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On local Hamiltonians and dissipative systems

M. Castagnino [a], M. Gadella [b,c,*], L.P. Lara [b]

[a] CONICET-Institutos de Física Rosario y de Astronomía y Física del Espacio Casilla de Correos 67, Sucursal 28, 1428, Buenos Aires, Argentina
[b] Facultad de Ciencias Exactas, Ingeniería y Agrimensura UNR, Rosario, Argentina
[c] Departamento de Física Teórica. Facultad de Ciencias c. Real de Burgos, s.n., 47011 Valladolid, Spain

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Abstract

We study a type of one-dimensional dynamical systems on the corresponding two-dimensional phase space. By using arguments related to the existence of integrating factors for Pfaff equations, we show that some one-dimensional non-Hamiltonian systems like dissipative systems, admit a Hamiltonian description by sectors on the phase plane. This picture is not uniquely defined and is coordinate dependent. A simple example is exhaustively discussed. The method, is not always applicable to systems with higher dimensions.

1. Introduction

In a recent publication on the classical limit and its relation with quantum chaos [1], attention was drawn to a characteristic of a type of Sinai billiard: the existence of local constants of motion. Then, it arises the question whether there are other non-Hamiltonian systems on phase space that have local constants of motion and admit a Hamiltonian formalism. Although this problem has been discussed with some generality elsewhere [2,3], it is rather unknown to the physics community. In this paper, we want to present some very handy examples that illustrate this problem in a way that be accessible to a wide audience.

We shall see that, under very general conditions, two-dimensional (in phase space) dynamical systems have local constants of motion, although this result cannot be extended to $2N$ dimensions with $N > 1$ using the arguments of the present work. As a consequence, we can define a sort of locally defined Hamiltonian for some type of dissipative systems, whose Hamilton equations provide local solutions for the equations of motion in phase space.

As an illustrative example, we begin with the discussion of a one-dimensional particle subjected to a dissipative force such as friction. The position of this particle obeys the following equation of motion:

$$\ddot{x}(t) + a\dot{x}(t) = 0, \tag{1}$$

* Corresponding author. Address: Departamento de Física Teórica. Facultad de Ciencias c. Real de Burgos, s.n., 47011 Valladolid, Spain.
E-mail address: manuelgadella@yahoo.com.ar (M. Gadella).

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where a is a constant, which we can assume positive ($a > 0$) without loss of generality. In order to translate Eq. (1) to the phase space language, let us construct from (1) the following system:

$$\dot{x} = y; \qquad \dot{y} = -ay. \tag{2}$$

The solutions of both (1) and (2) are trivial:

$$\begin{aligned} y(t) &= y_0 \mathrm{e}^{-at}, \\ x(t) &= x_\infty - \frac{y_0}{a}\mathrm{e}^{-at}. \end{aligned} \tag{3}$$

We note that the system given by (2) has a constant of motion which can be straightforwardly obtained. If we divide the second row of (2) by the former, we readily obtain:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -a \Rightarrow y = -ax + C, \tag{4}$$

where C is an arbitrary constant. From (4), it is obvious that

$$F_+(x,y) = y + ax \tag{5}$$

is a constant of motion for the system that fulfills (1).

Let us study the motion given by system (2) on phase space. First of all, from either (2), (3) or (5), we note that all points in the real axis are fixed points. For any initial condition (x_0, y_0), we have that:

$$\lim_{t\to+\infty} y(t) = 0; \qquad \lim_{t\to+\infty} x(t) = x_\infty. \tag{6}$$

However, if $t \mapsto -\infty$, both $x(t)$ and $y(t)$ go to the infinity. We note that the solutions of system (2) are all half lines, pointing in the direction of the real axis, plus the fixed points in the real axis. This situation is depicted in Fig. 1.

Going further with our discussion, note that system (2) cannot have been obtained from a system of Hamilton equations, since no function $H \equiv H(x,y)$ can satisfy the following system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y = \frac{\partial H}{\partial y}; \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -ay = -\frac{\partial H}{\partial x}. \tag{7}$$

This is equivalent to say that the differential form

$$ay\,\mathrm{d}x + y\,\mathrm{d}y \tag{8}$$

is not exact. However, we know that any differential form like $M(x,y)\mathrm{d}x + N(x,y)\mathrm{d}y$ always admit an integrating factor, under the assumption of certain regularity conditions on $M(x,y)$ and $N(x,y)$ (see Appendix). In our case, the integrating

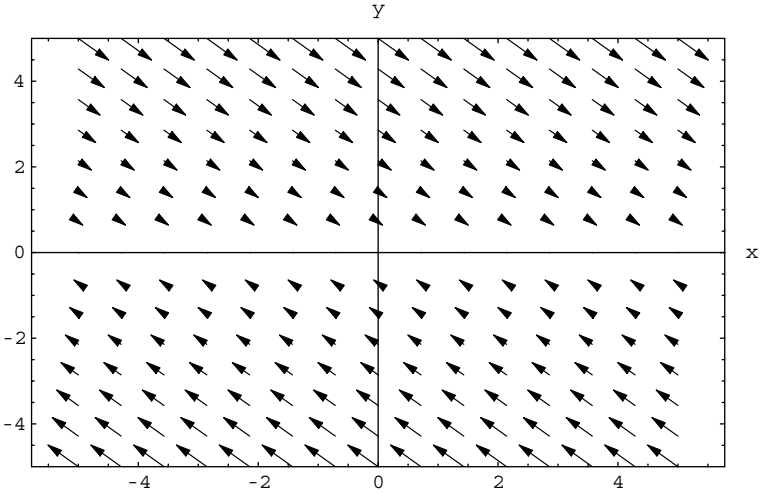

Fig. 1. Plot of solutions of system (2).

factor is given by Cy^{-1}, where C is an arbitrary constant. If we set $C = 1$, and multiply (8) by the integrating factor, we obtain the differential form:

$$dF = a\,dx + dy, \tag{9}$$

which is obviously exact. Its potential function is precisely (5). If we use $F(x,y) = ax + y$ as a Hamiltonian and use the Hamilton equations, we obtain the following system:

$$\frac{dx}{dT} = 1; \qquad \frac{dy}{dT} = -a \tag{10}$$

for which the solutions are of the form (5). Note that we have introduced a new parameter T that replaces time. The relation of T with time will be discussed later.

Obviously, if we divide the second equation in (10) by the former, we obtain the Pfaff equation $dF = a\,dx + dy = 0$. Then, we can compare system (2) and system (10) to conclude that they cannot have the same set of solutions because

(i) System (10) has no fixed points. On the other hand, the system given by (2) has an infinite number of fixed points that cover the real axis.
(ii) If $(x(T), y(T))$ are solutions for (10), we always have:

$$\lim_{T\to\pm\infty} x(T) = \pm\infty; \qquad \lim_{T\to\pm\infty} y(T) = \pm\infty. \tag{11}$$

Note on the big difference between (11) and the behavior as $t \mapsto \pm\infty$ of the solutions of (2). This situation is surprising because both systems have analogous Pfaff equations. The sole difference between their corresponding Pfaff differential forms ((8) and (9)) come from the multiplication by the integrating factor. We summarize this fact in the following table:

System	Pfaff equation
$\dot{x} = y;\ \dot{y} = -ay$	$ay\,dx + y\,dy = 0$
$\dot{x} = 1;\ \dot{y} = -a$	$a\,dx + dy = 0$

Then, although both Pfaff equations only differ by the integrating factor, we see through their corresponding systems that their solutions are not the same. However, there is a way to make both Pfaff equations dynamically equivalent in such a way that their solutions in the phase space (x,y) coincide. We have seen that the integrating factor depends on an arbitrary constant. We can make this constant equal to $+1$ on the upper half plane and equal to -1 on the lower half plane. Thus, the potential function or Hamiltonian remains to be (5) on the upper half plane and

$$F_{-}(x,y) = -ax - y \tag{12}$$

on the lower half plane. In this latter case, the Hamilton equations for (12) give

$$\frac{dx}{dT} = -1; \qquad \frac{dy}{dT} = a. \tag{13}$$

The solutions of (10) and (13) are respectively:

$$x(T) = T + C_x; \qquad y(T) = -aT + C_y \tag{14}$$

and

$$x(T) = -T + D_x; \qquad y(T) = aT + D_y, \tag{15}$$

where C_x, D_x, C_y and D_y are arbitrary constants. If we want $y(0) = 0$, then $C_y = D_y = 0$. Note that $x(0) = C_x$ in (13) and $x(0) = D_x$ in (14), so that we may want $C_x = D_x$ to match solutions above and below the real axis. If Eq. (14) is the solution in the upper half plane, one needs that $T \leqslant 0$ (with $T = 0$ on the real line only), since $a > 0$. Also, if (15) gives the solution in the lower half plane, T has to be negative as well. Thus, the solutions are now straight half lines oriented towards the real axis, i.e., as T grows to zero the flux is directed towards the real axis as in Fig. 1. Note that $F(x,y)$ is discontinuous on the real axis. In order to maintain the character of fixed points for the points on $\mathbb{R}$, we may assume that for each $x_0 \in \mathbb{R}$ we have that $F(x_0, 0) = x_0$.

Thus, in order to produce the same flow as the system given by (2), we have to divide the phase plane into three sectors and provide a different Hamiltonian function for each sector. This gives the following table:

Region in phase space	Hamiltonian function
$y > 0$	$y + ax$
$y = 0$	x_0 (initial condition)
$y < 0$	$-y - ax$

Note that, properly speaking the real line is not a sector, but each point in the real line should be considered as a sector because their character of fixed points.

We can compare solutions (3) of system (2) with (14,15). If $y_0 > 0$, i.e., the initial condition is in the upper half plane, comparison between (3) and (14) gives:

$$\mathrm{d}T = -y_0 \mathrm{e}^{-at}\,\mathrm{d}t. \tag{16}$$

If $y_0 < 0$, initial condition in the lower half plane, we have that

$$\mathrm{d}T = +y_0 \mathrm{e}^{-at}\,\mathrm{d}t. \tag{17}$$

In both cases, we have a complete agreement with previous results as the parameter T is always negative and goes to zero as $t \mapsto \infty$. Also this confirms the phase space diagram given in Fig. 1.

In conclusion, we have studied the behavior of the free particle with friction (or viscosity) in one dimension and we have observed the following facts:

- Although this system is not conservative, in the sense that the system (2) does not represent a system of Hamilton equations, we can find constants of motion, which are not globally defined. They are constants of motion in the sense that they are constant along each curve solution.
- As the curves solution are defined in sectors of the phase plane, the constants of motion are different on each of these sectors. The constants of motion are given by different values of the potential function $F(x, y)$ in different sectors. These different values appear because of the need of different integrating factors, in each sector, of a Pfaff differential form related to the equations of motion.
- The potential function $F(x, y)$ works in each sector as a Hamiltonian, in the sense that Hamilton equations, when applied to $F(x, y)$ in each sector, give the equations of motion of the system. Thus, we have seen that *a non-Hamiltonian, non-conservative system could be Hamiltonian and conservative at least by sectors*.
- In order to complete this equivalence between Hamiltonian and non-Hamiltonian systems, we have to replace the time scale of a non-Hamiltonian system, given by t by a new parameter T. The relation between both is given, in the present case, by (16) and (17). Note that the time scale changes within each region.
- Nevertheless, it is not strictly correct to say that we have replaced a dissipative system by a local Hamiltonian system on phase plane in the usual sense [4]. First of all, the Hamiltonian $F(x, y)$ is defined on regions which are not all open sets (see above diagram). In the open sets given by the upper and the lower half plane, the values of the Hamiltonian $F(x, y)$ are different, but these open sets do not cover the whole phase plane. It remains the real line, in which every point could be considered as a "sector".
- Note that in the limit $a \mapsto 0$, we recover the behavior of the one-dimensional free particle, for which the constant of motion is just the momentum y, i.e., $F(x, y) \equiv y$. Also, if we take the limit in (25) below, as $a \mapsto 0$, since $\pm y^{-1}$ is the integrating factor (see next section), we have that

$$T = \pm \int_0^t y(t)\mathrm{d}t = \pm \frac{y_0}{a}(\mathrm{e}^{-at} - 1) \underset{a \to 0}{\to} |y_0|t.$$

Thus in the limit $a \mapsto 0$, we obtain $T = |y_0|t$. This shows that the new parameter T does not have the dimensions of a time, although it serves equally well as a time parameter. To see the relation of T with a time, we first note that, after (3), a must have dimensions of the inverse of a time. Then, if we write (16) and (17) as $T = \pm y_0 \tau$, with $\tau = a^{-1}\mathrm{e}^{-at}$, we see that τ is a parameter with dimensions of time. In terms of τ, the trajectories can be written as:

$$x(\tau) = x_0 \pm y_0 \tau; \qquad y(\tau) = \mp y_0 a \tau.$$

Also, note that in the limit $a \mapsto 0$, we have here that $\tau \mapsto t$, where t is the time in (2) and (3). This is an expected result because in the non-dissipative limit, we have to obtain the free particle.

1.1. A second form for the sectorial Hamiltonians of the free particle with friction

The Hamiltonian functions on the above result strongly depend on the choice of the integrating factor. On the other hand, for any choice we must recover in the limit $a \mapsto 0$ the Hamiltonian of the free particle. Is this correct? What we can expect in general if we use another choice?

Let us go back to Eq. (8) and consider the following integrating factor:

$$\frac{y+ax}{y}. \tag{18}$$

If we carry (18) into (8), we obtain the following Pfaff equation:

$$\mathrm{d}F(x,y) = a(y+ax)\mathrm{d}x + (y+ax)\mathrm{d}y = 0, \tag{19}$$

whose solution is:

$$F(x,y) = \frac{1}{2}(ax+y)^2. \tag{20}$$

After (20), we note the following facts:

1. First of all if $a \mapsto 0$, we recover the usual Hamiltonian of the free particle.
2. As the Hamiltonian function in (20) is a function of the Hamiltonian functions $F_{\pm}(x,y) = \pm(ax+y)$ of the previous section, in fact the square divided by two, we know that both give the same curves solution. This is obvious. Assume that $\phi(F)$ is a differentiable function of the Hamiltonian F. Then,

$$\mathrm{d}\phi(F) = \frac{\mathrm{d}\phi(F)}{\mathrm{d}F}\frac{\partial F}{\partial x}\mathrm{d}x + \frac{\mathrm{d}\phi(F)}{\mathrm{d}F}\frac{\partial F}{\partial y}\mathrm{d}y = \frac{\mathrm{d}\phi(F)}{\mathrm{d}F}[\mathrm{d}F] = 0.$$

   Then, if $\phi(\cdot)$ has a non-vanishing derivative, the Pfaff equation $\mathrm{d}F = 0$ is equivalent to $\mathrm{d}\phi(F) = 0$, which proves our assertion.
3. Also from (20), we can derive the equations of motion:

$$\frac{\mathrm{d}x}{\mathrm{d}S} = y + ax; \qquad \frac{\mathrm{d}y}{\mathrm{d}S} = -a(y+ax), \tag{21}$$

where S is the new "time" parameter. The set of fixed points is given by the straight line $ax + y = 0$. Thus, we cannot recover completely the picture given by the Hamiltonians $F_{\pm}(x,y) = \pm(ax+y)$. If we want to recover a picture like the given by (10), we have to state that each point in the real axis is fixed. In addition, in order to obtain the same direction of the flux in each half plane, we have to make a proper choice of the signs in the integrating factor, and therefore in the Hamiltonian, as indicated in the following table and in Fig. 2.

Region	Hamiltonian
$y > -ax$, $y > 0$	$\frac{1}{2}(ax+y)^2$
$y < -ax$, $y > 0$	$-\frac{1}{2}(ax+y)^2$
$y < -ax$, $y < 0$	$\frac{1}{2}(ax+y)^2$
$y > -ax$, $y < 0$	$-\frac{1}{2}(ax+y)^2$

Note that, after Eq. (10), and taking into account that $ax + y$ is a constant of motion if $y > 0$, i.e., this amount is constant on each curve solution in the upper half plane, we have that:

$$1 = \frac{\mathrm{d}x}{\mathrm{d}T} = \frac{\mathrm{d}x}{\mathrm{d}T} = \frac{\mathrm{d}x}{\mathrm{d}S}\frac{\mathrm{d}S}{\mathrm{d}T} = (ax+y)\frac{\mathrm{d}S}{\mathrm{d}T} \tag{22}$$

and, then, on the upper half plane,

$$\mathrm{d}T = (ax+y)\mathrm{d}S. \tag{23}$$

If $ax + y > 0$, the parameters T and S have the same sign and their flows are oriented in the same direction, which in this case is the real axis. Therefore, the dynamics produced by (21) is totally equivalent to the dynamics produced by (10) in the sector given by $ax + y > 0$, $y > 0$, i.e., in the upper half plane to the right of the line $y = -ax$. However, on the left of this line, we have that $ax + y < 0$ and therefore S and T have opposite signs. The only possibility for recovering the right direction of the flow is producing the dynamics in this sector by using the Hamiltonian function $-\frac{1}{2}(ax+y)^2$. The sign

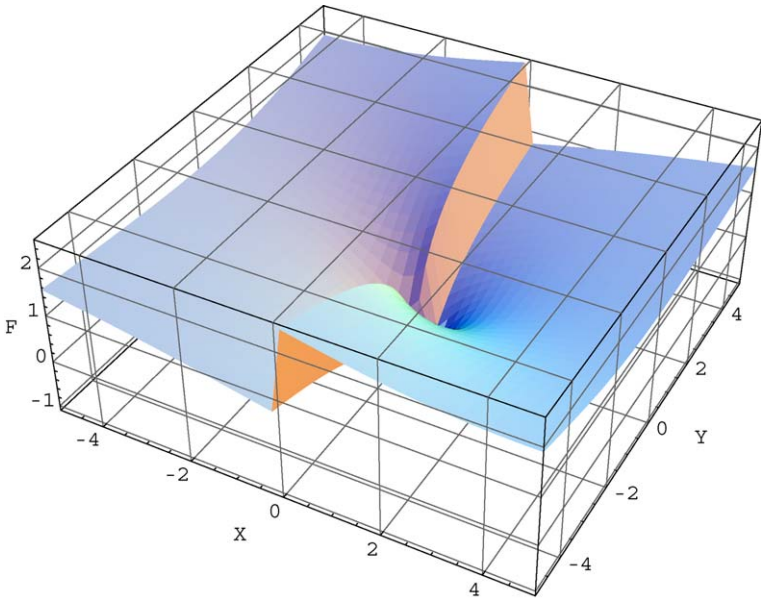

Fig. 2. Plot of the form of the sectorial Hamiltonian $F(x,y)$ as introduced in the table above.

minus is obtained by multiplying the integrating factor (18) by minus. In the lower half plane a similar situation arises. Note that the fixed points are those in the line $y + ax = 0$.

In the next section, we generalize the above ideas and we explore the limits of this generalization.

2. Dynamical systems in two-dimensional phase space. A general discussion

Let us consider the following system of equations in phase space:

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = f(x,y); \qquad \dot{y} = \frac{\mathrm{d}y}{\mathrm{d}x} = g(x,y). \tag{24}$$

We are assuming that the functions $f(x,y)$ and $g(x,y)$ satisfy sufficient conditions for the existence and uniqueness of local solutions with given initial conditions. In particular, they satisfy the conditions of the Picard–Lindelöf theorem that assumes that both $f(x,y)$ and $g(x,y)$ are continuous on a certain region and satisfy a Lipschitz condition for the second variable y [5,6].

System (24) yields to the following Pfaff equation:

$$-g(x,y)\mathrm{d}x + f(x,y)\mathrm{d}y = 0. \tag{25}$$

System (24) is a set of Hamilton equations if and only if the differential form $g(x,y)\mathrm{d}x - f(x,y)\mathrm{d}y$ is exact. In this case, there exists a function $H(x,y)$ such that

$$\dot{x} = f(x,y) = \frac{\partial H(x,y)}{\partial y}; \qquad \dot{y} = g(x,y) = -\frac{\partial H(x,y)}{\partial x}. \tag{26}$$

In general, this is not the case. The solution of (24) can be found. However, this dynamical system is not in general Hamiltonian. We have seen in the example developed in Section 1, that sometimes it is possible to define a kind of local Hamiltonian, defined by sectors, such that the equations of motion in phase space are equivalent within these sectors to the equations of motion of a Hamiltonian system in which we have changed the scale of times. We want to explore a generalization of this result.

This generalization comes from the fact that any sufficiently regular two-dimensional differential form has always an integrating factor, at least locally (see Appendix). In fact, let us assume that the differential form in (25) is not exact. Then it exists an integrating factor which is a function $h(x,y)$ such that the form given by

$$dF := -h(x,y)g(x,y)dx + h(x,y)f(x,y)dy \tag{27}$$

is now exact. The solution of the Pfaff equation $dF = 0$ is obviously given by $F(x,y) = C$. Note that the integrating factor is not unique, as we can multiply it by an arbitrary constant and obtaining a new integrating system. We have seen in the example given in Section 1, that this change is responsible of the locality of the Hamiltonian character of the dynamical system given by (25). Thus, for any dynamical system (with simple regularity conditions) in two dimensions in phase space, we always have sectors in which the equations of motion can be derived from a potential function which plays the role of a Hamiltonian. Consequently, there exists in these sectors at least a constant of motion along the curves solution, even if the system is dissipative.

As we have shown for the simplest case of the free particle with friction, the transition from a non-Hamiltonian to a Hamiltonian system is made also at the price of changing the time scale on each of the sectors. We want to discuss the role of the integrating factor in this change of scale. In fact, let us apply the Hamilton equations to $F(x,y)$ as obtained after the integration of (27). These are:

$$\begin{aligned} \frac{dx(T)}{dT} &= h(x,y)f(x,y) = \frac{\partial F(x,y)}{\partial y}, \\ \frac{dy(T)}{dT} &= -h(x,y)g(x,y) = -\frac{\partial F(x,y)}{\partial x}. \end{aligned} \tag{28}$$

Combining (28) with (26), we realize that

$$\frac{dx(T)}{dT} = h(x,y)\frac{dx(t)}{dt}; \qquad \frac{dy(T)}{dT} = h(x,y)\frac{dy(t)}{dt}, \tag{29}$$

which implies that

$$dT = \frac{dt}{h(x,y)}. \tag{30}$$

By integrating:

$$T = \int \frac{dt}{h(x(t),y(t))}. \tag{31}$$

This gives the change of scale in each sector. Sectors obviously depend on each particular situation. In the case studied in Section 1, we have that $h(x,y) = y^{-1}$ in the upper half plane and $h(x,y) = -y^{-1}$ in the lower half plane. Then, (31) and (3) give

$$T = \pm \int_{t_0}^{t} y(t)dt' = \pm \frac{y_0}{a}(e^{-at} - e^{-at_0}), \tag{32}$$

which obviously coincides with (16) and (17).

These results are valid for a two-dimensional phase space, although it cannot be generalized to higher dimensions. The reason is clear: a Pfaff differential form in n dimensions:

$$X_1(x_1,\ldots,x_n)dx_1 + \cdots + X_n(x_1,\ldots,x_n)dx_n \tag{33}$$

is not always exact and therefore no integrating factor may exist. A necessary and sufficient condition for the form (28) be exact is that for each three-dimensional vector field $Y := (X_{k_1}, X_{k_2}, X_{k_3})$, where X_{k_i} are three different functions out of $X_1, X_2, \ldots, X_n$, we have that

$$Y \cdot \mathrm{rot} Y = 0,$$

where 'rot' means rotational. Then, for dynamical systems in four and more dimensions in phase space, this discussion only makes sense for those systems that can lead to an integrable Pfaff equation.

2.1. The harmonic oscillator with friction

As an example of all of above, we propose the one-dimensional oscillator with friction, sometimes called the Caldirola–Kanai oscillator after work by these authors [7,8]. The discussion on this example is not new. A Lagrangian presentation of this system is well known [9]. An attempt to introduce a quantum dissipative equivalent system can be seen for instance in the work of El Naschie and Ord [10,11].

In its simplest form, this oscillator satisfies the following differential equation:

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0. \tag{34}$$

Eq. (34) is equivalent to the following system:

$$\dot{x} = y; \qquad \dot{y} = -ay - bx. \tag{35}$$

Both (33) and (34) are easily solvable. For the sake of completeness, we here study a slightly more general case and, instead of (35), we consider the system

$$\begin{aligned} \dot{x} &= a_{11}x + a_{12}y, \\ \dot{y} &= a_{21}x + a_{22}y. \end{aligned} \tag{36}$$

This system is easily solvable [5,6] and it yields to the following Pfaff equation:

$$-(a_{21}x + a_{22}y)\mathrm{d}x + (a_{11}x + a_{12}y)\mathrm{d}y = 0. \tag{37}$$

This Pfaff equation is also easily solvable by making the change of variable given by $y = xu$, where u is the new variable. Eq. (37) is then transformed into a differential equation with separate variables solvable by quadratures. However, after the above discussion, we are more interested in finding the integrating factor for the differential form in (37). This is

$$h(x,y) = \{-a_{21}x^2 + (a_{11} - a_{22})xy + a_{12}y^2\}^{-1}. \tag{38}$$

The potential function $F(x,y)$ such that $\mathrm{d}F = -h(x,y)(a_{21}x + a_{22}y)\mathrm{d}x + h(x,y)(a_{11}x + a_{12}y)\mathrm{d}y$ is given by

$$F(x,y) = C\left\{\frac{1}{2}\log[-a_{21}x^2 + (a_{11} - a_{22})xy + a_{12}y^2] + \frac{1}{\sqrt{\Delta}}\arctan\left[\frac{(a_{11} - a_{22})x + 2a_{12}y}{x\sqrt{\Delta}}\right](a_{11} + a_{22})\right\}, \tag{39}$$

where

$$\Delta = -(a_{11} + a_{22})^2 + 4D \quad \text{with } D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}. \tag{40}$$

We note that, due to the presence of the logarithm, the potential function $F(x,y)$ blows up at the singular points of the integrating factor. These points satisfy the following equation:

$$-a_{21}x^2 + (a_{11} - a_{22})xy + a_{12}y^2 = 0, \tag{41}$$

that gives

$$x = \frac{(a_{11} - a_{22})y \pm y\sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}}{2a_{21}} = A_{\pm}y, \tag{42}$$

where the values of $A_{\pm}$ can be immediately derived from (42). If both $A_{\pm}$ are real, the singular points of the integrating factor lie on two straight lines crossing the origin, except if the discriminant $(a_{11} - a_{22})^2 + 4a_{12}a_{21}$ is negative. In this case, the set of singular points drawn a parabola passing through the origin.

If we apply the general case to (35), we see that (42) takes the form

$$x = -\frac{a \pm \sqrt{a^2 - 4b^2}}{2b}y. \tag{43}$$

The discriminant is real if and only if $a^2 > 4b^2$ and (43) is the equation of two straight lines crossing the origin.

This procedure divides the phase plane into four different regions having a noteworthy property: If Ω is one of these four regions and $(x,y) \in \Omega$, the curve solution passing through (x,y) remains in Ω. Points in the straight lines (43) are not fixed points and the only fixed point is the origin.

Contrarily to the case studied in Section 1 concerning the free particle with friction (or viscosity), the fact that the only fixed point is the origin for system (35) implies that it is possible to describe the dynamics on phase space of the damped oscillator (34) with the aid of only one Hamiltonian function, which is given by

$$F(x,y) = \left\{\frac{1}{2}\log[bx^2 + axy + y^2] - \frac{a}{b^{1/2}}\arctan\left[\frac{ax + 2y}{xb^{1/2}}\right]\right\}. \tag{44}$$

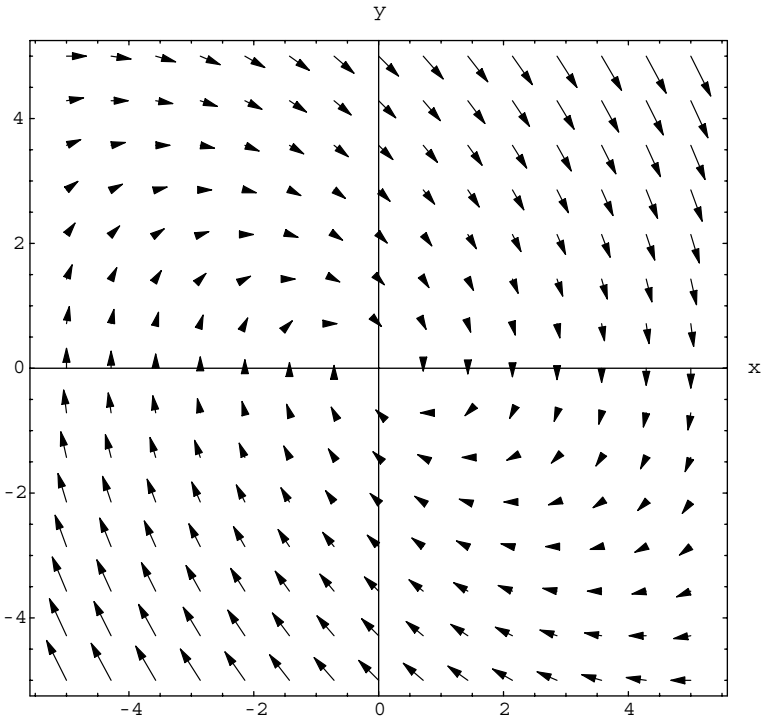

Fig. 3. Curves solution for the damped oscillator for $a^2 - 4b^2 > 0$.

Note that these Hamiltonian function has singularities that divide the phase plane into sectors. If the discriminant $a^2 - 4b^2$ is positive, the above Hamiltonian is well defined except on the straight lines (43). If the discriminant is negative, the values which make the logarithm vanish in (44) form a parabola. If the discriminant vanish, Eq. (43) shows that all the singular points belong to a straight line crossing the origin.

The conclusion is that the damped oscillator can be looked also as a Hamiltonian system by sectors. Here, the sectors are separated by the curves (or curve) $bx^2 + axy + y^2 = 0$, where the Hamiltonian function is not defined, although the Hamiltonian function can be defined to be the same for all sectors.

This Hamiltonian function gives a constant of motion along each curve solution. Curves solution for the case $a^2 - 4b^2 > 0$ are depicted in Fig. 3.

We could have also discussed the behavior of solutions of system (35) versus the behavior of the following system:

$$\begin{aligned} \dot{x} &= \frac{y}{bx^2 + axy + y^2}, \\ \dot{y} &= -\frac{bx + ay}{bx^2 + axy + y^2}, \end{aligned} \tag{45}$$

in which we have introduced the integrating factor. We also may have discussed the transformation that leads into the change of the parameter, which is an obvious consequence of (31). However, this discussion will increase the length of the present article and it does not give any new insight.

3. Concluding remarks

- When represented on phase space, one-dimensional dissipative systems satisfying an equation of motion of the form $\ddot{x} + F(x, \dot{x}) = 0$ are sectorially Hamiltonian. This means that we can divide the phase space into disjoint sectors such that the behavior of the dissipative system can be obtained from a Hamiltonian function defined on each sector. Consequently, in the given sector the solutions carry a constant of motion given by the Hamiltonian function.
- The sectors may change with a change of variables as we discuss in an example.
- As examples, we discussed the free particle with viscosity and the damped oscillator. Some more examples can be studied in detail are those for which the integrating factor is easily found.

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Appendix A. Two-dimensional differential forms always admit an integrating factor

Let us consider the following differential form

$$dF = A(x,y)dx + B(x,y)dy. \tag{46}$$

Does it admit an integrating factor? We know that a three-dimensional form like

$$dF = A(x,y,z)dx + B(x,y,z)dy + C(x,y,z)dz \tag{47}$$

admits an integrating factor if and only if

$$\mathbf{X} \cdot \mathbf{rotX} = 0, \tag{48}$$

where the vector field $\mathbf{X}$ is given by

$$\mathbf{X} \equiv (A(x,y,z), B(x,y,z), C(x,y,z)). \tag{49}$$

In the case of (46), the vector field $\mathbf{X}$ has components:

$$\mathbf{X} = (A(x,y), B(x,y), 0), \tag{50}$$

so that

$$\mathbf{rot}\ \mathbf{X} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A(x,y) & B(x,y) & 0 \end{vmatrix} = \frac{\partial B(x,y)}{\partial z}\mathbf{i} + \frac{\partial A(x,y)}{\partial z}\mathbf{j} + \left(\frac{\partial B(x,y)}{\partial x} - \frac{\partial A(x,y)}{\partial y}\right)\mathbf{k} = \left(0, 0, \frac{\partial B(x,y)}{\partial x} - \frac{\partial A(x,y)}{\partial y}\right). \tag{51}$$

Since the third component of $\mathbf{X}$ vanishes, the scalar product $\mathbf{X} \cdot \mathbf{rot}\ \mathbf{X}$ vanishes and, therefore, (48) holds.

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