a non-integrable quantum system (i.e., with no global $(N+1)$-CSCO), but let us suppose that, as usual, $H(\phi) = symb\,\widehat{H}$ is given and it is globally defined over $\mathscr{M}$ (this means that any non-global CSCO has at least one global observable: $\widehat{H}$). Now we can try to find N constants of the motion $\{O_I(\phi)\}$ $(I = 1,2,\ldots,N)$ satisfying

$$\{H(\phi), O_I(\phi)\}_{pb} = \sum_{j=1}^{N} \frac{\partial H}{\partial p_{qj}}\frac{\partial O_I}{\partial q_j} - \frac{\partial H}{\partial q_j}\frac{\partial O_I}{\partial p_{qj}} = 0. \tag{3.3}$$

This is a system of N partial differential equations which, with adequate boundary conditions, has a unique solution in a *maximal domain of integration* $\mathscr{D}_{\phi_i}$ *around any point* $\phi_i \in \mathscr{M}$ (provided that the functions involved satisfy reasonable—e.g., Lipschitzian—mathematical conditions and that certain determinant Δ, defined in [42, p. 72], be $\Delta \neq 0$ in this domain).

But we would like to obtain a set of constants of the motion in *involution*. Then, let us suppose that the initial conditions for Eq. (3.3) are given in a $2N+1$ dimensional hypersurface containing ϕ_i, that we will call $\mathscr{D}^N_{\phi_i}$. Integrating (3.3) we will obtain N constants of the motion $O_I(\phi)$. Moreover, we can easily show that, if these solutions are in involution in $\mathscr{D}^N_{\phi_i}$, they will remain in involution in the domain $\mathscr{D}_{\phi_i} = \mathscr{D}^{N+1}_{\phi_i}$ of $2(N+1)$ dimensions. In fact, according to the Jacobi property of the Poisson brackets we have

$$\{H(\phi), \{O_I(\phi), O_J(\phi)\}_{pb}\}_{pb} + \{O_I(\phi), \{O_J(\phi), H(\phi)\}_{pb}\}_{pb} + \{O_J(\phi), \{H(\phi), O_I(\phi)\}_{pb}\}_{pb} = 0. \tag{3.4}$$

[3] Observe that, if the CSCO has $<N+1$ operators, we have not *good* quantum numbers enough to label the eigenvectors.

[4] This fact can be considered as almost evident, but since it is not demonstrated in usual textbooks, we will give a complete demonstration below.

Then, since $O_I(\phi)$ and $O_J(\phi)$ are constants of the motion in $\mathscr{D}_{\phi_i}$, the $\{O_I(\phi),O_J(\phi)\}_{\mathrm{pb}}$ will also be so. As a consequence, if we could define N constants of the motion in involution such that

$$\{O_I(\phi), O_J(\phi)\}_{pb} = 0 \tag{3.5}$$

at each point $\phi \in \mathscr{D}^N_{\phi_i}$ (where $\mathscr{D}^N_{\phi_i}$ is the just defined domain of $2N+1$ dimensions around ϕ_i), by using these functions as initial conditions we can obtain a complete set of constants of the motion in involution in the domain $\mathscr{D}_{\phi_i} = \mathscr{D}^{N+1}_{\phi_i}$ of dimension $2(N+1)$, as promised.

Now the problem is reduced to prove the existence of the $O_I(\phi)$, $O_J(\phi)$ satisfying Eq. (3.5) in $\mathscr{D}^N_{\phi_i}$. Again, the existence of such a set can be easily proved by using the same strategy as above, but now *recursively*. We can begin with an arbitrary function $O_1(\phi)$ defined in a domain $\mathscr{D}^0_{\phi_i}$ of $N+1$ dimensions. Then, we consider another $O_2(\phi)$ (defined in a $N+2$ dimension domain $\mathscr{D}^1_{\phi_i}$ containing $\mathscr{D}^0_{\phi_i}$) as the Hamiltonian of Eq. (3.3) and obtain by integration a function $O_1(\phi)$, defined in the domain $\mathscr{D}^1_{\phi_i}$ of $N+2$ dimensions, such that in this domain $\{O_2(\phi), O_1(\phi)\}_{\mathrm{pb}} = 0$. Finally, we iterate the procedure up to find the set of functions in involution in the $\mathscr{D}^N_{\phi_i}$ of dimensions $2N+1$, which can be taken as initial conditions of Eq. (3.3). In this way, the proof is completed.[5]

(c) Now, in order to go from classical to quantum, we can also extend these local $O_I(\phi)$, defined in $\mathscr{D}_{\phi i} = \mathscr{D}^{N+1}_{\phi i}$ of dimensions $2(N+1)$, to all $\mathscr{M}$ by defining $O_I(\phi) = 0$ for $\phi \in \mathscr{M} \setminus \mathscr{D}_{\phi_i}$. In this case, there will be a jump in the frontier of $\mathscr{D}_{\phi_i}$, and the definition will be only continuous a.e. (almost everywhere). Or, on physical grounds, we can take the precaution of joining these zero functions with functions $O_I(\phi)$ inside $\mathscr{D}_{\phi_i}$ in an smooth way (e.g., by using C^r functions with an adequate r, as we explain in detail below).

Therefore, we have proved the existence of local complete systems of constants of the motion in involution $\{O_I(\phi)\} = \{H(\phi), O_1(\phi), \ldots, O_N(\phi)\}$ that we can extend to all $\mathscr{M}$, at least a.e., by adding null functions in $\mathscr{M} \setminus \mathscr{D}^{N+1}_{\phi_0}$ as explained. Since they belong to $\mathscr{D}_{\phi_i}$, we will call them $\{H(\phi), O_{\phi_i 1}(\phi), \ldots, O_{\phi_i N}(\phi)\}$. Each system $\{H(\phi), O_{\phi_i 1}(\phi), \ldots, O_{\phi_i N}(\phi)\}$ can be considered as a *local* (*approximate*) $(N+1)$-CSCO in $\mathscr{D}_{\phi_i} = \mathscr{D}^{N+1}_{\phi_i}$ in the sense that, even if it is not an exact CSCO, we can compute their Weyl transformations obtaining

$$\{\widehat{H}_{\phi_i}, \widehat{O}_{\phi_i 1}, \ldots, \widehat{O}_{\phi_i N}\}$$

and their Wigner transformations are a complete set of constants of the motion in involution in $\mathscr{D}_{\phi_i}$. In fact, from Eq. (2.10) we see that

$$\{O_{\phi_i I}(\phi), O_{\phi_j J}(\phi)\}_{mb} = 0(\hbar^2) \quad \text{or} \quad [\widehat{O}_{\phi_i I}, \widehat{O}_{\phi_j J}] = 0(\hbar^2), \tag{3.6}$$

namely, they only commute approximately.

Let us now consider in more detail the joining zone where we have used C^r-functions that do not satisfy the required differential Eqs. (3.3)–(3.5), in such a way that the higher order terms $\frac{i\hbar}{2}\overleftarrow{\partial}_a \omega^{ab} \overrightarrow{\partial}_b$ of Eq. (2.8) produce unwanted contributions of the order of $\hbar/PQ$, where P and Q are of the order of magnitude of the jumps in the momentum and configuration variables in the joining zone. Since $PQ = \varepsilon^2$ is an action measuring the joining zone (where ε is the characteristic mean width of the joining zone, precisely $\varepsilon^{2(N+1)} \cong V_\varepsilon$, the volume of the joining zones $\mathscr{F}$), the unwanted terms are of the order of $\hbar/\varepsilon^2$, that is, they are another contribution $0\left(\frac{\hbar^2}{\varepsilon^4}\right)$, or simply $0(\hbar^2)$, to add to (3.6). Anyhow, these terms will vanish when we make the limit $\hbar \to 0$ in Section 4.[6]

(d) Let us observe that natural global coordinates $\phi = (q, p_p)$ of phase space $\mathscr{M}$ can be (locally) substituted, by using a (local) canonical transformation, with (local) coordinates $(\theta_{\phi_i I}, O_{\phi_i I})$, with $i = 0, 1, \ldots, N$ and $H = O_{\phi_i 0}$, where the $\theta_{\phi_i I}(\phi)$ are the coordinates canonically conjugated to the $O_{\phi_i I}(\phi)$ defined in point (b) in each $\mathscr{D}_{\phi_i}$. The $(\theta_{\phi_i I}, O_{\phi_i I})$ is clearly a chart of $\mathscr{M}$ in the domain $\mathscr{D}_{\phi_i}$.[7] Since the system is endowed with adequate smooth properties (let us say C^r), another similarly constructed chart $(\theta_{\phi_j I}, O_{\phi_j I})$ in the domain $\mathscr{D}_{\phi_j}$ is smoothly connected with the previous one at any $\phi \in \mathscr{D}_{\phi_i} \cap \mathscr{D}_{\phi_j}$ (see demonstration in Section 5). Then, the set of all these charts is a C^r-atlas in $\mathscr{M}$. This will be the atlas we will primarily concerned with.

[5] An example of this phenomenon is the Sinai billiard discussed in Appendix A. Other examples are classical scattering systems: in fact, they have an 'in' CSCO and an 'out' CSCO, which are different since the values of the constants of the motion are not the same in these CSCOs. More complex examples are the so-called pseudointegrable systems [43, p. 98; 44; 45]; tori become spheres with 'handles' that cannot be covered with a single chart. A further example is Robnik's billiard [46].

[6] The term $0(\hbar)$ tends to zero only *counterfactually*, since $\hbar$ is a constant. The term tends to zero *factually* when the dimension is explicit, i.e., when we write it as $0(\hbar/S)$, where S is the characteristic action: we can neglect $0(\hbar/S)$ when $S \gg \hbar$. This means that $0(\hbar^2/\varepsilon^4)$ tends to zero if the action S of the system is large enough, $S \gg \hbar$, and, therefore, we choose an ε such that $\hbar \ll \varepsilon^2 \ll S$ (we shall come back to this argument in footnote 8).

[7] This is not a generic chart, but a very peculiar one, since coordinates $O_{\phi_i I}$ are constants of the motion satisfying Eqs. (3.3) and (3.5).

(e) We can also define a (*ad hoc*) *positive partition of the identity* (see [47, Section 3.4]) in the following sense. Let us define

$$1 = I(\phi) = \sum_i B_{\phi_i}(\phi), \tag{3.7}$$

where $B_{\phi_i}(\phi)$ are 'bump' smooth functions such that

$$B_{\phi_i}(\phi)\begin{cases} = 1 & \text{if } \phi \in D_{\phi_i}, \\ \in [0,1] & \text{if } \phi \in F_{\phi_i}, \\ = 0 & \text{if } \phi \notin D_{\phi_i} \cup F_{\phi_i}, \end{cases} \tag{3.8}$$

where D_{ϕ_i} is a domain and F_{ϕ_i} is the frontier zone around D_{ϕ_i} (all the F_{ϕ_i} belong to the joining zone $\mathscr{F}$) defined in such a way that $D_{\phi_i} \cup F_{\phi_i} \subset \mathscr{D}_{\phi_i}$ and the intersection zones of the $D's$ vanish: $D_{\phi_i} \cap D_{\phi_j} = \emptyset$. Let us stress that the $B_{\phi_i}(\phi)$ in the frontier zones satisfy Eq. (3.7). Now, for any $A(\phi)$ we can define a

$$A_{\phi_i}(\phi) = A(\phi) B_{\phi_i}(\phi)$$

and for any $A(\phi)$ we have

$$A(\phi) = A(\phi) \sum_i B_{\phi_i}(\phi) = \sum_i A_{\phi_i}(\phi).$$

With the mapping $symb^{-1}$ we find

$$\widehat{A} = symb^{-1} A(\phi) = \sum_i symb^{-1} A_{\phi_i}(\phi) = \sum_i \widehat{A}_{\phi_i}, \tag{3.9}$$

where $\widehat{A}_{\phi_i} = symb^{-1} A_{\phi_i}(\phi)$ can be considered as a *localization*of $\widehat{A}$ in D_{ϕ_i}. Moreover, since we have a local $(N+1)$-CSCO in each $D_{\phi_i} \cup F_{\phi_i} \subset \mathscr{D}_{\phi_i}$, we can decompose

$$\widehat{A}_{\phi_i} = \sum_j A_{j\phi_i} |j\rangle^{(A)}_{\phi_i} \langle j|^{(A)}_{\phi_i}, \tag{3.10}$$

where the $\{|j\rangle^{(A)}_{\phi_i}\}$ are the corresponding eigenvectors of $\widehat{A}_{\phi_i}$. If the $\widehat{A}_{\phi_i}$ also commute with $\widehat{H}$, the set $\{\widehat{A}_{\phi_i}\}$ can be considered as a local $(N+1)$-CSCO of $D_{\phi_i} \subset \mathscr{D}_{\phi_i}$.

Now we can prove that $symb|j\rangle^{(A)}_{\phi_i}\langle j|^{(A)}_{\phi_i}$ has the same support $D_{\phi_i} \cup F_{\phi_i}$ than $symb\,\widehat{A}_{\phi_i}$. In fact, from Eq. (3.10) we have

$$\widehat{A}_{\phi_i} |j\rangle^{(A)}_{\phi_i} = A_{j\phi_i} |j\rangle^{(A)}_{\phi_i} \quad \text{or} \quad \widehat{A}_{\phi_i} |j\rangle^{(A)}_{\phi_i} \langle j|^{(A)}_{\phi_i} = A_{j\phi_i} |j\rangle^{(A)}_{\phi_i} \langle j|^{(A)}_{\phi_i}.$$

Then,

$$symb\,\widehat{A}_{\phi_i} * symb|j\rangle^{(A)}_{\phi_i}\langle j|^{(A)}_{\phi_i} = A_{j\phi_i}\, symb|j\rangle^{(A)}_{\phi_i}\langle j|^{(A)}_{\phi_i}. \tag{3.11}$$

But $symb\,\widehat{A}_{\phi_i}$ and all its derivatives vanish for $\phi \notin D_{\phi_i} \cup F_{\phi_i}$. Therefore, if $A_{j\phi_i} \neq 0$, this also must happen for $symb|j\rangle^{(A)}_{\phi_i}\langle j|^{(A)}_{\phi_i}$, and the support of this function is contained in $D_{\phi_i} \cup F_{\phi_i}$. If $A_{j\phi_i} = 0$, we can repeat the argument with the operator $\widehat{A}_{\phi_i} + \alpha\widehat{B}_{\phi_i}$, with $\widehat{B}_{\phi_i} \neq 0$, and take the limit $\alpha \to 0$, and we will find the same result.

From Eqs. (3.9) and (3.10) we have

$$\widehat{A} = \sum_{ij} A_{j\phi_i} |j\rangle^{(A)}_{\phi_i} \langle j|^{(A)}_{\phi_i} \tag{3.12}$$

all over $\mathscr{M}$. Moreover, from Eq. (3.10) we also have

$$symb\,\widehat{A}_{\phi_i} = \sum_j A_{j\phi_i}\, symb|j\rangle^{(A)}_{\phi_i}\langle j|^{(A)}_{\phi_i}$$

and, as we have just proved,

$$symb|j\rangle^{(A)}_{\phi_i}\langle j|^{(A)}_{\phi_i}(\phi) = \begin{cases} symb|j\rangle^{(A)}_{\phi_i}\langle j|^{(A)}_{\phi_i}(\phi) & \text{if } \phi \in D_{\phi_i}, \\ 0 & \text{if } \phi \notin D_{\phi_i} \cup F_{\phi_i}. \end{cases}$$

Then, since for $i \neq k$, $D_{\phi_i} \cap D_{\phi_k} = \emptyset$ (but $F_{\phi_i} \cap F_{\phi_j} \neq 0$), we have

$$\left|\langle j|^{(A)}_{\phi_i}|j'\rangle^{(A)}_{\phi_k}\right|^2 = \langle j|^{(A)}_{\phi_i}|j'\rangle^{(A)}_{\phi_k}\langle j'|^{(A)}_{\phi_k}|j\rangle^{(A)}_{\phi_i} = \left(|j'\rangle^{(A)}_{\phi_k}\langle j'|^{(A)}_{\phi_k}||j\rangle^{(A)}_{\phi_i}\langle j|^{(A)}_{\phi_i}\right)$$
$$= \int_{\mathscr{M}} symb|j'\rangle^{(A)}_{\phi_k}\langle j'|^{(A)}_{\phi_k} symb|j\rangle^{(A)}_{\phi_i}\langle j|^{(A)}_{\phi_i}\,\mathrm{d}\phi^{2(N+1)} = \int_{\mathscr{F}} symb|j'\rangle^{(A)}_{\phi_k}\langle j'|^{(A)}_{\phi_k} symb|j\rangle^{(A)}_{\phi_i}\langle j|^{(A)}_{\phi_i}\,\mathrm{d}\phi^{2(N+1)}$$
$$= 0(\varepsilon^{2(N+1)}), \tag{3.13}$$

where $\mathscr{F}$ is the frontier or joining zone and ε is the characteristic width of the joining zone. Therefore, for $i \neq k$ and $\varepsilon \to 0$,[8] we obtain

$$\langle j|^{(A)}_{\phi_i}|j'\rangle^{(A)}_{\phi_k} = 0. \tag{3.14}$$

This means that, in the limit $\varepsilon \to 0$, decomposition (3.12) is an *orthogonormal decomposition* in the $|j\rangle^{(A)}_{\phi_i}$.

3.2. Decoherence in the energy

(a) We will now introduce decoherence according to the self-induced approach. Let us define, *in each* D_{ϕ_i}, a local $(N+1)$-CSCO where, as in Eq. (3.12) now applied for the $(N+1)$-CSCO $\{\widehat{H}, \widehat{O}_{\phi_i I}\}$, the observables can be expressed as

$$\widehat{H} = \int_0^\infty \omega \sum_{im} |\omega, m\rangle_{\phi_i}\langle\omega, m|_{\phi_i}\,\mathrm{d}\omega, \qquad \widehat{O}_{\phi_i I} = \int_0^\infty \sum_m O_{m_{I\phi_i}}|\omega, m\rangle_{\phi_i}\langle\omega, m|_{\phi_i}\,\mathrm{d}\omega, \tag{3.15}$$

where the energy spectrum is continuous, precisely $0 \leqslant \omega < \infty$, and $m_{I\phi_i} = \{m_{1\phi_i}, \ldots, m_{N\phi_i}\}$, $m_{I\phi_i} \in \mathbb{N}$ (the spectra of the $\widehat{O}_{\phi_i I}$ are discrete for simplicity).[9] Therefore

$$\widehat{H}|\omega, m\rangle_{\phi_i} = \omega|\omega, m\rangle_{\phi_i}, \qquad \widehat{O}_{\phi_i I}|\omega, m\rangle_{\phi_i} = O_{m_{I\phi_i}}|\omega, m\rangle_{\phi_i} \tag{3.16}$$

where the $|\omega, m\rangle_{\phi_i}$ are the eigenvectors of the observables $\widehat{O}_{\phi_i}$ (such that $symb\,\widehat{O}_{\phi_i} = O_{\phi_i}(\phi) \neq 0$ only in $D_{\phi_i} \cap F_{\phi_i}$) and m is a shorthand for $m_{\phi_i I} = \{m_{\phi_i 1}, \ldots, m_{\phi_i N}\}$. The set $\{|\omega, m\rangle_{\phi_i}\}$ is orthonormal in ω and in m, in the usual eigenvalue indices and in i, as proved in Eq. (3.14):

$$\langle\omega, m|_{\phi_i}|\omega', m'\rangle_{\phi_j} = \delta(\omega - \omega')\delta_{mm'}\delta_{ij}. \tag{3.17}$$

(b) Now we can define our relevant algebra of observables. This choice *will play the role of coarse-graining* in our approach. A generic observable reads, in the orthonormal basis just defined,

$$\widehat{O} = \sum_{imm'} \int_0^\infty \int_0^\infty \mathrm{d}\omega\,\mathrm{d}\omega'\,\widetilde{O}(\omega, \omega')_{\phi_i mm'}|\omega, m\rangle_{\phi_i}\langle\omega', m'|_{\phi_i}, \tag{3.18}$$

---

[8] Precisely: let us call $V_{\mathscr{M}}$ the volume of phase space: $V_{\mathscr{M}} \sim S^{N+1}$. Moreover,

$$I_{\mathscr{M}} = \int_{\mathscr{M}} symb|j'\rangle^{(A)}_{\phi_k}\langle j'|^{(A)}_{\phi_k} symb|j\rangle^{(A)}_{\phi_i}\langle j|^{(A)}_{\phi_i}\,\mathrm{d}\phi^{2(N+1)} \sim V_{\mathscr{M}} \sim S^{N+1}.$$

Let us also define

$$I_\varepsilon = \int_{\mathscr{F}} symb|j'\rangle^{(A)}_{\phi_k}\langle j'|^{(A)}_{\phi_k} symb|j\rangle^{(A)}_{\phi_i}\langle j|^{(A)}_{\phi_i}\,\mathrm{d}\phi^{2(N+1)} \sim V_\varepsilon = \varepsilon^{2(N+1)}.$$

In order to prove Eq. (3.13), it is necessary that $I_\varepsilon \ll I_{\mathscr{M}}$ in such a way that I_ε could be neglected. But $I_\varepsilon \sim V_\varepsilon$ and $I_{\mathscr{M}} \sim S^{N+1}$; then, $\varepsilon^2 \ll S$.Therefore, ε must be (see footnote 6):

1. Such that the ratio $\frac{\hbar}{\varepsilon^2}$ be negligible to eliminate the unwanted terms $\frac{i\hbar}{2}\overleftarrow{\partial}_a\,\omega^{ab}\,\overrightarrow{\partial}_b$ in the joining zone.
2. As small as $\varepsilon^2 \ll S$ to satisfy Eq. (3.13).Since $\hbar \ll S$, we can satisfy both conditions with an adequate ε, namely, such that:

$$\hbar \ll \varepsilon^2 \ll S.$$

[9] Hamiltonians with continuous spectra are considered in papers [23,24]. We use this kind of spectra since they are the usual ones in the macroscopic limit $\hbar \to 0$ (see [43, Eq. (3.1.24), p. 67]). Strictly, we should call $|\omega, m\rangle^{(\widehat{H}, \widehat{O}_{\phi_i})}_{\phi_i}$ the vectors $|\omega, m\rangle_{\phi_i}$, but we will just call them $|\omega, m\rangle_{\phi_i}$ for simplicity.

where $\widetilde{O}(\omega,\omega')_{\phi_i mm'}$ is a generic kernel or distribution in ω, ω'.[10] But we have to restrict this set of observables since it is too large for our purposes; furthermore, it is not easy to work with generic kernels or distributions. However, we cannot make the algebra too small either. In fact, let us suppose that, in order to make computation easier, we postulate that the $\widetilde{O}(\omega,\omega')_{\phi_i mm'}$ be just regular functions. Then, the states read

$$\hat{\rho} = \sum_{imm'} \int_0^\infty \int_0^\infty \mathrm{d}\omega\,\mathrm{d}\omega' \overline{\tilde{\rho}(\omega,\omega')_{\phi_i mm'}} |\omega,m\rangle_\phi \langle\omega',m'|_{\phi_i},$$

where the $\tilde{\rho}(\omega,\omega')_{\phi_i mm'}$ in the dual space are also regular functions. Then,

$$\langle\widehat{O}\rangle_{\hat{\rho}(t)} = \sum_{imm'} \int_0^\infty \int_0^\infty \mathrm{d}\omega\,\mathrm{d}\omega' \overline{\tilde{\rho}(\omega,\omega')_{\phi_i mm'}} \mathrm{e}^{\mathrm{i}(\omega-\omega')t} \widetilde{O}(\omega,\omega')_{\phi_i mm'}$$

and, since the product $\overline{\tilde{\rho}(\omega,\omega')_{\phi_i mm'}}\widetilde{O}(\omega,\omega')_{\phi_i mm'}$ is a regular function (i.e., $\mathbb{L}_1$ in $\nu = \omega - \omega'$), as a result of the Riemann–Lebesgue theorem the mean value $\langle\widehat{O}\rangle_{\hat{\rho}(t)}$ would vanish for $t \to \infty$: we would obtain destructive interference not only for the off-diagonal terms, *but for all of them*. This is obviously an *unacceptable result*. On the contrary, if $\overline{\tilde{\rho}(\omega,\omega')_{\phi_i mm'}}$ and $\widetilde{O}(\omega,\omega')_{\phi_i mm'}$ were generic kernels, we could not use the Riemann–Lebesgue theorem, and there would be no destructive interference. This means that $\widetilde{O}(\omega,\omega')_{\phi_i mm'}$ cannot be *so regular* nor *so non-regular:* we must choose something in between. In order to avoid these unacceptable results, the simplest choice is the van Hove choice; so, as in paper [26], we will take:

$$\widetilde{O}(\omega,\omega')_{\phi_i mm'} = O(\omega)_{\phi_i mm'}\delta(\omega-\omega') + O(\omega,\omega')_{\phi_i mm'}, \tag{3.19}$$

where the $O(\omega,\omega')_{\phi_i mm'}$ are ordinary functions of the real variables ω and ω' (these functions must have some mathematical properties in order to develop the theory; these properties are listed in paper [24]; moreover, this choice is theoretically explained in papers [21,26,36,38,39]). The addition of the term $\delta(\omega-\omega')$ is necessary in order that the members of the $(N+1)$-CSCO of Eq. (3.15) be contained in the space of observables. So our operator belongs to an algebra $\widehat{\mathscr{A}}$ (defined by Eq. (3.19) and the properties just required for the $O(\omega,\omega')_{\phi_i mm'}$), and reads

$$\widehat{O} = \sum_{imm'} \int_0^\infty \mathrm{d}\omega\, O(\omega)_{\phi_i mm'} |\omega,m\rangle_{\phi_i}\langle\omega,m'|_{\phi_i} + \sum_{imm'} \int_0^\infty \int_0^\infty \mathrm{d}\omega\,\mathrm{d}\omega' O(\omega,\omega')_{\phi_i mm'} |\omega,m\rangle_{\phi_i}\langle\omega',m'|_{\phi_i}. \tag{3.20}$$

The first term in the r.h.s. will be called $\widehat{O}_S$, the *singular* component, and the second term will be called $\widehat{O}_R$, the *regular* component,[11] and $[\hat{H},\widehat{O_S}] = 0$. The *observables* are the self-adjoint $O^\dagger = O$ operators. We will say that these observables belong to a space $\widehat{\mathscr{O}}$ (which is contained in the operator algebra $\widehat{\mathscr{A}}$); $\{|\omega,m,m'\rangle_{\phi_i}, |\omega,\omega',m,m'\rangle_{\phi_i}\}$ is a basis of this space, where

$$|\omega,m,m'\rangle_{\phi_i} \doteq |\omega,m\rangle_{\phi_i}\langle\omega,m'|_{\phi_i}, \qquad |\omega,\omega',m,m'\rangle_{\phi_i} \doteq |\omega,m\rangle_{\phi_i}\langle\omega',m'|_{\phi_i}. \tag{3.21}$$

(c) The quantum states $\hat{\rho}$ are measured by the observables just defined, leading to the mean values of these observables; in the usual notation: $\langle\widehat{O}\rangle_{\hat{\rho}} = \mathrm{Tr}(\hat{\rho}^\dagger\widehat{O})$. We can conceive that mean values as the more primitive objects of the quantum theory (see [48]). These mean values, generalized as in paper [24] and symbolized as $(\hat{\rho}|\widehat{O})$, can be considered as the result of the action of the linear functionals $\hat{\rho}$ on the observables of the vector space $\widehat{\mathscr{O}}$. Then, $\hat{\rho} \in \widehat{\mathscr{S}} \subset \widehat{\mathscr{O}}'$, where $\widehat{\mathscr{S}}$ is a convenient (i.e., satisfying Eqs. (3.24) and (3.25)) convex set contained in $\widehat{\mathscr{O}}'$, the space of linear functionals over $\widehat{\mathscr{O}}$. The basis of $\widehat{\mathscr{O}}'$ (that is, the *co-basis* of $\widehat{\mathscr{O}}$ in each D_{ϕ_i}) is $\{(\omega,m,m'|_{\phi_i}, (\omega,\omega',m,m'|_{\phi_i}\}$, and it is defined in terms of its functionals by the equations

$$\begin{aligned}
&(\omega,m,m'|_{\phi_i}|\eta,n,n')_{\phi_j} = \delta(\omega-\eta)\delta_{mn}\delta_{m'n'}\delta_{ij},\\
&(\omega,\omega',m,m'|_{\phi_i}|\eta,\eta',n,n')_{\phi_j} = \delta(\omega-\eta)\delta(\omega'-\eta')\delta_{mn}\delta_{m'n'}\delta_{ij},
\end{aligned} \tag{3.22}$$

and all other $(\cdot|\cdot)$ are zero. The orthogonality in $i,j,\ldots$ is a consequence of Eqs. (3.17) and (3.21). Let us observe that $(\omega,\omega',m,m'|_{\phi_i} \doteq |\omega,m\rangle_{\phi_i}\langle\omega',m'|_{\phi_i}$ but $(\omega,m,m'|_{\phi_i} \neq |\omega,m\rangle_{\phi_i}\langle\omega,m'|_{\phi_i}$.[12] Then, a generic quantum state reads

[10] As explained at the end of the last subsection, the index i in projector $|\omega,m\rangle_{\phi_i}\langle\omega',m'|_{\phi_i}$ corresponds to the fact that the decomposition is done in the $\mathscr{D}_{\phi_i}$ and, therefore, the index is repeated in $|\omega,m\rangle_{\phi_i}$ and in $\langle\omega',m'|_{\phi_i}$.

[11] The component $\widehat{O}_S$ is called singular because it contains a hidden distribution $\delta(\omega-\omega')$. In fact, it can be obtained from the regular part by making $O(\omega,\omega')_{\phi_i mm'} = O(\omega)_{\phi_i mm'}\delta(\omega-\omega')$.

[12] If $(\omega,m,m'|_{\phi_i} = |\omega,m\rangle_{\phi_i}\langle\omega,m'|_{\phi_i}$, it is easy to show that a divergence appears.

$$\hat{\rho} = \sum_{imm'} \int_0^\infty \mathrm{d}\omega\, \overline{\rho(\omega)}_{\phi_i mm'}(\omega, m, m'|_{\phi_i} + \sum_{imm'} \int_0^\infty \mathrm{d}\omega \int_0^\infty \mathrm{d}\omega'\, \overline{\rho(\omega,\omega')}_{\phi_i mm'}(\omega,\omega', m, m'|_{\phi_i}. \tag{3.23}$$

As before, the first term in the r.h.s. of Eq. (3.23) will be called $\hat{\rho}_S$, the *singular* component, and the second term will be called $\hat{\rho}_R$, the *regular* component. Functions $\rho(\omega,\omega')_{\phi_i mm'}$ are regular (see [24] for details). We also require that $\hat{\rho}^\dagger = \hat{\rho}$, i.e.,

$$\overline{\rho(\omega,\omega')}_{\phi_i mm'} = \rho(\omega',\omega)_{\phi_i m'm} \tag{3.24}$$

and that $\overline{\rho(\omega)}_{\phi_i mm'}$ be *real and non-negative*, satisfying the total probability condition

$$(\hat{\rho}|\widehat{I}) = \sum_{im} \int_0^\infty \mathrm{d}\omega\, \rho(\omega)_{\phi_i} = 1, \tag{3.25}$$

where $\widehat{I} = \int_0^\infty \mathrm{d}\omega \sum_{im} |\omega, m\rangle_{\phi_i}\langle\omega, m|_{\phi_i}$ is the identity operator in $\widehat{\mathscr{O}}$ represented in each D_{ϕ_i}. Eq. (3.25) is the extension to state functionals of the usual condition $\mathrm{Tr}\,\hat{\rho}^\dagger = 1$, when $\hat{\rho}$ is a density operator. Thus, from now on, $\mathrm{Tr}\,\hat{\rho} \doteq (\hat{\rho}|\widehat{I})$. For these reasons, $\hat{\rho}$ belongs to the just defined convex set $\widehat{\mathscr{S}} \subset \widehat{O}'$. The time evolution of the quantum state $\hat{\rho}$ reads

$$\hat{\rho}(t) = \sum_{imm'} \int_0^\infty \mathrm{d}\omega\, \overline{\rho(\omega)}_{\phi_i mm'}(\omega, m, m'|_{\phi_i} + \sum_{imm'} \int_0^\infty \mathrm{d}\omega \int_0^\infty \mathrm{d}\omega'\, \overline{\rho(\omega,\omega')}_{\phi_i mm'} \mathrm{e}^{\mathrm{i}(\omega-\omega')t/\hbar}(\omega,\omega', m, m'|_{\phi_i}. \tag{3.26}$$

As we have already said, at the statistical quantum level we essentially measure mean values of observables in quantum states

$$\begin{aligned}\langle\widehat{O}\rangle_{\hat{\rho}(t)} &= (\hat{\rho}(t)|\widehat{O}) \\ &= \sum_{imm'} \int_0^\infty \mathrm{d}\omega\, \overline{\rho(\omega)}_{\phi_i mm'} O(\omega)_{\phi_i mm'} + \sum_{imm'} \int_0^\infty \mathrm{d}\omega \int_0^\infty \mathrm{d}\omega'\, \overline{\rho(\omega,\omega')}_{\phi_i mm'} \mathrm{e}^{\mathrm{i}(\omega-\omega')t/\hbar} O(\omega,\omega')_{\phi_i mm'}.\end{aligned} \tag{3.27}$$

If we take into account that $O(\omega,\omega')$ and $\overline{\rho(\omega,\omega')}_{\phi_i mm'}$ are regular (as regular as needed to use the Riemann–Lebesgue theorem, i.e., $O(\omega,\omega')\overline{\rho(\omega,\omega')}_{\phi_i mm'} \in \mathbb{L}_1(\omega-\omega')$, see [24]), we can take the limit $t \to \infty$ and use the Riemann–Lebesgue theorem. As the result, we see that the fluctuating-regular part vanishes and we arrive to the weak (quantum and classical) limits

$$W \lim_{t\to\infty} \hat{\rho}(t) = \hat{\rho}_S = \hat{\rho}_* = \sum_{imm'} \int_0^\infty \mathrm{d}\omega\, \overline{\rho(\omega,p)}_{\phi_i mm'}(\omega, m, m'|_{\phi_i} \tag{3.28}$$

Since only the singular diagonal terms remain, we have obtained decoherence in the energy variable ω. Precisely, any quantum state weakly tends to a linear combination of the energy diagonal states $(\omega, m, m'|_{\phi_i}$ (the energy 'off-diagonal' states $(\omega,\omega', m, m'|_{\phi_i}$ are not present in $\hat{\rho}_*$). This is the case when we observe and measure the system evolution with *any possible observable of space* $\widehat{\mathscr{O}}$. Therefore, from the observational point of view, we have decoherence of the energy levels in spite of the fact that, from the strong limit point of view, the off-diagonal terms *never vanish:* they just oscillate since we cannot directly use the Riemann–Lebesgue theorem in the operator Eq. (3.26).

3.2.1. Important remarks

(i) It might be supposed that this kind of decoherence takes place without a coarse-graining. It is no so: the choice of the algebra $\widehat{\mathscr{A}}$ among all possible algebras (see under Eq. (3.19)) and the systematic use of mean values $\langle\widehat{O}\rangle_{\hat{\rho}(t)} = (\hat{\rho}(t)|\widehat{O})$ (Eq. (3.27)), restrict the available information and produce the effect of a coarse-graining. In fact, we can define the projector $\Pi = |\widehat{O})(\hat{\rho}_0|$, with $|\widehat{O}) \in \widehat{\mathscr{A}}$ and $(\hat{\rho}_0|\widehat{O}) = 1$, that projects $(\hat{\rho}(t)|$ as $(\hat{\rho}(t)|\Pi = \langle\widehat{O}\rangle_{\hat{\rho}(t)}(\hat{\rho}_0|$, and translates everything in projectors language: then we obtain, from Eq. (3.28), $\lim_{t\to\infty}(\hat{\rho}(t)|\Pi = (\hat{\rho}_*|\Pi$. This projection will obviously break the unitarity of the primitive evolution. In this way we could develop a formalism closer to the usual one (see a detailed explanation in [13,34]).

(ii) Theoretically, decoherence takes place at $t \to \infty$. But, in practice, decoherence appears at a decoherence time, as we have defined in [49]: the decoherence time can be easily computed from the poles of the resolvent or the initial conditions in the complex extension of the $\widehat{H}$ spectrum. Trivial $\widehat{H}$ (e.g., free particle $\widehat{H}$) and trivial initial conditions (e.g., zero temperature ones) do not have poles and the decoherence time is infinite. This means that, to reach equilibrium in a finite characteristic time, $\widehat{H}$ must be non-trivial (e.g., the sum of a free Hamiltonian plus an interaction Hamiltonian) and/or the initial conditions must be non-trivial (e.g., $T \neq 0$). For details, see [35], where decoherence times are estimated of the order of 10^{-37}–$10^{-39}s$ for macroscopic bodies at room temperature.

3.3. Decoherence in the remaining variables

Having obtained decoherence in the energy levels, we must consider decoherence in the other dynamical variables $O_{\phi_i I}$ of the set of local CSCOs we are using. We will call these variables 'momentum variables'. Since the expression of $\hat{\rho}_*$, given in Eq. (3.28), only involves the time independent components of $\hat{\rho}(t)$, it is impossible that a further decoherence process eliminates the off-diagonal terms in the remaining N dynamical momentum variables. Therefore, the only alternative is to find the basis where these off-diagonal components $\rho(\omega)_{\phi_i mm'}$ vanish at any time.

Let us consider the following unitary change of basis:

$$|\omega, p\rangle_{\phi_i} = \sum_m U(\omega)_{mp} |\omega, m\rangle_{\phi_i}, \tag{3.29}$$

where p and m are shorthand notations for $p \doteq \{p_1, \ldots, p_N\}$ and $m \doteq \{m_1, \ldots, m_N\}$, and $[U(\omega)^{-1}]_{mp} = \overline{U(\omega)}_{pm}$. We choose the new basis $\{|\omega, p\rangle_{\phi_i}\}$ such that it verifies the generalized orthogonality condition

$$\langle \omega, p|_{\phi_i} |\omega', p'\rangle_{\phi_i} = \delta(\omega - \omega')\delta_{pp'}.$$

Since $\overline{\rho(\omega)}_{\phi_i} = \rho(\omega)_{\phi_i}$, it is possible to choose $U(\omega)$ in such a way that the off-diagonal parts of $\rho(\omega)_{\phi_i pp'}$ vanish, i.e.,

$$\rho(\omega)_{\phi_i pp'} = \rho_{\phi_i}(\omega)_p \delta_{pp'}. \tag{3.30}$$

This means that there is a *final local pointer basis in* D_{ϕ_i} for the observables, given by $\{|\omega, p, p'\rangle_{\phi_i}, |\omega, \omega', p, p'\rangle_{\phi_i}\}$ and defined as in Eq. (3.21) but now with the p. The corresponding final pointer basis for the states, $\{(\omega, p, p'|_{\phi_i}, (\omega, \omega', p, p'|_{\phi_i}\}$, diagonalizes the time independent part of $\hat{\rho}(t)$ and, therefore, diagonalizes the final state $\hat{\rho}_*$.

Now, we have diagonalized the $\overline{\rho(\omega)}_{\phi_i mm'}$ in m and m', obtaining

$$W \lim_{t\to\infty} \hat{\rho}(t) = \hat{\rho}_S = \hat{\rho}_* = \sum_{ip} \int_0^\infty \mathrm{d}\omega \, \overline{\rho_{\phi_i}(\omega)}_p (\omega, p, p|_{\phi_i}. \tag{3.31}$$

Here we are using a local pointer $(N+1)$-CSCO $\{\widehat{H}, \widehat{P}_{\phi_i 1}, \ldots, \widehat{P}_{\phi_i N}\}$ at each D_{ϕ_i}, where the $\widehat{P}_{\phi_i I}$ are

$$\widehat{P}_{\phi_i I} = \sum_i \int_0^\infty \mathrm{d}\omega \sum_p p_{\phi_i I}(\omega) |\omega, p, p)_{\phi_i}, \tag{3.32}$$

where $|\omega, p, p)_{\phi_i} = |\omega, p\rangle_{\phi_i} \langle \omega, p|_{\phi_i}$ or simply $\{|\omega, p\rangle_{\phi_i}\}$ is the *local pointer basis* in D_{ϕ_i}; so, we can write Eq. (3.20) in this new basis (see Eq. (4.2) below).[13] Now all the operators and matrices involved are diagonal, and decoherence is complete. We can define all the observables $\widehat{O}$ of Eq. (3.20) in this new local pointer basis.

Since in the limit $\hbar \to 0$ we usually have $\widehat{P}$ with continuous spectra, instead of the last equations we would have the natural analogues of Eq. (3.31) (see [31,33] for details)

$$W \lim_{t\to\infty} \hat{\rho}(t) = \hat{\rho}_S = \hat{\rho}_* = \sum_i \int_0^\infty \mathrm{d}\omega \int_{p\in D_{\phi_i}} \mathrm{d}p^N \, \overline{\rho(\omega)_{\phi_i}}(\omega, p, p|_{\phi_i}. \tag{3.33}$$

In the next section we will consider the classical limit and, then, we will only use continuous spectra[14]; then, we will rewrite some equations in the new basis for the sake of completeness.

4. The classical statistical limit

4.1. Quantum and classical operators

(a) From now on, we will consider a system from the point of view of the local pointer complete set of $(N+1)$-commuting observables $\{\widehat{H}, \widehat{P}_{\phi_i 1}, \ldots, \widehat{P}_{\phi_i N}\}$, defined by Eqs. (3.15) and (3.32). As above, to simplify the notation we will just

[13] The complexity of these formulae demonstrates why it was so difficult to define the pointer basis in a general case. As we can see, the pointer basis depends on H and the initial conditions, but obviously there are some particular cases where it only depends on H.

[14] If we use the Heisenberg picture, the $\widehat{A}$ would become diagonal in the $\widehat{H}$ eigenbasis. So, heuristically

$$\lim_{t\to\infty} (\hat{\rho}_* | [\widehat{A}(t), \widehat{B}]) = \lim_{t\to\infty} \mathrm{Tr}(\hat{\rho}_* \widehat{A}(t) \widehat{B} - \hat{\rho}_* \widehat{B} \widehat{A}(t)) = \mathrm{Tr}(\hat{\rho}_* \widehat{A}_* \widehat{B} - \hat{\rho}_* \widehat{B} \widehat{A}_*) = \mathrm{Tr}(\hat{\rho}_* \widehat{A}_* \widehat{B} - \widehat{A}_* \hat{\rho}_* \widehat{B}) = 0,$$

where $\widehat{A}_*$ is the diagonal weak limit of $\widehat{A}$ and, therefore, commutes with the diagonal $\hat{\rho}_*$. As a consequence, the evolution is *weakly asymptotically abelian* [47, Definition 4.11] since, in the limit $t \to \infty$, $\widehat{\mathscr{A}}$ can be considered commutative. Therefore, a quantum system with continuous spectrum is *weakly asymptotically abelian*.

call $\{\widehat{H},\widehat{P}_{\phi_i}\}$ the set $\{\widehat{H},\widehat{P}_{\phi_i 1},\ldots,\widehat{P}_{\phi_i N}\}$. Thus, we will consider the orthonormal eigenbasis $\{|\omega,p\rangle_{\phi_i}\}$ of $\{\widehat{H},\widehat{P}_{\phi_i}\}$, and write the Hamiltonian and $\widehat{P}$ as

$$\widehat{H}=\sum_i\int_{p\in D_{\phi_i}}\mathrm{d}p^N\int_0^\infty \omega|\omega,p\rangle_{\phi_i}\langle\omega,p|_{\phi_i}\,\mathrm{d}\omega,\qquad \widehat{P}_{\phi_i}=\int_{p\in D_{\phi_i}}\mathrm{d}p^N\int_0^\infty p|\omega,p\rangle_{\phi_i}\langle\omega,p|_{\phi_i}\,\mathrm{d}\omega. \tag{4.1}$$

Furthermore, we will consider the algebra $\widehat{\mathscr{A}}$ of the operators (3.20), which now reads

$$\widehat{O}=\sum_i\int_{p\in D_{\phi_i}}\mathrm{d}p^N\int_0^\infty O_{\phi_i}(\omega,p)|\omega,p\rangle_{\phi_i}\langle\omega,p|_{\phi_i}\,\mathrm{d}\omega+\sum_i\int_{p\in D_{\phi_i}}\int_{p'\in D_{\phi_i}}\mathrm{d}p^N\,\mathrm{d}p'^N\int_0^\infty \times\int_0^\infty O_{\phi_i}(\omega,\omega',p,p')|\omega,p\rangle_{\phi_i}\langle\omega,p'|_{\phi_i}\,\mathrm{d}\omega\,\mathrm{d}\omega'. \tag{4.2}$$

As before, the first term in the r.h.s. will be called $\widehat{O}_S$, the *singular* component, and the second term will be called $\widehat{O}_R$, the *regular* component. Also as before, functions $O_{\phi_i}(\omega,\omega',p,p')$ are regular (see [24] for details), $[\widehat{H},\widehat{O}_S]=0$, $\widehat{O}_S\in\widehat{\mathscr{L}}_S$, where $\widehat{\mathscr{L}}_S$ is the singular space, $\widehat{O}_R\in\widehat{\mathscr{L}}_R$, where $\widehat{\mathscr{L}}_R$ is the regular space, and $\widehat{\mathscr{A}}=\widehat{\mathscr{L}}_S\oplus\widehat{\mathscr{L}}_R$. The observables are the self-adjoint operators of $\widehat{\mathscr{A}}$, and they belong to a space $\widehat{\mathscr{O}}$.

(b) Let us now consider the Wigner transformation of these objects. The operators of $\widehat{\mathscr{L}}_R$ are regular; so, their transformation is obtained as explained in Section 2. Then, we have to consider only the singular space $\widehat{\mathscr{L}}_S$, the space of the operators that commute with $\widehat{H}$. This is not a regular space of operators on a Hilbert space $\mathscr{H}$ as $\widehat{\mathscr{L}}_R$, since it contains a hidden $\delta(\omega-\omega')$ (see Eq. (3.19)), but the mapping *symb* given by Eq. (2.4) can also be well defined for the observables in $\widehat{\mathscr{L}}_S$. In fact, from Eq. (4.2) we know that

$$\widehat{O}_S=\sum_i\int_{p\in D_{\phi_i}}\mathrm{d}p^N\int_0^\infty O_{\phi_i}(\omega,p)|\omega,p\rangle_{\phi_i}\langle\omega,p|_{\phi_i}\,\mathrm{d}\omega. \tag{4.3}$$

If we consider, as usual, first O_{ϕ_i} as a polynomial, and then O_{ϕ_i} as a function of a certain space where the polynomials are dense,[15] by using Eq. (4.1) we can conclude that

$$\widehat{O}_S=\sum_i O_{\phi_i}(\widehat{H},\widehat{P}_{\phi_i})=\sum_i\widehat{O}_{S\phi_i}, \tag{4.4}$$

where, for the last equality, we have used Eq. (3.9); so $\widehat{O}_{S\phi_i}$ is related with D_{ϕ_i}. But, when $\widehat{H},\widehat{P}_{\phi_i}$ commute, we can use Eq. (2.11); then, by means of the same procedure as before and Eq. (2.7)) we have

$$symb\,\widehat{O}_S=O_S(\phi)=\sum_i O_{\phi_i}(H(\phi),P_{\phi_i}(\phi))+0(\hbar^2)=\sum_i symb\,\widehat{O}_{S\phi_i}, \tag{4.5}$$

where $H(\phi)$, $P_{\phi_i}(\phi)$ can be computed as usually (see [31] for details). In this way, we have succeeded in computing all the *symb* of the observables of $\widehat{\mathscr{L}}_S$ up to $0(\hbar^2)$, which are just the $O_{\phi_i}(H(\phi),P(\phi))$, and we have defined the mapping

$$symb:\widehat{\mathscr{L}}_S\to\mathscr{L}_{Sq},\qquad symb\,\widehat{O}_S=O_S(\phi)=\sum_i O_{\phi_i}(H(\phi),P_{\phi_i}(\phi))+0(\hbar^2). \tag{4.6}$$

Moreover, since decompositions D_{ϕ_i} or $\mathscr{D}_{\phi_i}$ are arbitrary (because they depend on the initial conditions of Section 3.1), from Eqs. (4.4) and (4.5) we obtain (up to $0(\hbar^2)$)

$$\widehat{O}_{S\phi_i}=O_{\phi_i}(\widehat{H},\widehat{P}_{\phi_i}),\quad O_{S\phi_i}(\phi)=symb\,\widehat{O}_{S\phi_i}=O_{\phi_i}(H(\phi),P_{\phi_i}(\phi)). \tag{4.7}$$

Let us observe that, if $O_{\phi_i}(\omega,p)=\delta(\omega-\omega')\delta(p-p')$, we have (also up to $0(\hbar^2)$)

$$symb|\omega',p'\rangle_{\phi_i}\langle\omega',p'|_{\phi_i}=\delta(H(\phi)-\omega')\delta(P_{\phi_i}(\phi)-p) \tag{4.8}$$

an equation that we will use below.

Summing up, from Eqs. (2.2) and (4.6) we have defined a classical space $\mathscr{A}_q=\mathscr{L}_R\oplus\mathscr{L}_S$ and a mapping

$$symb:\widehat{\mathscr{A}}\to\mathscr{A}_q,\qquad symb\,\widehat{O}=O(\phi), \tag{4.9}$$

where Eqs. (2.9) and (2.10) are also valid. Then, we can repeat what we have said below Eq. (2.10), but now for the algebra $\widehat{\mathscr{A}}$ defined as in this section, with its *regular and singular parts*.

If now we take the limit $\hbar\to 0$, we obtain $\mathscr{A}_q\to\mathscr{A}$, where $\mathscr{A}$ is the usual algebra of observables on phase space. Then, in this limit we have a correspondence $\widehat{\mathscr{A}}\to\mathscr{A}$. However, even if this limit is well defined and can be considered

[15] These polynomials have several variables, but there is no problem since all these variables commute.

as *the classical limit of the algebra of operators*, it is only the limit of the *equations* of the system, since these are a consequence of the algebra. Therefore, this is just a 'formal' limit. The limit $\hbar \to 0$ will be completely studied when we deal with the state space.

For the sake of simplicity, from now on we will systematically eliminate all the $0(\hbar^2)$ from the equations and call the $\mathscr{A}_q$ just $\mathscr{A}$. This is a rigorous simplification. In fact, when $\hbar = 0$, we can make the $0(\hbar) = 0$ everywhere since, from Eq. (2.7), when $\hbar = 0$ we have $\exp 0 = 1$ in that equation; in other words, the $\lim_{\hbar \to 0}$ is continuous.

4.2. Quantum and classical states

(a) Let us remember that $|\omega,p)_{\phi_i} = |\omega,p,p)_{\phi_i} = |\omega,p\rangle_{\phi_i}\langle\omega,p|_{\phi_i}$ and $|\omega,\omega',p,p')_{\phi_i} = |\omega,p\rangle_{\phi_i}\langle\omega',p'|_{\phi_i}$ as in Eq. (3.21). $\{|\omega,p,p')_{\phi_i}\}$ is the basis of $\widehat{\mathscr{L}}_S$ and $\{|\omega,\omega\prime,p,p\prime)_{\phi_i}\}$ is the basis of $\widehat{\mathscr{L}}_R$. Then, Eq. (4.2) reads

$$\begin{aligned}\widehat{O} &= \sum_i \widehat{O} \\ &= \sum_i \int_{p\in D_{\phi_i}} \mathrm{d}p^N \int_0^\infty O_{\phi_i}(\omega,p)|\omega,p)_{\phi_i}\,\mathrm{d}\omega + \sum_i \int_{p\in D_{\phi_i}}\int_{p'\in D_{\phi_i}} \mathrm{d}p^N\,\mathrm{d}p'^N \int_0^\infty \\ &\quad\times \int_0^\infty O_{\phi_i}(\omega,\omega',p,p')|\omega,\omega',p,p')_{\phi_i}\,\mathrm{d}\omega\,\mathrm{d}\omega'. \end{aligned} \tag{4.10}$$

Since the states are functionals over the space $\widehat{\mathscr{A}} = \widehat{\mathscr{L}}_S \oplus \widehat{\mathscr{L}}_R$, let us consider the dual space $\widehat{\mathscr{A}}' = \widehat{\mathscr{L}}'_S \oplus \widehat{\mathscr{L}}'_R$. We will call $\{(\omega,p|_{\phi_i}\}$ the local bases of $\widehat{\mathscr{L}}'_S$ and $\{(\omega,\omega',p,p'|_{\phi_i}\}$ the local bases of $\widehat{\mathscr{L}}'_R$. Let us remember that $(\omega,\omega',p,p'|_{\phi_i} = |\omega,p\rangle_{\phi_i}\langle\omega',p'|_{\phi_i}$ but $(\omega,p|_{\phi_i} \neq |\omega,p\rangle_{\phi_i}\langle\omega,p|_{\phi_i}$. Moreover, as in Eq. (3.22),

$$\begin{aligned}&(\omega,p|_{\phi_i}|\omega',p')_{\phi j} = \delta(\omega-\omega')\delta^N(p-p')\delta_{ij}, \\ &(\omega,\sigma,p,s|_{\phi_i}|\omega',\sigma',p',s')_{\phi_j} = \delta(\omega-\omega')\delta(\sigma-\sigma')\delta^N(p-p')\delta^N(s-s')\delta_{ij}, \\ &(\omega,\sigma,|_{\phi_i}|\omega',\sigma',p',s')_{\phi_j} = (\omega,\sigma,p,s|_{\phi_i}|\omega',\sigma')_{\phi_i} = 0.\end{aligned} \tag{4.11}$$

Then, a generic functional of $\widehat{\mathscr{A}}'$ reads

$$\begin{aligned}\hat{\rho} &= \sum_i \int_{p\in D_{\phi_i}} \mathrm{d}p^N \int_0^\infty \overline{\rho_{\phi_i}(\omega,p,)}(\omega,p|_{\phi_i}\,\mathrm{d}\omega + \sum_i \int_{p\in D_{\phi_i}}\int_{p'\in D_{\phi_i}} \mathrm{d}p^N\,\mathrm{d}p'^N \int_0^\infty \\ &\quad\times \int_0^\infty \overline{\rho_{\phi_i}(\omega,\omega',p,p')}(\omega,\omega',p,p'|_{\phi_i}\,\mathrm{d}\omega\,\mathrm{d}\omega'. \end{aligned} \tag{4.12}$$

Like functions $O_{\phi_i}(\omega,\omega',p,p')$, functions $\rho_{\phi_i}(\omega,\omega',p,p')$ are regular and have all the mathematical properties necessary to make the formalism successful (see [24]). Moreover, the $\hat{\rho}$ must be self-adjoint, and their diagonal $\rho_{\phi_i}(\omega,p)$ must represent probabilities; thus, $\sum_{i,p}\int_0^\infty \rho_{\phi_i}(\omega,p)\,\mathrm{d}\omega = 1$ (as in Eq. (3.25)) and, *most important*,

$$\rho_{\phi_i}(\omega,p) \geqslant 0. \tag{4.13}$$

The $\hat{\rho}$ with such properties belong to a convex set $\widehat{\mathscr{S}}$, the set of states. Also, as in Eq. (3.27),

$$\begin{aligned}(\hat{\rho}|\widehat{O}) &= \sum_i \int_{p\in D_{\phi_i}} \int_0^\infty \overline{\rho_{\phi_i}(\omega,p,)}O_{\phi_i}(\omega,p)\,\mathrm{d}\omega\,\mathrm{d}p^N + + \sum_i \int_{p\in D_{\phi_i}}\int_{p'\in D_{\phi_i}} \int_0^\infty \\ &\quad\times \int_0^\infty \overline{\rho_{\phi_i}(\omega,\omega',p,p')}O_{\phi_i}(\omega,\omega',p,p')\,\mathrm{d}\omega\,\mathrm{d}\omega'\,\mathrm{d}p^N\,\mathrm{d}p'^N. \end{aligned} \tag{4.14}$$

(b) Since $\widehat{\mathscr{L}}_R$ and $\widehat{\mathscr{L}}'_R$ are spaces of operators on a Hilbert space $\mathscr{H}$, the symbol for any $\hat{\rho}_R \in \widehat{\mathscr{L}}'_R$ is defined as in Eq. (2.14).[16] From this definition, Eq. (2.15) can be proved for the regular parts with the usual demonstration in the bibliography (see [40, Eq. (2.13)]):

$$(\hat{\rho}_R|\widehat{O}_R) = (symb\,\hat{\rho}_R|symb\widehat{O}_R) = \sum_i \int_{D_{\phi_i}} \mathrm{d}\phi^{2(N+1)}\,\rho_{\phi_i R}(\phi)O_{\phi_i R}(\phi). \tag{4.15}$$

Then, in $\widehat{\mathscr{L}}_R$ and $\widehat{\mathscr{L}}'_R$ all the equations are the usual ones (i.e., those of papers [40,41]).

[16] We repeat that, in the case of states, we must add a new factor $(2\pi\hbar)^{-(N+1)}$ to definition (2.4) in order to maintain the usual normalization of $\rho(\phi)$.

Let us now consider the singular dual space $\widehat{\mathscr{L}}'_S$, the case not treated in the bibliography. In this space we will *define* $symb\,\hat{\rho}_S$ as the function on $\mathscr{M}$ that satisfies an equation similar to Eqs. (2.15) or (4.15) for any $\widehat{O}_S \in \widehat{\mathscr{L}}_S$, namely,

$$(\hat{\rho}_S|\widehat{O}_S) \doteq (symb\,\hat{\rho}_S|symb\,\widehat{O}_S),$$

precisely,

$$\sum_i \int_{p\in D_{\phi_i}} \int_0^\infty \rho_{\phi i}(\omega,p,)O_{\phi_i}(\omega,p)\,\mathrm{d}\omega\,\mathrm{d}p^N = \sum_i \int_{D_{\phi_i}} \mathrm{d}\phi^{2(N+1)}\,\rho_{\phi_i S}(\phi)O_{\phi_i S}(\phi), \tag{4.16}$$

where the unknown density function $\rho_S(\phi) = symb\,\hat{\rho}_S$ can be decomposed as

$$symb\,\hat{\rho}_S = \rho_S(\phi) = \sum_i \rho_{\phi_i S}(\phi) \tag{4.17}$$

in each D_{ϕ_i}. Thus, since we know $\rho_{\phi i}(\omega,p,)$, $O_{\phi_i}(\omega,p)$, and $O_{\phi_i S}(\phi)$, we can compute $\rho_{\phi_i S}(\phi)$ to obtain $\rho_S(\phi) = symb\,\hat{\rho}_S$. Now, $\hat{\rho}_S$, being time invariant, must be a function of the constants of the motion; therefore (as in Section 4.1) its Weyl-transformed $\rho_S(\phi)$ must be endowed with the same property, but now in the classical case. Since the $\{H(\phi), P_{\phi_i}(\phi)\}$ are locally a complete set of constants of the motion in involution, we must have

$$\rho_{\phi_i}(\phi) = F(H(\phi), P_{\phi_i}(\phi)). \tag{4.18}$$

We will find the function F. The system has a local pointer CSCO of $N+1$ operators and the dimension of its phase space is $2(N+1)$, i.e., it is *locally* an integrable system.[17] Then, we can define *locally at* D_{ϕ_i} the action angle variables $(\theta^0,\theta^1,\ldots,\theta^N,J^0_{\phi_i},J^1_{\phi_i},\ldots,J^N_{\phi_i})$, where $J^0_{\phi_i},J^1_{\phi_i},\ldots,J^N_{\phi_i}$ would be just $H,P_{\phi_i 1},\ldots,P_{\phi_i N}$ (multiplied by adequate constants in such a way that the $\theta^{i0}_{\phi_i}$ variables belong to an interval $0 \leqslant \theta^{i0}_{\phi_i} \leqslant 2\pi$ in the integrable case). We will call 'J' just the 'H,P_{ϕ_i}'. Thus, we can make the canonical transformation $\phi^a \to \theta^0_{\phi_i},\theta^1_{\phi_i},\ldots,\theta^N_{\phi_i},H,P_{\phi_i 1},\ldots,P_{\phi_i N}$, and we obtain

$$\mathrm{d}\phi^{2(N+1)} = \mathrm{d}q^{(N+1)}\,\mathrm{d}p^{(N+1)} = \mathrm{d}\theta^{(N+1)}_{\phi_i}\,dH\,\mathrm{d}P^N_{\phi_i}, \tag{4.19}$$

since the Jacobian of a canonical transformation is one.

In order to compute the r.h.s. of Eq. (4.16), we must know how to integrate functions $f(H(\phi),P_{\phi_i}(\phi)) = F(H(\phi),P_{\phi_i}(\phi))O_{\phi_i S}(\phi)$ (see Eqs. (4.7) and (4.16)), which are just functions of the constants of motion, precisely,

$$\int_{D_{\phi_i}} \mathrm{d}\phi^{2(N+1)} f(H,P_{\phi_i}) = \int_{D_{\phi_i}} \mathrm{d}\theta^{(N+1)}_{\phi_i}\,\mathrm{d}H\,\mathrm{d}P^N_{\phi_i} f(H,P_{\phi_i}) = \int_{D\phi_i} \mathrm{d}H\,\mathrm{d}P^N_{\phi_i}\,C_{\phi_i}(H,P_{\phi_i})f(H,P_{\phi_i}), \tag{4.20}$$

where we have integrated the angular variables $\theta^0_{\phi_i},\theta^1_{\phi_i},\ldots,\theta^N_{\phi_i}$ and obtained the configuration volume $C_{\phi_i}(H,P_{\phi_i})$ of the portion of the hypersurface defined by ($H=$ constant, $P_{\phi_i}=$ constant) and contained in D_{ϕ_i}. So, from Eqs. (4.16) and (4.20) we have that

$$\int_{p\in D_{\phi_i}} \int_0^\infty \rho_{\phi i}(\omega,p,)O_{\phi_i}(\omega,p)\,\mathrm{d}\omega\,\mathrm{d}p^N = \int \mathrm{d}H\,\mathrm{d}P^N_{\phi_i}\,C_{\phi_i}(H,P_{\phi_i})\rho_{\phi_i S}(H,P_{\phi_i})O_{\phi_i S}(H,P_{\phi_i}), \tag{4.21}$$

for all $O_{\phi_i}(H,P_{\phi_i}) = O_{S\phi_i}(H,P_{\phi_i})$ (see Eq. (4.7)). The last equation defines $\rho_{S\phi_i}(H,P) = \frac{1}{C_{\phi_i}}\rho_{\phi_i}(H,P)$ for $\phi \in \mathscr{D}_{\phi_i}$,[18] but not for $\phi \in \mathscr{M} \setminus \mathscr{D}_{\phi_i}$; then, as in the case of $O_{S\phi_i}(\phi)$, we will consider that $\rho_{S\phi_i}(\phi) = 0$ for $\phi \in \mathscr{M} \setminus \mathscr{D}_{\phi_i}$ and that they are

[17] We have discussed this fact in detail at the beginning of Section 3. The constants J are global or isolating in the case of an integrable system, but not in the non-integrable case. Nevertheless, they are locally defined. Moreover, we will only consider the cases where action-angle variables can be locally defined.

[18] In the integrable case, where there is just one $\rho(H,P)$, it would be $\rho_{\phi_i}(H,P) = \frac{C_{\phi_i}(H,P)}{2\pi^{N+1}}\rho(H,P)$ and the results of paper [26] would be reobtained. In fact, by integrating over a torus in the θ, we have $(2\pi)^{N+1}\rho(H,P) = \sum_i C_{\phi_i}(H,P)\rho_{\phi_i}(H,P)$.An example to fix the ideas: let us consider the harmonic oscillator and the plane q, p in radial coordinates θ, H. Let us define two $D\phi_i$: D_1 with $0 \leqslant \theta < \Theta(H)$ and D_2 with $\Theta(H) \leqslant \theta < 2\pi$, where $\Theta(H)$ is an arbitrary function. Then,

$$\rho(\phi) = \rho_1(\phi)I_1(\phi) + \rho_2(\phi)I_2(\phi).$$

If $\rho(\phi) = \rho(H)$, by integrating over the θ we obtain

$$2\pi\rho(H) = \int_0^{\Theta(H)} \rho_1(H)I_1(\phi)\,\mathrm{d}\theta + \int_{\Theta(H)}^{2\pi} \rho_2(H)I_2(\phi)\,\mathrm{d}\theta = \rho_1(H)\Theta(H) + (2\pi - \Theta(H))\rho_2(H) = \rho_1(H)C_1(\phi) + \rho_2(H)C_2(\phi),$$

namely, the equation $(2\pi)^{N+1}\rho(H,P) = \sum_i C_{\phi_i}(H,P)\rho_{\phi_i}(H,P)$ for this particular case.

defined all over $\mathscr{M}$ (this causes no problem because in the last equation $O_{S\phi_i}(\phi)$ is multiplied by $\rho_{S\phi_i}(\phi)$, and $O_{S\phi_i}(\phi)$ has this property). In this way, we can arrive from Eq. (4.17) to our final result

$$\rho_S(\phi) = \rho_*(\phi) = \sum_i \frac{1}{C_{\phi_i}(H, P_{\phi_i})} \rho_{\phi_i}(H(\phi), P_{\phi_i}(\phi)). \tag{4.22}$$

Now, from Eq. (4.13) we obtain that

$$\rho_S(\phi) = \rho_*(\phi) \geqslant 0. \tag{4.23}$$

This means that the Wigner transformation of the singular part *can be considered a density function since it is non-negatively defined* (of course, this is not the case for the regular part).[19]

Always working in the domain $\mathscr{D}_{\phi_i}$ and making $\rho_{\phi_i}(\omega, p) = \delta(\omega - \omega')\delta^N(p - p')$, we also have[20]

$$symb(\omega', p', (\phi)|_{\phi_i} = \frac{1}{C_{\phi_i}(H, P_{\phi_i})} \delta(H(\phi) - \omega')\delta^{(N)}(P(\phi) - p'_{\phi_i}). \tag{4.24}$$

(c) From Eqs. (3.33) and (4.24) we obtain

$$\rho_S(\phi) = \rho_*(\phi) = \sum_i \int_{p \in D_{\phi_i}} \mathrm{d}p \int_0^\infty \rho_{\phi_i}(\omega, p) \frac{1}{C_{\phi_i}(H, P_{\phi_i})} \delta(H(\phi) - \omega)\delta^{(N)}(P(\phi) - p_{\phi_i})\, \mathrm{d}\omega. \tag{4.25}$$

The continuity of the function $\rho_*(\phi)$, when it goes from one D_{ϕ_i} to another D_{ϕ_j} $(i \neq j)$, will be proved in Section 5. Therefore, we have obtained a decomposition of $\rho_*(\phi) = \rho_S(\phi)$ in classical hypersurfaces ($H = \omega$, $P_{\phi_i}(\phi) = p_{\phi_i}$), containing classical trajectories, summed with different positive weight coefficients $\rho_{\phi_i}(\omega, p)/C_{\phi_i}(H, P_{\phi_i})$, and represented in different ways in each domain D_{ϕ_i}, but still with the same interpretation as in the integrable case.

(d) Since now we know how to deal with the singular part, we have defined the mapping of the quantum space of states $\widehat{\mathscr{A}}'$ on the 'classical' space of states $\mathscr{A}'$

$$symb : \widehat{\mathscr{A}}' \to \mathscr{A}'. \tag{4.26}$$

In the limit $\hbar \to 0$, Eqs. (2.9) and (2.10) are always valid; then, it might be supposed that we have arrived to the classical limit for the states. But *this is not so* because, in general, even for $\hbar \to 0$ the obtained $\rho(\phi)$ *does not satisfy* the condition

$$\rho(\phi) \geqslant 0. \tag{4.27}$$

This is due to the fact that the regular part is still present and this part does not satisfy the last condition (on the contrary, from Eq. (4.23) we can see that the singular part satisfies the last inequality). As a consequence, $\rho(\phi)$ is not a density function and, therefore, the mapping (4.26) is not a mapping of quantum mechanics on classical statistical mechanics. This mapping does not give us the classical world, but a deformed classical world where 'density functions' can be negative. In other words, when $\hbar \to 0$ the isomorphism (4.26) is a mapping of quantum mechanics on a certain quantum mechanics 'alla classica', namely, formulated in phase space $\mathscr{M}$. This clearly shows that $\hbar \to 0$ is not the classical limit.[21] In order to obtain this limit, we have to introduce decoherence, as previously studied, both at the quantum and the classical level.

[19] We can verify the normalization:

$$\int \rho_S(\phi)\, \mathrm{d}\phi^{2(N+1)} = \sum_i \int \rho_S(\phi)\, \mathrm{d}H\, \mathrm{d}P_{\phi_i}^N\, \mathrm{d}\theta_{\phi_i}^{N+1} = \sum\nolimits_i \int \mathrm{d}H\, \mathrm{d}P_{\phi_i}^N \frac{\rho_S(H, P_{\phi_i})}{C_{\phi_i}(H, P_{\phi_i})} \int \mathrm{d}\theta_{\phi_i}^{N+1} = \sum\nolimits_i \int \mathrm{d}H\, \mathrm{d}P_{\phi_i}^N\, \rho_S(H, P_{\phi_i}) = 1.$$

[20] In the chaotic, homogeneous, ergodic case, we have a $(N+1)$-CSCO with just $\widehat{H}$ and, classically, just H as a constant of motion. In this case (see [50, p. 247]),

$$\rho_S(\phi) = \rho_*(\phi) = \int_0^\infty \delta(\omega - E) \frac{1}{C(H)} \delta(H(\phi) - E) = \frac{\delta(H(\phi) - E)}{\int \mathrm{d}q\, \mathrm{d}p\, \delta(H(\phi) - E)}.$$

[21] It might be thought that, since the evolution factor is $\mathrm{e}^{\frac{i(\omega-\omega')t}{\hbar}}$, the limit $t \to \infty$ is equivalent to $\hbar \to 0$ and, therefore, by means of the Riemann–Lebesgue theorem the limit $\hbar \to 0$ eliminates the off-diagonal terms making $\rho(\phi) \geqslant 0$. But we must recall that $\hbar \to 0$ is a counterfactual limit: $\hbar$ never tends to zero because it is a constant (see footnote 6). The factual, physical limit is $\hbar/S \to 0$ $(S \gg \hbar)$, where now $S = (\omega - \omega')t$, so $S \to \infty$ either if $t \to \infty$ or $(\omega - \omega')$ can be consider very large (eight energy limit).

4.3. Time evolution and decoherence

As we have seen, the only thing that prevents us from having a good isomorphism (4.26) is that the regular parts do not satisfy condition (4.27). But the Wigner transform of Eq. (3.33) is

$$W \lim_{t\to\infty} \rho(\phi,t) = \rho_S(\phi) = \rho_*(\phi) = \sum_i \int_0^\infty d\omega \int_{p\in D_{\phi_i}} dp^N \overline{\rho(\omega)_{\phi_i}}(\omega,p,p,(\phi)|_{\phi_i}$$

Then for $t \to \infty$, the regular part vanishes and only the singular part remains, which does satisfy this condition. As a consequence, after decoherence and $\hbar \to 0$ (that is, the elimination of all the $0(\hbar^2)$ that we have omitted), we finally obtain the classical statistical limit since the classical densities obtained obey all the laws of classical statistical mechanics. In fact, as we will see in the next section in detail, Eq. (4.25) shows that these distributions are the result of classical pointlike-states moving in phase space and following classical trajectories. The usual classical limit is obtained by choosing one of these trajectories; we will explain this procedure in the next section.

5. The classical limit

From what we have learnt above, we can explain with more detail the three steps involved in the classical limit, presented in the introduction and shown in the following diagram:

Quantum Mechanics—(decoherence) → Boolean Quantum Mechanics—(*symb* and $\hbar \to 0$)
→ Classical Statistical Mechanics—(choice of a trajectory)
→ Classical Mechanics

Let us comment these three steps:

(i) *Quantum Mechanics—(decoherence) → Boolean Quantum Mechanics*. Decoherence transforms non-Boolean quantum mechanics into Boolean quantum mechanics[22] since it eliminates the off-diagonal terms, as we have shown in Eq. (3.33).

(ii) *Boolean Quantum Mechanics—(symb and $\hbar \to 0$) → Classical Statistical Mechanics*. The Wigner transformation *symb* and the limit $\hbar \to 0$ are defined with no problems in the singular part remaining after decoherence. They map Boolean quantum mechanics onto classical statistical mechanics: this is what we have essentially shown above. Our demonstration culminates in Section 4.3, where we have proved that the transformed quantum Boolean states are really positively defined densities. From Eq. (4.25) we also know that these densities are the sums of densities strongly peaked on the classical hypersurfaces defined by the constants of the motion $H(\phi) = \omega$, $P_{\phi_i}(\phi) = p_{\phi_i}$. In the next step we will see that such classical hypersurfaces contain classical trajectories averaged by the coefficients $\rho_{\phi_i}(\omega,p)$.

(iii) *Classical Statistical Mechanics—(choice of a trajectory) → Classical Mechanics* (*Localization or Actualization*). After step (ii), we are still in classical statistical mechanics but not in proper classical mechanics. To perform the last step we have to pass from classical densities to classical trajectories (i.e., to consider the localization effect[23]). For this purpose, let us observe that, after the two first steps, the formalism of Boolean quantum mechanics is isomorphic with the formalism of statistical classical mechanics:

- *For the observables*: After *symb* and $\hbar \to 0$, we obtain the correspondence $symb : \widehat{\mathscr{A}} \sim \mathscr{A}$ (see Section 4.1), namely,

$$A_{\phi_i}(\widehat{H},\widehat{P}_{\phi_i}) \sim A_{\phi_i}(H(\phi),P_{\phi_i}(\phi)).$$

- *For the states*: After decoherence, *symb* and $\hbar \to 0$, again $symb : \widehat{\mathscr{A}}' \sim \mathscr{A}'$ (see Section 4.2), namely,

$$\rho_{\phi_i}(\widehat{H},\widehat{P}_{\phi_i}) \sim \rho_{\phi_i}(H(\phi),P_{\phi_i}(\phi)) \geqslant 0,$$

and the states $\rho_*(\widehat{H},\widehat{P})$ and $\rho_*(H(\phi),P(\phi))$ are time invariant (see Eqs. (3.33) and (4.25)).

[22] Namely, quantum mechanics in the local CSCO $\{\widehat{H},\widehat{P}\}$ using only diagonal states.
[23] See [43, Chapter 4], for a different view.

Moreover, since $\Delta A \Delta B \geqslant \frac{\hbar}{2}|\langle [A,B]\rangle_\rho|$, in the limit $\hbar \to 0$ there are no uncertainty relations and the algebras $\widehat{\mathscr{A}}$ and $\mathscr{A}$ can be considered commutative (remember that, according to the uncertainty principle, $\hbar \to 0$ has the same effect that $[A,B]=0$). In other words, in the limit $\hbar \to 0$ all the picture is classical in such a way that the trajectories, contained in the hypersurfaces $H(\phi)=\omega$, $P_{\phi_i}(\phi)=p_{\phi_i}$, could be interpreted as *real classical trajectories*. However, the $\delta(H(\phi)-\omega)\delta^N(P_{\phi_i}(\phi)-p_{\phi_i})$ still represent states strongly peaked around these hypersurfaces. Therefore, if we want to obtain an equation clearly showing the classical trajectories, we have to introduce the initial conditions of each trajectory.

Let us consider a classical trajectory in phase space $\mathscr{M}=\mathscr{M}_{2(N+1)}$, expressed in the classical coordinates $(\tau,\theta_{\phi_i},H,P_{\phi_i})$, where τ is the coordinate canonically conjugated to H and the θ_{ϕ_i} are the coordinates canonically conjugated to the P_{ϕ_i}. The constants of the motion in involution are $\{H,P_{\phi_i}\}$; but, for conciseness and generality, let us consider that the constants of the motion in involution are $\{\Pi_{\phi_i}\}$ with conjugated coordinates $\{A_{\phi_i}\}$, and that $H=H(\Pi_{\phi_i})$. From the von Neumann–Liouville equation in the Heisenberg representation,

$$\mathrm{i}\hbar\frac{\mathrm{d}\widehat{A}}{\mathrm{d}t}=[A,H],$$

we obtain

$$\frac{\mathrm{d}A(\phi)}{\mathrm{d}t}=\{A,H\}_{mb}=\{A,H\}_{pb}+0(\hbar^2).$$

Then, the Hamiltonian equations in the limit $\hbar \to 0$ read[24]

$$\frac{\mathrm{d}A_{\phi_i}}{\mathrm{d}t}=\frac{\partial H}{\partial \Pi_{\phi_i}}=\Omega_{\phi_i}(\Pi_{\phi_i})=\text{constant},\qquad \frac{\mathrm{d}\Pi_{\phi_i}}{\mathrm{d}t}=-\frac{\partial H}{\partial A_{\phi_i}}=0. \tag{5.1}$$

The classical trajectories are

$$A_{\phi_i}(t)=A^{(0)}_{\phi_i}+\Omega_{\phi_i}(\Pi_{\phi_i})t,\qquad \Pi_{\phi_i}=\Pi^{(0)}_{\phi_i}=\text{constant}, \tag{5.2}$$

where the $A^{(0)}_{\phi_i}$ and $\Pi^{(0)}_{\phi_i}$ are integration constants. A distribution strongly peaked on this trajectory reads

$$\delta\left[A_{\phi_i}(t)-A^{(0)}_{\phi_i}-\Omega_{\phi_i}(\Pi_{\phi_i})t\right]\delta\left(\Pi_{\phi_i}-\Pi^{(0)}_{\phi_i}\right),$$

and a general classical distribution evolving according to the motion (5.2) reads[25]

$$\rho_C(t,\phi)=\sum_i\int_{D_{\phi_i}}\rho^{(C)}_{\phi_i}\left(A^{(0)}_{\phi_i},\Pi^{(0)}_{\phi_i}\right)\delta\left[A_{\phi_i}(t)-A^{(0)}_{\phi_i}-\Omega_{\phi_i}(\Pi_{\phi_i})t\right]\delta\left(\Pi_{\phi_i}-\Pi^{(0)}_{\phi_i}\right)\mathrm{d}^{N+1}A^{(0)}_{\phi_i}\,\mathrm{d}^{N+1}\Pi^{(0)}_{\phi_i}, \tag{5.3}$$

where $\rho^{(C)}_{\phi_i}(A^{(0)}_{\phi_i},\Pi^{(0)}_{\phi_i})$ is a generic classical coefficient (undefined up to now). If we want that this density (evolving according to a Frobenius–Perron evolution, see [51]) be an equilibrium density, we have to eliminate the variable t. For this purpose, it is sufficient to choose the initial distribution $\rho^{(C)}_{\phi_i}(A^{(0)}_{\phi_i},\Pi^{(0)}_{\phi_i})$ as just a function of $\Pi^{(0)}_{\phi_i}$, namely, $\rho^{(C)}_{\phi_i}(\Pi^{(0)}_{\phi_i})$, which is still free to represent different $\rho_C(\phi)$. Then, we obtain

$$\rho_C(\phi)=\sum_i\int_{D_{\phi_i}}\rho^{(C)}_{\phi_i}(\Pi^{(0)}_{\phi_i})\delta(\Pi_{\phi_i}-\Pi^{(0)}_{\phi_i})\,\mathrm{d}^{N+1}\Pi^{(0)}_{\phi_i}, \tag{5.4}$$

since, for any fixed t, we have

$$\sum_i\int_{D_{\phi_i}}\delta\left(A_{\phi_i}(t)-A^{(0)}_{\phi_i}-\Omega_{\phi_i}(\Pi_{\phi_i})t\right)\mathrm{d}^{N+1}A^{(0)}_{\phi_i}=1.$$

Going back to our primitive variables, Eq. (5.3) reads

$$\begin{aligned}\rho_C(\phi)=\sum_i\int\rho^{(C)}_{\phi_i}(\omega,p)\delta(H(\phi)-\omega)\delta(P_{\phi_i}-p_{\phi_i})\delta(\tau(\phi)-\tau_0-\omega t)\delta(\theta_{\phi_i}(\phi)-\theta_{\phi_i 0}\\ -p_{\phi_i}t)\,\mathrm{d}\omega\,\mathrm{d}^N p_{\phi_i}\,\mathrm{d}\tau_0\,\mathrm{d}\theta_{\phi_i 0},\end{aligned} \tag{5.5}$$

[24] These equations correspond to the system of differential equation (3.1) of [51].

[25] If the evolution S^t of [51] were the (5.2), the corresponding density would be $f(t,x)\equiv P^t f(x)$ (see [51, Eq. (3.2)]) where P^t would represent a Frobenius–Perron evolution. Moreover, it is easy to show that $\rho(t,\phi)$ satisfies the Liouville equation.

while Eq. (5.4) reads

$$\rho_C(\phi) = \sum_i \int \rho_{\phi_i}^{(C)}(\omega, p)\delta(H(\phi) - \omega)\delta(P_{\phi_i} - p_{\phi_i})\,\mathrm{d}\omega\,\mathrm{d}^N p_{\phi_i}. \tag{5.6}$$

Then, from Eq. (4.25) and making the undefined coefficient $\rho_{\phi_i}^{(C)}(\omega, p) = \frac{\rho_{\phi_i}(\omega,p)}{C_{\phi_i}(\omega,p_{\phi_i})}$, we have

$$\rho_*(\phi) = \rho_C(\phi). \tag{5.7}$$

The function $\rho_C(\phi)$ can be interpreted as the equilibrium density of a Frobenius–Perron evolution of particle-like states $(\tau, \theta_{\phi_i}, H, P_{\phi_i})$, *as if* these states would move in phase space $\mathscr{M} = \mathscr{M}_{2(N+1)}$ according to the classical motions (5.2).

Moreover, each term of the sum $\sum_i$ of Eq. (5.5) is valid in the chart $\mathscr{D}_{\phi_i}(D_{\phi_i} \subset \mathscr{D}_{\phi_i})$. In a different chart $\mathscr{D}_{\phi j}(D_{\phi_j} \subset \mathscr{D}_{\phi j})$, the equation is also valid and, then, at $\phi \in \mathscr{D}_{\phi_i} \cap \mathscr{D}_{\phi_j}$ we have

$$\begin{aligned}&\int \frac{\rho_{\phi_i}(\omega,p)}{C_{\phi_i}(\omega,p_{\phi_i})}\delta(H(\phi) - \omega)\delta(P_{\phi_i} - p_{\phi_i})\delta(\tau(\phi) - \tau_0 - \omega t)\delta(\theta_{\phi_i}(\phi) - \theta_{\phi_i 0} - p_{\phi_i} t)\,\mathrm{d}\omega\,\mathrm{d}^N p_{\phi_i}\,\mathrm{d}\tau_0\,\mathrm{d}A_{\phi_i 0}\\ &= \int \frac{\rho_{\phi_i}(\omega,p)}{C_{\phi_i}(\omega,p_{\phi_j})}\delta(H(\phi) - \omega)\delta(P_{\phi_j} - p_{\phi_j})\delta(\tau(\phi) - \tau_0 - \omega t)\delta(\theta_{\phi_j}(\phi) - \theta_{\phi_j 0} - p_{\phi_j} t)\,\mathrm{d}\omega\,\mathrm{d}^N p_{\phi_j}\,\mathrm{d}\tau_0\,\mathrm{d}A_{\phi_j 0}.\end{aligned} \tag{5.8}$$

Here it is worth emphasizing that the trajectories $H = \omega$, $P_{\phi_i}(\phi) = p_{\phi_i}$, $\tau(\phi) = \tau_0 + \omega t$, $\theta_{\phi_i}(\phi) = \theta_{\phi_i 0} + p_{\phi_i} t$ in chart $\mathscr{D}_{\phi_i}$ are *continuously connected* with those $H = \omega$, $P_{\phi_j}(\phi) = p_{\phi_j}$, $\tau(\phi) = \tau_0 + \omega t$, $\theta_{\phi_j}(\phi) = \theta_{\phi_j 0} + p_{\phi_j} t$ in chart $\mathscr{D}_{\phi_j}$, because these charts are not generic but constructed using the solution of Eqs. (3.3) or (5.1). Since $D_{\phi_i} \subset \mathscr{D}_{\phi_i}$ and $D_{\phi_j} \subset \mathscr{D}_{\phi_j}$, the same holds for the trajectories going from D_{ϕ_i} to D_{ϕ_j}. Thus, the continuous connection follows from the fact that *one and only one* solution of the trajectory equation passes for each point of $\mathscr{M}$ (and, therefore, for each $\phi \in \mathscr{D}_{\phi_i} \cap \mathscr{D}_{\phi_j}$).

Summing up, we have obtained a decomposition of $\rho_*(\phi) = \rho_S(\phi)$ in classical trajectories $H = \omega$, $P_{\phi_i}(\phi) = p_{\phi_i}$, $\tau(\phi) = \tau_0 + \omega t$, $\theta_{\phi_j}(\phi) = \theta_{\phi_j 0} + p_{\phi_j} t$, summed with different weight coefficients $\rho_{\phi_i}(\omega, p)/C_{\phi_i}(H, P_{\phi_i})$ and represented in different ways in each domain $\mathscr{D}_{\phi_i}$, but still with the same interpretation as in the integrable case. Moreover, as announced in Section 3.1(c), we see that chart $\mathscr{D}_{\phi_i}$ is continuously connected with chart $\mathscr{D}_{\phi_j}$, for any $\mathscr{D}_{\phi_i}$, $\mathscr{D}_{\phi_j}$. Therefore, we have finally obtained the classical limit to the extent that we have described each one of the classical trajectories. But, since from the very beginning our system was a non-integrable one, we have obtained *the classical limit of a non-integrable system*, where the tori are broken and the trajectories are *potentially chaotic trajectories*.

Finally, we must remark that:

- Each one of the described processes, decoherence, route to macroscopicity, i.e., $\hbar \to 0$ (e.g., the macroscopicity obtained when the two rays of an Stern-Gerlach experiment gradually separate), and eventually localization (e.g., by a localizing potential, see [26, Appendix A]), has its own characteristic time; in particular, the decoherence time is computed in [35].
- We have explained the classical limit as if each process (decoherence, macroscopicity, and localization) took place one after the other, only for didactical reasons. But this is an oversimplified picture of the phenomenon. In fact, this may be not the case if the different processes overlap. Considering that they have different characteristic times, there are different possibilities according to the order in which the processes finish.

6. Conclusions

We want to conclude the paper proposing some suggestions for future research.

(i) We have essentially presented a *minimal formalism for quantum chaos*, to the extent that our quantum formalism satisfies a minimal requirement for such a theory: by definition, a quantum chaotic system has, at least, a classical non-integrable system as its classical limit. In fact, this is a necessary but not a sufficient condition that any proposed theory of quantum chaos must fulfil (see, e.g., [52]). Our next task is to address the question of whether the set of phenomena known under the name of 'quantum chaos' [43,47,50,53] can be explained by means of our theoretical structure.

(ii) Quantum contexts are clearly related with $(N + 1)$-CSCOs. We have seen that generic $(N + 1)$-CSCOs are local. This might have a relation with well known physical questions, as the EPR problem and the Kochen–Speker and Bell theorems (see [48]), where paradoxes arise when we try to describe the quantum system with just one CSCO.

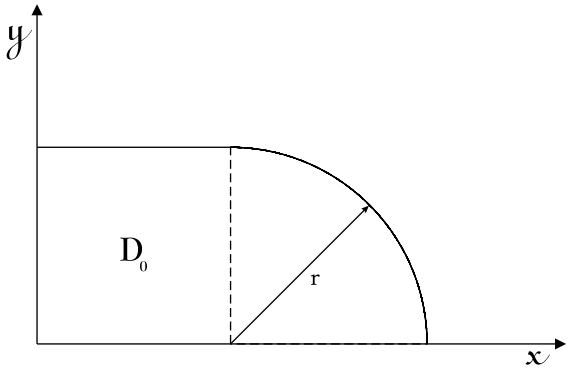

Fig. 1. A Sinai billiard.

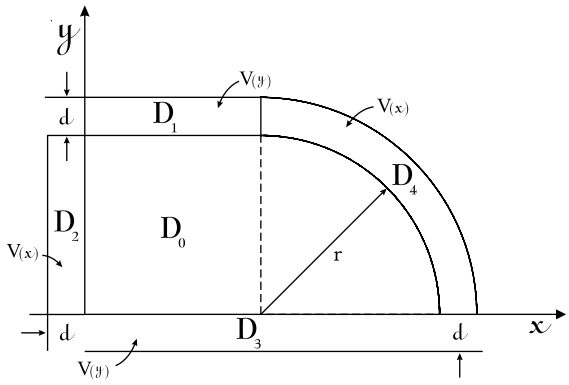

Fig. 2. A Sinai billiard with potential barriers.

(iii) In some sense, the equations of quantum physics have a local character [54–56]; we have found that this is also the case of the CSCOs: it might be useful to explore this analogy.

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Appendix A. Sinai billiard

Let us consider the Sinai billiard of Fig. 1 [43]. It is clear that, when the ball is confined to the interior of the billiard, the trajectories are defined by two independent constants of the motion, H and P_x (or H and P_y, or P_x and P_y), which constitute a complete set of local (i.e., in the interior D_0 of the billiard) constants of the motion in involution. When the ball strikes the boundaries, it is symmetrically reflected, i.e., the incident angle is equal to the reflected angle, and the value of some of the constants of the motion changes: for the two horizontal boundaries, H and P_x still constitute a complete set of local constants of the motion in involution, but P_y changes its sign; for the vertical boundary, H and P_y still constitute a complete set of local constants of the motion in involution, but P_x changes its sign.

Without modifying the physical characterization of the example, we can replace the rigid walls with infinitely high potential barriers of width d, namely, the potentials $V(x), V(y)$ and $V(r)$ of Fig. 2, connected in a smooth way with the

interior D_0 (e.g., $V(x)$ behaves as $V(0) = 0$, $V'(0) = 0$, $V(-d) \to \infty$). Due to the symmetry of the potentials (translation symmetry for $V(x)$ and $V(y)$, rotation symmetry for $V(r)$), the reflections are still symmetric, i.e., the ball climbs the potential walls and then falls down with symmetrical motion. Calling D_1 and D_3 the domains in the potential of the x walls, D_2 that of the y wall, and D_4 that of the curved wall, we see that x is a cyclic variable in D_1 and D_3, y is a cyclic variable in D_2, and θ is a cyclic angular variable in D_4. Therefore, we have the following local constants of the motion in each domain:

$$D_0: \quad H \quad P_x \quad (\text{or } P_y)$$

$$D_1: \quad H \quad P_x$$

$$D_2: \quad H \quad P_y$$

$$D_3: \quad H \quad P_x$$

$$D_4: \quad H \quad P_\theta$$

In summary, we have found five domains, each one with two local constants of the motion in involution. If $d \to 0$, we go from Fig. 2 to Fig. 1.

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