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Lipschitz type smoothness of the fractional integral on variable exponent spaces

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ABSTRACT

We give conditions on $p(\cdot)$ in order to ensure boundedness of the fractional integral operator I_α from strong and weak $L^{p(\cdot)}$ spaces into suitable integral Lipschitz-type spaces.

1. Introduction

It is well known that the fractional integral defined as

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy, \tag{1.1}$$

for $\alpha \in (0, n)$ given and $f \in L^1_{\text{loc}}$, is a bounded operator from L^p into L^q with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, whenever $1 < p < \frac{n}{\alpha}$, and from $L^{\frac{n}{\alpha}}$ into BMO. Less known is its behavior on L^p for $p > \frac{n}{\alpha}$.

In regards to this, Gatto and Vagi [17] proved that, when $p > \frac{n}{\alpha}$, an extension of I_α is bounded from L^p into Lipschitz spaces whose smoothness is controlled by p and α. Moreover the result holds with L^p replaced by $L^{p,\infty}$ and in the general setting of the spaces of homogeneous type. In addition they proved an extension of another well known result: that I_α takes Lipschitz spaces of order β into Lipschitz spaces of order $\alpha + \beta$.

It is worth mentioning that those results have a one-weight extension in [20] but this time in $\mathbb{R}^n$. Also, this weighted result was proved for the case of the fractional integral related to Schrödinger operators in [6].

The aim of this work is to extend some of the above mentioned unweighted boundedness properties in another direction. More precisely, to consider Lebesgue space with a variable exponent instead of L^p with constant p. These are particular cases of the Musielak–Orlicz spaces (see [26]), which have been generating interest in recent years because of its connection with the study of variational integrals and partial differential equations with a non-standard growth condition (see, for instance, [1,2,4,5,14,19,21,25,29]).

In order to state our main results we begin by giving some basic definitions related to the spaces we are going to deal with.

Given an exponent function, that is a non-negative measurable function $p(\cdot)$ from $\mathbb{R}^n$ to $[1, \infty)$, we say that a measurable function f belongs to $L^{p(\cdot)}$, the Lebesgue space with variable exponent $p(\cdot)$, if it satisfies

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx < \infty.$$

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The formula

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \le 1\right\}$$

defines a norm in $L^{p(\cdot)}$. Topics related to general properties of these spaces are treated in [24]. In particular, associated to $p(\cdot)$, we consider the following constants:

$$p_-(\Omega) := \inf_{x\in\Omega} p(x); \qquad p_+(\Omega) := \sup_{x\in\Omega} p(x), \tag{1.2}$$

where Ω is a measurable set in $\mathbb{R}^n$. For simplicity, when $\Omega = \mathbb{R}^n$ we only denote p_- and p_+. Whenever $p_-(\Omega) > 1$ we denote $p'(x) = \frac{p(x)}{p(x)-1}$, $x \in \Omega$. It can be proved (see [24]) that the following generalization of Hölder's inequality holds

$$\int_{\mathbb{R}^n} |f(x)\, g(x)|\, dx \le C \ \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)} . \tag{1.3}$$

This inequality is an important tool, as the classical one in the case $p \equiv$ constant, in order to prove the following relationship (see [24] again)

$$\|f\|_{p(\cdot)} \approx \sup_{\|g\|_{p'(\cdot)}\le 1} \int_{\mathbb{R}^n} f(x)g(x)\, dx. \tag{1.4}$$

In many articles related to boundedness of classical operators in variable exponent spaces it is common to assume that $p(\cdot)$ satisfies

$$\mathrm{LH}_0 : \ |p(x) - p(y)| \le \frac{C}{\log\left(e + \frac{1}{|x-y|}\right)}, \quad x, y \in \mathbb{R}^n \tag{1.5}$$

$$\mathrm{LH}_\infty : \ |p(x) - p_\infty| \le \frac{C}{\log(e + |x|)}, \quad x \in \mathbb{R}^n \tag{1.6}$$

for some positive constants C and p_∞. In our work we will refer to these properties by saying that $p(\cdot) \in \mathrm{LH}_0$ and $p(\cdot) \in \mathrm{LH}_\infty$ respectively.

Assuming this type of hypothesis on $p(\cdot)$, different extensions of the classical result about the boundedness of I_α from L^p to L^q have been given by several authors. For instance, in [28] it is considered a variable order for I_α, that is the constant α is replaced by a function $\alpha(\cdot)$, and the boundedness is proved on a bounded domain Ω in $\mathbb{R}^n$. In [23], the variable order is considered again, L^q is replaced by a certain weighted $L^{q(\cdot)}$ and the setting is $\mathbb{R}^n$. On the other hand, in [10] a result on unbounded domains is proved for I_α with constant order α and p constant outside a large ball. Another extension for $\mathbb{R}^n$ and constant α but p variable in all $\mathbb{R}^n$ appears in [8], where an original approach using ideas taken from the extrapolation theory is adopted.

However, these articles do not consider the behavior of I_α when $p_+ > \frac{n}{\alpha}$, that is the range where some kind of smoothness can be expected. This is precisely, as we said before, our objective. To this aim we introduce a class of spaces that generalizes the integral Lipschitz-type spaces $\mathfrak{L}_{p,\lambda}$ considered in [27].

Definition 1.7. Given $0 < \alpha < n$ and an exponent function $p(\cdot)$ with $1 \le p_- \le p_+ < \infty$ we say that a locally integrable function f belongs to $\mathfrak{L}_{\alpha,p(\cdot)} = \mathfrak{L}_{\alpha,p(\cdot)}(\mathbb{R}^n)$ if there exists a constant C such that

$$\frac{1}{|B|^{\frac{\alpha}{n}} \, \|\chi_B\|_{p'(\cdot)}} \int_B |f - m_B f|\, dx \le C, \tag{1.8}$$

for every ball $B \subset \mathbb{R}^n$, with $m_B f = \frac{1}{|B|}\int_B f$. The least constant C in (1.8) will be denoted by $\|f\|_{\mathfrak{L}_{\alpha,p(\cdot)}}$.

Remark 1.9. It is easy to see that in Definition 1.7 the average can be replaced by a constant in the following sense

$$\frac{1}{2}\, \|f\|_{\mathfrak{L}_{\alpha,p(\cdot)}} \le \sup_{B\in\mathbb{R}^n} \inf_{a\in\mathbb{R}} \frac{1}{|B|^{\frac{\alpha}{n}} \, \|\chi_B\|_{p'(\cdot)}} \int_B |f(x) - a|\, dx \le \|f\|_{\mathfrak{L}_{\alpha,p(\cdot)}} .$$

Remark 1.10. In [15] p. 4155, for a bounded open set Ω in $\mathbb{R}^n$ and functions $p(\cdot) : \overline{\Omega} \to [1, \infty)$ and $\lambda(\cdot) : \overline{\Omega} \to [0, \infty)$ such that $p(\cdot), \lambda(\cdot) \in \mathrm{LH}_0 \cap \mathrm{LH}_\infty$, a definition of variable exponent Campanato spaces is given as follows:

$$\mathcal{L}^{q(\cdot),\lambda(\cdot)}(\Omega) = \left\{f \in L^{q(\cdot)}(\Omega) : \sup_{x_0\in\Omega,\rho>0} \rho^{-\lambda(x_0)} \int_{\Omega(x_0,\rho)} |f(x) - f_{\Omega(x_0,\rho)}|^{q(x)}\, dx < +\infty\right\},$$

where $\Omega(x_0, \rho) = \Omega \cap Q(x_0, \rho)$ and $Q(x_0, \rho)$ denotes the cube of $\mathbb{R}^n$ with center at x_0 and side length 2ρ.

Since for small balls it holds that $|B|^{p_-(B)} \approx |B|^{p(x)} \approx |B|^{p_+(B)}$ for each $x \in B$ whenever $p(\cdot) \in \mathrm{LH}_0$, then restricting our spaces $\mathfrak{L}_{\alpha,p(\cdot)}$ to a similar Ω they coincide with $\mathcal{L}^{1,\alpha+n/p'(x)}(\Omega)$.

With these basic details we are in a position to state our first theorem. It establishes, under certain hypotheses, a characterization of the exponent functions $p(\cdot)$ such that an extension of I_α is bounded from $L^{p(\cdot)}$ into $\mathfrak{L}_{\alpha,p(\cdot)}$. It is important to note that in relation to the boundedness of operators on variable exponent Lebesgue spaces is not so common to get results like this. In general, only sufficient conditions on $p(\cdot)$ are given.

Theorem 1.11. *Given $0 < \alpha < n$ let $p(\cdot)$ be an exponent function with $1 < p_- \le p_+ < \infty$. Then the following statements are equivalent.*

(a) *The operator I_α can be extended to a linear bounded operator $\widetilde{I_\alpha}$ from $L^{p(\cdot)}$ into $\mathfrak{L}_{\alpha,p(\cdot)}$ as follows,*

$$\widetilde{I_\alpha} f(x) = \int_{\mathbb{R}^n} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1-\chi_{B(0,1)}(y)}{|y|^{n-\alpha}} \right) f(y)\, dy.$$

(b) *There exists a positive constant C such that*

$$(P_\alpha) \qquad \left\| \frac{\chi_{\mathbb{R}^n - B}}{|x_B - \cdot|^{n-\alpha+1}} \right\|_{p'(\cdot)} \le C\ |B|^{\frac{\alpha}{n} - \frac{1}{n} - 1} \|\chi_B\|_{p'(\cdot)},$$

holds for every ball B, where x_B denotes its center.

For our next theorem we consider the following definition of weak Lebesgue spaces with a variable exponent.

Definition 1.12. Given an exponent function $p(\cdot)$ we say that a measurable function f belongs to $L^{p(\cdot),\infty}$ if there exists a constant C such that

$$\int_{\mathbb{R}^n} t^{p(x)} \chi_{\{|f|>t\}}(x)\, dx \le C,$$

for every $t > 0$.

It is not difficult to see that

$$[f]_{p(\cdot),\infty} = \inf \left\{ \lambda > 0 \,:\, \sup_{t>0} \int_{\mathbb{R}^n} \left(\frac{t}{\lambda}\right)^{p(x)} \chi_{\{|f|>t\}}(x)\, dx \le 1 \right\}$$

is a quasi-norm in $L^{p(\cdot),\infty}$.

We remark that the spaces $L^{p(\cdot),\infty}$ were before considered by Capone, Cruz-Uribe and Fiorenza in [7], although the authors there did not define them explicitly.

Our second theorem gives sufficient conditions on $p(\cdot)$ under which an extension of I_α is bounded from $L^{p(\cdot),\infty}$ into $\mathfrak{L}_{\alpha,p(\cdot)}$. As we will see in our examples, the condition on $p(\cdot)$ is stronger than the ones considered in Theorem 1.11.

Theorem 1.13. *Given $0 < \alpha < n$, let $p(\cdot)$ be an exponent function with $1 < p_- \le p_+ < \frac{n}{(\alpha-1)^+}$. Assume that $p(\cdot) \in LH_0 \cap LH_\infty$ and there exists a positive r_0 such that $p(x) \le p_\infty$ for $|x| > r_0$. Then the extension $\widetilde{I_\alpha}$, defined as in Theorem 1.11, is bounded from $L^{p(\cdot),\infty}$ into $\mathfrak{L}_{\alpha,p(\cdot)}$.*

Throughout this paper, we denote for a ball $B \subset \mathbb{R}^n$, aB with $a > 0$ the ball concentric with B and radius a times the radius of B. For a subset A in $\mathbb{R}^n$, $|A|$ denote the Lebesgue measure of A. Also, we denote by C a constant that may be different in each occurrence. The structure of the paper is as follows. Section 2 is devoted to a study of the condition we assume on $p(\cdot)$ and the proof of some properties of $\|\cdot\|_{p(\cdot)}$ derived from them. Section 3 contains some technical lemmas that will be useful in Section 4 for the proof of our main results.

2. On the conditions on $p(\cdot)$

The following estimates show very useful relations between the norm of a characteristic function of a ball and its Lebesgue measure.

Lemma 2.1. *Given a ball $B = B(x_0, r)$, we have the following.*

(a) *If $r < 1$, there exist constants a_1 and a_2 such that*

$$a_1\, |B|^{\frac{1}{p_-(B)}} \le \|\chi_B\|_{p(\cdot)} \le a_2\, |B|^{\frac{1}{p_+(B)}}.$$

(b) *If $r > 1$, there exist constants b_1 and b_2 such that*

$$b_1\, |B|^{\frac{1}{p_+(B)}} \le \|\chi_B\|_{p(\cdot)} \le b_2\, |B|^{\frac{1}{p_-(B)}}.$$

We can find the proof in [16,11].

Definition 2.2. We say that the function $p(\cdot)$ satisfies a doubling condition D, if there exists a positive constant C such that

$$\|\chi_{2B}\|_{p(\cdot)} \le C \ \|\chi_B\|_{p(\cdot)},$$

for every ball $B \subset \mathbb{R}^n$. In this case, we write $p(\cdot) \in D$.

It is important to note that depending on the behavior of $p(\cdot)$, the functions belonging to $\mathfrak{L}_{\alpha,p(\cdot)}$ can get some kind of local smoothness. More precisely, using Lemma 2.1 and the definition above we deduce the following lemma.

Lemma 2.3. *Let α be such that $0 < \alpha < n$ and $p(\cdot)$ be an exponent function with $p_- > 1$. Let Ω be an open set such that $p_-(\Omega) > \frac{n}{\alpha}$. In addition, if $p(\cdot) \in D$, then for all $f \in \mathfrak{L}_{\alpha,p(\cdot)}$, there exist a constant C verifying*

$$|f(x) - f(y)| \le C \ \|f\|_{\mathfrak{L}_{\alpha,p(\cdot)}} |x-y|^{\alpha - \frac{n}{p_-(2B)}},$$

for all ball B such that $2B \subset \Omega$, $|B| \le 1$ and almost everywhere $x, y \in B$.

Proof. Let $f \in \mathfrak{L}_{\alpha,p(\cdot)}$. Let us take x and y Lebesgue points of f in B, where B is such that $2B \subset \Omega$ and $|B| \le 1$. Now, considering the balls $B^1 = B(x, |x-y|)$ and $B^2 = B(y, |x-y|)$, we can write

$$|f(x) - f(y)| \le |f(x) - f_{B^1}| + |f(y) - f_{B^2}| + |f_{B^1} - f_{B^2}|.$$

We are going to estimate the first term of the sum. Note that the estimate of the second one is quite similar. For $B_i = B(x, 2^{-i}|x-y|)$, we have

$$\begin{aligned}
|f(x) - f_{B^1}| &\le \lim_{m\to\infty} \left(|f(x) - f_{B_m}| + \sum_{i=0}^{m-1} |f_{B_{i+1}} - f_{B_i}| \right) \\
&= \sum_{i=0}^{\infty} |f_{B_{i+1}} - f_{B_i}| \\
&\le C \ \sum_{i=0}^{\infty} |B_i|^{-1} \int_{B_i} |f(z) - f_{B_i}| dz \\
&\le C \ \|f\|_{\mathfrak{L}_{\alpha,p(\cdot)}} \sum_{i=0}^{\infty} |B_i|^{-1+\frac{\alpha}{n}} \left\|\chi_{B_i}\right\|_{p'(\cdot)}.
\end{aligned}$$

Then, from Lemma 2.1(a), we get

$$\begin{aligned}
|f(x) - f_{B^1}| &\le C \ \|f\|_{\mathfrak{L}_{\alpha,p(\cdot)}} \sum_{i=0}^{\infty} |B_i|^{-1+\frac{\alpha}{n}+\frac{1}{(p')_+(B_i)}} \\
&\le C \ \|f\|_{\mathfrak{L}_{\alpha,p(\cdot)}} |x-y|^{\alpha-n+\frac{n}{(p')_+(B)}} \sum_{i=0}^{\infty} (2^{-i})^{\alpha-n+\frac{n}{(p')_+(B)}} \\
&\le C \ \|f\|_{\mathfrak{L}_{\alpha,p(\cdot)}} |x-y|^{\alpha-\frac{n}{p_-(B)}},
\end{aligned}$$

since $\alpha - n + \frac{n}{(p')_+(B)} = \alpha - \frac{n}{p_-(B)} > 0$. Finally, for the third term, using again Lemma 2.1 (a), we have

$$\begin{aligned}
|f_{B^1} - f_{B^2}| &\le |f_{B^1} - f_{2B^1}| + |f_{B^2} - f_{2B^1}| \\
&\le |B^1|^{-1} \int_{B^1} |f(z) - f_{2B^1}| dz + |B^2|^{-1} \int_{B^2} |f(z) - f_{2B^1}| dz \\
&\le C \ |B^1|^{-1} \int_{2B^1} |f(z) - f_{2B^1}| dz \\
&\le C \ \|f\|_{\mathfrak{L}_{\alpha,p(\cdot)}} \left\|\chi_{2B^1}\right\|_{p'(\cdot)} |B^1|^{\alpha-1} \\
&\le C \ \|f\|_{\mathfrak{L}_{\alpha,p(\cdot)}} |x-y|^{\alpha-\frac{n}{p_-(2B)}}.
\end{aligned}$$

Then the proof is complete. □

Remark 2.4. In Theorem 4.3 of [15] the authors proved that functions in $\mathcal{L}^{q(\cdot),\lambda(\cdot)}(\Omega)$ (see Remark 1.10) satisfy some kind of local pointwise smoothness under the assumption that "Ω has no external cusps", i.e. there exists a constant $A > 0$ such that for every $x_0 \in \overline{\Omega}$ and for every $\rho \in (0, \operatorname{diam} \Omega]$, $|\Omega(x_0, \rho)| \ge A\,|Q(x_0, \rho)|$ holds. In the case $q(x) \equiv 1$ and $\lambda(x) = \alpha + n/p'(x)$, this result has an obvious relation with the lemma above (see also Theorem 6 of [3] p. 493).

As a corollary of our lemma we can get local smoothness result similar to that contained in Theorem 5.4 in [13].

Corollary 2.5. *Let Ω be a bounded open set in $\mathbb{R}^n$. Let $p(\cdot) \in LH_0$ with $p(x) > n$ for all $x \in \Omega$. Given $f \in L^{p(\cdot)}(\Omega) \cap C^1(\Omega)$ with $\nabla f \in L^{p(\cdot)}(\Omega)$, there exists a constant $c > 0$ such that*

$$|f(x) - f(y)| \le c \, \| |\nabla f| \|_{p,\Omega} \, |x-y|^{1-\frac{n}{p(x)}},$$

for all $x, y \in B$ such that $2B \subset \Omega$, $|B| \le 1$.

Proof. Note that from Lemma 2.3 and Lemma 3.6 in [22], we have

$$|f(x) - f(y)| \le C \, \|f\|_{\mathfrak{L}_{1,p(\cdot)}} \, |x-y|^{1-\frac{n}{p(x)}}.$$

So, we only need to prove that $\|f\|_{\mathfrak{L}_{1,p(\cdot)}} \le C \, \| |\nabla f| \|_{p,\Omega}$. In order to do this, using the well known pointwise inequality

$$|f(x) - f_B| \le C \int_B \frac{|\nabla f(z)|}{|x-z|^{n-1}}\, dz,$$

(see [18], Lemma 7.16), we get

$$\begin{aligned}\int_B |f(x) - f_B|\, dx &\le C \int_B \int_B \frac{|\nabla f(z)|}{|x-z|^{n-1}}\, dz\, dx \\ &\le C \int_B |\nabla f(z)| \left(\int_{|x-z|<2r} \frac{dx}{|x-z|^{n-1}} \right) dz \\ &\le C \, |B|^{\frac{1}{n}} \, \| |\nabla f| \|_{p(\cdot)} \, \|\chi_B\|_{p'(\cdot)}\end{aligned}$$

where we use the Hölder inequality (1.3). □

In the proof of the corollary above we use a result about the LH_0 condition. This estimation will be useful for other results in this paper. So, for the sake of completeness we state it (see Lemma 3.6 in [22] for the proof).

Lemma 2.6. *Let $p_+ < \infty$. Then the following conditions are equivalent.*

(a) *The function $p \in LH_0$.*
(b) *There exists a constant C such that*

$$|B|^{p_-(B)-p_+(B)} \le C,$$

for every ball $B \subset \mathbb{R}^n$.

The next proposition is a particular case of a result contained in [9], Lemma 3.6 (see also [12], p. 7–8).

Proposition 2.7. *Let $p(\cdot) \in LH_\infty$. Then*

$$\|\chi_B\|_{p(\cdot)} \simeq |B|^{1/p_\infty},$$

for every ball B with radius $r_B \ge 1/4$.

Remark 2.8. If $p(\cdot) \in LH_0 \cap LH_\infty$, Proposition 2.7, Lemmas 2.1 and 2.6 imply that $p(\cdot) \in D$.

In the following we will see some properties that can be deduced from the hypothesis on $p(\cdot)$ considered in Theorems 1.11 and 1.13. In addition we will study the relation between them. First, in order to see that P_α also implies D (see Definition 2.2), we notice that P_α is equivalent to the following condition

$$|B|^{\frac{1}{n}-\frac{\alpha}{n}+1} \left\| \frac{1}{(|B|^{\frac{1}{n}} + |x_B - \cdot|)^{n-\alpha+1}} \right\|_{p'(\cdot)} \le C \, \|\chi_B\|_{p'(\cdot)}.$$

In fact, if $p(\cdot)$ satisfies P_α we have

$$\begin{aligned}|B|^{\frac{1}{n}-\frac{\alpha}{n}+1} \left\| \frac{1}{(|B|^{\frac{1}{n}} + |x_B - \cdot|)^{n-\alpha+1}} \right\|_{p'(\cdot)} &\le |B|^{\frac{1}{n}-\frac{\alpha}{n}+1} \left\| \frac{\chi_{\mathbb{R}^n - B}(\cdot)}{|x_B - \cdot|^{n-\alpha+1}} \right\|_{p'(\cdot)} + |B|^{\frac{1}{n}-\frac{\alpha}{n}+1} \left\| \frac{\chi_B(\cdot)}{|B|^{\frac{1}{n}-\frac{\alpha}{n}+1}} \right\|_{p'(\cdot)} \\ &= C \, \|\chi_B\|_{p'(\cdot)}.\end{aligned}$$

The converse is clear.

Lemma 2.9. *If $p(\cdot)$ satisfies P_α, then $p(\cdot) \in D$.*

Proof. Using the remark above, we get

$$\begin{aligned}\|\chi_B\|_{p(\cdot)} &\geq C\ |B|^{\frac{1}{n}-\frac{\alpha}{n}+1}\left\|\frac{1}{(|B|^{\frac{1}{n}}+|x_B-\cdot|)^{n-\alpha+1}}\right\|_{p'(\cdot)}\\ &\geq C\ |B|^{\frac{1}{n}-\frac{\alpha}{n}+1}\left\|\frac{\chi_{2B}(\cdot)}{|B|^{\frac{1}{n}-\frac{\alpha}{n}+1}}\right\|_{p'(\cdot)}\\ &\geq C\ \|\chi_{2B}\|_{p'(\cdot)}\,,\end{aligned}$$

for all ball B. This proves the lemma. □

Using Lemmas 2.1 and 2.9 and Proposition 2.7 we can prove the following proposition.

Proposition 2.10. *Let $p(\cdot)$ be an exponent function belonging to $LH_0\cap LH_\infty$ with $p_+<\frac{n}{(\alpha-1)^+}$. Then*

$$\int_r^\infty \frac{\left\|\chi_{B(x_0,t)}\right\|_{p'(\cdot)}}{t^{n-\alpha+1}}\,\frac{dt}{t}\leq C\ |B|^{\frac{\alpha}{n}-\frac{1}{n}-1}\,\|\chi_B\|_{p'(\cdot)}\,, \tag{2.11}$$

for all ball B with center x_0 and radius r.

Proof. Let $B=B(x_0,r)$ be a ball in $\mathbb{R}^n$. First, assuming $r>1$, from Proposition 2.7, since $p_+<\frac{n}{(\alpha-1)^+}$ we get

$$\begin{aligned}\int_r^\infty \frac{\left\|\chi_{B(x_0,t)}\right\|_{p'(\cdot)}}{t^{n-\alpha+1}}\,\frac{dt}{t} &\leq C\int_r^\infty \frac{t^{n-\frac{n}{p_\infty}}}{t^{n-\alpha+1}}\,\frac{dt}{t}\\ &= C\ r^{\alpha-1-\frac{n}{p_\infty}}\\ &= C\ r^{\alpha-1-n+\frac{n}{(p_\infty)'}}\\ &\leq C\ |B|^{\frac{\alpha}{n}-\frac{1}{n}-1}\,\|\chi_B\|_{p'(\cdot)}\,,\end{aligned} \tag{2.12}$$

where $1/(p_\infty)'=1-1/p_\infty$. Now, if $r\leq 1$, we have

$$\begin{aligned}\int_r^\infty \frac{\left\|\chi_{B(x_0,t)}\right\|_{p'(\cdot)}}{t^{n-\alpha+1}}\,\frac{dt}{t} &\leq \int_r^1 \frac{\left\|\chi_{B(x_0,t)}\right\|_{p'(\cdot)}}{t^{n-\alpha+1}}\,\frac{dt}{t}+\int_1^\infty \frac{\left\|\chi_{B(x_0,t)}\right\|_{p'(\cdot)}}{t^{n-\alpha+1}}\,\frac{dt}{t}\\ &= I_1+I_2.\end{aligned} \tag{2.13}$$

From (2.12), (b) of Lemma 2.1 and the fact that $p_+<\frac{n}{(\alpha-1)^+}$ again, it follows that

$$I_2\leq C\leq C\ r^{\alpha-1-n+\frac{n}{p'_-}}=C\ |B|^{\frac{\alpha}{n}-\frac{1}{n}-1+\frac{1}{(p_+)'}} \tag{2.14}$$

$$\leq C\ |B|^{\frac{\alpha}{n}-\frac{1}{n}-1}\,\|\chi_B\|_{p'(\cdot)}\,.$$

On the other hand, using again (b) of Lemma 2.1 and the fact that $p(\cdot)\in \mathrm{LH}_0$, we obtain

$$\begin{aligned}I_1 &\leq C\int_r^1 \frac{|B(x_0,t)|^{1-\frac{1}{p_-(B(x_0,t))}}}{t^{n-\alpha+1}}\,\frac{dt}{t}\\ &\leq C\int_r^1 \frac{t^{n-\frac{n}{p_+(B(x_0,t))}}}{t^{n-\alpha+1}}\,t^{\frac{n}{p_+(B(x_0,t))}-\frac{n}{p_-(B(x_0,t))}}\,\frac{dt}{t}\\ &\leq C\int_r^1 \frac{t^{n-\frac{n}{p_+(B(x_0,t))}}}{t^{n-\alpha+1}}\,\frac{dt}{t}\\ &\leq C\int_r^1 t^{\alpha-\frac{n}{p_+(B(x_0,r))}-1}\,\frac{dt}{t}\\ &\leq C\ r^{\alpha-\frac{n}{p_+(B(x_0,r))}-1}\\ &\leq C\ |B|^{\frac{\alpha}{n}-\frac{1}{n}-1}\,\|\chi_B\|_{p'(\cdot)}\,.\end{aligned} \tag{2.15}$$

Finally, collecting (2.12)–(2.15) the proof is done. □

From this proposition the following result is obvious.

Corollary 2.16. *Under the hypothesis of Proposition 2.10, $p(\cdot)$ satisfies P_α.*

The next lemma states some alternative characterizations for the exponents that satisfy inequality (2.11). In particular, the one appearing in (c) will be one of the main tools in proving Theorem 1.13.

Lemma 2.17. *Let $p(\cdot)$ be an exponent function such that $p_+ < \frac{n}{(\alpha-1)^+}$. Then the following conditions are equivalent.*

(a) *The function $p(\cdot)$ satisfies (2.11).*
(b) *There exists $a > 1$ such that*

$$\left\| \chi_{B(x,at)} \right\|_{p'(\cdot)} \leq \frac{1}{2} a^{n-\alpha+1} \left\| \chi_{B(x,t)} \right\|_{p'(\cdot)},$$

for every $t > 0$ and $x \in \mathbb{R}^n$.
(c) *There exist positive constants C and ϵ such that*

$$\left\| \chi_{B(x,\theta t)} \right\|_{p'(\cdot)} \leq C\ \theta^{n-\alpha+1-\epsilon} \left\| \chi_{B(x,t)} \right\|_{p'(\cdot)},$$

for every $\theta \geq 1$, $t > 0$ and $x \in \mathbb{R}^n$.

This result is analogous to that in Lemma 3.3 in [20] (p. 241).

Remark 2.18. Given $0 < \alpha < n$, let p_∞ and b be such that $1 < p_\infty < \frac{n}{(\alpha-1)^+}$ and $1 - p_\infty < b < \frac{n}{(\alpha-1)^+} - p_\infty$. Then, it is not difficult to see that the function

$$p(x) = p_\infty + \frac{b}{\log(e + |x|)},$$

satisfies the hypothesis of Proposition 2.10 above. Note that $p(\cdot)$ is a radial function increasing or decreasing depending on the sign of b.

Remark 2.19. There exist α and $p(\cdot)$ such that $p(\cdot)$ satisfies P_α and $p(\cdot) \notin \mathrm{LH}_\infty$. In fact, consider the set $D \subset \mathbb{R}^n$, $n \geq 2$, given by

$$D = \bigcup_{j\in\mathbb{Z}} B_j,$$

where $B_j = B(x_j, 1/4)$ with $x_j = (j, 0, \ldots, 0)$, $j \in \mathbb{Z}$. Let $p(\cdot)$ be a continuous exponent function such that $p(x) = p_-$ for all $x \in D^c$. If we also assume that

(a) $p(\cdot)$ satisfies LH_0,
(b) $1 < p_- < p_+ < \frac{n}{(n-\alpha)^+}$ with $0 < \alpha < n$,
(c) $\frac{n}{p_-} - \frac{1}{p_+} \leq n - 1$,

and the supremum is attained on every ball B_j, then it is not hard to see that $p(\cdot)$ satisfies P_α but $p(\cdot) \notin \mathrm{LH}_\infty$.

3. Technical lemmas

The following two lemmas will be used in the proof of Theorem 1.13.

In the first we use the notation $\|\cdot\|_{L^{q(\cdot)}(\mathbb{R}^n, \frac{dx}{|B|})}$ to mean the norm of $L^{q(\cdot)}$ with respect to the measure defined through $\frac{dx}{|B|}$.

Lemma 3.1. *Let $p(\cdot)$ be an exponent function in $LH_0 \cap LH_\infty$ such that $1 < p_- \leq p_+ < \frac{n}{(\alpha-1)^+}$ and $p(x) \leq p_\infty$ for $|x| \geq r_0$ with $r_0 > 1$. Then there exists a positive constant C depending on r_0 and the constants associated to LH_0 and LH_∞, such that*

$$\int_B |f(x)|\, dx \leq C\ \|\chi_B\|_{p'(\cdot)} [f]_{p(\cdot),\infty}, \tag{3.2}$$

for every ball B and $f \in L^{p(\cdot),\infty}$.

Proof. Without loss of generality, we can assume that $[f]_{p(\cdot),\infty} = 1$. Choosing $\varepsilon > 0$ such that $0 < \varepsilon < p_- - 1$, we take $q(x) = p(x) - \varepsilon$. From the generalized Hölder inequality, we get

$$\int_B |f(x)|\, dx \leq C\ |B|\ \|f\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n, \frac{dx}{|B|})}. \tag{3.3}$$

Now, taking into account that

$$|B| \leq C \ \|\chi_B\|_{p(\cdot)} \, \|\chi_B\|_{p'(\cdot)} \, ,$$

which easily follows from applying again the generalized Hölder inequality, (3.3) lead us to (3.2) whenever we prove that

$$\|f\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n, \frac{dx}{|B|})} \leq \frac{C}{\|\chi_B\|_{p(\cdot)}}. \tag{3.4}$$

To this aim we write

$$\begin{aligned} I &\doteq \int_B |\, \|\chi_B\|_{p(\cdot)} \, f(x)|^{q(x)} \frac{dx}{|B|} \\ &= \int_0^\infty \int_{\mathbb{R}^n} q(x) t^{q(x)-1} \chi_{\{|\|\chi_B\|_{p(\cdot)} f\chi_B| > t\}}(x) \, \frac{dx}{|B|} \, dt \\ &= \int_0^\infty \int_{\mathbb{R}^n} q(x) u^{q(x)-1} \frac{\|\chi_B\|_{p(\cdot)}^{q(x)}}{|B|} \chi_{\{|f\chi_B| > u\}}(x) \, dx \, du. \end{aligned} \tag{3.5}$$

Now, if we prove that

$$\frac{\|\chi_B\|_{p(\cdot)}^{p(x)}}{|B|} = \|\chi_B\|_{p(\cdot)}^{\varepsilon} \, \frac{\|\chi_B\|_{p(\cdot)}^{q(x)}}{|B|} \leq C,$$

holds for all $x \in \mathbb{R}^n$, where C is a constant independent of B, from (3.5) recalling that $[f]_{p(\cdot),\infty} = 1$, we have

$$\begin{aligned} I &\leq C \ \|\chi_B\|_{p(\cdot)}^{-\varepsilon} \int_0^\infty \int_{\mathbb{R}^n} q(x) u^{q(x)-1} \chi_{\{|f\chi_B|>u\}}(x) \, dx \, du \\ &\leq C \ \|\chi_B\|_{p(\cdot)}^{-\varepsilon} \left[\int_B \left(\int_0^a q(x) u^{q(x)-1} \, du \right) dx + \int_a^\infty u^{-\varepsilon-1} \int_{\mathbb{R}^n} u^{p(x)} \chi_{\{|f\chi_B|>u\}}(x) \, dx \, du \right] \\ &\leq C \ \|\chi_B\|_{p(\cdot)}^{-\varepsilon} \left(\int_B a^{q(x)} \, dx + \int_a^\infty u^{-\varepsilon-1} \, du \right) \\ &\leq C \ \|\chi_B\|_{p(\cdot)}^{-\varepsilon} \left(\int_B a^{q(x)} \, dx + a^{-\varepsilon} \right). \end{aligned}$$

Choosing $a = \|\chi_B\|_{p(\cdot)}^{-1}$ and noting that $\int_B a^{q(x)} \, dx \leq a^{-\varepsilon}$ we clearly get

$$I \leq C \ \|\chi_B\|_{p(\cdot)}^{-\varepsilon} \, a^{\varepsilon} \leq C \tag{3.6}$$

which is (3.4). In order to prove the claim we consider three cases.

Case 1. Balls $B = B(x_B, r)$ with $r \leq 2r_0$. From Lemmas 2.1 and 2.6, for $x \in B$, we get

$$\begin{aligned} \frac{\|\chi_B\|_{p(\cdot)}^{p(x)}}{|B|} &\leq C \ |B|^{\frac{p(x)}{p_+(B)} - 1} \\ &\leq C \ |B|^{\frac{p_-(B) - p_+(B)}{p_+(B)}} \\ &\leq C \ . \end{aligned}$$

Case 2. Balls $B = B(x_B, r)$ with $r > 2r_0$ and $|x_B| \geq 2r$. From Lemmas 2.1 and 2.6 again, for $x \in B$, we have

$$\begin{aligned} \frac{\|\chi_B\|_{p(\cdot)}^{p(x)}}{|B|} &\leq C \ |B|^{\frac{p(x)}{p_-(B)} - 1} \\ &\leq C \ |B|^{\frac{p_+(B) - p_-(B)}{p_-(B)}} \\ &\leq C \ . \end{aligned}$$

Case 3. Balls $B = B(x_B, r)$ with $r > 2r_0$ and $|x_B| < 2r$. In this situation, it is enough to consider the case $x_B = 0$. In fact, if $x_B \neq 0$, we get $B \subset \widetilde{B} = B(0, 3r)$. Therefore, from Proposition 2.7, we obtain

$$\begin{aligned} \int_B |f(x)| \, dx &\leq \int_{\widetilde{B}} |f(x)| \, dx \leq C \ \left\| \chi_{\widetilde{B}} \right\|_{p'(\cdot)} \\ &\leq C \ |\widetilde{B}|^{\frac{1}{p'_\infty}} \leq C \ |B|^{\frac{1}{p'_\infty}} \leq C \ \|\chi_B\|_{p'(\cdot)} \, . \end{aligned}$$

which proves our thesis. So, let us take $B = B(0, r)$, with $r > 2r_0$ and $B_0 = B(0, r_0)$. Hence

$$\int_B |f(x)|\,dx = \int_{B_0} |f(x)|\,dx + \int_{B_0^c \cap B} |f(x)|\,dx.$$

From *Case* 1. the first integral satisfies our thesis. In order to estimate the second one, from (3.3)–(3.6) it is enough to prove that

$$\frac{\|\chi_B\|_{p(\cdot)}^{p(x)}}{|B|} \leq C,$$

is true for $x \in B_0^c \cap B$.

To prove this, we apply the hypothesis $p(x) \leq p_\infty$ for all $x \in B_0^c$ and Proposition 2.7 again. So we get

$$\frac{\|\chi_B\|_{p(\cdot)}^{p(x)}}{|B|} \leq |B|^{\frac{p(x)}{p_\infty}-1} \leq C.$$

Then, the proof of the lemma is complete. □

Lemma 3.7. *Under the same hypothesis of the lemma above on $p(\cdot)$, there exists a positive constant C such that*

$$\int_{\mathbb{R}^n - B} \frac{|f(y)|}{|x_0 - y|^{n-\alpha+1}}\,dy \leq C\ |B|^{\frac{\alpha}{n}-\frac{1}{n}-1} \left\|\chi_{B(x_0,r)}\right\|_{p'(\cdot)} [f]_{p(\cdot),\,\infty}\,, \tag{3.8}$$

for every ball $B = B(x_0, r)$ and all $f \in L^{p(\cdot),\,\infty}$.

Proof. Using Lemma 3.1 and (c) of Lemma 2.17, and denoting $B_k = B(x_0, 2^k r)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n - B} \frac{|f(y)|}{|x_0 - y|^{n-\alpha+1}}\,dy &= \sum_{k=0}^{\infty} \int_{B_{k+1}-B_k} \frac{|f(y)|}{|x_0 - y|^{n-\alpha+1}}\,dy \\
&\leq C\ |B|^{\frac{\alpha}{n}-\frac{1}{n}-1} \sum_{k=0}^{\infty} (2^k)^{\alpha-n-1} \int_{B_{k+1}} |f(y)|\,dy \\
&\leq C\ [f]_{p(\cdot),\,\infty}\,|B|^{\frac{\alpha}{n}-\frac{1}{n}-1} \sum_{k=0}^{\infty} (2^k)^{\alpha-n-1} \left\|\chi_{B_{k+1}}\right\|_{p'(\cdot)} \\
&\leq C\ [f]_{p(\cdot),\,\infty}\,|B|^{\frac{\alpha}{n}-\frac{1}{n}-1} \|\chi_B\|_{p'(\cdot)} \sum_{k=0}^{\infty} (2^k)^{-\varepsilon} \\
&\leq C\ [f]_{p(\cdot),\,\infty}\,|B|^{\frac{\alpha}{n}-\frac{1}{n}-1} \left\|\chi_{B(x_0,r)}\right\|_{p'(\cdot)}\,,
\end{aligned}$$

and the proof is complete. □

4. Proofs of Theorems 1.11 and 1.13

Proof of Theorem 1.11. Assuming (b) we will prove (a). In order to extend the classical operator I_α to $\widetilde{I_\alpha}$ we first note an application of Tonelli's Theorem and the generalized Hölder inequality, I_α is finite a.e. $x \in \mathbb{R}^n$ for $f \in L^{p(\cdot)}$ with compact support. In fact, let $B_0 = B(0, r_0)$ be such that $\operatorname{supp} f \subset B_0$ and $r > 0$, we have

$$\begin{aligned}
\int_{B(0,r)} |I_\alpha f(x)|\,dx &\leq \int_{B(0,r)} \int_{B_0} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy\,dx \\
&\leq \int_{B_0} |f(y)| \left(\int_{B(0,r)} \frac{dx}{|x-y|^{n-\alpha}} \right) dy \\
&\leq C\ \|f\|_{p(\cdot)} \left\|\chi_{B_0}\right\|_{p'(\cdot)} r^\alpha.
\end{aligned}$$

Thus, $|I_\alpha f(x)| < \infty$ for a.e. $x \in B(0, r)$, and, consequently, for a.e. $x \in \mathbb{R}^n$. Now, let us see that $\widetilde{I_\alpha} f(x)$ is well defined for every $f \in L^{p(\cdot)}$. In fact, for a ball $B = B(x_B, r)$ with $\widetilde{B} = 2B$, we take

$$a_B = \int_{\mathbb{R}^n} \left(\frac{1 - \chi_{\widetilde{B}}(y)}{|x_B - y|^{n-\alpha}} - \frac{1 - \chi_{B(0,1)}(y)}{|y|^{n-\alpha}} \right) f(y)\,dy.$$

Choosing $\rho = \max\{|x_B|, 1, r\}$, from the generalized Hölder inequality and the mean value Theorem, we get

$$\begin{aligned} |a_B| &\le \int_{B(0,2\rho)} \left| \frac{1-\chi_{\widetilde{B}}(y)}{|x_B-y|^{n-\alpha}} - \frac{1-\chi_{B(0,1)}(y)}{|y|^{n-\alpha}} \right| |f(y)|\, dy + \int_{\mathbb{R}^n - B(0,2\rho)} \left| \frac{1-\chi_{\widetilde{B}}(y)}{|x_B-y|^{n-\alpha}} - \frac{1-\chi_{B(0,1)}(y)}{|y|^{n-\alpha}} \right| |f(y)|\, dy \\ &\le C\ r^{\alpha-n} \int_{\substack{|y|<2\rho \\ |x_B-y|>2r}} |f(y)|\, dy + C \int_{\substack{|y|<2\rho \\ |y|>1}} |f(y)|\, dy + C\ |x_B| \int_{\mathbb{R}^n - B(0,2\rho)} \frac{|f(y)|}{|x_0-y|^{n-\alpha+1}}\, dy \\ &\le C\ \|f\|_{p(\cdot)} \left\{ (r^{\alpha-n}+1) \left\| \chi_{B(0,2\rho)} \right\|_{p'(\cdot)} + |x_B| \left\| \frac{\chi_{\mathbb{R}^n-B}}{|x_B-\cdot|^{n-\alpha+1}} \right\|_{p'(\cdot)} \right\} \\ &< \infty, \end{aligned}$$

where the conclusion follows from the hypothesis on $p(\cdot)$. On the other hand, if we are able to prove that

$$\begin{aligned} T(x) &= \int_{\mathbb{R}^n} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1-\chi_{\widetilde{B}}(y)}{|x_0-y|^{n-\alpha}} \right) f(y)\, dy \\ &= \int_{\widetilde{B}} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy + \int_{\mathbb{R}^n-\widetilde{B}} \left(\frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x_0-y|^{n-\alpha}} \right) f(y)\, dy \\ &= T_1(x) + T_2(x), \end{aligned}$$

satisfies the inequality

$$\int_B |T(x)|\, dx \le C\ |B|^{\frac{\alpha}{n}}\ \|\chi_B\|_{p'(\cdot)}\ \|f\|_{p(\cdot)},$$

then, since

$$\widetilde{I}_\alpha f(x) = a_B + T(x),$$

we get that $\widetilde{I}_\alpha f$ is finite for a.e. x in B and

$$\int_B |\widetilde{I}_\alpha f(x) - a_B|\, dx \le C\ |B|^{\frac{\alpha}{n}}\ \|\chi_B\|_{p'(\cdot)}\ \|f\|_{p(\cdot)}.$$

Then, taking into account Remark 1.9 we get (a).

Let us estimate $T_1(x)$. From Tonelli's theorem and the generalized Hölder inequality, we have

$$\int_B |T_1(x)|\, dx \le C\ |B|^{\frac{\alpha}{n}}\ \|\chi_B\|_{p'(\cdot)}\ \|f\|_{p(\cdot)}. \tag{4.1}$$

On the other hand, applying once again, the mean value theorem, the generalized Hölder inequality and the hypothesis on $p(\cdot)$, for $x \in B$, we obtain

$$\begin{aligned} |T_2(x)| &\le C\ |B|^{\frac{1}{n}} \int_{\mathbb{R}^n-\widetilde{B}} \frac{|f(y)|}{|x_B-y|^{n-\alpha+1}}\, dy \\ &\le C\ |B|^{\frac{\alpha}{n}-1} \left\| \chi_{\widetilde{B}} \right\|_{p'(\cdot)} \|f\|_{p(\cdot)}. \end{aligned}$$

Combining this estimate with (4.1) the inequality for $T(x)$ follows immediately. Now, in order to see that (a) implies (b), we will reason in a similar way to that made at [20]; that is, given a ball $B = B(x_B, r)$ and denoting $\widetilde{x}_B = x_B - \frac{r}{3\sqrt{n}}(1, \dots, 1)$ we consider the following three regions:

$$\begin{aligned} K &= \{x_B + h : h \in \mathbb{R}^n, |h| > r, h_i \ge 0, i = 1, \dots, n\}; \\ G_1 &= B\left(x_B, \frac{r}{6\sqrt{n}}\right) \cap \{x_B + h : h \in \mathbb{R}^n, h_i \le 0, i = 1, \dots, n\}; \\ G_2 &= B \cap \{\widehat{x}_B + h : h \in \mathbb{R}^n, h_i \le 0, i = 1, \dots, n\}. \end{aligned}$$

It is clear that $|G_1| \approx |B|$, $|G_2| \ge C_n |B|$ and $|x - z| \ge C_n |B|^{\frac{1}{n}}$ for every $x \in G_1$ and $z \in G_2$. Then, taking a non negative function $f \in L^{p(\cdot)}$ and denoting $f_m = f\chi_{K_m}$, where $K_m = K \cap B(0, m)$, for $m \in \mathbb{N}$, we get

$$\frac{1}{|B|^{\frac{\alpha}{n}-\frac{1}{n}-1}\ \|\chi_B\|_{p'(\cdot)}} \int_K \frac{f_m(y)}{|x_B-y|^{n-\alpha+1}}\, dy \le \frac{1}{|B|^{\frac{\alpha}{n}-1}\ \|\chi_B\|_{p'(\cdot)}} \frac{1}{|G_1|} \frac{1}{|G_2|} \int_{G_1} \int_{G_2} \left| \int_K \frac{|B|^{\frac{1}{n}}}{|x_B-y|^{n-\alpha+1}} f_m(y)\, dy \right| dz\, dx.$$

Since $|x_B - y| \approx |x_\xi - y|$, where x_ξ is a point in a segment connecting x and z, applying the mean value theorem we obtain the following estimates

$$\begin{aligned}\frac{1}{|B|^{\frac{\alpha}{n}-\frac{1}{n}-1}\|\chi_B\|_{p'(\cdot)}}\int_K \frac{f_m(y)}{|x_B-y|^{n-\alpha+1}}\,dy &\leq \frac{C}{|B|^{\frac{\alpha}{n}+1}\|\chi_B\|_{p'(\cdot)}}\int_{G_1}\int_{G_2}\left|\int_K\left[\frac{1}{|x-y|^{n-\alpha}}-\frac{1}{|z-y|^{n-\alpha}}\right]f_m(y)\,dy\right|dz\,dx\\ &\leq \frac{C}{|B|^{\frac{\alpha}{n}}\|\chi_B\|_{p'(\cdot)}}\int_B |\widetilde{I}_\alpha f_m(x) - m_B\widetilde{I}_\alpha f_m|\,dx\\ &\leq C\ \left\|\widetilde{I}_\alpha f_m\right\|_{\mathfrak{L}_{\alpha,p(\cdot)}}\\ &\leq C\ \|f_m\|_{p(\cdot)}\\ &\leq C\ \|f\|_{p(\cdot)},\end{aligned}$$

where C is independent of m and f. Letting $m \to \infty$ we have

$$\int_K \frac{f(y)}{|x_B-y|^{n-\alpha+1}}\,dy \leq C\ |B|^{\frac{\alpha}{n}-\frac{1}{n}-1}\|\chi_B\|_{p'(\cdot)}\|f\|_{p(\cdot)}.$$

Let us observe that K is the complement of B relative to the first quadrant of the Cartesian system with center at x_B. Proceeding as above with the complement of B with respect to the other quadrants we get similar estimates for each region. Adding all these inequalities we can write

$$\int_{\mathbb{R}^n - B} \frac{f(y)}{|x_B-y|^{n-\alpha+1}}\,dy \leq C\ |B|^{\frac{\alpha}{n}-\frac{1}{n}-1}\|\chi_B\|_{p'(\cdot)}\|f\|_{p(\cdot)}.$$

Finally, from (1.4), we get $p(\cdot)$ satisfies P_α. □

Proof of Theorem 1.13. As in Theorem 1.11, we observe an application of Tonelli's Theorem and Lemma 3.1, I_α is finite a.e. $x \in \mathbb{R}^n$ for $f \in L^{p(\cdot),\infty}$ with compact support. Now, considering the definition of $\widetilde{I}_\alpha$, we will proceed as in the proof of Theorem 1.11 but using this time the information given by Lemmas 3.1 and 3.7. First, given a ball $B = B(x_B, r)$ and defining a_B as there, an application of Lemma 3.1 and Corollary 2.16 allows us to get $|a_B| < \infty$ again. In a similar way, using Lemma 3.1 again, for the estimate of $T_1(x)$ and Lemma 3.7 for $T_2(x)$ we have

$$\int_B |T(x)|\,dx \leq C\ |B|^{\frac{\alpha}{n}}\left\|\chi_{B(x_B,2r)}\right\|_{p'(\cdot)}[f]_{p(\cdot),\infty}.$$

Then, taking into account that $\widetilde{I}_\alpha f(x) = a_B + T(x)$, the final conclusion follows as in the proof of Theorem 1.11. □

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