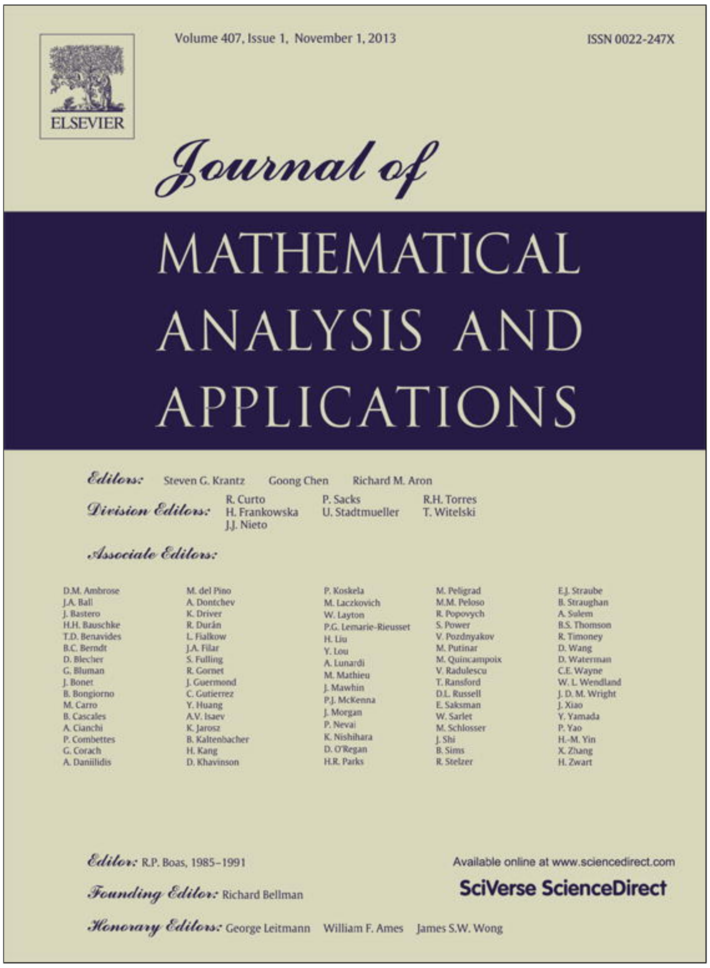
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On dyadic nonlocal Schrödinger equations with Besov initial data ☆

Hugo Aimar [a,b], Bruno Bongioanni [a,c], Ivana Gómez [a,b,*]

[a] Instituto de Matemática Aplicada del Litoral (IMAL), CONICET-UNL, Güemes 3450, S3000GLN Santa Fe, Argentina

[b] Facultad de Ingeniería Química, UNL, Santa Fe, Argentina

[c] Facultad de Ingeniería y Ciencias Hídricas, UNL, Santa Fe, Argentina

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ABSTRACT

In this paper we consider the pointwise convergence to the initial data for the Schrödinger–Dirac equation $i\frac{\partial u}{\partial t} = D^{\beta}u$ with $u(x, 0) = u^0$ in a dyadic Besov space. Here D^{β} denotes the fractional derivative of order β associated to the dyadic distance δ on $\mathbb{R}^+$. The main tools are a summability formula for the kernel of D^{β} and pointwise estimates of the corresponding maximal operator in terms of the dyadic Hardy–Littlewood function and the Calderón sharp maximal operator.

1. Introduction

In quantum mechanics time dependent Schrödinger type equations with space derivatives of order less than two, have been considered since the introduction of the Dirac operator which is actually local and of first order [6]. More recently some fractional nonlocal Riemann–Liouville calculus, and some other nonlocal cases, have also been considered in the literature, [8]. See also [11,9,2].

The differential operator in the space variable that we shall consider is an analogous of the nonlocal fractional derivative of order $\beta > 0$

$$\int \frac{f(x) - f(y)}{|x - y|^{1+\beta}} dy. \tag{1.1}$$

The basic difference is given by the fact that we substitute the Euclidean distance $|x - y|$ by the dyadic distance δ from x to y. To introduce our main result let us start by defining the basic metric δ and the Besov type spaces induced by δ on the interval $\mathbb{R}^+ = (0, \infty)$.

Let $\mathfrak{D} = \cup_{j\in\mathbb{Z}} \mathfrak{D}^j$ be the family of the standard dyadic intervals in $\mathbb{R}^+$. In other words $I \in \mathfrak{D}$ if $I = I_k^j = [(k-1)2^{-j}, k2^{-j})$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^+$. Each $\mathfrak{D}^j$ contains the intervals of the j-th level, for $I \in \mathfrak{D}^j$, $|I| = 2^{-j}$. We shall write $\mathfrak{D}^+$ to denote the intervals I in $\mathfrak{D}$ with $|I| \leq 1$. For $I \in \mathfrak{D}^j$ we shall denote by I^+ and I^- the right and left halves of I, which belong to $\mathfrak{D}^{j+1}$. Given two points x and y in $\mathbb{R}^+$ its dyadic distance $\delta(x, y)$, is defined as the length of the smallest dyadic interval $J \in \mathfrak{D}$ which contains x and y. On the diagonal Δ of $\mathbb{R}^+ \times \mathbb{R}^+$, δ vanishes.

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* Corresponding author.
E-mail addresses: haimar@santafe-conicet.gov.ar (H. Aimar), bbongio@santafe-conicet.gov.ar (B. Bongioanni), ivanagomez@santafe-conicet.gov.ar (I. Gómez).

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Since for x fixed $\delta^{-1-\beta}(x,y)$ is not integrable, the analog to (1.1) with $\delta(x,y)$ instead of $|x-y|$ in $\mathbb{R}^+$ is well defined as an absolutely convergent integral, only on a subspace of functions which have certain regularity with respect to the distance δ. For $0<\lambda<1$, with $B^{\lambda}_{2,dy}$ we denote the class of all L^2 complex valued functions f defined on $\mathbb{R}^+$ such that

$$\iint_Q \frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\lambda}}dxdy<\infty,$$

with $Q=\left\{(x,y)\in\mathbb{R}^2:\delta(x,y)<2\right\}$. For f and g both in $B^{\lambda}_{2,dy}$, the inner product

$$\int_{\mathbb{R}^+} f\overline{g}dx+\iint_Q \frac{f(x)-f(y)}{\delta(x,y)^{\lambda}}\frac{\overline{g(x)-g(y)}}{\delta(x,y)^{\lambda}}\frac{dxdy}{\delta(x,y)},$$

gives a Hilbert structure on $B^{\lambda}_{2,dy}$.

Since, as it is easy to check from the definition of δ, $|x-y|\le\delta(x,y)$ when $(x,y)\in Q$, we have that the standard Besov space B^{λ}_2 on $\mathbb{R}^+$ is a subspace of $B^{\lambda}_{2,dy}$. See [12] for the classical theory of Besov spaces.

For $I\in\mathfrak{D}$ we shall write h_I to denote the Haar wavelet adapted to I. In other words $h_I=|I|^{-\frac{1}{2}}(\mathcal{X}_{I^-}-\mathcal{X}_{I^+})$ where, as usual $\mathcal{X}_E$ is the indicator function of the set E. Sometimes, when the parameters of scale and position j and k, need to be emphasized, we shall write h^j_k to denote h_I for $I=I^j_k$. In the sequel the scale parameter j of I will be denoted by $j(I)$. As it is well known $\{h_I:I\in\mathfrak{D}\}$ is an orthonormal basis for L^2. As usual we write V_0 to denote the subspace of L^2 of those functions which are constant on each interval between integers. With P_0 we denote the projector of L^2 onto V_0.

As a consequence of Theorem 9 in Section 3, we shall obtain the next result.

Theorem 1. *Let $0<\beta<1$ and $u^0\in L^2$ with $P_0u^0=0$, be given. Assume that u^0 is a function in B^{λ}_2 with $\beta<\lambda<1$, then the function defined by*

$$u(x,t)=\sum_{I\in\mathfrak{D}} e^{it|I|^{-\beta}}\left\langle u^0,h_I\right\rangle h_I(x)$$

solves the problem

$$(P)\quad\begin{cases} i\dfrac{\partial u}{\partial t}(x,t)=\dfrac{2^{\beta}-1}{2^{\beta}}\displaystyle\int_{\mathbb{R}^+}\frac{u(x,t)-u(y,t)}{\delta(x,y)^{1+\beta}}\,dy & x\in\mathbb{R}^+,t>0\\ u(x,0)=u^0(x) & x\in\mathbb{R}^+;\end{cases}$$

where the initial condition is verified pointwise almost everywhere.

The corresponding problem in the classical case of the free particle is hard. Some fundamental steps in this direction are contained in [1,4,7,3,13,15,14].

The paper is organized as follows. In Section 2 we introduce the basic operator and the corresponding Besov space and its wavelet characterization in terms of the Haar system. In Section 3 we prove the main result, which contains a detailed formulation of Theorem 1.

2. Nonlocal dyadic differential operators and dyadic Besov spaces

Let $0<\beta<1$ be given. We shall deal with the operator D^{β} whose spectral form in the Haar system is given by $D^{\beta}h_I=|I|^{-\beta}h_I$ for $I\in\mathfrak{D}$.

Let $\mathcal{S}(\mathcal{H})$ be the linear span of the Haar system $\mathcal{H}=\{h_I:I\in\mathfrak{D}\}$. The space $\mathcal{S}(\mathcal{H})$ is dense in L^2.

The operator D^{β} is well defined from $\mathcal{S}(\mathcal{H})$ into itself and is given by

$$D^{\beta}f=\sum_{I\in\mathfrak{D}}|I|^{-\beta}\langle f,h_I\rangle h_I$$

for $f\in\mathcal{S}(\mathcal{H})$. Observe that D^{β} is unbounded in the L^2 norm.

In the next result we show that D^{β} has the structure of a nonlocal differential operator if we change the Euclidean distance by the dyadic distance on $\mathbb{R}^+$.

Theorem 2. *Let $0<\beta<1$ be given, then for $f\in\mathcal{S}(\mathcal{H})$ we have*

$$D^{\beta}f(x)=\frac{2^{\beta}-1}{2^{\beta}}\int_{\mathbb{R}^+}\frac{f(x)-f(y)}{\delta(x,y)^{1+\beta}}dy, \tag{2.1}$$

where the integral on the right hand side is absolutely convergent.

Before proving Theorem 2, we collect some basic properties of δ.

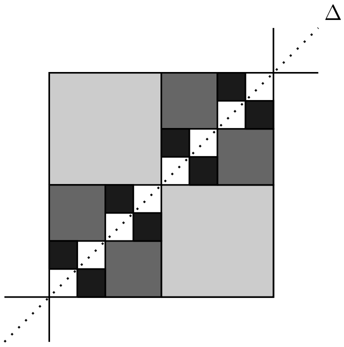

Fig. 1. The picture depicts schematically the level sets Λ_j of δ for $j = 0$ (lightgray), for $j = 1$ (darkgray) and $j = 2$ (black).

Lemma 3. (3.a) *$\mathbb{R}^+ \times \mathbb{R}^+$ is the disjoint union of the diagonal Δ and the level sets $\Lambda_j = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ : \delta(x, y) = 2^{-j}\}$ of δ for $j \in \mathbb{Z}$, and $Q = \cup_{j\geq 0} \Lambda_j$ (see Fig. 1).*

(3.b) *For $\gamma \in \mathbb{R}$, $\delta^\gamma = \sum_{j\in\mathbb{Z}} 2^{-j\gamma} \mathcal{X}_{\Lambda_j}$.*

(3.c) *Each Λ_j is the disjoint union of the sets $B(I) = (I^+ \times I^-) \cup (I^- \times I^+)$ for $I \in \mathfrak{D}^j$.*

(3.d) *For $f \in \mathcal{S}(\mathscr{H})$, set $F(x, y) = f(x) - f(y)$, then $\inf\{\delta(x, y) : (x, y) \in \text{supp}F\} > 0$.*

(3.e) *Let $\alpha > -1$. Then for every $x \in \mathbb{R}^+$, $\delta(x, y)^\alpha$ is locally integrable as a function of y. Moreover, $\int_{l-1}^{l} \delta(x, y)^\alpha dy$ is bounded by $(2^{1+\alpha} - 1)^{-1}$ for every $l \in \mathbb{Z}^+$.*

Proof of Lemma 3. Proof of (3.a). Given a point $(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+$ which does not belong to Δ, since for some $J \in \mathfrak{D}$, $(x, y) \in J \times J$ and since $x \neq y$, there exists one and only one subinterval I of J such that x and y belong both to I but not both to the same half of I. In other words $(x, y) \in B(I)$. Since $I \subset J$ then, $j(I) \geq 0$ and $\delta(x, y) = 2^{-j(I)}$, so that $(x, y) \in \Lambda_{j(I)}$. □

Proof of (3.b). Follows directly from (3.a). □

Proof of (3.c). Notice first that if I and J are two different intervals on $\mathfrak{D}^j$, then $I^+ \cap J^+ = \emptyset$ and $I^- \cap J^- = \emptyset$ hence $B(I) \cap B(J) = \emptyset$. On the other hand, if $(x, y) \in B(I)$ for some $I \in \mathfrak{D}^j$, then $x \in I^+$ and $y \in I^-$ or $x \in I^-$ and $y \in I^+$, so that the smallest dyadic interval containing both x and y is I itself. This means that $\delta(x, y) = 2^{-j}$, in other words $(x, y) \in \Lambda_j$. Assume now that (x, y) is any point in Λ_j, then $\delta(x, y) = 2^{-j}$. This means that there exists $I \in \mathfrak{D}^j$ such that $(x, y) \in I \times I$ but x and y do not belong to the same half of I. In other words $(x, y) \in I \times I$ but $(x, y) \notin (I^- \times I^-) \cup (I^+ \times I^+)$. Hence $(x, y) \in B(I)$. □

Proof of (3.d). Since any $f \in \mathcal{S}(\mathscr{H})$ is finite linear combination of some of the h_I's, all we need to prove is that $\inf\{\delta(x, y) : (x, y) \in \text{supp}H_I\} > 0$ for every $I \in \mathfrak{D}$, where $H_I(x, y) = h_I(x) - h_I(y)$. Take $I \in \mathfrak{D}$, then $I \in \mathfrak{D}^j$ for some $j \geq 0$ and H_I vanishes on $(I^- \times I^-) \cup (I^+ \times I^+)$ hence $\delta(x, y) \geq 2^{-j}$ for every $(x, y) \in \text{supp}H_I$. □

Proof of (3.e). The desired properties are trivial for $\alpha \geq 0$. Assume then that $-1 < \alpha < 0$ and $x \in \mathbb{R}^+$. Then $\int_{l-1}^{l} \delta(x, y)^\alpha dy$ vanishes when $x \notin (l - 1, l)$. If $x \in (l - 1, l)$ then

$$\begin{aligned}\int_{l-1}^{l} \delta(x, y)^\alpha dy &= \sum_{k=0}^{\infty} \int_{\{y\in(l-1,l):2^{-k-1}<\delta(x,y)<2^{-k}\}} \delta(x, y)^\alpha dy \\ &\leq \sum_{k=0}^{\infty} 2^{-\alpha(k+1)} \left|\{y \in (l - 1, l) : \delta(x, y) < 2^k\}\right| \\ &\leq \sum_{k=0}^{\infty} 2^{-(1+\alpha)(k+1)} = (2^{1+\alpha} - 1)^{-1}. \quad \square\end{aligned}$$

Proof of Theorem 2. It is enough to check (2.1) for $f = h_I$. From (3.b) and (3.c) we have

$$\begin{aligned}\int_{\mathbb{R}^+} \frac{h_I(x) - h_I(y)}{\delta(x, y)^{1+\beta}} dy &= \int_{\mathbb{R}^+} \left(\sum_{j\in\mathbb{Z}} 2^{j(1+\beta)} \mathcal{X}_{\Lambda_j}(x, y)\right) (h_I(x) - h_I(y))dy \\ &= \sum_{j\in\mathbb{Z}} 2^{j(1+\beta)} \int_{\mathbb{R}^+} \mathcal{X}_{\Lambda_j}(x, y)(h_I(x) - h_I(y))dy \\ &= \sum_{j\in\mathbb{Z}} 2^{j(1+\beta)} \sum_{J\in\mathfrak{D}^j} \int_{\mathbb{R}^+} \mathcal{X}_{B(J)}(x, y)(h_I(x) - h_I(y))dy.\end{aligned}$$

Now, since the support of $h_I(x) - h_I(y)$ and $B(J)$ are disjoint when $j(I) < j$ the last sum of j reduces to the sum for $j \leq j(I)$. On the other hand, for $j \leq j(I)$ there exists a unique $J_j \in \mathfrak{D}^j$ such that the support of $(h_I(x) - h_I(y))$ intersects $B(J)$. Actually that unique J_j is the only ancestor of I in the generation j. With these remarks in mind we have, for $x \in I^-$, that

$$\int_{\mathbb{R}^+} \frac{h_I(x) - h_I(y)}{\delta(x,y)^{1+\beta}} dy = |I|^{-\frac{1}{2}} \sum_{j \leq j(I)} 2^{j(1+\beta)} \int_{J_j} [1 - \mathcal{X}_{I^-}(y) + \mathcal{X}_{I^+}(y)] dy$$
$$= |I|^{-\frac{1}{2}} \sum_{j \leq j(I)} 2^{j(1+\beta)} \left|J_j\right| = \frac{2^\beta}{2^\beta - 1} |I|^{-\beta} |I|^{-\frac{1}{2}}.$$

In a similar way, with $x \in I^+$, we get that $\int_{\mathbb{R}^+} \frac{h_I(x)-h_I(y)}{\delta(x,y)^{1+\beta}} dy = -\frac{2^\beta}{2^\beta-1} |I|^{-\beta} |I|^{-\frac{1}{2}}$. In other words, $\int_{\mathbb{R}^+} \frac{h_I(x)-h_I(y)}{\delta(x,y)^{1+\beta}} dy = \frac{2^\beta}{2^\beta-1} |I|^{-\beta} h_I(x)$, as desired. □

A basic identity to obtain a characterization of the Besov type spaces in terms of the Haar system is contained in Theorem 4.

Theorem 4. *Let $0 < \lambda < 1$, be given, then the identity*

$$\iint_Q \frac{|f(x) - f(y)|^2}{\delta(x,y)^{1+2\lambda}} dxdy = \sum_{I \in \mathfrak{D}^+} |\langle f, h_I \rangle|^2 \left[(2 + c_\lambda) |I|^{-2\lambda} - c_\lambda\right] \tag{2.2}$$

holds for every function $f \in \mathcal{S}^+(\mathcal{H})$ and $c_\lambda = 2(2^{2\lambda} - 1)^{-1}$, where $\mathcal{S}^+(\mathcal{H})$ is the linear span of $\left\{h_I : I \in \mathfrak{D}^+\right\}$.

Theorem 4 will be a consequence of some elementary geometric properties of the dyadic system and the distance δ.

Lemma 5. (5.a) *Set $C(J) = [(J \times \mathbb{R}^+) \cup (\mathbb{R}^+ \times J)] \setminus (J \times J)$ for $J \in \mathfrak{D}^+$, then $B(I) \cap C(J) = \emptyset$ for $j(I) \geq j(J)$.*
(5.b) *For every $I \in \mathfrak{D}^+$ and every $j = 0, 1, \ldots, j(I) - 1$ there exists one and only one $J \in \mathfrak{D}^j$ for which $B(J)$ intersects $C(I)$.*
(5.c) *For each $I \in \mathfrak{D}^+$ we have*

$$\sum_{j \geq 0} 2^{j(1+2\lambda)} \sum_{J \in \mathfrak{D}^j} m(B(J) \cap C(I)) = c_\lambda |I| \left(|I|^{-2\lambda} - 1\right),$$

where m is the area measure in $\mathbb{R}^2$, $\lambda > 0$ and $c_\lambda = 2(2^{2\lambda} - 1)^{-1}$.
(5.d) *For $j \geq 0$, $I \in \mathfrak{D}^+$ and $J \in \mathfrak{D}^+$, with $I \neq J$,*

$$\mathcal{I}(j, I, J) := \iint_Q \mathcal{X}_{\Lambda_j}(x,y)[h_I(x) - h_I(y)][h_J(x) - h_J(y)] dxdy = 0.$$

(5.e) *For each $I \in \mathfrak{D}^+$,*

$$\sum_{j \geq 0} 2^{j(1+2\lambda)} \mathcal{I}(j, I, I) = (2 + c_\lambda) |I|^{-2\lambda} - c_\lambda.$$

Let us start by proving Theorem 4 assuming the results in Lemma 5.

Proof of Theorem 4. Let f be a finite linear combination of some of the Haar functions h_I for $I \in \mathfrak{D}^+$, i.e. $f = \sum_{I \in \mathfrak{D}^+} \langle f, h_I \rangle h_I$ with $\langle f, h_I \rangle = 0$ except for a finite number of I in $\mathfrak{D}^+$. From (3.a) and (3.b) in Lemma 3, (5.c), (5.d) and (5.e) in Lemma 5 we get

$$\iint_Q \frac{|f(x) - f(y)|^2}{\delta(x,y)^{1+2\lambda}} dxdy = \iint_Q \left(\sum_{j \geq 0} 2^{j(1+2\lambda)} \mathcal{X}_{\Lambda_j}(x,y)\right)$$
$$\times \left(\sum_{I \in \mathfrak{D}^+} \sum_{J \in \mathfrak{D}^+} \langle f, h_I \rangle \langle f, h_J \rangle [h_I(x) - h_I(y)][h_J(x) - h_J(y)]\right) dxdy$$
$$= \sum_{j \geq 0} 2^{j(1+2\lambda)} \sum_{I \in \mathfrak{D}^+} \sum_{J \in \mathfrak{D}^+} \langle f, h_I \rangle \langle f, h_J \rangle \iint_Q \mathcal{X}_{\Lambda_j}(x,y)[h_I(x) - h_I(y)][h_J(x) - h_J(y)] dxdy$$
$$= \sum_{j \geq 0} 2^{j(1+2\lambda)} \sum_{I \in \mathfrak{D}^+} |\langle f, h_I \rangle|^2 \iint_{\Lambda_j} [h_I(x) - h_I(y)]^2 dxdy$$
$$= \sum_{I \in \mathfrak{D}^+} |\langle f, h_I \rangle|^2 \sum_{j \geq 0} 2^{j(1+2\lambda)} \iint_{\Lambda_j} [h_I(x) - h_I(y)]^2 dxdy$$

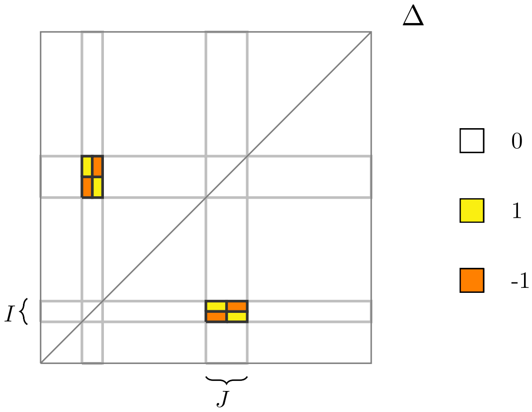

Fig. 2. Values of $\frac{k_{IJ}}{\sqrt{|I||J|}}$.

$$= \sum_{I\in\mathfrak{D}^+} |\langle f, h_I\rangle|^2 \sum_{j\geq 0} 2^{j(1+2\lambda)} \mathcal{I}(j, I, I)$$
$$= \sum_{I\in\mathfrak{D}^+} |\langle f, h_I\rangle|^2 \left[(2 + c_\lambda)\, |I|^{-2\lambda} - c_\lambda\right]. \quad \square$$

Proof of Lemma 5. Proof of (5.a). Since $j(I) \geq j(J)$ we have $I \subseteq J$ or $I \cap J = \emptyset$, we divide our analysis in these two cases. When $I \cap J = \emptyset$, then $I^+ \cap J = \emptyset$ and $I^- \cap J = \emptyset$ and $B(I) \cap C(J) = \emptyset$. Assume now that $I \subseteq J$, then $B(I) \subset J \times J$ which is disjoint from $C(J)$. $\square$

Proof of (5.b). Let $I \in \mathfrak{D}^+$ and $j = 0, 1, \ldots, j(I) - 1$ be given. Let J be the only dyadic interval in $\mathfrak{D}^j$ such that $J \supsetneqq I$. Then $C(I) \cap B(J) \neq \emptyset$. In fact, since $J \supsetneqq I$, then $I \subset J^+$ or $I \subset J^-$. Assume for example that $I \subset J^+$, then any point (x, y) with $x \in I$ and $y \in J^-$ belongs to both $C(I)$ and $B(J)$. So that, since for $J \in \mathfrak{D}^j$ and $j < j(I)$, arguing as in the proof of (5.a), the condition $J \supset I$ is necessary for $B(J) \cap C(I) \neq \emptyset$, we get the result. $\square$

Proof of (5.c). Let $I \in \mathfrak{D}^+$ be given. For $j = 0, 1, \ldots, j(I) - 1$ set $J(j, I)$ to denote the only $J \in \mathfrak{D}^j$ for which $B(J) \cap C(I) \neq \emptyset$, provided by (5.b). Now from (5.a) we have

$$\sum_{j\geq 0} 2^{j(1+2\lambda)} \sum_{J\in\mathfrak{D}^j} m(B(J) \cap C(I)) = \sum_{j=0}^{j(I)-1} 2^{j(1+2\lambda)} \sum_{J\in\mathfrak{D}^j} m(B(J) \cap C(I))$$
$$= \sum_{j=0}^{j(I)-1} 2^{j(1+2\lambda)} m(B(J(j, I)) \cap C(I)).$$

But, as it is easy to see, $m(B(J(j, I)) \cap C(I)) = 2\, |I|\, 2^{-j}$. Hence

$$\sum_{j\geq 0} 2^{j(1+2\lambda)} \sum_{J\in\mathfrak{D}^j} m(B(J) \cap C(I)) = 2\, |I| \sum_{j=0}^{j(I)-1} 2^{j(1+2\lambda)} 2^{-j} = c_\lambda\, |I| \left(|I|^{-2\lambda} - 1\right). \quad \square$$

Proof of (5.d). From (3.c) it is enough to show that $\iint_{B(K)} k_{IJ}(x, y)dxdy = 0$ for every I, J and $K \in \mathfrak{D}^+$ with $I \neq J$, where $k_{IJ}(x, y) = (h_I(x) - h_I(y))(h_J(x) - h_J(y))$. We shall divide our analysis into two cases according to the relative positions of I and J.

Assume first that $I \cap J = \emptyset$; more precisely, assume that I is to the left of J. Then $k_{IJ}(x, y) = \sqrt{|I|\,|J|}\big[(\mathcal{X}_{I^-\times J^+}(x, y) - \mathcal{X}_{I^-\times J^-}(x, y) + \mathcal{X}_{I^+\times J^-}(x, y) - \mathcal{X}_{I^+\times J^+}(x, y)) + (\mathcal{X}_{J^-\times I^+}(x, y) - \mathcal{X}_{J^-\times I^-}(x, y) + \mathcal{X}_{J^+\times I^-}(x, y) - \mathcal{X}_{J^+\times I^+}(x, y))\big]$ whose support is $(I \times J) \cup (J \times I)$. See Fig. 2.

Notice that while $I \times J$ lies above the diagonal, $J \times I$ is contained in $\{y < x\}$. When $B(K)$ does not intersect $(I \times J) \cup (J \times I)$ then $\iint_{B(K)} k_{IJ}(x, y)dxdy = 0$. Assume now that $B(K) \cap [(I \times J) \cup (J \times I)] \neq \emptyset$. Since $\iint_Q k_{IJ}(x, y)dxdy = 0$, if we show that $B(K) \cap [(I \times J) \cup (J \times I)] \neq \emptyset$ implies $(I \times J) \cup (J \times I) \subseteq B(K)$ we have $\iint_{B(K)} k_{IJ}dxdy = 0$. Since the set $B(K) \cap [(I \times J) \cup (J \times I)] = [(K^- \times K^+) \cup (K^+ \times K^-)] \cap [(I \times J) \cup (J \times I)]$ is nonempty, we see that $(K^- \times K^+) \cap (I \times J) \neq \emptyset$. Since $K^- \cap I \neq \emptyset$ and $K^+ \cap J \neq \emptyset$ and K, I and J are dyadic intervals with $I \cap J = \emptyset$, we must have that $K^- \supset I$ and $K^+ \supset J$. Therefore $B(K) \supset [(I \times J) \cup (J \times I)]$.

Let us assume now that I and J are nested. For example that $I \subsetneqq J$. Fig. 3 depicts in this situation the normalized kernel $\frac{k_{IJ}}{\sqrt{|I||J|}}$.

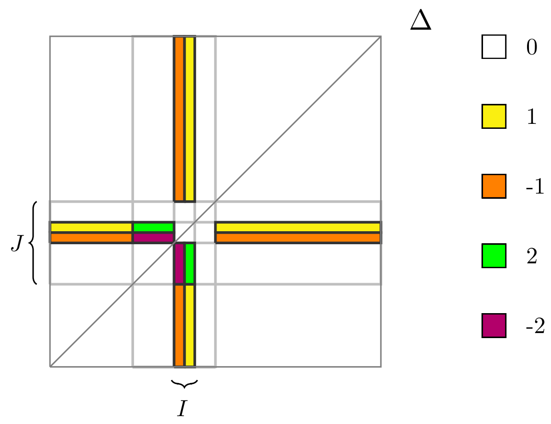

Fig. 3. Values of $\frac{k_{IJ}}{\sqrt{|I||J|}}$.

Since $k_{IJ}(x,y) = k_{IJ}(y,x)$ and $B(K)$ is symmetric, we only need to show that $\iint_{K^+\times K^-} k_{IJ}(x,y)dxdy = 0$. When $j(K) \geq j(J)+1$, $\text{supp}k_{IJ} \cap B(K) = \emptyset$ and $\iint_{B(K)} k_{IJ}(x,y)dxdy = 0$. Assume on the other hand that $0 \leq j(K) \leq j(J)$. In this case the intersection of the support of k_{IJ} and $B(K)$ can still be empty or, if not, the kernel $k_{IJ}(x,y)$ on $K^+\times K^-$ takes only two opposite constant non trivial values on subsets of the same area. Hence, again, $\iint_{B(K)} k_{IJ}(x,y)dxdy = 0$. See Fig. 4 where two possible positions of K when $I \subsetneq J$ are illustrated. □

Proof of (5.e). Let us start by computing $\mathcal{I}(j,I,I)$ for $j \geq 0$ and $I \in \mathfrak{D}^+$. From (3.c) we get

$$\begin{aligned}\mathcal{I}(j,I,I) &= \iint_{\Lambda_j} [h_I(x)-h_I(y)]^2 dxdy\\ &= |I|^{-1}\sum_{J\in\mathfrak{D}^j}\iint_{B(J)}[4\mathcal{X}_{B(I)}+\mathcal{X}_{C(I)}]dxdy\\ &= 4\,|I|^{-1}\sum_{J\in\mathfrak{D}^j} m(B(J)\cap B(I)) + |I|^{-1}\sum_{J\in\mathfrak{D}^j} m(B(J)\cap C(I))\end{aligned}$$

for $j\geq 0$ and $I\in\mathfrak{D}^+$. Hence, since from (3.a) and (3.c), $B(J)\cap B(I)=\emptyset$ for $I\neq J$ and then applying (5.c)

$$\begin{aligned}\sum_{j\geq 0} 2^{j(1+2\lambda)}\mathcal{I}(j,I,I) &= \sum_{j\geq 0} 2^{j(1+2\lambda)}\left\{4\,|I|^{-1}\sum_{J\in\mathfrak{D}^j} m(B(J)\cap B(I)) + |I|^{-1}\sum_{J\in\mathfrak{D}^j} m(B(J)\cap C(I))\right\}\\ &= 4\,|I|^{-1}\,2^{j(I)(1+2\lambda)}\frac{|I|^2}{2} + c_\lambda\left(|I|^{-2\lambda}-1\right)\\ &= (2+c_\lambda)\,|I|^{-2\lambda} - c_\lambda. \quad \Box\end{aligned}$$

For $0<\lambda<1$, a function $f\in L^2$ is said to belong to the *Besov space* $B^\lambda_{2,dy}$ if the function $\frac{f(x)-f(y)}{\delta(x,y)^\lambda}$ belongs to $L^2(Q, \frac{dxdy}{\delta(x,y)})$. In other words, $f\in B^\lambda_{2,dy}$ if and only if

$$\|f\|^2_{B^\lambda_{2,dy}} = \|f\|^2_{L^2} + \iint_Q \frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\lambda}}dxdy$$

is finite.

For our purposes the main result concerning $B^\lambda_{2,dy}$ is the following Haar wavelet characterization of the Besov space. For the classical nondyadic Euclidean case see for example [10].

Theorem 6. *Let $0<\lambda<1$ be given. The space $B^\lambda_{2,dy}$ coincides with the set of all square integrable functions on $\mathbb{R}^+$ for which*

$$\sum_{I\in\mathfrak{D}^+}\left|\frac{\langle f,h_I\rangle}{|I|^\lambda}\right|^2<\infty.$$

Moreover, $\|f\|_{L^2} + \left(\sum_{I\in\mathfrak{D}^+}\left|\frac{\langle f,h_I\rangle}{|I|^\lambda}\right|^2\right)^{\frac{1}{2}}$ *is equivalent to* $\|f\|_{B^\lambda_{2,dy}}$.

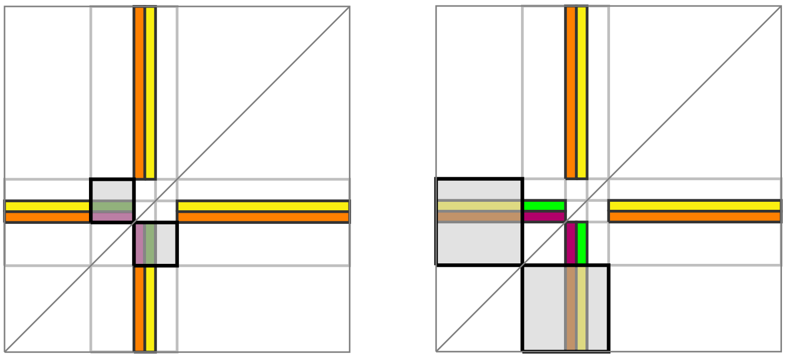

Fig. 4. On the left, K equals J, and on the right, K is the father of J.

Proof. We start by noticing that, from the definition of Q as a union of the squares $(k-1,k)^2$, $k\in\mathbb{Z}^+$, there is no interference between blocks corresponding to different values of k and then it is enough to prove that $\|f\|_{L^2(0,1)}+\left(\sum_{I\in\mathfrak{D}^+_{(0,1)}}\left(\frac{|\langle f,h_I\rangle|}{|I|^\lambda}\right)^2\right)^{\frac12}$ is equivalent to $\|f\|_{L^2(0,1)}+\left(\iint_{(0,1)^2}\frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\lambda}}\right)^{\frac12}$, with $\mathfrak{D}^+_{(0,1)}=\left\{I\in\mathfrak{D}^+: I\subset(0,1)\right\}$.

Assume then that f is an $L^2(0,1)$ function such that $\sum_{I\in\mathfrak{D}^+_{(0,1)}}\frac{|\langle f,h_I\rangle|^2}{|I|^{2\lambda}}<\infty$. Let $\mathcal{F}_n$ be an increasing sequence of finite subfamilies of $\mathfrak{D}^+_{(0,1)}$ with $\cup_{n=1}^\infty\mathcal{F}_n=\mathfrak{D}^+_{(0,1)}$ and if $f_n=\sum_{I\in\mathcal{F}_n}\langle f,h_I\rangle h_I$ we have both the $L^2(0,1)$ and a.e. pointwise convergence of f_n to f. Then from Fatou's Lemma we have that

$$\begin{aligned}\iint_{(0,1)^2}\frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\lambda}}dxdy&=\iint_{(0,1)^2}\lim_{n\to\infty}\frac{|f_n(x)-f_n(y)|^2}{\delta(x,y)^{1+2\lambda}}dxdy\\&\le\liminf_{n\to\infty}\iint_{(0,1)^2}\frac{|f_n(x)-f_n(y)|^2}{\delta(x,y)^{1+2\lambda}}dxdy.\end{aligned}$$

Now, since each $f_n\in\mathcal{S}(\mathcal{H})$, from Theorem 4 we get

$$\begin{aligned}\iint_{(0,1)^2}\frac{|f_n(x)-f_n(y)|^2}{\delta(x,y)^{1+2\lambda}}dxdy&=\sum_{I\in\mathcal{F}_n}|\langle f,h_I\rangle|^2\left[(2+c_\lambda)\,|I|^{-2\lambda}-c_\lambda\right]\\&\le 2\sum_{I\in\mathfrak{D}^+_{(0,1)}}\frac{|\langle f,h_I\rangle|^2}{|I|^{2\lambda}},\end{aligned}$$

hence

$$\iint_{(0,1)^2}\frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\lambda}}dxdy\le 2\sum_{I\in\mathfrak{D}^+_{(0,1)}}\frac{|\langle f,h_I\rangle|^2}{|I|^{2\lambda}}.$$

In order to prove the opposite inequality let us start by noticing that the identity (2.2) in Theorem 4 provides, by polarization, the following formula which holds for every φ and $\psi\in\mathcal{S}(\mathcal{H})$

$$\iint_{(0,1)^2}\frac{\varphi(x)-\varphi(y)}{\delta(x,y)^\lambda}\frac{\psi(x)-\psi(y)}{\delta(x,y)^\lambda}\frac{dxdy}{\delta(x,y)}=\sum_{I\in\mathfrak{D}^+_{(0,1)}}\langle\varphi,h_I\rangle\langle\psi,h_I\rangle\left[(2+c_\lambda)\,|I|^{-2\lambda}-c_\lambda\right].\tag{2.3}$$

Assume that $f\in B^\lambda_{2,dy}$. Since for any $\psi\in\mathcal{S}(\mathcal{H})$ by (3.d), the function $\frac{\psi(x)-\psi(y)}{\delta(x,y)^{1+2\lambda}}$ has support at a positive δ-distance of the diagonal Δ, we have that it is bounded on $(0,1)^2$. Hence $\frac{\psi(x)-\psi(y)}{\delta(x,y)^{1+2\lambda}}\in L^2((0,1)^2,dxdy)$. Taking in (2.3) $f_n=\sum_{I\in\mathcal{F}_n}\langle f,h_I\rangle h_I$ instead of φ with $\mathcal{F}_n$ as before, we get

$$\iint_{(0,1)^2}\frac{f_n(x)-f_n(y)}{\delta(x,y)^\lambda}\frac{\psi(x)-\psi(y)}{\delta(x,y)^\lambda}\frac{dxdy}{\delta(x,y)}=\sum_{\substack{I\in\mathcal{F}_n,\\\langle\psi,h_I\rangle\neq0}}\langle f,h_I\rangle\langle\psi,h_I\rangle\left[(2+c_\lambda)\,|I|^{-2\lambda}-c_\lambda\right].$$

Now since $f_n(x)-f_n(y)$ tends $f(x)-f(y)$ in $L^2((0,1)^2,dxdy)$ and $\psi\in\mathcal{S}(\mathcal{H})$ we get

$$\iint_{(0,1)^2}\frac{f(x)-f(y)}{\delta(x,y)^\lambda}\frac{\psi(x)-\psi(y)}{\delta(x,y)^\lambda}\frac{dxdy}{\delta(x,y)}=\sum_{I\in\mathfrak{D}^+_{(0,1)}}\langle f,h_I\rangle\langle\psi,h_I\rangle\left[(2+c_\lambda)\,|I|^{-2\lambda}-c_\lambda\right].$$

We have to prove that $\sum \frac{|\langle f,h_I\rangle|^2}{|I|^{2\lambda}}$ is finite. This quantity can be estimated by duality, since

$$\left(\sum_{I\in\mathfrak{D}^+_{(0,1)}} \frac{|\langle f,h_I\rangle|^2}{|I|^{2\lambda}}\right)^{\frac{1}{2}} = \sup \sum_{I\in\mathfrak{D}^+_{(0,1)}} \frac{\langle f,h_I\rangle}{|I|^{\lambda}} b_I$$

where the supremum is taken on the family of all sequences (b_I) with $\sum_{I\in\mathfrak{D}^+_{(0,1)}} b_I^2 \le 1$ and $b_I = 0$ except for a finite number of I's in $\mathfrak{D}^+_{(0,1)}$. Notice that every such sequence (b_I) can be uniquely determined by the sequence of Haar coefficients of the function $\psi = \sum b_I\, |I|^{-\lambda}\, [(2+c_\lambda)\, |I|^{-2\lambda} - c_\lambda]^{-1} h_I \in \mathcal{S}(\mathscr{H})$. In fact, $b_I = \langle \psi, h_I\rangle\, |I|^{\lambda}\, [(2+c_\lambda)\, |I|^{-2\lambda} - c_\lambda]$. Hence the condition $\sum_{I\in\mathfrak{D}^+_{(0,1)}} b_I^2 \le 1$ becomes $\sum_{I\in\mathfrak{D}^+_{(0,1)}} \langle \psi, h_I\rangle^2\, |I|^{2\lambda} \left[(2+c_\lambda)\, |I|^{-2\lambda} - c_\lambda\right]^2 \le 1$. So

$$\begin{aligned}
\sum_{I\in\mathfrak{D}^+_{(0,1)}} \frac{\langle f,h_I\rangle}{|I|^{\lambda}} b_I &= \sum_{I\in\mathfrak{D}^+_{(0,1)}} \langle f,h_I\rangle\, \langle \psi,h_I\rangle \left[(2+c_\lambda)\, |I|^{-2\lambda} - c_\lambda\right] \\
&= \iint_{(0,1)^2} \frac{f(x)-f(y)}{\delta(x,y)^{\lambda}} \frac{\psi(x)-\psi(y)}{\delta(x,y)^{\lambda}} \frac{dxdy}{\delta(x,y)} \\
&\le \left[\iint_{(0,1)^2} \left(\frac{|f(x)-f(y)|}{\delta(x,y)^{\lambda}}\right)^2 \frac{dxdy}{\delta(x,y)}\right]^{\frac{1}{2}} \iint_{(0,1)^2} \left(\frac{|\psi(x)-\psi(y)|}{\delta(x,y)^{\lambda}}\right)^2 \frac{dxdy}{\delta(x,y)} \\
&\le \|f\|_{B^{\lambda}_{2,dy}},
\end{aligned}$$

since $\iint_{(0,1)^2} \left(\frac{|\psi(x)-\psi(y)|}{\delta(x,y)^{\lambda}}\right)^2 \frac{dxdy}{\delta(x,y)} = \sum_{I\in\mathfrak{D}^+_{(0,1)}} \langle \psi,h_I\rangle^2\, |I|^{2\lambda}\, [(2+c_\lambda)\, |I|^{-2\lambda} - c_\lambda]^2 \le 1$ for $\psi \in \mathcal{S}(\mathscr{H})$. □

As a corollary of Theorem 6 we easily obtain the following density result.

Corollary 7. *For $f \in B^{\lambda}_{2,dy}$ with $P_0 f = 0$ and $f_n = \sum_{I\in\mathcal{F}_n} \langle f,h_I\rangle\, h_I$ with $\mathcal{F}_n \subset \mathcal{F}_{n+1}$, $\mathcal{F}_n$ finite and $\cup_{n=1}^{\infty} \mathcal{F}_n = \mathfrak{D}^+$, we have $f_n \to f$ in $B^{\lambda}_{2,dy}$ as $n \to \infty$.*

The above result allows to extend Theorem 2 to dyadic Besov functions with vanishing means between integers.

Theorem 8. *Let $0 < \beta < \lambda < 1$ be given. Then for each $f \in B^{\lambda}_{2,dy}$ with $P_0 f = 0$, we have*

$$\sum_{I\in\mathfrak{D}^+} |I|^{-\beta}\, \langle f,h_I\rangle\, h_I(x) = \frac{2^{\beta}-1}{2^{\beta}} \int_{\mathbb{R}^+} \frac{f(x)-f(y)}{\delta(x,y)^{1+\beta}} dy \tag{2.4}$$

as functions in L^2.

Proof. For f_n as in Corollary 7, Theorem 2 provides the identity (2.4). Hence to prove (2.4) in our new situation, it suffices to prove that both sides in (2.4) define bounded operators with respect to the norms $B^{\lambda}_{2,dy}$ in the domain and L^2 in its image. For the left hand side, we see that

$$\left\|\sum_{I\in\mathfrak{D}^+} |I|^{-\beta}\, \langle f,h_I\rangle\, h_I\right\|_2^2 = \sum_{I\in\mathfrak{D}^+} |I|^{2(\lambda-\beta)} \left(\frac{|\langle f,h_I\rangle|}{|I|^{\lambda}}\right)^2$$

which is bounded by the $B^{\lambda}_{2,dy}$ norm of f from Theorem 6. For the operator on the right hand side of (2.4) we start by splitting the integral in the following way

$$\int_{\mathbb{R}^+} \frac{f(x)-f(y)}{\delta(x,y)^{1+\beta}} dy = \int_{\delta(x,y)<2} \frac{f(x)-f(y)}{\delta(x,y)^{1+\beta}} dy + \int_{\delta(x,y)\ge 2} \frac{f(x)-f(y)}{\delta(x,y)^{1+\beta}} dy.$$

Applying (3.e) in Lemma 3 we obtain that the L^2 norm of the first term in the right is bounded by

$$\int_{\mathbb{R}^+} \left(\int_{[x]}^{[x]+1} \frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\lambda}} dy\right) \left(\int_{[x]}^{[x]+1} \frac{dy}{\delta(x,y)^{1-2(\lambda-\beta)}}\right) dx \le C \iint_{Q} \frac{|f(x)-f(y)|^2}{\delta(x,y)^{1+2\lambda}} dydx,$$

as desired. On the other hand the square of the L^2 norm of the second term is bounded by

$$\int_{\mathbb{R}^+}\left(\int_{\delta(x,y)\geq 2}\frac{|f(x)|+|f(y)|}{\delta(x,y)^{\frac{1+\beta}{2}}}\frac{dy}{\delta(x,y)^{\frac{1+\beta}{2}}}\right)^2 dx \leq \int_{\mathbb{R}^+}\left(\int_{\delta(x,y)\geq 2}\frac{(|f(x)|+|f(y)|)^2}{\delta(x,y)^{1+\beta}}dy\right)\left(\int_{\delta(x,y)\geq 2}\frac{dy}{\delta(x,y)^{1+\beta}}\right)dx$$
$$\leq C\int_{\mathbb{R}^+}\int_{\delta(x,y)\geq 2}\frac{|f(x)|^2}{\delta(x,y)^{1+\beta}}dydx \leq \overline{C}\,\|f\|^2_{L^2} \leq \overline{C}\,\|f\|^2_{B^\lambda_{2,dy}}. \quad \square$$

3. The main result

In this section we state and prove a detailed formulation of Theorem 1. With the operator D^β and the spaces $B^\lambda_{2,dy}$ introduced in Section 2 the problem can now be formally written in the following way

$$(P)\quad \begin{cases} i\dfrac{\partial u}{\partial t} = D^\beta u & \text{in } \mathbb{R}^+\times\mathbb{R}^+ \\ u(0)=u^0 & \text{in } \mathbb{R}^+. \end{cases}$$

Theorem 9. *For $0<\beta<\lambda<1$ and $u^0\in B^\lambda_2$ with $P_0u^0=0$, define*

$$u(t) = -\sum_{I\in\mathfrak{D}} e^{it|I|^{-\beta}}\left\langle u^0, h_I\right\rangle h_I, \quad t\geq 0. \tag{3.1}$$

Then,

(9.a) *u is continuous as a function of $t\in[0,\infty)$ with values in $B^\lambda_{2,dy}$ and $u(0)=u^0$. In other words, $\|u(t)-u(s)\|_{B^\lambda_{2,dy}}\to 0$ for $s\to t$ and $t\geq 0$;*

(9.b) *u is differentiable as a function of $t\in(0,\infty)$ with respect to the norm $\|\cdot\|_{B^{\lambda-\beta}_{2,dy}}$, and $\frac{du}{dt}=-iD^\beta u$; precisely, $\left\|\frac{u(t+h)-u(t)}{h}+iD^\beta u\right\|_{B^{\lambda-\beta}_{2,dy}}\to 0$ when $h\to 0$;*

(9.c) *there exists $Z\subset\mathbb{R}^+$ with $|Z|=0$ such that the series (3.1) defining $u(t)$ converges pointwise for every $t\in[0,1)$ outside Z;*

(9.d) *$u(t)\to u^0$ pointwise almost everywhere on $\mathbb{R}^+$ when $t\to 0$.*

Notice that pointwise convergence is not a consequence of convergence in the $B^\lambda_{2,dy}$ norm. In fact, with the standard notation for the Haar system $h^j_k(x)=2^{\frac{j}{2}}h(2^jx-k)$, we define a sequence of functions supported in $(0,1)$ in the following way. Let n be a given positive integer. Then there exists one and only one $j=0,1,2,\ldots$ such that $2^j\leq n<2^{j+1}$. Set $f_n=2^{-\frac{j}{2}}h^j_{n-2^j}$. Then $\|f_n\|_{L^2}=2^{-\frac{j}{2}}$ which tends to 0 as $n\to\infty$. Since $D^\lambda f_n = 2^{-\frac{j}{2}}D^\lambda h^j_{n-2^j} = 2^{j\lambda}2^{-\frac{j}{2}}h^j_{n-2^j} = 2^{-j(\frac{1}{2}-\lambda)}h^j_{n-2^j}$, we see that for $0<\lambda<\frac{1}{2}$, $\left\|D^\lambda f_n\right\|_{L^2}\to 0$ as $n\to\infty$. Hence $f_n\to 0$ in the $B^\lambda_{2,dy}$ sense. Nevertheless f_n does not converge pointwise.

Before proving Theorem 9 we shall obtain some basic maximal estimates involved in the proofs of (9.c) and (9.d). With M_{dy} we denote the Hardy–Littlewood dyadic maximal operator given by

$$M_{dy}f(x) = \sup \frac{1}{|I|}\int_I |f(y)|\,dy$$

where the supremum is taken on the family of all dyadic intervals $I\in\mathfrak{D}$ for which $x\in I$. Calderón's sharp maximal operator of order λ is defined by

$$M^\#_\lambda f(x) = \sup_J \frac{1}{|J|^{1+\lambda}}\int_J |f(y)-f(x)|\,dy,$$

where the supremum is taken on the family of all subintervals (dyadic or not) J of $\mathbb{R}^+$ such that $x\in J$. In [5], see Corollary 11.6, DeVore and Sharpley prove that the L_p norm of $M^\#_\lambda f$ is bounded by the B^λ_p norm of f. For our purposes the case $p=2$ is of particular interest,

$$\left\|M^\#_\lambda f\right\|_{L^2} \leq A\,\|f\|_{B^\lambda_2}. \tag{3.2}$$

When dealing with (9.c) and (9.d) two maximal operators related to the series (3.1) are also relevant. For $t>0$ set

$$S^*_t f(x) = \sup_{N\in\mathbb{N}}\left|S^N_t f(x)\right|, \quad \text{where } S^N_t f(x) = \sum_{j=0}^N\sum_{k\in\mathbb{Z}^+} e^{it2^{j\beta}}\left\langle f, h^j_k\right\rangle h^j_k(x).$$

Set

$$S^*f(x) = \sup_{0<t<1} S_t^* f(x).$$

The next result contains the basic estimates of S_t^* and S^* in terms of M_{dy} and $M_\lambda^\#$.

Lemma 10. *Let $f \in B_2^\lambda$ with $0 < \beta < \lambda < 1$ and $P_0 f = 0$. Then with $C := 2^{\lambda-\beta+1}(2^{\lambda-\beta} - 1)$ we have*

(10.a) $S_t^* f(x) \le CtM_\lambda^\# f(x) + 2M_{dy} f(x)$ *for* $t \ge 0$ *and* $x \in \mathbb{R}^+$;
(10.b) $S^* f(x) \le CM_\lambda^\# f(x) + 2M_{dy} f(x)$ *for* $x \in \mathbb{R}^+$;
(10.c) $\|S^* f\|_{L^2} \le (AC + 2)\, \|f\|_{B_2^\lambda}$, *where A is the constant in* (3.2).

Proof. For $f \in B_2^\lambda$, $t \ge 0$ and $N \in \mathbb{N}$, we have

$$\left|S_t^N f(x)\right| \le \left|S_t^N f(x) - S_0^N f(x)\right| + \left|S_0^N f(x)\right|. \tag{3.3}$$

Since $S_0^N f(x) = P_N f(x)$, where P_N is the projection over the space V_N of functions which are constant on each $I \in \mathfrak{D}^N$, we have $\sup_N \left|S_0^N f(x)\right| \le M_{dy} f(x)$. Let us now estimate the first term on the right hand side of (3.3). For $x \in \mathbb{R}^+$ and $j \in \mathbb{N}$, let $k(x,j) \in \mathbb{Z}^+$, be the only index for which $x \in I^j_{k(x,j)}$,

$$\begin{aligned}
\left|S_t^N f(x) - S_0^N f(x)\right| &\le \left|\sum_{j=0}^N \sum_{k\in\mathbb{Z}^+} (e^{it2^{j\beta}} - 1)\left\langle f, h_k^j\right\rangle h_k^j(x)\right| \\
&= \left|\sum_{j=0}^N (e^{it2^{j\beta}} - 1)\left(\int_{I^j_{k(j,x)}} [f(y) - f(x)] h^j_{k(j,x)}(y)dy\right) h^j_{k(j,x)}(x)\right| \\
&\le \sum_{j=0}^\infty \left|e^{it2^{j\beta}} - 1\right| \frac{1}{\left|I^j_{k(j,x)}\right|} \int_{I^j_{k(j,x)}} |f(y) - f(x)|\, dy \\
&= t\sum_{j=0}^\infty \frac{\left|e^{it2^{j\beta}} - 1\right|}{t2^{j\lambda}} \frac{1}{\left|I^j_{k(j,x)}\right|^{1+\lambda}} \int_{I^j_{k(j,x)}} |f(y) - f(x)|\, dy \\
&\le 2t\left(\sum_{j=0}^\infty 2^{-(\lambda-\beta)j}\right) M_\lambda^\# f(x),
\end{aligned} \tag{3.4}$$

which proves (10.a). The estimate (10.b) follows from (10.a) by taking supremum for $t < 1$. To show (10.c) we invoke (3.2), and the L^2 boundedness of the Hardy–Littlewood dyadic maximal operator. □

The next lemma gives the pointwise convergence of $S_t^N g(x)$ for every $x \in \mathbb{R}^+$ in a dense subspace of B_2^λ.

Lemma 11. *Let g be a Lipschitz function defined on $\mathbb{R}^+$. Then*

$$S_t^N g(x) = \sum_{j=0}^N \sum_{k\in\mathbb{Z}^+} e^{it2^{j\beta}} \left\langle g, h_k^j\right\rangle h_k^j(x)$$

converges when $N \to \infty$, for every $x \in \mathbb{R}^+$ and every $t \ge 0$.

Proof. Fix $t \ge 0$ and $x \in \mathbb{R}^+$. We shall prove that $(S_t^N g(x) : N = 1, 2, \ldots)$ is a Cauchy sequence of complex numbers. In fact, for $1 \le M \le N$,

$$\begin{aligned}
\left|S_t^N g(x) - S_t^M g(x)\right| &= \left|\sum_{j=M+1}^N \sum_{k\in\mathbb{Z}^+} e^{it2^{j\beta}} \left\langle g, h_k^j\right\rangle h_k^j(x)\right| \\
&= \left|\sum_{j=M+1}^N \sum_{k\in\mathbb{Z}^+} e^{it2^{j\beta}} \left(\int_{\mathbb{R}^+} [g(y) - g(x)] h_k^j(y)dy\right) h_k^j(x)\right| \\
&\le \sum_{j=M+1}^N \sum_{k\in\mathbb{Z}^+} \left\|g'\right\|_\infty 2^j \int_{I_k^j} |x-y|\, dy\, \mathcal{X}_{I_k^j}(x) \\
&= \left\|g'\right\|_\infty \sum_{j=M+1}^N 2^j \int_{I^j_{k(j,x)}} |x-y|\, dy \le \left\|g'\right\|_\infty \sum_{j=M+1}^N 2^{-j}. \quad \square
\end{aligned}$$

Proof of Theorem 9. Proof of (9.a). From Theorem 6 we see that for each $t > 0$, $u(t) \in B^{\lambda}_{2,dy}$, since $u^0 \in B^{\lambda}_2 \subset B^{\lambda}_{2,dy}$. Moreover, for $t, s \geq 0$,

$$\|u(t) - u(s)\|_{B^{\lambda}_{2,dy}} = \left\| \sum_{I \in \mathfrak{D}^+} \left(e^{it|I|^{-\beta}} - e^{is|I|^{-\beta}} \right) \langle u^0, h_I \rangle h_I \right\|_{B^{\lambda}_{2,dy}}$$

$$= \sum_{I \in \mathfrak{D}^+} \left| e^{it|I|^{-\beta}} - e^{is|I|^{-\beta}} \right|^2 \left| \langle u^0, h_I \rangle \right|^2 + \sum_{I \in \mathfrak{D}^+} \left| e^{it|I|^{-\beta}} - e^{is|I|^{-\beta}} \right|^2 \frac{\langle u^0, h_I \rangle^2}{|I|^{2\lambda}}$$

which tends to zero if $s \to t$. □

Proof of (9.b). Let us prove that the formal derivative of $u(t)$ is actually the derivative in the sense of $B^{\lambda-\beta}_{2,dy}$. In fact, for $t > 0$ and h such that $t + h > 0$

$$\left\| \frac{u(t+h) - u(t)}{h} - i \sum_{I \in \mathfrak{D}^+} e^{it|I|^{-\beta}} |I|^{-\beta} \langle u^0, h_I \rangle h_I \right\|^2_{B^{\lambda-\beta}_{2,dy}} = \left\| \sum_{I \in \mathfrak{D}^+} e^{it|I|^{-\beta}} \left[\frac{e^{ih|I|^{-\beta}} - 1}{h} - i\,|I|^{-\beta} \right] \langle u^0, h_I \rangle h_I \right\|^2_{B^{\lambda-\beta}_{2,dy}}$$

$$\leq c \left\{ \left\| \sum_{I \in \mathfrak{D}^+} e^{it|I|^{-\beta}} \left[\frac{e^{ih|I|^{-\beta}} - 1}{h} - i\,|I|^{-\beta} \right] \langle u^0, h_I \rangle h_I \right\|^2_{L^2} + \sum_{I \in \mathfrak{D}^+} \left| \frac{e^{ih|I|^{-\beta}} - 1}{h} - i\,|I|^{-\beta} \right|^2 \frac{\left|\langle u^0, h_I \rangle\right|^2}{|I|^{2(\lambda-\beta)}} \right\}$$

$$\leq c \sum_{I \in \mathfrak{D}^+} |I|^{2\beta} \left| \frac{e^{ih|I|^{-\beta}} - 1}{h} - i\,|I|^{-\beta} \right|^2 \frac{\left|\langle u^0, h_I \rangle\right|^2}{|I|^{2\lambda}}.$$

Since, from Theorem 6, $\sum_{I \in \mathfrak{D}^+} \frac{\left|\langle u^0, h_I \rangle\right|^2}{|I|^{2\lambda}} < \infty$, and $|I|^{2\beta} \left| \frac{e^{ih|I|^{-\beta}} - 1}{h} - i\,|I|^{-\beta} \right|^2 = \left| \frac{e^{ih|I|^{-\beta}} - 1}{|I|^{-\beta} h} - i \right|^2 \to 0$ as $h \to 0$ for each $I \in \mathfrak{D}^+$, we obtain the result.

On the other hand since $u(t) \in B^{\lambda}_{2,dy}$ and since $\lambda > \beta$, $D^{\beta} u(t)$ is well defined and it is given by

$$D^{\beta} u(t) = D^{\beta} \left(\sum_{I \in \mathfrak{D}^+} e^{it|I|^{-\beta}} \langle u^0, h_I \rangle h_I \right) = \sum_{I \in \mathfrak{D}^+} e^{it|I|^{-\beta}} |I|^{-\beta} \langle u^0, h_I \rangle h_I = -i \frac{du}{dt}.$$

Hence $u(t)$ is a solution of the nonlocal equation and (9.b) is proved. □

Proof of (9.c). The boundedness properties of S^*_t and S^* and the pointwise convergence on a dense subset of B^{λ}_2 allow us to use standard arguments for the a.e. pointwise convergence of $S^N_t u^0$ for general $u^0 \in B^{\lambda}_2$. We shall prove that the set Z of all points x in $\mathbb{R}^+$ such that for some $t \in (0, 1)$

$$\overline{L_t}(x) := \inf_N \sup_{n,m \geq N} \left| S^n_t u^0(x) - S^m_t u^0(x) \right| > 0$$

has measure zero. It is enough to show that for each $\varepsilon > 0$, the Lebesgue measure of the set $\{x \in \mathbb{R}^+ : \overline{L_t}(x) > \varepsilon \text{ for some } t \in (0, 1)\}$ vanishes. Since, for any Lipschitz function v defined on $\mathbb{R}^+$ and every $t \in (0, 1)$,

$$\left| S^n_t u^0(x) - S^m_t u^0(x) \right| \leq \left| S^n_t (u^0 - v)(x) \right| + \left| S^n_t v(x) - S^m_t v(x) \right| + \left| S^m_t (v - u^0)(x) \right|,$$

from Lemma 11, we have $\overline{L_t}(x) \leq 2S^*(u^0 - v)(x)$. So that, from (10.c) we obtain

$$\left| \left\{ x \in \mathbb{R}^+ : \overline{L_t}(x) > \varepsilon \text{ for some } t \in (0, 1) \right\} \right| \leq \left| \left\{ x \in \mathbb{R}^+ : S^*(u^0 - v)(x) > \frac{\varepsilon}{2} \right\} \right|$$

$$\leq \frac{4}{\varepsilon^2} \left\| S^*(u^0 - v) \right\|^2_{L^2} \leq \frac{4(AC+2)^2}{\varepsilon^2} \left\| u^0 - v \right\|^2_{B^{\lambda}_2}.$$

Since v is an arbitrary Lipschitz function in $\mathbb{R}^+$ we get that $|Z| = 0$. Hence for every $t \in [0, 1)$ and every $x \notin Z$, $(S^n_t u^0(x) : n = 1, 2, \ldots)$ is a Cauchy sequence which must converge to its L^2 limit, i.e. $u(t)(x)$ for $x \notin Z$ and $t \in [0, 1)$. □

Proof of (9.d). For $x \notin Z$, taking the limit as $N \to \infty$ in (3.4) we get the maximal estimate

$$\sup_{t \in (0,1)} \frac{\left| u(t)(x) - u^0(x) \right|}{t} \leq 2 \frac{2^{\lambda-\beta}}{1 - 2^{-(\lambda-\beta)}} M^{\#}_{\lambda} u^0(x).$$

Since $M^{\#}_{\lambda} u^0$ belongs to L^2, the left hand side is finite almost everywhere, hence $u(t)(x) \to u^0(x)$ as $t \to 0$ almost everywhere. □

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